Schemes of rational curves on Del Pezzo surfaces

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Abstract

The main objects of study in this thesis are schemes parametrizing morphisms from the projective line to projective varieties.

The local properties of these schemes are well understood and a thorough treatment can be found in [Kol96]. Moreover, they behave nicely with respect to morphisms, *i.e.*, if $X \to Y$ is a morphism, then there is a natural morphism $\mathrm{Mor}(\mathbb{P}^1_k, X) \to \mathrm{Mor}(\mathbb{P}^1_k, Y)$. The first part of this thesis is dedicated to understand properties obtained by this behaviour, for instance we see this association preserves open an closed immersions of schemes.

Furthermore, if X is a projective variety, then there is a natural partition of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ on closed subschemes in terms of the degrees of the morphisms. In the second part of this thesis, we use these properties of the schemes of morphisms to find a natural partition of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ when X is the blow-up of projective spaces at finitely many points. This reflects the intersection of rational curves on X with the exceptional divisors of the blow-up, and it is different from the former partition in terms of degrees. In particular, we can use it to refine the usual partition. We fully characterize this refinement on the case that X is a Del Pezzo surface obtained by blowing up $\mathbb{P}^2_k$ at up to eight points in general position. For a Del Pezzo surface obtained by the blow-up $\sigma : X \to \mathbb{P}^2_k$, we use this to characterize irreducible components of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ that parametrizes rational curves which are resolutions of singularities of plane curves which are singular at the blown-up points of σ. We also compute their dimension. When X is the classical example of a cubic surface in $\mathbb{P}^3_k$, we provide a complete list of them.

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Contents

Introduction

Historical Background

The idea of a parameter space has been present in algebraic geometry for a very long time. In projective geometry, even the most basic definition, that of a projective space, is an example of this, *i.e*, the points of a projective space $\mathbb{P}^n_k$ over a field k are the lines through the origin of a vector space of dimension $n+1$ over k.

Other classical examples are given by the set of all linear spaces of dimension m in the projective space $\mathbb{P}^n_k$, which have the structure of varieties. These varieties are known as Grassmannians and by definition, they parametrize all linear spaces of dimension m in $\mathbb{P}^n_k$.

Yet another example is the group $\mathrm{PGL}_{n+1}(k)$ of $(n+1) \times (n+1)$ invertible matrices with entries in the field k. Elements of this group correspond to the automorphisms of the projective space $\mathbb{P}^n_k$ and it also has a natural structure of a variety *i.e.*, it parametrizes automorphisms of $\mathbb{P}^n_k$.

Parameter spaces arise very naturally, as shown in the examples above, which appear by considering basic definitions on the projective space. They are also very useful: the idea of using a space parametrizing objects to solve concrete geometrical questions was already used by the end of the nineteenth century. For instance, Schubert [Sch79] describes many enumerative problems on algebraic geometry in terms of subspaces of Grassmannians. However, even if the existence and usefulness of spaces parametrizing objects was widely known, their construction would be *ad hoc* and there could be more than one way of constructing a space parametrizing the same objects. It was not until much later that parameter spaces were canonically defined. As a consequence, any two constructions of a given parameter space need to yield spaces which are isomorphic.

This was achieved by the work of Grothendieck and his students in the sixties by heavily using the language of categories and functors as a foundational approach to algebraic geometry. According to Grothendieck, one of the key ideas of this approach is that "the functor of points should be taken as the most fundamental definition of a scheme" (cf. [Law03]). Using this idea to define a parameter space, the starting point is a functor $\mathcal{F}$ from the category of varieties (or schemes) to sets. Then X will be a parameter space (parametrizing the sets on the essential image of the functor $\mathcal{F}$) if its functor of points is isomorphic to $\mathcal{F}$. The precise way to state this is to say that $\mathcal{F}$ is *representable* by the scheme X. Grothendieck [Gro60] succeeded in proving the representability of very general functors, such as the Hilbert functor, which would encompass the construction of many parameter spaces.

The main space studied in this thesis parametrizes morphisms from the projective line to a given variety X over an algebraically closed field k. However, these spaces exist in more generality: Grothendieck [Gro60, §4.c] proved that there is a scheme parametrizing morphisms of schemes, say from Y to X over a base scheme S, under natural assumptions on Y, X and S. We denote this scheme by $\operatorname{Mor}_S(Y, X)$.

In general, the schemes $\operatorname{Mor}_S(Y, X)$ have infinitely many components and can have many singularities, which makes understanding their global behaviour challenging. However, much is known about their local behaviour. In fact, Grothendieck [Gro60, §5] has also characterized their tangent spaces. Moreover, Mori [Mor79] gives a lower bound for the local dimension of this scheme at a point.

As remarked before, parameter spaces are useful to solve many kinds of different problems. For instance, when Y is a curve over $S = \operatorname{Spec} k$, Mori [Mor79] uses the schemes $\operatorname{Mor}_S(Y, X)$ to introduce a technique of deforming curves on a variety to produce rational curves on it (which is known today as "bend-and-break"). He used this to prove his famous characterization of the projective spaces: a nonsingular projective variety with ample tangent bundle is isomorphic to a projective space. This was known then as Hartshorne's conjecture.

After Mori's breakthrough, the schemes $\operatorname{Mor}_S(Y, X)$ have been a prolific tool for understanding the geometry of curves on X and in addition to that were also essential on birational classification of varieties. In particular, they are used in [KMM92] to prove that Fano varieties are rationally connected. An overview of some of the methods used in the aforementioned papers can be found the books [Kol96] and [Deb13].

When $S = \operatorname{Spec} k$, where k is a field, and $Y = \mathbb{P}^1_k$, the scheme $\operatorname{Mor}(\mathbb{P}^1_k, X) := \operatorname{Mor}_{\operatorname{Spec} k}(\mathbb{P}^1_k, X)$ parametrizes morphisms whose images are rational curves on X. As mentioned before, in most cases $\operatorname{Mor}(\mathbb{P}^1_k, X)$ has countably many irreducible components. However, if X is projective, then there exists a natural partition of $\operatorname{Mor}(\mathbb{P}^1_k, X)$ by degrees of the polynomials defining the morphisms from $\mathbb{P}^1_k$ to X, and we define $\operatorname{Mor}_e(\mathbb{P}^1_k, X)$ to be the subscheme parametrizing morphisms whose defining polynomials have degree e.

There has been considerable progress in finding irreducible components of schemes parametrizing curves of a given degree e and computing their dimension for certain varieties X, especially for hypersurfaces in the projective space, see for instance [JS04; HRS04; HRS05; HS05]. These papers focus on the spaces parametrizing *smooth* rational curves of degree e. However, these spaces are not exactly the same as the schemes $\mathrm{Mor}_e(\mathbb{P}^1_k, X)$. The latter ones parametrize morphisms, and in particular, they also parametrize morphisms whose images are singular curves. Moreover, in the aforementioned papers varieties are defined over the field of complex numbers $\mathbb{C}$ and their results are often deduced from compactifications of the parameter spaces which hold only over characteristic zero.

On the other hand, each $\mathrm{Mor}_e(\mathbb{P}^1_k, X)$ can also have subschemes parametrizing rational curves with specific properties. Other authors have focused their attention on determining the dimension of subschemes parametrizing rational curves with a specified splitting type of their normal bundles, see for example [EV82; CR18; CR19].

The first results of this thesis follow a different strategy than all of the aforementioned papers. We do not use compactifications of the schemes $\mathrm{Mor}_e(\mathbb{P}^1_k, X)$ and do not use directly computations of the normal bundles of curves. Instead we were motivated by the following: let $Y \to X$ be a morphism and suppose we know the dimensions of components $\mathrm{Mor}(\mathbb{P}^1_k, X)$, can we use the induced morphism $\mathrm{Mor}(\mathbb{P}^1_k, Y) \to \mathrm{Mor}(\mathbb{P}^1_k, X)$ to find components of $\mathrm{Mor}(\mathbb{P}^1_k, Y)$ and deduce their dimension?

Of course, when the morphism $Y \to X$ is arbitrary, there is little chance we can find reasonable geometric interpretation for the induced morphism. However, this motivates a follow-up question: under which conditions on $Y \to X$ can we obtain meaningful geometric properties of $\mathrm{Mor}(\mathbb{P}^1_k, Y) \to \mathrm{Mor}(\mathbb{P}^1_k, X)$? To the best of the author's knowledge these questions have not been explored in the literature.

Our first investigation along these lines is with respect to open and closed immersions of schemes. We find that when W is an open (resp. closed) subscheme of Y, then $\mathrm{Mor}(\mathbb{P}^1_k, W)$ is an open (resp. closed) subscheme of $\mathrm{Mor}(\mathbb{P}^1_k, Y)$.

We also apply this idea to the the blow-up of finitely many points in $\mathbb{P}^n_k$ denoted $\sigma : X \to \mathbb{P}^n_k$. We already find interesting geometric behaviour for $\mathrm{Mor}(\mathbb{P}^1_k, X)$ in this case. We then specialize to $n = 2$ and when σ is the blow-up of up to eight points in general position. We find that the previous behaviour connects naturally with many results on rational curves on Del Pezzo surfaces, linear systems and resolutions of singularities of rational curves on the projective plane and allows us to find a collection of irreducible components in $\mathrm{Mor}(\mathbb{P}^1_k, X)$ and compute their dimension.

Main results and strategies

We present the main results contained in this thesis and give brief descriptions of the strategies for their proofs. Most material of Chapter 1 is known. However, we prove a proposition that, to the best of the author's knowledge, has no proof in the literature:

1.5.5 Proposition. *Let S be a noetherian scheme and suppose the scheme $\mathrm{Mor}_S(X,Y)$ exists. Let $i : Z \hookrightarrow Y$ and $j : U \hookrightarrow Y$ be a closed and an open immersion of schemes over S respectively. Then the canonical morphisms*

$$\mathrm{Mor}_S(X,i) : \mathrm{Mor}_S(X,Z) \to \mathrm{Mor}_S(X,Y),$$
$$\mathrm{Mor}_S(X,j) : \mathrm{Mor}_S(X,U) \to \mathrm{Mor}_S(X,Y)$$

are also a closed and an open immersion respectively.

Strategy of proof. First of all, we notice that $\mathrm{Mor}_S(X,Y)$ is an open subscheme of $\mathrm{Hilb}_S(X \times Y)$. Then we prove that for any open or closed immersion $W \hookrightarrow Y$ we have a morphism

$$\mathrm{Hilb}_S(X \times_S W) \to \mathrm{Hilb}_S(X \times_S Y)$$

and that the scheme $\mathrm{Mor}_S(X,W)$ is isomorphic to the fibered product

$$\mathrm{Mor}_S(X,Y) \times_{\mathrm{Hilb}_S(X \times_S Y)} \mathrm{Hilb}_S(X \times_S W).$$

It suffices then to prove that $\mathrm{Hilb}(X \times_S W) \to \mathrm{Hilb}(X \times_S Y)$ is an open (resp. closed) immersion. ◇

On Chapter 2, we explore the behaviour of schemes of morphisms on blow-ups and the first result on this direction is the following.

2.2.4 Proposition. *Let X be a projective scheme over an algebraically closed field k, Z be a closed subscheme of X and $\sigma : \mathrm{Bl}_Z(X) \to X$ be the blow-up of X along Z. Let*

$$\sigma_M : \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \to \mathrm{Mor}(\mathbb{P}^1_k, X)$$

be the induced morphism and let $N := \mathrm{Mor}(\mathbb{P}^1_k, X) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, Z)$ be the open subscheme parametrizing rational curves intersecting $X \smallsetminus Z$ and $N' := \sigma_M^{-1}(N)$. Then the restriction

$$\sigma_M|_{N'} : N' \to N$$

is locally quasi-finite. More specifically, it is a bijection on k-points.

Strategy of proof. We notice that any morphism $f : \mathbb{P}^1_k \to X$ corresponding to a point in N gives rise to a rational map $g : \mathbb{P}^1_k \dashrightarrow \mathrm{Bl}_Z(X)$ such that the diagram

$$\begin{tikzcd} & \mathrm{Bl}_Z(X) \arrow[d, "\sigma"] \\ \mathbb{P}^1_k \arrow[ur, dashed, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

is commutative. We notice that it g is actually a morphism and is unique, yielding the desired bijection. $\diamond$

Next, we define for each morphism $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ and point $p \in \mathbb{P}^n_k$ the parametric multiplicity $m_p(f)$. This parametric multiplicity coincides with the multiplicity of the scheme theoretic image of f when it is birational to its image, see Definition 2.2.9. We use this definition and we combine the functorial behaviour given in Proposition 1.5.5, the description of the schemes $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k)$ and properties of blow-ups to prove the following.

2.2.11 Theorem. *Let k be an algebraically closed field, $\{p_1, \dots, p_r\} \subset \mathbb{P}^n_k$ be a finite collection of points in a projective space. Let $\sigma : X \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ along $\{p_1, \dots, p_r\}$ with exceptional divisor E. Let $\mathbf{m} := (m_1, \dots, m_r)$ denote an r-tuple of non-negative integers. Then we have the partition in closed subschemes*

$$\mathrm{Mor}(\mathbb{P}^1_k, X) \cong \mathrm{Mor}_0(\mathbb{P}^1_k, X) \amalg \left(\coprod_{d>0} \coprod_{m_i \le d} M_{d,\mathbf{m}} \right) \amalg \mathrm{Mor}_{>0}(\mathbb{P}^1_k, E),$$

where

- $\mathrm{Mor}_0(\mathbb{P}^1_k, X)$ *parametrizes constant morphisms;*
- *a k-point $[g]$ belongs to $M_{d,\mathbf{m}}$ if and only if*

$$\deg(\sigma \circ g) = d \quad \textit{and}$$
$$m_{p_i}(\sigma \circ g) = m_i \quad \textit{for} \quad 1 \le i \le r;$$

- $\mathrm{Mor}_{>0}(\mathbb{P}^1_k, E) \cong \coprod_{i=1}^r \coprod_{e>0} \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)$.

In particular, for $\mathbf{0} := (0, \dots, 0)$ and each positive integer d, the subschemes $M_{d,\mathbf{0}}$ are nonsingular of dimension $nd + d + n$ and parametrize curves which do not intersect E.

Strategy of proof. We recall that σ induces a morphism

$$\sigma_M : \mathrm{Mor}(\mathbb{P}^1_k, X) \to \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k).$$

We use an embedding

$$\iota : X \hookrightarrow \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)$$

to construct an auxiliary morphism

$$\iota_M : \mathrm{Mor}(\mathbb{P}^1_k, X) \to \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)).$$

We find suitable partitions on the targets of σ_M and ι_M to define the partition above by intersecting preimages of irreducible components of the targets. $\diamond$

Recall that if we have an embedding $\iota : X \hookrightarrow \mathbb{P}^N_k$, then there is a natural partition

$$\mathrm{Mor}(\mathbb{P}^1_k, X) = \coprod_{e \geq \mathbb{N}} \mathrm{Mor}_e(\mathbb{P}^1_k, X),$$

where $\mathrm{Mor}_e(\mathbb{P}^1_k, X)$ is the closed subscheme of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ parametrizing morphisms of degree e. Thus, once we fix the embedding ι, Theorem 2.2.11 yields a refinement of this partition. In particular, when $n = 2$ and σ is the blow-up of at most eight points in general position, then X is a Del Pezzo surface and we prove the following.

2.4.1 Corollary. *Let k be an algebraically closed field and let $r \leq 8$ be a positive integer. Let $\sigma : X \to \mathbb{P}^2_k$ be the blow-up of $\mathbb{P}^2_k$ in r points in general position. Let $M_{d,\mathbf{m}}$ denote the closed subschemes defined in Theorem 2.2.11 and $\mathrm{Mor}_e(\mathbb{P}^1_k, E_i)$ denote the scheme parametrizing morphisms of degree e to the exceptional line E_i. Let a be a positive integer such that $-aK_X$ is very ample and let*

$$\iota_{-aK_X} : X \hookrightarrow \mathbb{P}^N_k$$

be the corresponding embedding. Then for each integer $e > 0$, the scheme parametrizing morphisms of degree e from $\mathbb{P}^1_k$ to X with respect to ι_{-aK_X} is given by

$$\mathrm{Mor}_e(\mathbb{P}^1_k, X) = \left(\coprod_{e = a(3d - |\mathbf{m}|)} M_{d,\mathbf{m}} \right) \amalg \left(\coprod_{i=1}^{r} \mathrm{Mor}_e(\mathbb{P}^1_k, E_i) \right).$$

Strategy of proof. Once we fix the embedding $\iota_{-aK_X} : X \hookrightarrow \mathbb{P}^N_k$, we notice that for each pair d and $\mathbf{m}$ and each point in $M_{d,\mathbf{m}}$ we will have a corresponding plane curve. We can compute the degree of the strict transform of this plane curve under σ by computing the intersection number with very ample line bundle $-aK_X$. This degree will correspond to a unique e such that $M_{d,\mathbf{m}} \subset \mathrm{Mor}_e(\mathbb{P}^1_k, X)$. ◇

Finally, we provide further characterization of the closed subschemes $M_{d,\mathbf{m}}$ in Corollary 2.4.1. We prove that if they contain points $[f] \in M_{d,\mathbf{m}}$ corresponding to rational curves that are resolutions of singularities of rational plane curves, then we can compute the dimension of the irreducible components $M^0_{d,\mathbf{m}} \subset M_{d,\mathbf{m}}$ containing $[f]$. In order to do so, we define in Section 2.3 a morphism

$$\varXi_d : \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \to \mathbb{P}(H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d)))$$

for each d. We prove the following.

2.4.8 Theorem. *Let k be an algebraically closed field of characteristic 0. Let*

$$\sigma : X \to \mathbb{P}^2_k$$

be the blow-up of $\mathbb{P}^2_k$ at r points. Suppose there exists a rational curve C of degree d in $\mathbb{P}^2_k$ passing through these points with multiplicities $\mathbf{m} = (m_1, \ldots, m_r)$ such that its strict transform under σ is nonsingular. Consider $M_{d,\mathbf{m}}$ to be the closed subscheme of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ defined in Theorem 2.2.11. Then, there exists an irreducible component $M^0_{d,\mathbf{m}} \subset M_{d,\mathbf{m}}$ such that

$$\dim M^0_{d,\mathbf{m}} = \max\left\{d^2 + 1 - \textstyle\sum_{i=1}^r m_i^2, 0\right\} + 3.$$

Moreover, a general point of $M^0_{d,\mathbf{m}}$ corresponds to a generically one-to-one morphism.

Strategy of proof. We prove that the image of the composition $\sigma_M|_{M_{d,\mathbf{m}}} \circ \varXi_d$ is a locally closed subscheme of dimension

$$\max\left\{d^2 + 1 - \textstyle\sum_{i=1}^r m_i^2, 0\right\}.$$

We use a variant of the theorem of dimension of fibers on this composition to find the desired irreducible component with the aforementioned properties. ◇

Structure of the thesis

Chapter 1: Hilbert schemes and schemes of morphisms. This chapter is meant to be an introduction to the parameter spaces $\mathrm{Mor}_S(X, Y)$ and to close a gap in the literature concerning the properties that can be deduced from its functor of points.

First we recall the definition of Hilbert schemes as schemes representing the Hilbert functor. We will review some literature about them and also highlight properties that can be deduced from its functor of points.

We then recall the definition of schemes parametrizing morphisms between two schemes over a base. We prove that it is an open subscheme on a Hilbert scheme and deduce the aforementioned properties, in particular, we prove Lemma 1.5.5.

Chapter 2: Schemes of rational curves. We start the chapter by giving a well known heuristic description of the scheme $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k)$ including its partition in terms of the degrees of the morphisms it parametrizes. We use this partition to prove Theorem 2.2.11. We specialize to $n = 2$ and fix the classical embeddings of Del Pezzo surfaces in $\mathbb{P}^N$ to obtain the refined partition of Corollary 2.4.1. Finally, we define the regular morphisms Ξ_d used in the proof of Theorem 2.4.8 and use the results of [DM12] on rational linear systems to describe the image of the composition

$$M_{d,\mathbf{m}} \to \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \to \mathbb{P}(H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d))).$$

We also prove that Ξ_d is $\mathrm{Aut}(\mathbb{P}^1_k)$-invariant and that all its fibers are irreducible and have dimension 3.

To conclude the chapter, we compare Theorem 2.4.8 with the classification of Gimigliano, Harbourne and Idà [GHI13] of rational curves whose singularities are resolved by blowing-up points in general position. This classification allows us to find all the possible components $M^0_{d,\mathbf{m}}$ given in Theorem 2.4.8 when $r \leq 7$. As an example, we provide a complete list of these components on a smooth cubic surface in Example 2.4.10.

Appendix A: Categorical remarks. We summarise some basic notions in category theory that are used throughout the thesis and in particular in Chapter 1. We recall the definitions of representability, base change of functors and open and closed subfuntors. We also recall a useful criterion to determine when a subfunctor is open or closed.

Appendix B: Linear systems on surfaces. We recall some basic definitions and language of linear systems on projective surfaces over an algebraically closed field. This is mainly used to define the morphism Ξ_d appearing in Chapter 2. In particular, we recall that for each invertible sheaf $\mathcal{L}$ on a projective surface, the complete linear system corresponding to $\mathcal{L}$ is the scheme representing a functor $\mathcal{L}in\mathcal{S}ys_{\mathcal{L}}$. This functor takes each algebraic scheme S over k to the base change of $\mathcal{L}$ tensored with the base change of an invertible sheaf of S.

A note on the required background

Our intention was to make the thesis followable by a graduate student in algebraic geometry with some familiarity with three topics:

- some theory of schemes: sheaves of modules, existence of fibered product of schemes, generic points, reduced schemes. This material is covered, for instance, in Chapter II of [Har77] or Chapters 3-6 of [Vak17];

- classical geometric constructions: for instance, regular morphisms to projective spaces, blow-ups, linear systems on surfaces and geometry of Del Pezzo surfaces. This material can be found in the first chapters of [Sha13a] and Chapter V of [Har77];

- basic definitions on category theory: such as functors, natural transformations, limits and colimits. This material is covered, for instance, in Chapter 1 of [Vak17].

We try our best to provide references in the books by Hartshorne [Har77], Vakil [Vak17] or the Stacks project [Stacks] to well known results in algebraic geometry that cannot be deduced directly from our exposition.

Conventions

Ground field. We use k to denote the ground field which is assumed to be algebraically closed unless explicitly stated otherwise.

Schemes and functors over the ground field. Throughout the text we employ notation which usually emphasizes the dependency on a base scheme S such as $\mathrm{Hilb}_S(X)$, $\mathrm{Mor}_S(X, Y)$ and $X \times_S Y$. When $S = \operatorname{Spec} k$ is the spectrum of the ground field we will simplify this notation by omitting S. For instance, in the examples just mentioned we will use the notation $\mathrm{Hilb}(X)$, $\mathrm{Mor}(X, Y)$ and $X \times Y$.

Immersions. An *open immersion* or *open embedding* is a morphism of schemes $j : U \to X$ if it is a homeomorphism between U and an open subset of X and the maps of sheaves $j^{-1}\mathcal{O}_X \to \mathcal{O}_U$ is an isomorphism. A morphism of schemes $i : Z \to X$ is a *closed immersion* or *closed embedding* if i is a homeomorphism between a closed subset of X and the induced morphism of sheaves $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. We say a morphism $\iota : W \to X$ is an *immersion* if it is a composition of open or closed immersions (cf. [Stacks, 01IO]).

Projective and quasi-projective morphisms. There are many notions of projectivity. For this thesis we fix the following one: for any morphism of schemes $f : X \to Y$ over a base S, f is said to be *projective* if there exists a non-negative integer n and a closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_S$. The morphism f is said to be *quasi-projective* if ι is a quasi-compact immersion instead. These are the notions of H-projective and H-quasi-projective morphisms in [Stacks, 01W8] and [Stacks, 01VW] respectively. We denote the categories of projective (resp. quasi-projective) schemes over S, that is projective (resp. quasi-porjective) morphisms of schemes with target S, as $\mathbf{PrSch}/S$ (resp. $\mathbf{QPrSch}/S$).

Images and scheme theoretic images. The *set theoretic image* of a morphism of schemes $f : X \to Y$ is denoted by $\mathrm{im}(f)$ and the *scheme theoretic image* of f is denoted $\overline{\mathrm{im}\,(f)}$, which is defined to be the smallest closed subscheme of Y such that f factors through $\overline{\mathrm{im}\,(f)}$. In most situations of the text the scheme theoretic image $\overline{\mathrm{im}\,(f)}$ and the closure of $\mathrm{im}(f)$ equipped with the induced reduced subscheme structure on Y will coincide.

Scheme theoretic intersection. Suppose X is a scheme and W and Z are open or closed subschemes of X, then the *scheme theoretic intersection* of W and Z, denoted $W \cap Z$, is the fibered product $W \times_X Z$.

Chapter 1

Hilbert schemes and schemes of morphisms

Hilbert schemes are important parameter spaces in algebraic geometry. In this chapter we introduce them and point out some directions on how useful they can actually be. Roughly speaking, if X is a projective flat scheme over a locally noetherian scheme S, the Hilbert scheme $\mathrm{Hilb}_S(X)$ is the scheme parametrizing proper and flat subschemes of X.

This heuristic description already tells us that $\mathrm{Hilb}_S(X)$ is a very large scheme in general. Therefore, it is very natural to expect that we should be able to restrict our attention to subschemes of $\mathrm{Hilb}_S(X)$ parametrizing a more restricted family of subschemes. In fact, the Hilbert scheme of points and Hilbert schemes of curves are examples of these subschemes which received the attention of many authors over the years due to their applications. We highlight some aspects of those.

We also prove properties of $\mathrm{Hilb}_S(X)$ that can be deduced from the functor of points. In a nutshell, we prove that Hilbert schemes behave well under base change, and if $W \hookrightarrow X$ is an open (resp. closed) immersion of schemes, then $\mathrm{Hilb}_S(W)$ exists and we have a natural immersion

$$\mathrm{Hilb}_S(W) \hookrightarrow \mathrm{Hilb}_S(X)$$

which is open (resp. closed).

The second part of this chapter recalls the definition of schemes parametrizing morphisms between two schemes over a base: if X is a projective flat scheme over S and Y is quasi-projective over S, then the scheme $\mathrm{Mor}_S(X, Y)$ parametrizes morphisms $X \to Y$ over S. We will recall that $\mathrm{Mor}_S(X, Y)$ is an open subscheme of a Hilbert scheme $\mathrm{Hilb}_S(X \times_S Y)$.

We also recall that $\mathrm{Mor}_S(X, Y)$ behaves well under base change. Moreover, if X is flat and projective over S we have a functor

$$\mathrm{Mor}_S(X, -) : \mathbf{QPrSch}/S \to \mathbf{Sch}/S.$$

We prove that this functor preserves open and closed immersions. These properties will be useful to provide examples of schemes of morphisms from the projective line to blow-ups of schemes on Chapter 2. These schemes parametrize morphisms whose images are rational curves on a given variety and we will see they differ slightly from a Hilbert scheme of curves.

Many of the proofs in this chapter depend on the functor of points of the schemes above. We naturally use some jargon coming from category theory such as "universal section", "representability" and "open subfunctor". The precise definitions of these terms and the needed statements for the proofs on this chapter (which depend exclusively of category theory) have been gathered in Appendix A for the convenience of the reader.

1.1 Hilbert polynomials

1.1.1 Let X be a proper scheme over a field k and $\mathcal{F}$ a coherent sheaf on X. The *Euler characteristic* of $\mathcal{F}$ is defined as

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}),$$

where $H^i(X, \mathcal{F})$ is the i-th cohomology group of $\mathcal{F}$ over X. Recall that since X is proper, each $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space ([Stacks, 02O6]) and by Grothendieck's Vanishing Theorem [Har77, Theorem III.2.7] only finitely many $H^i(X, \mathcal{F})$ are positive dimensional. It follows that this sum is finite. Moreover, if $\mathcal{L}$ is a line bundle on X, we define the *Hilbert function* of $\mathcal{F}$ with respect to $\mathcal{L}$ as

$$\begin{aligned} H_{\mathcal{L},\mathcal{F}} : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ t &\longmapsto \chi(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes t}). \end{aligned} \tag{1.1}$$

1.1.2 Theorem-Definition. ([FGI+05, p. 109]) Let $\mathcal{F}$ be a coherent sheaf on a finite type scheme X over $\operatorname{Spec} k$. If $t \gg 0$, then the function $H_{\mathcal{L},\mathcal{F}}$ is a polynomial in $\mathbb{Q}[t]$ (see Snapper's Lemma ([FGI+05, Part 5, B.7]). This polynomial is said to be the *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{L}$ and is denoted by $P_{\mathcal{L},\mathcal{F}}(t)$. We will be interested in a few particular cases:

- If $\mathcal{F} = \mathcal{O}_X$ we denote $P_{\mathcal{L},X}(t) := P_{\mathcal{L},\mathcal{O}_X}(t)$ and say it is the *Hilbert polynomial of X with respect to $\mathcal{L}$*;

- if X is projective, let $\mathcal{L}$ be a very ample line bundle inducing an immersion

$\iota : X \hookrightarrow \mathbb{P}^n_k$, that is, $\mathcal{L} = \mathcal{O}_X(1) = \iota^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. Then for any coherent sheaf $\mathcal{F}$ we say that

$$P_{\mathcal{F}}(t) := P_{\mathcal{O}_X(1),\mathcal{F}}(t)$$

is the *Hilbert polynomial of* $\mathcal{F}$. By [Vak17, Theorem 18.6.1], we have that

$$P_{\mathcal{F}}(t) = \dim_k H^0(X, \mathcal{F}(t))$$

for $t \gg 0$, where $\mathcal{F}(t) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$.

- if X is projective, we say

$$P_X(t) := P_{\mathcal{O}_X(1),\mathcal{O}_X}(t)$$

is the *Hilbert polynomial of* X. By the previous item we have that

$$P_X(t) = \dim_k H^0(X, \mathcal{O}_X(t))$$

for $t \gg 0$.

1.1.3 Example. The example of a Hilbert polynomial that is the simplest to compute is that of the projective space. Indeed, we have that

$$\dim_k H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(t)) \cong \dim_k k[x_0, \dots, x_n]_t,$$

where $k[x_0, \dots, x_n]_t$ is just the k-vector space of polynomials of degree t generated by the monomials $\left\{x_0^{t_0} \cdots x_n^{t_n}\right\}_{t_0+\cdots+t_n=t}$. Therefore $P_{\mathbb{P}^n_k}(t) = \binom{n+t}{n}$ for $t \gg 0$.

1.1.4 Hilbert polynomials and invariants. For any projective variety X of dimension m, its Hilbert polynomial $P_X(t)$ is known to be a source of many invariants. The main ones we will point out are the following.

1. The dimension m coincides with the degree of $P_X(t)$;

2. the *degree* of X is defined to be the leading coefficient of $P_X(t)$ multiplied by $m!$. This notion coincides with the number of points of intersection of X with a general linear subspace of dimension $n - m$ in $\mathbb{P}^n_k$ counted with appropriate multiplicities, see [Vak17, Exercise 18.6.N];

3. the *virtual arithmetic genus* of X is defined by

$$p_a(X) := (-1)^m(P_X(0) - 1).$$

This is an invariant proposed by Severi for any projective variety (not necessarily non-singular) and coincides with the usual accepted notion of arithmetic genus when X is an irreducible nonsingular curve or surface, see discussion in [Pop16, Chapter 37] and [Bal56, Chapter V].

1.1.5 Remark. Notice that, from 1.1.4, if C is an irreducible projective curve over k, then its Hilbert polynomial is linear given by

$$P_C(t) = dt - p_a(C) + 1$$

for some $d > 0$. If C is non-singular, the arithmetic genus $p_a(C)$ coincides with the *geometric genus* $p_g(C) := \dim_k H^0(C, \Omega_{C/k})$. If C is arbitrary, the arithmetic genus is a non-negative integer.

1.1.6 In the next few paragraphs we will compute the Hilbert polynomial of a curve which is the image of a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$. These curves will be the main curves studied on Chapter 2. Before that we recall a few important definitions.

1.1.7 Definition. A projective curve C over k is said to be *rational* if it is birational to $\mathbb{P}^1_k$, that is, there is an open subset of C isomorphic to an open subset of $\mathbb{P}^1_k$.

1.1.8 Definition. A scheme X is said to be *normal* if for any point $p \in X$ the local ring $\mathcal{O}_{X,p}$ is integrally closed on its field of fractions. For any integral scheme X, its *normalization* consists of a normal scheme X^ν and a dominant integral morphism $\nu : X^\nu \to X$ inducing an isomorphism on the function fields of X and X^ν (see [GW10, Proposition 12.44]).

1.1.9 Remark. We highlight three important properties of normalization:

- for any scheme integral scheme X, its normalization satisfies the following universal property: for any normal scheme X' and morphism $f : X' \to X$ there exists a unique morphism $g : X' \to X^\nu$ such that

$$\begin{tikzcd} & X^\nu \arrow[d, "\nu"] \\ X' \arrow[ur, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

  is commutative, see [GW10, Proposition 12.44];

- the morphism $\nu : X^\nu \to X$ is a birational, see [Stacks, 0BXC];

- if $\dim X \leq 1$, X is normal if and only if X is nonsingular, see [Stacks, 0569] and [Stacks, 0BX2].

1.1.10 Lemma. *A projective curve $C \hookrightarrow \mathbb{P}^n_k$ is rational if and only if it is the scheme theoretic image of a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$.*

Proof. Clearly if $C \hookrightarrow \mathbb{P}^n$ is rational there is a birational map $g : \mathbb{P}^1_k \dashrightarrow C$, since $\mathbb{P}^1_k$ is non-singular, by [Sil09, Chapter II, Proposition 2.1.], we have that g is regular, therefore C is the scheme theoretic image of $f : \mathbb{P}^1_k \xrightarrow{g} C \hookrightarrow \mathbb{P}^n_k$.

Conversely, let $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be a morphism and $C \coloneqq \overline{\mathrm{im}\,(f)}$. Let $\nu : C^\nu \to C$ be its normalization. By definition there exists a morphism $g : \mathbb{P}^1 \to C^\nu$ such that $f = \nu \circ g$. In particular since $\mathbb{P}^1_k$ and C^ν are nonsingular, by [Har77, Theorem II.6.8] the morphism is finite and by [Har77, Example IV.2.5.4] we have the inequality

$$0 = p_g(\mathbb{P}^1_k) \geq p_g(C^\nu)$$

thus the geometric genus of C^ν is zero. It follows from [HS00, Theorem A.4.3.1] that $C^\nu \cong \mathbb{P}^1_k$ and since ν is birational, the curve C is rational. $\square$

1.1.11 Example (Hilbert polynomial and finite morphisms)**.** Let $g : X \to Z$ be a finite flat surjective morphism between integral projective schemes over k. Let $\mathcal{O}_Z(1)$ be the very ample line bundle inducing a closed embedding $Z \hookrightarrow \mathbb{P}^n_k$. We claim that the Hilbert polynomial of Z can be computed by the Hilbert polynomial of $\mathcal{O}_X$ with respect to $g^*\mathcal{O}_Z(1)$.

Since g is finite and flat it is equivalent to say that $g_*\mathcal{O}_X$ is locally free of finite rank on X (see [Stacks, 02KB]). Let m_0 be this rank, we claim that

$$g_*g^*\mathcal{O}_Z(t) \cong \mathcal{O}_Z(m_0 t).$$

Indeed, just notice that if $\mathcal{O}_Z(t)$ corresponds to a Cartier divisor D in Z (see [Har77, Proposition II.6.13]) then $g_*g^*\mathcal{O}_Z(t)$ corresponds to the divisor g_*g^*D, that is, the image of D under the flat pullback and proper pushforward. By [Stacks, 02RH], we have that $g_*g^*D = m_0 D$ and therefore the claim follows.

Furthermore, recall that $(g^*\mathcal{O}_Z(1))^{\otimes t} \cong g^*\mathcal{O}_Z(t)$ and that since g is a finite morphism, it is affine. Then by [Har77, Chapter III, Ex.4.1] and the definition of the Hilbert functions (1.1):

$$\begin{aligned} H_{g^*\mathcal{O}_Z(1),\mathcal{O}_X}(t) &= \chi(X, g^*\mathcal{O}_Z(t)) = \chi(Z, g_*g^*(\mathcal{O}_Z(t))) \\ &= \chi(Z, \mathcal{O}_Z(m_0 t)) = H_{\mathcal{O}_Z(1),\mathcal{O}_Z}(m_0 t) \end{aligned}$$

for all integers t. Therefore, we conclude

$$P_{g^*(\mathcal{O}_Z(1)),X}(t) = P_Z(m_0 t). \tag{1.2}$$

1.1.12 Remark. The rank m_0 in Example 1.1.11 can also be understood in terms of field extensions. By [GW10, Proposition 12.21], for any point $q \in Z$ we have that

$$m_0 = \dim_{\kappa(q)} \Gamma(X_q, \mathcal{O}_{X_q})$$

where X_q denotes the fiber of X over the point q. Let ξ and ζ be the generic points of X and Z respectively. Since we assumed g surjective (which implies in particular g dominant) we have that $X_\zeta \cong \operatorname{Spec} \kappa(\xi)$ (see [Stacks, 0CC1]) and it follows that

$$m_0 = [\kappa(\xi) : \kappa(\zeta)], \tag{1.3}$$

where $[\kappa(\xi) : \kappa(\zeta)]$ denotes the degree of the field extension $\kappa(\xi)/\kappa(\zeta)$ induced by g.

1.1.13 Example (Hilbert polynomials of rational curves)**.** Let

$$f : \mathbb{P}^1_k \to \mathbb{P}^n_k$$

be a non-constant morphism. We want to compute the Hilbert polynomial of the scheme theoretic image $C \coloneqq \overline{\operatorname{im}(f)}$. By [Har77, Theorem II.7.1] the morphism f is uniquely determined by the line bundle $f^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ and $n+1$ global sections on $\Gamma(\mathbb{P}^1_k, f^*\mathcal{O}_{\mathbb{P}^n_k}(1))$ with no zeros in common. It is well known that the only line bundles on the projective line are precisely the twisting sheaves, hence

$$f^*\mathcal{O}_{\mathbb{P}^n_k}(1) \cong \mathcal{O}_{\mathbb{P}^1_k}(d)$$

for some non-negative integer d (if d was negative $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d)) = 0$).

Notice that f factors as

$$f : \mathbb{P}^1_k \xrightarrow{g} C \xhookrightarrow{\iota} \mathbb{P}^n_k,$$

where $g : \mathbb{P}^1_k \to C$ is a surjective morphism of projective curves. By [Har77, Theorem II.6.8], g is finite and by [Stacks, 0CCK] it is flat. Let ζ be the generic point of C and $m_0 = [k(t) : \kappa(\zeta)]$ be the degree of the field extension induced by g. We denote $\mathcal{O}_C(1) \coloneqq \iota^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. We have that

$$g^*\mathcal{O}_C(1) = f^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(d).$$

Therefore, by (1.2) we have

$$P_C(m_0 t) = P_{g^*(\mathcal{O}_C(1)),\mathbb{P}^1_k}(t) = P_{\mathcal{O}_{\mathbb{P}^1_k}(d),\mathbb{P}^1_k}(t) = P_{\mathbb{P}^1_k}(dt).$$

Since $P_{\mathbb{P}^1_k}(t) = \dim_k k[u,v]_t = t+1$, we obtain

$$P_C(t) = d_0 t + 1,$$

where $d_0 = d/m_0$. In particular, notice that f is birational onto its image, i.e. g is birational, if and only if $k(t) \cong \kappa(C)$. This is equivalent to say $m_0 = 1$ and $\deg C = d$.

1.1.14 Example (Hilbert polynomial of a graph of curves)**.** Let $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be a morphism.

We compute the Hilbert polynomial of the graph of f with respect to the following line bundle

$$\mathcal{O}_{\mathbb{P}^1_k\times\mathbb{P}^n_k}(1) \coloneqq \mathcal{O}_{\mathbb{P}^1_k}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n_k}(1) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1_k}(1) \otimes_{\mathcal{O}_{\mathbb{P}^1_k\times\mathbb{P}^n_k}} \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^n_k}(1).$$

By [Har77, Ex. II.5.11], the line bundle $\mathcal{O}_{\mathbb{P}^1_k\times\mathbb{P}^n_k}(1)$ induces the Segre embedding

$$\begin{aligned} \alpha : \mathbb{P}^1_k \times \mathbb{P}^n_k &\longrightarrow \mathbb{P}^{2n+1}_k \\ ((u:v),(x_0:\dots:x_n)) &\longmapsto (ux_0 : ux_1 : \ldots : vx_{n-1} : vx_n). \end{aligned}$$

Let $\Gamma_f : \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^n_k$ be the graph morphism and $Z = \overline{\mathrm{im}\,(\Gamma_f)}$. Since $\mathbb{P}^1_k$ is separated, this morphism is a closed immersion. We have the following commutative diagram

$$\begin{tikzcd} \mathbb{P}^1_k \arrow[r, "\Gamma_f"'] \arrow[rr, bend left, "f"] \arrow[dr, equal] & \mathbb{P}^1_k \times \mathbb{P}^n_k \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & \mathbb{P}^n_k \arrow[d] \\ & \mathbb{P}^1_k \arrow[r] & \mathrm{Spec}\, k. \end{tikzcd}$$

Notice that

$$\begin{aligned} \Gamma_f^*(\mathcal{O}_{\mathbb{P}^1_k\times\mathbb{P}^n_k}(t)) &\cong \Gamma_f^* \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1_k}(t) \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \Gamma_f^* \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^n_k}(t) \\ &\cong \mathcal{O}_{\mathbb{P}^1_k}(t) \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} f^*(\mathcal{O}_{\mathbb{P}^n_k}(t)) \end{aligned}$$

for any integer t. Now, we know that $f^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$ is a line bundle on $\mathbb{P}^1_k$, therefore $f^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) \cong \mathcal{O}_{\mathbb{P}^1_k}(d)$ for some non-negative integer d. Thus, $\Gamma_f^*(\mathcal{O}_{\mathbb{P}^1_k\times_k\mathbb{P}^n_k}(t)) \cong$

$\mathcal{O}_{\mathbb{P}^n_k}((d+1)t)$ and we obtain

$$P_Z(t) = P_{\Gamma_f^*(\mathcal{O}_{\mathbb{P}^1_k \times_k \mathbb{P}^n_k}(1)),\mathbb{P}^1_k}(t) = P_{\mathcal{O}_{\mathbb{P}^1_k}(d+1),\mathbb{P}^1_k}(t) = P_{\mathbb{P}^1_k}((d+1)t) = (d+1)t+1.$$

1.1.15 Comment. A first approach to parametrize rational curves on $\mathbb{P}^n_k$ involves classifying them by their Hilbert polynomials. By Remark 1.1.5 this depends on two invariants. Their degree and their arithmetic genus. However this presents an inconvenience: the Hilbert polynomial of a rational curve could coincide with the Hilbert polynomial of a curve of higher geometric genus if this rational curve is singular. If one is interested only in rational curves, one needs to exclude this situation.

Another approach to the same problem consists in regarding a rational curve in $\mathbb{P}^n_k$ as the image of a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$, as we have seen in Lemma 1.1.10. From Example 1.1.13 we know that the Hilbert polynomial of the image f depends on $f^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ and the degree m_0 between the function fields of $\mathbb{P}^1_k$ and the scheme theoretic image of f.

On the other hand on Example 1.1.14 we have seen that the Hilbert polynomial of the image graph $\Gamma_f : \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^n_k$ depends only on $f^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. In fact, we will see later on this chapter that the graphs of morphisms provide a natural way to parametrize morphisms and in Section 2.1 we will see that the line bundles $\mathcal{O}_{\mathbb{P}^1_k}(d)$, with $d \geq 0$, define a natural partition on the parameter spaces of morphisms $\mathbb{P}^1_k \to \mathbb{P}^n_k$.

Notice also that trying to parametrizing rational curves from morphisms has a downside which needs attention: if $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is any automorphism of $\mathbb{P}^1_k$, then $\overline{\mathrm{im}\,(f \circ g)} = \overline{\mathrm{im}\,(f)}$. An approach to get rid of this situation is to take a quotient by algebraic group $\mathrm{Aut}(\mathbb{P}^1_k)$, however this comes at a cost as describing the resulting space becomes more technical. For many applications, for instance the ones described on the introduction, spaces parametrizing morphisms are sufficient.

1.2 Hilbert functors and Hilbert schemes

In this section we recall the definition of the Hilbert functor, its stratification via Hilbert polynomials and recall the result of its representability.

1.2.1 Remark. Recall that there is a difference between a closed immersion $\iota : V \hookrightarrow X$ and a closed subscheme of X. Namely, ι is a *closed immersion* if the underlying topological subspace $\iota(V) \subset X$ is isomorphic to a closed subset and the induced morphism of sheaves $\iota^{\#} : \mathcal{O}_X \to \iota_*\mathcal{O}_V$ is surjective. A *closed subscheme* of X is an equivalence class of closed immersions where $\iota : V \hookrightarrow X$ is equivalent to $\iota' : V' \hookrightarrow X$ if there exists an isomorphism $\alpha : V \xrightarrow{\sim} V'$ such that $\iota = \iota' \circ \alpha$.

However, for every closed immersion ι in a given equivalence class we have that the associated ideal sheaf is unique, see [Stacks, 01QP] for instance. If there is no risk of confusion, we will simply refer to a closed immersion $\iota : V \hookrightarrow X$ as a closed subscheme.

1.2.2 Notation. Let S be a fixed base scheme and X and S' be S-schemes. When there is no risk of confusion we denote the fiber product $X \times_S S'$ by $X_{S'}$.

1.2.3 Definition. Let X be a scheme over a noetherian scheme S and let $\mathbf{Noe}/S$ be the category of locally noetherian schemes over S. We define the *Hilbert functor*

$$\mathcal{H}ilb_S(X) : (\mathbf{Noe}/S)^{op} \to \mathbf{Set}$$

on the objects by

$$S' \mapsto \left\{ \begin{array}{c} V \hookrightarrow X_{S'} \text{ closed subscheme} \\ \text{such that } V \to S' \text{ is flat and proper} \end{array} \right\}$$

In other words, for each section in $\mathcal{H}ilb_S(X)(S')$ there is an equivalence class of flat and proper morphisms $V \to S'$ fitting in the following commutative diagram:

$$\begin{tikzcd} V \arrow[d, hook] \arrow[dd, bend right] & \\ X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

where $V \to S'$ and $V' \to S'$ are equivalent if V and V' define the same subscheme on $X_{S'}$, that is $V \cong V'$.

Moreover, to define the Hilbert functor on morphisms just notice that if $f : S'' \to S'$ is a morphism over S and $V \to S'$ corresponds to a section of

$\mathcal{H}ilb_S(X)(S')$ then we have a base change diagram

$$\begin{tikzcd}
V \times_{S'} S'' \arrow[r] \arrow[d, hook] \arrow[dd, bend right] & V \arrow[d, hook] \arrow[dd, bend left] & \\
X_{S''} \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\
S'' \arrow[r, "f"] & S' \arrow[r] & S,
\end{tikzcd}$$

where all squares are cartesian. Since being a closed embedding, flatness and properness are properties stable under base change this implies that $V \times_{S'} S''$ is a section of $\mathcal{H}ilb_S(X)(S'')$. Hence, the Hilbert functor associates every morphism $f : S'' \to S'$ in $\mathbf{Sch}/S$ to the morphism

$$\begin{aligned} \mathcal{H}ilb_S(X)(f) : \mathcal{H}ilb_S(X)(S') &\longrightarrow \mathcal{H}ilb_S(X)(S'') \\ V &\longmapsto V \times_{S'} S''. \end{aligned} \tag{1.4}$$

1.2.4 Notation. Let X be a projective scheme over an integral noetherian scheme S, *i.e.* X is embedded in a projective space $\mathbb{P}^n_S$ as a closed subscheme. For every point $p \in S$ let $\kappa(p)$ denote the residue field of the point p and consider the morphism

$$\operatorname{Spec} \kappa(p) \to S$$

whose image is the point p. This map induces the pullback diagram

$$\begin{tikzcd}
X_p \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\
\mathbb{P}^n_{\kappa(p)} \arrow[r] \arrow[d] & \mathbb{P}^n_S \arrow[d] \\
\operatorname{Spec} \kappa(p) \arrow[r] & S,
\end{tikzcd}$$

where every square is cartesian.

1.2.5 Theorem. *[Har77, Theorem III.9.9.] Let S be an irreducible locally noetherian scheme and let X be a projective scheme over S. For each point $p \in S$, let $P_{X_p}(t)$ be the Hilbert polynomial of X_p considered as a closed subscheme of $\mathbb{P}^n_{\kappa(p)}$. If X is flat over S then the Hilbert polynomial $P_{X_p}(t)$ is independent of the point* *p. In other words, the map*

$$\begin{aligned} \Phi : S &\longrightarrow \mathbb{Q}[t] \\ p &\longmapsto P_{X_p}(t) \end{aligned}$$

is constant.

1.2.6 Remark. The original statement of theorem above in [Har77] assumes that S is an integral noetherian scheme. However, the reducedness hypothesis is used only to prove a converse statement about the morphism $X \to S$: if the function Φ is constant then X is flat over S.

When the local rings of S are reduced this statement follows from [Eis95, Exercise 20.14.b] or [Har77, Lemma II.8.9].

1.2.7 Remark. Let S be locally noetherian scheme, let $f : X \to S$ be a flat morphism. Theorem 1.2.5 implies that $p \mapsto P_{X_p}(t)$ is a locally constant function, that is, it is constant on each irreducible component of S. Thus we can write

$$X = \coprod_{P \in \mathbb{Q}[t]} X_P,$$

where $X_P = \{q \in X \mid P_{X_{f(q)}}(t) = P(t)\}$.

1.2.8 Definition. Let X be a projective scheme of finite type over S, $P(t) \in \mathbb{Q}[t]$ be a polynomial and let $\mathbf{Noe}/S$ be the category of locally noetherian schemes over S. We define the *Hilbert functor*

$$\mathcal{H}ilb_S^P(X) : (\mathbf{Noe}/S)^{op} \to \mathbf{Set}$$

on objects by

$$S' \mapsto \left\{ \begin{array}{c} V \subset X_{S'} \text{ closed subscheme} \\ \text{such that } f : V \to S' \text{ is flat and proper} \\ \text{and } P_{V_{f(q)}}(t) = P(t) \text{ for all } q \in V. \end{array} \right\}$$

and via pullbacks on S-morphisms $S'' \to S'$ just as in (1.4).

Notice that for any polynomial $P \in \mathbb{Q}[t]$ there is an injective map of functors

$$\mathcal{H}ilb_S^P(X) \hookrightarrow \mathcal{H}ilb_S(X),$$

and moreover, by Remark 1.2.7 we have the equality of functors

$$\mathcal{H}ilb_S(X) = \coprod_{P \in \mathbb{Q}[t]} \mathcal{H}ilb_S^P(X). \tag{1.5}$$

1.2.9 Theorem. *Let S be a noetherian scheme and X be a flat projective scheme over S. The functor $\mathcal{H}ilb_S^P(X)$ is representable by a proper scheme over S denoted* $\mathrm{Hilb}_S^P(X)$. *Therefore, $\mathcal{H}ilb_S(X)$ is representable by a scheme*

$$\mathrm{Hilb}_S(X) = \coprod_{P \in \mathbb{Q}[t]} \mathrm{Hilb}_S^P(X)$$

which is locally of finite type over S. This scheme is called the Hilbert scheme *of X over S.*

Proof. See [Kol96, Section I.1] or [FGI+05, Chapter 5]. □

1.2.10 It is clear that in general the Hilbert schemes of a projective scheme over a base have infinitely many components. There has been extensive work in trying to find the components and dimension of $\mathrm{Hilb}_S^P(X)$, in particular when $S = \operatorname{Spec} k$.

Furthermore, even if the schemes X and S are particularly nice, the schemes $\mathrm{Hilb}_S^P(X)$ can be very pathological. Mumford was the first to show the existence of a component in $\mathrm{Hilb}(\mathbb{P}_k^3)$ which is nowhere reduced in [Mum62]. Even more so, Vakil provides examples for which $\mathrm{Hilb}_S^P(X)$ have arbitrarily bad singularities in [Vak06]. Below we provide a few examples of Hilbert schemes and some progress on their description.

1.2.11 Hilbert schemes of connected components. As remarked before, it is not a simple task to describe all the components of a Hilbert scheme as it is an incredibly rich parameter space. To illustrate this, let us consider S to be an arbitrary scheme and let us look for the components of the seemingly trivial example of $\mathrm{Hilb}_S(S)$. This scheme parametrizes arbitrary unions of connected components of S.

Recall that for any scheme S there is a bijective correspondence

$$\left\{ \begin{array}{c} V \hookrightarrow S \text{ closed subscheme} \\ \text{such that } V \hookrightarrow S \text{ is flat.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Unions of connected} \\ \text{components of } S \end{array} \right\},$$

see [Stacks, 04PW] and [Stacks, 04PX]. For any scheme S, let

$$S = \coprod_{i \in I_S} S_i$$

be its decomposition on connected components. Notice that if V is a union of connected components of X, then there exists a unique subset $J \subset I_S$ such that

$$V = \coprod_{i \in J} S_i.$$

Therefore, the set of unions of connected components of S is in bijective correspondence with

$$C_S := 2^{I_S} \setminus \{\emptyset\},$$

where 2^{I_S} stands for the power set of I_S, that is, the set of all subsets of I_S. It follows that, for any morphism $S' \to S$, the Hilbert functor is given by

$$\mathcal{H}ilb_S(S)(S') = \left\{ \begin{array}{c} V \hookrightarrow S' \text{ closed subscheme} \\ \text{such that } V \hookrightarrow S' \text{ is flat} \end{array} \right\} \cong C_{S'}$$

We claim that

$$\mathrm{Hilb}_S(S) \cong \coprod_{J \in C_S} S$$

and that its universal section is isomorphic to S. For simplicity denote

$$\mathfrak{S} = \coprod_{J \in C_S} S.$$

Recall that for each $J \in C_S$ there is a canonical inclusion $\iota_J : S \hookrightarrow \mathfrak{S}$. We warn that the embedding of the universal section $\iota : S \hookrightarrow \mathfrak{S}$ is obtained from the ι_J but it is not any of them. Roughly speaking the embedding ι is obtained by taking each connected component S_i of S to the component in the partition of $\mathfrak{S}$ indexed by the singleton subset $\{i\}$. Rigorously, it is obtained in the following way: let $S = \coprod_{i \in I_S} S_i$ be the decomposition of S in connected components, then

$$\iota = \coprod_{i \in I_S} \iota_{\{i\}}|_{S_i},$$

that is, ι fits the following commutative diagrams

$$\begin{tikzcd} S_i \arrow[r, hook] \arrow[d, hook] & S \arrow[r, hook, "\iota_{\{i\}}"] & \mathfrak{S} \arrow[d, equal] \\ S \arrow[rr, hook, "\iota"] & & \mathfrak{S} \end{tikzcd}$$

for all $i \in I_S$.

Now it suffices to prove that ι is actually a universal section. In other words, it suffices to check that for each morphism $f : S' \to S$ and each $V' \in \mathrm{Hilb}_S(S)(S')$

there exists a unique morphism $g : S' \to \mathfrak{S}$ such that $V' \cong S' \times_{\mathfrak{S}} S$, see Corollary A.1.6.

Once again, the morphism g is intuitively simple. For each connected component $S'_i \subset V'$ we have that $f(S'_i) \subset S_j$ for some connected component $S_j \subset S$. Then g is just the morphism coinciding with the restriction $f|_{S'_i}$ mapped on the copy of S on $\mathfrak{S}$ labelled by the singleton $\{j\}$. Rigorously, we have the following: we can write $S' = \coprod_{i \in I_{S'}} S'_i$ to be the decomposition of S' in connected components. For each morphism $f : S' \to S$ we have $f(S'_i) = S_{f^\sharp(i)}$ for some $f^\sharp(i) \in I_S$ that is we have a map of sets

$$f^\sharp : I_{S'} \longrightarrow I_S.$$

Moreover, for any $V' \in \mathrm{Hilb}_S(S)(S')$ there is a unique subset $J \subset I_{S'}$ such that $V' = \coprod_{i \in J} S'_i$. Then we can define

$$g = \coprod_{i \in J} \iota_{\{f^\sharp(i)\}} \circ f|_{S'_i}.$$

That is g fits the commutative diagrams

$$\begin{tikzcd}
S'_i \arrow[r, hook] \arrow[d, hook] & S' \arrow[r, "f"] & S \arrow[r, hook, "\iota_{\{f^\sharp(i)\}}"] & \mathfrak{S} \arrow[d, equal] \\
S' \arrow[rrr, "g"] \arrow[d, equal] & & & \mathfrak{S} \arrow[d] \\
S' \arrow[rrr, "f"] & & & S
\end{tikzcd}$$

for all $i \in J$. It is clear that g is unique by the uniqueness of the set J and it is straightforward to check that the definitions of the morphisms imply that

$$\begin{tikzcd}
V' \arrow[r, "f|_{V'}"] \arrow[d, hook] & S \arrow[d, hook, "\iota"] \\
S' \arrow[r, "g"] & \mathfrak{S}
\end{tikzcd}$$

is cartesian. It follows that $\mathrm{Hilb}_S(S) = \mathfrak{S}$.

We conclude by remarking that $\mathrm{Hilb}_S(S)$ has finitely many components if and only if S has finitely many connected (and therefore irreducible) components. More precisely if $\#I_S = n$ then $\mathrm{Hilb}_S(S)$ has $n.\#C_S = n(2^n - 1)$ components.

1.2.12 Hilbert schemes of points. Let X be a projective variety over k. The simplest Hilbert polynomials for subschemes of X are the constant ones. Recall that the degree of the Hilbert polynomial of a projective scheme coincides with its dimension. Thus, by definition of the Hilbert polynomial, if Z is a subscheme of dimension 0 on X, then its Hilbert polynomial must be the constant

$$P_Z(t) = \dim_k H^0(Z, \mathcal{O}_Z).$$

Let us look at examples of schemes corresponding to points on $\mathrm{Hilb}^m(X)$. Suppose Z is a collection of m distinct k-points $\{p_1, \dots, p_r\} \subset X$, that is

$$Z \cong \coprod_{i=1}^{m} \operatorname{Spec} k,$$

then clearly $H^0(\mathcal{O}_Z, Z) = \bigoplus_{i=1}^m k$ and $P_Z(t) = m$. In other words, Z defines a k-point in $\mathrm{Hilb}^m(X)$.

Now, suppose Z' is a non-reduced point of X, for example $Z' \cong \operatorname{Spec} k[t]/(t^m)$ for some m. Then since $H^0(Z', \mathcal{O}_{Z'}) = k[t]/(t^m)$ is generated as a k vector space by the classes of $\{1, \dots, t^{m-1}\}$ we clearly have

$$P_Z(t) = \dim_k k[t]/(t^m) = m$$

and Z' defines a k-point on $\mathrm{Hilb}^m(X)$.

Combining both of those examples we have subschemes $Z \hookrightarrow X$ such that

$$Z \cong \coprod_{m_1+\dots+m_r=m} \operatorname{Spec} k[t]/(t^{m_i})$$

defining k-points on $\mathrm{Hilb}^m(X)$.

The study of Hilbert schemes of points has been very prolific in many areas of mathematics. To illustrate how far these parameter spaces can reach we will mention an interesting application on complex geometry in two steps:

1. If $\operatorname{char} k = 0$, the Hilbert schemes $\mathrm{Hilb}^m(X)$ have a natural connection with the symmetric powers of X. Recall that the m-th symmetric power of a projective variety X over k is defined to be the quotient of X^m by the group of permutations of m elements Σ_m. We will denote it $\mathrm{Sym}^m(X)$. The k-points of the symmetric powers $\mathrm{Sym}^m(X)$ can be understood as unordered selections of k-points on X which can be simply denoted by $\{p_1, \dots, p_m\}$. Notice however that each p_i is not necessarily distinct from p_m. This repetition can be dealt with in the following way: since $\operatorname{char} k = 0$ the variety $\mathrm{Sym}^m(X)$ is isomorphic to a variety parametrizing effective 0-

cycles on X, so that any k-point in $\mathrm{Sym}^m(X)$ actually corresponds to a finite formal sum of points $\sum_{i=1}^r m_i p_i$ such that $\sum_{i=1}^r m_i = m$. There exists a regular morphism

$$\begin{aligned} \mathrm{Hilb}^m(X) &\longrightarrow \mathrm{Sym}^m(X) \\ [Z] &\longmapsto \sum_{p \in \mathrm{Supp}(Z)} \dim_k(\mathcal{O}_{Z,p})p \end{aligned}$$

called the *Hilbert-Chow morphism*. If $\dim X = 1$, this morphism is an isomorphism. If $\dim X = 2$ then it is the resolution of singularities of the symmetric power $\mathrm{Sym}^m(X)$, see for instance [FGI+05, Chapter 7].

2. If $k = \mathbb{C}$ and X is a $K3$ surface, then $\mathrm{Hilb}^m(X)$ are examples of irreducible holomorphic symplectic (IHS) manifolds. IHS manifolds are defined to be simply connected compact Kähler manifolds such that $H^0(X, \Omega^2_{X/\mathbb{C}})$ is generated by a non-degenarate two form.

   The existence of the Hilbert-Chow morphism was used by Beauville [Bea83] to prove that $\mathrm{Hilb}^m(X)$ satisfies these conditions. This deserves special attention since the list of known IHS manifolds so far is not particularly big: they consist of $K3$ surfaces, the collection of the Hilbert scheme of points on those $K3$ surfaces, generalized Kummer varieties and two examples of dimension 6 and 10 developed by Kieran O'Grady, commonly referred as $OG6$ and $OG10$. See for instance [Huy99] for a more detailed discussion on IHS manifolds.

1.2.13 Comment. The representability result of the functors $\mathcal{H}ilb_S(X)$ given in Theorem 1.2.9 provides a wide range of examples for which the Hilbert scheme exists. A natural question is whether the hypothesis that X is projective and flat over a noetherian base scheme S can be relaxed. We will also see below that in fact, if we take X to be quasi-projective over S, the functor $\mathcal{H}ilb_S(X)$ is still representable.

The explicit example in 1.2.11 of $\mathrm{Hilb}_S(S)$ is another example of representability when S is an arbitrary scheme. A much more interesting instance of relaxing the hypotheses of Theorem 1.2.9 is given by Gustavsen, Laksov and Skjelnes in [GLS07] where they prove that for an arbitrary scheme S and X projective over S, the functors $\mathcal{H}ilb^m_S(X)$ are representable and provide explicit constructions for the representing schemes.

1.2.14 Hilbert schemes of curves. Let X be a projective variety over k. Consider the linear polynomials $dt+m$. By Remark 1.1.5 the scheme $\mathrm{Hilb}^{dt+m}(X)$

parametrizes curves of degree d and arithmetic genus $1-m$.

An interesting particular case happens when $m=1$. That is, when $\mathrm{Hilb}^{dt+1}(X)$ parametrizes curves of degree d and arithmetic genus 0.

Recall that the arithmetic and geometric genera coincide for any nonsingular projective curve. Therefore, any nonsingular rational curve of degree d on X corresponds to a point on $\mathrm{Hilb}^{dt+1}(X)$. The converse might not be true, that is, not every k-point on $\mathrm{Hilb}^{dt+1}(X)$ corresponds necessarily to a smooth rational curve (or not even necessarily a rational curve). The points corresponding to smooth rational curves of degree d form an open subscheme of $\mathrm{Hilb}^{dt+1}(X)$.

There is plenty of literature about those schemes. A good example is provided by Piene and Schlessinger [PS85]: they prove that the space parametrizing smooth rational curves of degree 3 in $\mathrm{Hilb}^{3t+1}(\mathbb{P}^3_k)$ is a subvariety H_0 which is smooth and irreducible of dimension 12. Moreover they prove that $\mathrm{Hilb}^{3t+1}(\mathbb{P}^3_k)$ consists of two components: $H=\overline{H_0}$ and a component $H'=\overline{H'_0}$, where H'_0 is a subvariety parametrizing plane cubic curves union a point, these are, of course, singular curves. We have that $H_0\cap H'_0=\emptyset$, however $H\cap H'$ is a smooth rational variety of dimension 11.

Other interesting examples arise from hypersurfaces in projective spaces. For instance, Starr proves in his thesis [Sta00] that if X is a smooth cubic threefold in $\mathbb{P}^4_{\mathbb{C}}$ the subschemes of $\mathrm{Hilb}^{dt+1}(X)$ parametrizing nonsingular rational curves on X are irreducible of dimension $2d$. In the same direction there has also been progress on these schemes when $X\subset\mathbb{P}^5_{\mathbb{C}}$ is a smooth cubic fourfold: Starr and de Jong [JS04] study the birational geometry of the subscheme parametrizing nonsingular rational curves in $\mathrm{Hilb}^{dt+1}(X)$ and prove that for $d\geq 5$ and odd this scheme is not uniruled.

Further to that Harris, Roth and Starr [HRS05] prove that for a nonsingular complex cubic threefold $X\subset\mathbb{P}^4_{\mathbb{C}}$ the subschemes of $\mathrm{Hilb}^{dt+1-g}(X)$ parametrizing nonsingular curves of degree d and arithmetic genus g are irreducible of dimension $2d$ for $1\leq d\leq 5$.

1.2.15 Comment. Recall that, when we are working with projective curves as schemes over a field k one can usually study them by taking their normalization. In fact, the category of normal projective curves over a field and non-constant morphisms between them is equivalent to the category of smooth projective curves over a field. See for instance [Stacks, 0BY1].

However, if we want to study a projective variety X over k by the behaviour of its curves, most of them will be unavoidably singular. We will have to manage their singularities without altering significantly the geometry of X.

If we restrict ourselves to rational curves we can quickly see this even on plane curves: by the genus-degree formula for plane curves [Har77, Example V.1.5.1.] the only rational nonsingular curves in $\mathbb{P}^2_k$ are those of degree $d = 1, 2$, that is lines and conics; every rational curve of degree $d \geq 3$ will be singular.

1.3 Properties of the Hilbert schemes

We formulate a few properties of the Hilbert scheme which will be useful to deduce properties of different parameter spaces such as the scheme of morphisms. The first useful property is that the Hilbert scheme behaves well under base change, that is, the following holds:

1.3.1 Proposition. *Let $h : S' \to S$ be a morphism of noetherian schemes and let X be a flat projective scheme over S. Then we have a natural isomorphism $\mathrm{Hilb}_{S'}(X_{S'}) \cong \mathrm{Hilb}_S(X) \times_S S'$.*

Proof. For this proof, denote for simplicity

$$\mathrm{Hilb}_S := \mathrm{Hilb}_S(X) \ \text{ and } \ \mathrm{Hilb}_{S'} := \mathrm{Hilb}_{S'}(X_{S'}),$$

and let

$$U \in \mathcal{H}ilb_S(X)(\mathrm{Hilb}_S) \ \text{ and } \ U' \in \mathcal{H}ilb_{S'}(X_{S'})(\mathrm{Hilb}_{S'})$$

be the respective universal sections (A.1.7) of the Hilbert functors.

Let $g : \mathrm{Hilb}_S \to S$ and $g' : \mathrm{Hilb}_{S'} \to S'$ denote the structure morphisms. Notice that h defines $\mathrm{Hilb}_{S'}$ as an S-scheme, hence U' fits the following commutative diagram

$$\begin{tikzcd}
U' \arrow[d, hook] & & \\
X_{S'} \times_{S'} \mathrm{Hilb}_{S'} \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Hilb}_{S'} \arrow[r, "g'"'] & S' \arrow[r, "h"'] & S,
\end{tikzcd}$$

where all the squares are cartesian. In other words, we can consider U' as a section in $\mathcal{H}ilb_S(X)(\mathrm{Hilb}_{S'})$, and the representability of $\mathcal{H}ilb_S(X)$ implies that there exists an S-morphism $h' : \mathrm{Hilb}_{S'} \to \mathrm{Hilb}_S$ such that

$$\begin{tikzcd} \mathrm{Hilb}_{S'} \arrow[r, "h'"] \arrow[d, "g'"'] & \mathrm{Hilb}_S \arrow[d, "g"] \\ S' \arrow[r, "h"'] & S \end{tikzcd} \tag{1.6}$$

is commutative and such that $U' \cong U \times_{\mathrm{Hilb}_S} \mathrm{Hilb}_{S'}$. That is, there is a commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ X_{S'} \times_{S'} \mathrm{Hilb}_{S'} \arrow[r] \arrow[d] & X \times_S \mathrm{Hilb}_S \arrow[d] \\ \mathrm{Hilb}_{S'} \arrow[r, "h'"'] & \mathrm{Hilb}_S \end{tikzcd}$$

where all squares are cartesian. It suffices to prove that the square (1.6) is cartesian by checking that it satisfies the corresponding universal property. That is, if we suppose there exists an S-scheme S'' such that the diagram of full arrows

$$\begin{tikzcd} S'' \arrow[dr, dashed, "\exists!\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & \mathrm{Hilb}_{S'} \arrow[r, "h'"] \arrow[d, "g'"'] & \mathrm{Hilb}_S \arrow[d, "g"] \\ & S' \arrow[r, "h"'] & S \end{tikzcd}$$

is commutative, then there exists a unique dashed arrow $\gamma : S'' \to \mathrm{Hilb}_{S'}$ making everything commutative. Indeed, notice that by the representability of $\mathcal{H}ilb_S(X)$, the morphism $\alpha : S'' \to \mathrm{Hilb}_S$ corresponds to a unique section $V \in \mathcal{H}ilb_S(X)(S'')$. That is, V is a closed subscheme of $X_{S''}$, flat and proper over S'' and, moreover, such that $V \cong U \times_{\mathrm{Hilb}_S} S''$. Equivalently, we have that the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ S'' \arrow[r, "\alpha"] & \mathrm{Hilb}_S \end{tikzcd}$$

is cartesian. The morphism $\beta : S'' \to S'$ makes S'' into a S'-scheme, and since

$X_{S''} \cong X \times_{S'} S''$, we have that $V \in \mathcal{H}ilb_{S'}(X_{S'})(S'')$. In particular, the universal property of U' implies that there exists a unique S'-morphism $\gamma : S'' \to \mathrm{Hilb}_{S'}$ such that $\beta = g' \circ \gamma$ and such that $V \cong U' \times_{\mathrm{Hilb}_{S'}} S''$. Therefore, $h' \circ \gamma$ induces an isomorphism

$$V \cong U' \times_{\mathrm{Hilb}_{S'}} S'' \cong (U \times_{\mathrm{Hilb}_S} \mathrm{Hilb}_{S'}) \times_{\mathrm{Hilb}_{S'}} S'' \cong U \times_{\mathrm{Hilb}_S} S''.$$

We can summarise the above by saying that we have a commutative diagram

$$\begin{tikzcd}
V \arrow[dd, hook] \arrow[dr, bend left] & & & & \\
& U' \arrow[rr] \arrow[dd, hook] & & U \arrow[dd, hook] & \\
X_{S''} \arrow[rr] \arrow[dd] \arrow[dr, bend left] & & X_{S'} \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
& X_{S'} \times_{S'} \mathrm{Hilb}_{S'} \arrow[ur] \arrow[rr] \arrow[dd] & & X \times_S \mathrm{Hilb}_S \arrow[ur] \arrow[dd] & \\
S'' \arrow[rr, "\beta"'] \arrow[dr, bend right, "\gamma"'] & & S' \arrow[rr, "h"'] & & S \\
& \mathrm{Hilb}_{S'} \arrow[ur, "g'"'] \arrow[rr, "h'"'] & & \mathrm{Hilb}_S \arrow[ur, "g"'] &
\end{tikzcd}$$

for which every *vertical* square is cartesian. Since U is a universal section, it satisfies property (A.3), and therefore, α is the unique morphism inducing the isomorphism $V \cong U \times_{\mathrm{Hilb}_S} S''$. We conclude that $\alpha = h' \circ \gamma$. $\square$

1.3.2 . Let X be a projective scheme over a noetherian base scheme S, and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be an open and a closed immersion of schemes over S repectively. We will prove that there exists Hilbert schemes of U and Z, which will be respectively an open and a closed subscheme of the Hilbert scheme $\mathrm{Hilb}_S(X)$.

1.3.3 Lemma. *Let X be a separated scheme over S and $\iota : W \hookrightarrow X$ be an immersion of schemes over S. Then, ι induces a monomorphism of functors $\eta_\iota : \mathcal{H}ilb_S(W) \hookrightarrow \mathcal{H}ilb_S(X)$.*

Proof. We check it on sections: let S' be an S-scheme and let $V \in \mathcal{H}ilb_S(W)(S')$ be a section, and consider a closed immersion $V \hookrightarrow W_{S'}$ fitting in the commutative

diagram

$$\begin{tikzcd} & V \arrow[dl, hook] \arrow[dr, hook] & \\ W_{S'} \arrow[rr, hook, "\iota_{S'}"] \arrow[dr] & & X_{S'} \arrow[dl] \\ & S', & \end{tikzcd}$$

where $V \hookrightarrow W_{S'} \to S'$ is proper and flat, and the schemes $W_{S'}$ and $X_{S'}$ stand for the base changes of W and X respectively via $S' \to S$.

Since X is separated, it follows that the composition $V \hookrightarrow W_{S'} \overset{\iota_{S'}}{\hookrightarrow} X_{S'}$ is proper (see [Stacks, 01W6]). Moreover, since $V \hookrightarrow X_{S'}$ is an immersion of schemes and it is proper, it is a closed immersion of schemes (see [Stacks, 01IQ]). Therefore, V induces a closed subscheme of $X_{S'}$ which is flat and proper over S'. Thus we have a map

$$\mathcal{H}ilb_S(W)(S') \to \mathcal{H}ilb_S(X)(S')$$

taking the closed subscheme defined by $V \hookrightarrow W_{S'}$ to the closed subscheme defined by $V \hookrightarrow W_{S'} \overset{\iota_{S'}}{\hookrightarrow} X_{S'}$. This map is clearly natural on S' and injective since $\iota_{S'}$ is a monomorphism in the category of schemes (see [Stacks, 01L7]). $\square$

1.3.4 Proposition. *Let X be a proper scheme over S. Let $i : V \hookrightarrow X$ be a closed subscheme over S and $j : U \hookrightarrow X$ be an open subscheme of X. Then there exists an open subscheme $S_0 \subseteq S$ with the following property:*

$$\begin{aligned} &\textit{for any morphism of schemes } f : S' \to S, f \textit{ factors through } S_0 \\ &\textit{if and only if the base change } i_{S'} : V_{S'} \hookrightarrow X_{S'} \textit{ factors through } U_{S'}. \end{aligned} \tag{1.7}$$

Proof. First, let $U \cap V$ denote the scheme theoretic intersection of V and U defined by the cartesian square

$$\begin{tikzcd} U \cap V \arrow[r, hook, "i^U"] \arrow[d, hook, "j^V"'] & U \arrow[d, hook, "j"] \\ V \arrow[r, hook, "i"'] & X. \end{tikzcd}$$

If U and V are disjoint inside X, then $U \cap V$ is the empty scheme. Notice that for any morphism $f : S' \to S$, the base change $V_{S'}$ is disjoint from $U_{S'}$. In this case S_0 can also be taken to be the empty scheme, since clearly $V_{S'}$ factors through $U_{S'}$ if and only if S' is the empty scheme.

Therefore, we can assume U and V are not disjoint, that is, $U \cap V$ is non-empty. Then let $q \in U \cap V$ be a point and $p \in S$ be the image of q under the structure morphism $U \cap V \to S$. Then we have a commutative diagram

$$\begin{tikzcd}
(U \cap V)_p \arrow[rr, hook, "i_p^U"] \arrow[dr, hook, "j_p^V"] \arrow[dd] & & U_p \arrow[dr, hook, "j_p"] \arrow[dd] & & \\
& V_p \arrow[rr, hook, "i_p"] \arrow[dd] & & X_p \arrow[r] \arrow[dd] & \operatorname{Spec} \kappa(p) \arrow[dd] \\
U \cap V \arrow[rr, hook, "i^U"] \arrow[dr, hook, "j^V"] & & U \arrow[dr, hook, "j"] & & \\
& V \arrow[rr, hook, "i"] & & X \arrow[r] & S,
\end{tikzcd}$$

where $(U \cap V)_p, U_p, V_p$ and X_p correspond to the fibers over p, and all squares are cartesian.

Claim. The morphism $i_p : V_p \hookrightarrow X_p$ factors through U_p if and only if

$$j_p^V : (U \cap V)_p \hookrightarrow V_p$$

is surjective.

Proof of claim. Indeed, by the universal property of the fiber product, V_p factors through U_p if and only if the open immersion j_p^V admits a section, in particular, this implies that j_p^V is surjective.

Conversely, if j_p^V is surjective, since it is also an open immersion, it is a fortiori an isomorphism, therefore V_p factors through U_p. $\square$

We define the set

$$\begin{aligned} S_0 :=& \{p \in S \mid j_p^V : (U \cap V)_p \hookrightarrow V_p \text{ is surjective}\} \\ =& \{p \in S \mid i_p : V_p \hookrightarrow X_p \text{ factors through } U_p\}. \end{aligned}$$

We claim that S_0 is open in S and satisfies property (1.7). Indeed, consider the morphism $g : V \overset{i}{\hookrightarrow} X \to S$. Since X is proper over S and i is a closed immersion, g is a proper morphism. Consider the diagram

$$\begin{tikzcd} U \cap V \arrow[rr, hook, "j^V"] \arrow[dr] & & V \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

Claim. We have $S_0 = S \setminus g(V \setminus j^V(U \cap V))$. In particular, S_0 is open, since j^V is an open embedding and g is proper.

Proof of claim. Let $p \in S \setminus g(V \setminus j^V(U \cap V))$. Then, the underlying topological space of V_p, denoted $|V_p|$, is contained in the image $j^V(U \cap V)$. This is easy to see, since

$$\begin{aligned} |V_p| \subseteq & g^{-1}(S \setminus g(V \setminus j^V(U \cap V))) = V \setminus (g^{-1}(g(V \setminus j^V(U \cap V)))) \\ \subseteq & V \setminus (V \setminus j^V(U \cap V)) = j^V(U \cap V). \end{aligned}$$

Since j^V is an open embedding, that means that $V_p \to V$ factors uniquely through $j^V : U \cap V \to V$ (see [Stacks, 01HI]). By the universal property of fiber product, the morphism $j_p^V : (U \cap V)_p \to V_p$ admits a section, that is, it is surjective.

Conversely, if $p \in S_0$, then j_p^V is a surjective open embedding and thus it is an isomorphism. Therefore, $V_p \to V$ factors through $j^V : U \cap V \to V$, which happens if and only if

$$|V_p| = g^{-1}(p) \subseteq j^V(U \cap V).$$

That is,

$$g^{-1}(p) \cap \big(V \setminus j^V(U \cap V)\big) = \emptyset,$$

or equivalently,

$$p \in S \setminus g(V \setminus j^V(U \cap V)).$$

□

Claim. S_0 has the property (1.7).

Proof of claim. Let $f : S' \to S$ be a morphism. Suppose that $i_{S'} : V_{S'} \hookrightarrow X_{S'}$ factors through $U_{S'}$. Let $q \in S'$ be a point and $p = f(q)$. We prove that $p \in S_0$. Notice that, in particular, the morphism $i_{S',q} : V_{S',q} \hookrightarrow X_{S',q}$ induced at the fibers over q factors through the fiber $U_{S',q}$. Consider the canonical morphism

$\alpha : \operatorname{Spec}\kappa(q) \to \operatorname{Spec}\kappa(p)$ and the induced commutative diagram:

$$\begin{tikzcd}
 & V_{S',q} \arrow[dl] \arrow[dd, "\alpha_V"] \arrow[dr, hook, "i_{S',q}"] & & & \\
U_{S',q} \arrow[rr, hook, "j_{S',q}"'] \arrow[dd, "\alpha_U"'] & & X_{S',q} \arrow[r] \arrow[dd, "\alpha_X"] & \operatorname{Spec}\kappa(q) \arrow[r] \arrow[dd, "\alpha"] & S' \arrow[dd, "f"] \\
 & V_p \arrow[dr, hook, "i_p"] & & & \\
U_p \arrow[rr, hook, "j_p"] & & X_p \arrow[r] & \operatorname{Spec}\kappa(p) \arrow[r] & S.
\end{tikzcd}$$

Notice that since α is a morphism between points, it is surjective, and since surjectivity is stable under base change ([Stacks, 01S1]), the morphisms $\alpha_V, \alpha_U, \alpha_X$ are surjective.

It suffices to prove that the underlying topological space of the image $i_p(V_p)$ is contained in $j_p(U_p)$. Indeed, if $v \in V_p$, by surjectivity we have a point $v' \in V_{S',q}$ such that $\alpha_V(v') = v$. Then we have $i_{S',q}(v') = j_{S',q}(u')$ for some $u' \in U_{S',q}$. Denote $u = \alpha_U(u')$. We have

$$\begin{aligned} j_p(u) =& j_p \circ \alpha_U(u') = \alpha_X \circ j_{S',q}(u') \\ =& \alpha_X \circ i_{S',q}(v') = i_p \circ \alpha_V(v') = i_p(v). \end{aligned}$$

That is, $i_p(V_p)$ is contained in $j_p(U_p)$, as desired. Thus, $p = f(q) \in S_0$ for all $q \in S'$, and $f : S' \to S$ factors through $S_0 \hookrightarrow S$.

Conversely, suppose that $f : S' \to S$ factors through S_0. Let $i_{S_0} : V_{S_0} \hookrightarrow X_{S_0}$ be the base change of i with respect to the inclusion $S_0 \hookrightarrow S$, and consider the

diagram

$$\begin{tikzcd}
 & V_{S'} \arrow[dl] \arrow[dr, hook, "i_{S'}"] \arrow[dd] & & \\
U_{S'} \arrow[rr, hook, "j_{S'}"'] \arrow[dd] & & X_{S'} \arrow[r] \arrow[dd] & S' \arrow[dd] \arrow[dddd, bend left, "f"] \\
 & V_{S_0} \arrow[dl] \arrow[dr, hook, "i_{S_0}"] \arrow[dd] & & \\
U_{S_0} \arrow[rr, hook, "j_{S_0}"'] \arrow[dd] & & X_{S_0} \arrow[r] \arrow[dd] & S_0 \arrow[dd, hook] \\
 & V \arrow[dr, hook, "i"] & & \\
U \arrow[rr, hook, "j"] & & X \arrow[r] & S,
\end{tikzcd}$$

where all squares are cartesian.

By the definition of S_0, we have

$$i_{S_0,p}(V_{S_0,p}) \subseteq j_{S_0,p}(U_{S_0,p}) \subseteq j_{S_0}(U_{S_0})$$

for all $p \in S_0$. That is, i_{S_0} factors through j_{S_0}. It follows immediately that $i_{S'} : V_{S'} \to X_{S'}$ factors through $U_{S'} \cong U_{S_0} \times_{S_0} S'$ by the universal property of fiber product. Therefore, S_0 has property (1.7), which proves the claim and the lemma. $\square$

$\square$

1.3.5 Proposition. *Let $j : U \to X$ be an open immersion of schemes over S. Then the morphism $\eta_j : \mathcal{H}ilb_S(U) \hookrightarrow \mathcal{H}ilb_S(X)$ defined in 1.3.3 makes $\mathcal{H}ilb_S(U)$ into an open subfunctor of $\mathcal{H}ilb_S(X)$. In particular, if X is projective and flat over S, then $\mathcal{H}ilb_S(U)$ is representable by an open subscheme* $\mathrm{Hilb}_S(U) \hookrightarrow \mathrm{Hilb}_S(X)$.

Proof. By Lemma 1.3.3, we have a subfunctor $\eta_j : \mathcal{H}ilb_S(U) \hookrightarrow \mathcal{H}ilb_S(X)$. To prove it is an open subfunctor, by Proposition A.2.9, it suffices to prove that for every S-scheme S' and section $V \in \mathcal{H}ilb_S(X)(S')$, there exists an open subscheme $S_0 \hookrightarrow S'$ satisfying the property: for any morphism $f : S'' \to S'$, f factors through S_0 if and only if $\mathcal{H}ilb_S(X)(f)(V)$ belongs to $\eta_{j,S''}(\mathcal{H}ilb_S(U)(S''))$ or equivalently, the base change $V \times_{S'} S'' \hookrightarrow X_{S''}$ factors through $U_{S''}$.

Such S_0 is obtained by applying Lemma 1.3.4 to the closed immersion $V \hookrightarrow X_{S'}$ and open immersion $j_{S'} : U_{S'} \to X_{S'}$.

The second assertion follows from Proposition A.2.7. □

1.3.6 Remark. A similar statement can be proved for a closed embedding over S by different methods. That is, if $i : Z \hookrightarrow X$ is a closed immersion, then $\eta_i : \mathcal{H}ilb_S(Z) \hookrightarrow \mathcal{H}ilb_S(X)$ is a closed subfunctor. We provide a sketch of the reasoning.

Let S be a noetherian scheme, X be a projective S-scheme and $\mathcal{E}$ be a coherent sheaf over S. There exists a functor

$$\mathcal{Q}uot_{\mathcal{E}/X/S} : (\mathbf{Noe}/S)^{op} \to \mathbf{Set}$$

associating every morphism $f : S' \to S$ to the set of equivalence classes of families of quotients of $\mathcal{E}_{S'} := f^*\mathcal{E}$ which are flat over S' and have proper support.

Grothendieck [Gro60] proved that the functor $\mathcal{Q}uot_{\mathcal{E}/X/S}$ is representable by a scheme, denoted $\mathrm{Quot}_{\mathcal{E}/X/S}$. More recent and detailed proofs of this representability can be found in [AK80] and [FGI+05, Chapter 5]. In particular, when $\mathcal{E} = \mathcal{O}_X$, we have $\mathcal{Q}uot_{\mathcal{O}_X/X/S} \cong \mathcal{H}ilb_S(X)$, see [FGI+05, Section 5.1.3, page 109]. If $i : Z \hookrightarrow X$ is a closed immersion, we have a canonical surjection $\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z$. Moreover, by [FGI+05, Lemma 5.17, pg 127] we have a natural morphism

$$\mathcal{Q}uot_{i_*\mathcal{O}_Z/X/S} \to \mathcal{Q}uot_{\mathcal{O}_X/X/S}$$

which is a closed subfunctor. Since the support of $i_*\mathcal{O}_Z$ on X is Z and in addition, Z is flat if and only if $i_*\mathcal{O}_Z$ is flat over X, we have canonical isomorphisms

$$\mathcal{Q}uot_{i_*\mathcal{O}_Z/X/S} \cong \mathcal{Q}uot_{\mathcal{O}_Z/Z/S} \cong \mathcal{H}ilb_S(Z),$$

and the claim follows.

1.4 The functor and scheme of morphisms

In this section we present the main parameter spaces studied in Chapter 2 of the thesis. These are schemes parametrizing morphisms of schemes over a base scheme. As for any other parameter space, we should first define the functor of points.

1.4.1 Definition. Let X and Y be schemes over S. The *functor of morphisms*

$$\mathcal{M}or_S(X, Y) : (\mathbf{Noe}/S)^{op} \to \mathbf{Set}$$

from X to Y is defined on S-schemes by

$$\mathcal{M}or_S(X,Y)(S') = \mathrm{Hom}_{S'}(X_{S'}, Y_{S'})$$

where $\mathrm{Hom}_{S'}(X_{S'}, Y_{S'})$ is the set of morphisms from $X_{S'}$ to $Y_{S'}$ over S'. We define the functor on S-morphisms taking $S'' \to S'$ to the map

$$\begin{aligned} \mathrm{Hom}_{S'}(X_{S'}, Y_{S'}) &\longrightarrow \mathrm{Hom}_{S''}(X_{S''}, Y_{S''}) \\ f_{S'} &\longmapsto f_{S''} \end{aligned}$$

where $f_{S''} : X_{S''} \to Y_{S''}$ is the base change of $f_{S'}$ making the following diagram commutative

$$\begin{tikzcd}
& S'' \arrow[dd] & \\
X_{S''} \arrow[ur, bend left] \arrow[rr, "f_{S''}"'] \arrow[dd] & & Y_{S''} \arrow[ul, bend right] \arrow[dd] \\
& S' \arrow[dd] & \\
X_{S'} \arrow[ur, bend left] \arrow[rr, "f_{S'}"'] \arrow[dd] & & Y_{S'} \arrow[ul, bend right] \arrow[dd] \\
& S & \\
X \arrow[ur, bend left] & & Y \arrow[ul, bend right]
\end{tikzcd}$$

and for which every square is cartesian.

1.4.2 If the functor $\mathcal{M}or_S(X,Y)$ is representable by a scheme, then each of its S-points correspond to morphisms $f : X \to Y$ over S. We then denote the representing scheme $\mathrm{Mor}_S(X,Y)$. We give a few examples for which we can determine representability by the functor of points, and recall under which conditions on X and Y representability is guaranteed.

1.4.3 Lemma (Rigidity)**.** *Let S be a locally noetherian scheme. Consider the commutative diagram of schemes*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\
& S. &
\end{tikzcd}$$

Suppose that g is flat and proper. Let X_p denote the fibre of g at a point $p \in S$, *and suppose we have $H^0(X_p, \mathcal{O}_{X_p}) \cong \kappa(p)$ for all points $p \in S$. Then, if there exists a point $p \in S$ on each connected component of S such that $f(X_p)$ is set-theoretically a single point, then there exists a S-section $\eta : S \to Y$ such that $\eta \circ g = f$.*

Proof. When S is connected this is precisely [MFK94, Proposition 6.1, pg 115]. If S has several connected components, we can write $S = \coprod_{i \in I_S} S_i$ as its partition in connected components. Since S is locally noetherian, the collection $\{S_i\}_{i \in I_S}$ of connected components forms an open cover of S. Hence, we apply the same proposition to each component S_i and obtain the sections η_i fitting in the fibered diagram

$$\begin{tikzcd}
& S_i \arrow[dr, shift left, "\eta_i"] \arrow[dd, hook] & \\
X_i \arrow[ur, "g|_{X_i}"] \arrow[rr, "f|_{X_i}"'] \arrow[dd, hook] & & Y_i \arrow[ul, shift left, "h|_{Y_i}"] \arrow[dd, hook] \\
& S & \\
X \arrow[ur, "g"] \arrow[rr, "f"] & & Y. \arrow[ul, "h"']
\end{tikzcd}$$

Therefore, we can define $\eta = \coprod_{i \in I_S} \eta_i$, that is, η is the glueing of the η_i. $\square$

1.4.4 Lemma. *Let S be a scheme and X, Y be schemes over S. Then we have a canonical morphism*

$$h_Y \to \mathcal{M}or_S(X, Y).$$

If $X \to S$ has an S-section, then this morphism is injective. Moreover, if X and S satisfy the conditions of the Rigidity Lemma 1.4.3, then this morphism is surjective if, and only if, for every morphism $\operatorname{Spec} K \to S$ with K a field, all morphisms in $\operatorname{Hom}_K(X_K, Y_K)$ are constant, where $X_K := X \times_S \operatorname{Spec} K$ and $Y_K := Y \times_S \operatorname{Spec} K$. In such a situation we have a canonical isomorphism

$$Y \cong \operatorname{Mor}_S(X, Y).$$

Proof. Let S' be an S-scheme. Then any morphism $f \in h_Y(S') = \operatorname{Hom}_S(S', Y)$ induces a unique S'-section $S' \to Y_{S'}$ and we can define the composition $g : X_{S'} \to$

$S' \to Y_{S'}$, fitting on the diagram (in black)

$$\begin{tikzcd}
X_{S'} \arrow[r, shift left] \arrow[d] \arrow[rr, bend left, "g"] & S' \arrow[l, shift left] \arrow[r, shift left] \arrow[d] \arrow[dr, "f"] & Y_{S'} \arrow[l, shift left] \arrow[d] \\
X \arrow[r, shift left] & S \arrow[l, shift left] & Y \arrow[l]
\end{tikzcd}$$

and defining a map $h_Y(S') \to \mathcal{M}or_S(X,Y)(S')$ natural in S'.

Moreover, if $X \to S$ has a S-section $S \to X$, this morphism is injective: indeed, this S-section induces a unique S'-section $S' \to X_{S'}$ (in blue in the diagram above) so that we can define a map $\mathcal{M}or(X,Y)(S') \to h_Y(S')$ taking any morphism $g : X_{S'} \to Y_{S'}$ to the composition $S' \to X_{S'} \xrightarrow{g} Y_{S'} \to Y$. It is clear that this map is natural in S' and that the composition

$$h_Y \to \mathcal{M}or_S(X,Y) \to h_Y$$

is the identity. In particular, $h_Y \to \mathcal{M}or_S(X,Y)$ is injective.

Assuming that X and S satisfy the additional hypotheses of Lemma 1.4.3, we claim that $h_Y \to \mathcal{M}or_S$ is surjective if and only if for each morphism in $\mathrm{Hom}_S(\operatorname{Spec} K, Y)$ the morphisms in $\mathrm{Hom}_K(X_K, Y_K)$ are constant. Indeed, suppose all $\mathrm{Hom}_K(X_K, Y_K)$ are constant, let $g : X_{S'} \to Y_{S'}$ be a morphism in $\mathrm{Hom}_{S'}(X_{S'}, Y_{S'})$ and let $q \in S'$ be a point. Notice that fiber $X_{\kappa(q)}$ fits in the fibered diagram

$$\begin{tikzcd}
X_{\kappa(q)} \cong X_{S',q} \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \kappa(q) \arrow[r] & S' \arrow[r] & S.
\end{tikzcd}$$

We have an analogous diagram for $Y_{\kappa(q)} = Y_{S',q}$. Then, by assumption, the induced morphism on fibres

$$g_q : X_{\kappa(q)} \to Y_{\kappa(q)}$$

is constant. Therefore, by the Rigidity Lemma 1.4.3, the morphism g factors through S, hence $h_Y(S') \to \mathcal{M}or_S(X,Y)(S')$ is surjective for all S'.

Conversely, if $h_Y \to \mathcal{M}or_S(X,Y)$ is surjective, then for any $\operatorname{Spec} K \to S$, all morphisms $\mathrm{Hom}_K(X_K, Y_K)$ factor through $\operatorname{Spec} K$ and hence are constant. $\square$

1.4.5 Example. Let k be a field, not necessarily algebraically closed, and K/k

be any field extension. Denote $T = \operatorname{Spec} K$ and let $\alpha : T \to \operatorname{Spec} k$ be the morphism of schemes corresponding to the field extension. We claim $\mathcal{M}or(\mathbb{P}^n_k, T)$ is representable by T.

Recall that $\mathbb{P}^n_k \to \operatorname{Spec} k$ has k-sections, therefore by Lemma 1.4.4 it suffices to check that for any other field extension L/k, the morphisms

$$\operatorname{Hom}_L(\mathbb{P}^n_L, \operatorname{Spec} L \otimes_k K) \tag{1.8}$$

are constant. Denote $U = \operatorname{Spec} L$, hence $\operatorname{Spec} L \otimes_k K = T \times U$. Notice that every L-morphism $g : \mathbb{P}^n_L \to T \times U$ fits the following commutative diagram

$$\begin{tikzcd}
& U \arrow[dd, "\beta"] \arrow[dddr, dashed, "\varepsilon"] & \\
\mathbb{P}^n_L \arrow[ur, "\delta"] \arrow[rr, "g"] \arrow[dd] \arrow[ddrr, bend right, "\gamma"'] & & T \times U \arrow[ul] \arrow[dd] \\
& \operatorname{Spec} k & \\
\mathbb{P}^n_k \arrow[ur] & & T. \arrow[ul, "\alpha"']
\end{tikzcd} \tag{1.9}$$

Notice that, by the universal property of fiber products, the morphism g factors through $\delta : \mathbb{P}^n_L \to U$ if and only if there exists a dashed morphism ε making the following diagram commute

$$\begin{tikzcd}
\mathbb{P}^n_L \arrow[r, "\gamma"] \arrow[d, "\delta"'] & T \arrow[d, "\alpha"] \\
U \arrow[r, "\beta"'] \arrow[ur, dashed, "\varepsilon"] & \operatorname{Spec} k.
\end{tikzcd} \tag{1.10}$$

Next, we claim that in fact there exists a unique morphism $\varepsilon : U \to T$ such that (1.10) commutes. Recall that since the underlying topological spaces of T and U consist of points, the comorphisms $\gamma^\#$ and $\delta^\#$ are completely given by global sections, and we can write

$$\begin{aligned}
\gamma^\# : K = \mathcal{O}_T(T) \hookrightarrow \gamma_* \mathcal{O}_{\mathbb{P}^n_L}(T) \cong H^0(\mathbb{P}^n_L, \mathcal{O}_{\mathbb{P}^n_L}) \cong L \text{ and} \\
\delta^\# : L = \mathcal{O}_U(U) \hookrightarrow \delta_* \mathcal{O}_{\mathbb{P}^n_L}(U) \cong H^0(\mathbb{P}^n_L, \mathcal{O}_{\mathbb{P}^n_L}) \cong L.
\end{aligned}$$

Notice that the existence of sections for δ implies that $\delta^\#$ is actually an automorphism. Thus, we can define ε as the morphism of schemes associated to the morphism of rings

$$\varepsilon^{\#} := (\delta^{\#})^{-1} \circ \gamma^{\#}.$$

Notice then that $\gamma = \varepsilon \circ \delta$; this is clear at the level of topological spaces and by definition, we have

$$(\varepsilon \circ \delta)^{\#} : \mathcal{O}_T(T) \xrightarrow{\varepsilon^{\#}} \varepsilon_* \mathcal{O}_U(T) \xrightarrow{\varepsilon_*(\delta^{\#})} \varepsilon_* \delta_* \mathcal{O}_{\mathbb{P}^n_L}(T).$$

Since ε is a morphism between schemes whose underlying topological spaces are single points we clearly have $\varepsilon_*(\delta^{\#}) = \delta^{\#}$, in other words

$$(\varepsilon \circ \delta)^{\#} = \delta^{\#} \circ (\delta^{\#})^{-1} \circ \gamma^{\#} = \gamma^{\#}.$$

Moreover, we have $\alpha \circ \varepsilon = \beta$: indeed,

$$\alpha \circ \gamma = \alpha \circ \varepsilon \circ \delta = \beta \circ \delta$$

and hence $\alpha \circ \varepsilon = \beta$, since δ is an epimorphism[1] of schemes. Uniqueness of ε also follows from δ being an epimorphism.

Therefore, all morphisms $\mathrm{Hom}_L(\mathbb{P}^n_L, T \times U)$ are constant, and by Lemma 1.4.4 we conclude that $\mathcal{M}or(\mathbb{P}^n_k, T) \cong h_T$, or equivalently,

$$\mathrm{Mor}(\mathbb{P}^n_k, \mathrm{Spec}\, K) \cong \mathrm{Spec}\, K.$$

1.4.6 Example. Let S be a locally noetherian scheme and A be an abelian scheme over S. We claim that

$$\mathrm{Mor}_S(\mathbb{P}^1_S, A) \cong A.$$

Indeed, since $\mathbb{P}^1_S \to S$ always has S-sections, we have the injective morphism

$$h_A \to \mathcal{M}or_S(\mathbb{P}^1_S, A).$$

Thus, by Lemma 1.4.4, in order to prove it is also surjective, it suffices to prove that for every $\mathrm{Spec}\, K \to S$, the induced morphism of fibers in $\mathrm{Hom}_K(\mathbb{P}^1_K, A_K)$ is constant. Recall that $A_K = A \times_S \mathrm{Spec}\, K$ is an abelian variety over K, therefore every morphism $\mathbb{P}^1_K \to A_K$ is constant, see for instance [Mil08, Proposition 3.9].

[1] Recall that any surjective and flat morphism is an epimorphism of schemes, see [Stacks, 02VW].

1.4.7 Representability of $\mathcal{M}or_S(X,Y)$. Next, we prove that if X is projective and flat over S and Y is also projective, then $\mathcal{M}or_S(X,Y)$ is representable by an open subscheme of $\mathrm{Hilb}_S(X \times_S Y)$. This result is well known and the proof presented here is an expansion on proofs presented in [FGI+05, Theorem 5.23] and the course notes [Oss].

1.4.8 Lemma. *Let $f : X \to Y$ be a morphism of proper schemes over a locally noetherian scheme S. Suppose that X is flat over S and suppose there exists a point $p \in S$ such that the fiber $f_p : X_p \to Y_p$ is an isomorphism. Then there exists an open subscheme $S_0 \hookrightarrow S$ satisfying the following property:*

$$\begin{gathered}\text{for any morphism of locally noetherian schemes } h : S' \to S,\\ h \text{ factors through } S_0 \text{ if and only if the base change}\\ f_{S'} : X_{S'} \hookrightarrow Y_{S'} \text{ is an isomorphism .}\end{gathered} \tag{1.11}$$

Proof. Consider the set

$$A = \{p \in S \mid f_p : X_p \to Y_p \text{ is an isomorphism}\}.$$

If $A = \emptyset$ then the empty scheme trivially satisfy the conditions of the statement. Hence we can assume $A \neq \emptyset$.

It follows from [Gro61, Chapitre III, Proposition 4.6.7(ii)] that for each $p \in A$ there exists an open neighbourhood S_p such that $f|_{X_{S_p}} : X_{S_p} \to Y_{S_p}$ is an isomorphism. Therefore, we define $S_0 \coloneqq \bigcup_{p \in A} S_p$.

Let $h : S' \to S$ be a morphism of locally noetherian schemes and suppose $f'_S : X_{S'} \to Y_{S'}$ is an isomorphism. Then for every point $q \in S'$, we have a morphism

$$\operatorname{Spec} \kappa(q) \to S' \xrightarrow{h} S$$

such that $f_{S',q} : X_{S',q} \to Y_{S',q}$ is an isomorphism. The morphism h induces a morphism $\operatorname{Spec} \kappa(q) \to \operatorname{Spec} \kappa(h(q))$, which is clearly flat and surjective, and such that the diagram

$$\begin{tikzcd} X_q \arrow[r, "f_{S',q}"] \arrow[d] & Y_q \arrow[r] \arrow[d] & \operatorname{Spec} \kappa(q) \arrow[d] \\ X_{h(q)} \arrow[r, "f_{h(q)}"] & Y_{h(q)} \arrow[r] & \operatorname{Spec} \kappa(h(q)) \end{tikzcd}$$

is fibered. Hence, by faithfully flat descent [Gro65, Chapitre II, Proposition 2.7.1], $f_{S',q}$ is an isomorphism if and only if $f_{h(q)}$ is an isomorphism. In other words,

$h(q) \in A \subset S_0$ for all $q \in S'$, that is, $h(S') \subset S_0$. Therefore, h factors through S_0, see for instance [Stacks, 01HD]. The converse is clear since $f_{S'} : X_{S'} \to Y_{S'}$ will be the base change of the morphism $f_{S_0} : X_{S_0} \to Y_{S_0}$ and isomorphisms are stable under base change. $\square$

1.4.9 Theorem. *Let X and Y be schemes projective over S. Then $\mathcal{M}or_S(X,Y)$ is an open subfunctor of $\mathcal{H}ilb((X \times_S Y)/S)$. In particular, if X is flat over S, then $\mathcal{M}or_S(X,Y)$ is representable by an open subscheme of the Hilbert scheme* $\mathrm{Hilb}_S(X \times_S Y)$ *and each irreducible component of* $\mathrm{Mor}_S(X,Y)$ *is of finite type over S.*

Proof. Let $f \in \mathcal{M}or_S(X,Y)(S')$ and let

$$\Gamma_f : X_{S'} \to X_{S'} \times_S Y_{S'}$$

be the graph morphism of f, that is, the unique morphism making the diagram

$$\begin{tikzcd}
X_{S'} \arrow[drr, bend left, "f"] \arrow[dr, "\Gamma_f"] \arrow[ddr, equal, bend right, "\mathrm{id}_X"'] & & \\
& X_{S'} \times_{S'} Y_{S'} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y_{S'} \arrow[d] \\
& X_{S'} \arrow[r] & S'
\end{tikzcd} \tag{1.12}$$

commutative. Since Y is projective, it is in particular separated and therefore Γ_f is a closed immersion, see [GW10, Definition and Proposition 9.7] for instance. Therefore, it defines the closed subscheme $\overline{\mathrm{im}\,(\Gamma_f)}$ on $X_{S'} \times_S Y_{S'}$ isomorphic to $X_{S'}$. Since X is proper and flat over S, so is $X_{S'} \cong \overline{\mathrm{im}\,(\Gamma_f)}$ over S'. In other words, we have defined a map

$$\begin{aligned}
\Theta_{S'} : \mathcal{M}or_S(X,Y)(S') &\longrightarrow \mathcal{H}ilb_S(X \times_S Y)(S') \\
f &\longmapsto \overline{\mathrm{im}\,(\Gamma_f)}.
\end{aligned}$$

Injectivity follows since if f, f' are two morphisms defining the same closed subscheme of $X_{S'} \times_{S'} Y_{S'}$, we have an automorphism $\alpha : X_{S'} \xrightarrow{\sim} X_{S'}$ such that $\Gamma_f = \Gamma_{f'} \circ \alpha$ (see Remark 1.2.1). By the definition of the graph morphism in diagram (1.12) we deduce that

$$\begin{aligned}
f &= \mathrm{pr}_2 \circ \Gamma_f = \mathrm{pr}_2 \circ \Gamma_{f'} \circ \alpha = f' \circ \alpha \quad \text{and} \\
\mathrm{id}_{X_{S'}} &= \mathrm{pr}_1 \circ \Gamma_f = \mathrm{pr}_1 \circ \Gamma_{f'} \circ \alpha = \alpha.
\end{aligned}$$

That is, we deduce that $f = f'$. In other words, $\mathcal{M}or_S(X,Y)$ is a subfunctor of $\mathcal{H}ilb_S(X \times_S Y)$.

It remains to show that $\mathcal{M}or_S(X,Y)$ is open. Before that, notice the following.

Claim. A closed subscheme V in $X_{S'} \times_{S'} Y_{S'}$ is in the image of $\Theta_{S'}$ if and only if $\mathrm{pr}_1(V) \cong X_{S'}$.

Proof of claim. The direct implication is clear from the definition of the graph. For the converse, notice that if $\iota : V \hookrightarrow X_{S'} \times_{S'} Y_{S'}$ is a subscheme such that we have an isomorphism $\beta := \mathrm{pr}_1 \circ \iota : V \xrightarrow{\sim} X_{S'}$, then we define $f = \mathrm{pr}_2 \circ \iota \circ \beta^{-1}$ and it is easy to see that $\Gamma_f = \iota \circ \beta^{-1}$. Then the claim follows from Remark 1.2.1. □

Finally, to prove that $\mathcal{M}or_S(X,Y)$ is open, we use the criterion provided by Proposition A.2.9, which in our case translates to the following. Let S' be an S-scheme and

$$V \in \mathcal{H}ilb_S(X \times_S Y)(S')$$

be a section. Then it suffices to prove that there is an open subscheme $S_0 \hookrightarrow S'$ such that a morphism $h : S'' \to S'$ factors through S_0 if and only if the base change $V_{S''}$ of V with respect to h fitting in the fibered diagram

$$\begin{tikzcd}
V_{S''} \arrow[r] \arrow[d, hook] & V \arrow[d, hook] \\
X_{S''} \times_S Y_{S''} \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & X_{S'} \times_S Y_{S'} \arrow[d, "\mathrm{pr}_1"] \\
X_{S''} \arrow[r] \arrow[d] & X_{S'} \arrow[d] \\
S'' \arrow[r, "h"'] & S'
\end{tikzcd}$$

is in the image of $\Theta_{S''}$. By the claim above, this happens if and only if the composition $V_{S''} \hookrightarrow X_{S''} \times_{S''} Y_{S''} \xrightarrow{\mathrm{pr}_1} X_{S''}$ is an isomorphism.

Therefore, by Lemma 1.4.8 applied to $V \to X_{S'}$, an open subscheme $S_0 \hookrightarrow S'$ satisfying this condition exists, and thus $\mathcal{M}or_S(X,Y)$ is an open subfunctor of $\mathcal{H}ilb_S(X \times_S Y)$. Finally, by Proposition A.2.7, the functor $\mathcal{M}or_S(X,Y)$ is represented by an open subscheme $\mathrm{Mor}_S(X,Y)$ of $\mathrm{Hilb}_S(X \times_S Y)$.

In particular, if we fix a very ample line on $X \times_S Y$, each irreducible component of $\mathrm{Mor}_S(X,Y)$ is a locally closed subscheme contained in a proper subscheme

$\mathrm{Hilb}_S^P(X \times_S Y)$ for some polynomial $P \in \mathbb{Q}[t]$, and therefore it is of finite type over S. $\square$

1.4.10 Universal morphism. Whenever $\mathcal{M}or_S(X, Y)$ is representable by a scheme $\mathrm{Mor}_S(X, Y)$, there exists a universal section:

$$f^{un} \in \mathcal{M}or_S(X, Y)(\mathrm{Mor}_S(X, Y))$$

which we call *universal morphism*. Unwinding the definition in A.1.7, we have that a morphism

$$f^{un} : X \times_S \mathrm{Mor}_S(X, Y) \to Y \times_S \mathrm{Mor}_S(X, Y)$$

is universal if it satisfies the following property: for each S' in $\mathbf{Noe}/S$ and each S'-morphism $f' : X_{S'} \to Y_{S'}$, there exists a unique map $g : S' \to \mathrm{Mor}_S(X, Y)$ such that $\mathcal{M}or_S(X, Y)(g)(f^{un}) = f'$, *i.e.* such that the diagram

$$\begin{tikzcd}
 & & S' \arrow[dd, "g"] & & \\
X_{S'} \arrow[urr, bend left] \arrow[rrrr, "f'"] \arrow[dd] & & & & Y_{S'} \arrow[ull, bend right] \arrow[dd] \\
 & & \mathrm{Mor}_S(X, Y) & & \\
X \times_S \mathrm{Mor}_S(X, Y) \arrow[urr, bend left] \arrow[rrrr, "f^{un}"] & & & & Y \times_S \mathrm{Mor}_S(X, Y) \arrow[ull, bend right]
\end{tikzcd}$$

is commutative and all squares are cartesian.

1.4.11 Corollary. *Let X and Y be schemes projective over S with X flat over S and let $h : S' \to S$ be a morphism of locally noetherian schemes. Then we have a natural isomorphism*

$$\mathrm{Mor}_{S'}(X_{S'}, Y_{S'}) \cong \mathrm{Mor}_S(X, Y) \times_S S'.$$

Proof. The proof follows the same lines as the one of Proposition 1.3.1. Denote $\mathrm{Mor}_S := \mathrm{Mor}_S(X, Y)$ and $\mathrm{Mor}_{S'} := \mathrm{Mor}_{S'}(X_{S'}, Y_{S'})$ and let

$$f_S^{un} \in \mathcal{M}or_S(X, Y)(\mathrm{Mor}_S) \text{ and } f_{S'}^{un} \in \mathcal{M}or_{S'}(X_{S'}, Y_{S'})(\mathrm{Mor}_{S'})$$

be their respective universal morphisms. Since h makes $\mathrm{Mor}_{S'}$ into an S-scheme, the representability of $\mathcal{M}or_S(X, Y)$ implies there exists a unique S-morphism

$h' : \mathrm{Mor}_{S'} \to \mathrm{Mor}_S$ such that

$$\mathcal{M}or_S(X,Y)(h')(f_S^{un}) = f_{S'}^{un}$$

(in the sense of 1.4.10). Let g and g' be the structure morphisms of Mor_S and $\mathrm{Mor}_{S'}$ respectively, and let $\alpha : S'' \to \mathrm{Mor}_S$ and $\beta : S'' \to S'$ be two morphisms such that $h \circ \beta = g \circ \alpha$. In order to conclude the proof, it suffices to prove there is a unique $\gamma : S'' \to \mathrm{Mor}_{S'}$ such that

$$\begin{tikzcd}
S'' \arrow[dr, dashed, "\exists!\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& \mathrm{Mor}_{S'} \arrow[r, "h'"] \arrow[d, "g'"'] & \mathrm{Mor}_S \arrow[d, "g"] \\
& S' \arrow[r, "h"'] & S
\end{tikzcd}$$

is commutative.

By representability of $\mathcal{M}or_S(X,Y)$, the morphism α corresponds to a unique section of $\mathcal{M}or_S(X,Y)(S'')$, that is, a unique morphism $f'' : X_{S''} \to Y_{S''}$ such that $\mathcal{M}or_S(X,Y)(\alpha)(f_S^{un}) = f''$. On the other hand, since β makes both $X_{S''}$ and $Y_{S''}$ into S'-schemes, $f'' \in \mathcal{M}or_{S'}(X,Y)(S'')$ and therefore representability of $\mathcal{M}or_{S'}(X_{S'},Y_{S'})$ implies that there exists $\gamma : S'' \to \mathrm{Mor}_{S'}$ such that $\beta = g' \circ \gamma$ and f'' is given by the pullback of $f_{S'}^{un}$, that is

$$\mathcal{M}or_{S'}(X_{S'},Y_{S'})(\gamma)(f_{S'}^{un}) = f''.$$

Since $g \circ \alpha = h \circ g' \circ \gamma$ we have commutative diagrams

$$
\begin{tikzcd}
 & S'' \arrow[ddd, "\alpha"] & \\
X_{S''} \arrow[ur, bend left] \arrow[rr, "f''"] \arrow[ddd] & & Y_{S''} \arrow[ul, bend right] \arrow[ddd] \\
 & & \\
 & \mathrm{Mor}_S & \\
X \times_S \mathrm{Mor}_S \arrow[ur, bend left] \arrow[rr, "f_S^{un}"] & & Y \times_S \mathrm{Mor}_S \arrow[ul, bend right]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S'' \arrow[dd, "\gamma"] & \\
X_{S''} \arrow[ur, bend left] \arrow[rr, "f''"] \arrow[dd] & & Y_{S''} \arrow[ul, bend right] \arrow[dd] \\
 & \mathrm{Mor}_{S'} \arrow[dd, "h'"] & \\
X \times_S \mathrm{Mor}_{S'} \arrow[ur, bend left] \arrow[rr, "f_{S'}^{un}"] \arrow[dd] & & Y \times_S \mathrm{Mor}_{S'} \arrow[ul, bend right] \arrow[dd] \\
 & \mathrm{Mor}_S & \\
X \times_S \mathrm{Mor}_S \arrow[ur, bend left] \arrow[rr, "f_S^{un}"] & & Y \times_S \mathrm{Mor}_S, \arrow[ul, bend right]
\end{tikzcd}
$$

where every square is cartesian. In particular, we have proved that

$$\mathcal{M}or_S(X,Y)(h' \circ \gamma)(f_S^{un}) = \mathcal{M}or_S(X,Y)(\alpha)(f_S^{un}) = f'',$$

by universal property of f_S^{un} that gives $\alpha = h' \circ \gamma$. $\square$

1.4.12 Lemma. *Let S be a locally noetherian scheme and X be a projective flat scheme over S. Let $\mathbf{PrSch}/S$ be the category of projective schemes over S and Y be an object in $\mathbf{PrSch}/S$. Then the association $Y \mapsto \mathrm{Mor}_S(X,Y)$ is natural in Y. In other words, this association defines a functor*

$$\mathrm{Mor}_S(X,-) : \mathbf{PrSch}/S \to \mathbf{Sch}/S.$$

Proof. In fact, we prove a more general statement at the level of functors. Let $f : W \to Y$ be a morphism of S-schemes and $h : S' \to S$ a morphism in $\mathbf{Noe}/S$, then we have a map

$$\mu_{f,S'} : \mathrm{Hom}_{S'}(X_{S'}, W_{S'}) \longrightarrow \mathrm{Hom}_{S'}(X_{S'}, Y_{S'})$$

taking any $g : X_{S'} \to W_{S'}$ to the composition $f_{S'} \circ g : X_{S'} \to Y_{S'}$, where $f_{S'}$ is the pullback of f with respect to h fitting in the commutative diagram

$$\begin{tikzcd}
S' \arrow[dd, "h"'] & & X_{S'} \arrow[ll] \arrow[dl, "g"] \arrow[dd] \arrow[dr, bend left, "f_{S'}\circ g"] & \\
& W_{S'} \arrow[ul] \arrow[rr, "f_{S'}"'] \arrow[dd] & & Y_{S'} \arrow[dd] \\
S & & X \arrow[ll] & \\
& W \arrow[ul] \arrow[rr, "f"] & & Y.
\end{tikzcd}$$

In particular, by the definition of the functors $\mathcal{M}or_S(X, W)$ and $\mathcal{M}or_S(X, Y)$, the maps $\mu_{f,S'}$ define a natural transformation

$$\mu_f : \mathcal{M}or_S(X, W) \to \mathcal{M}or_S(X, Y). \tag{1.13}$$

That is, for any $S'' \to S'$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}or_S(X, W)(S') \arrow[r, "\mu_{f,S'}"] \arrow[d] & \mathcal{M}or_S(X, Y)(S') \arrow[d] \\
\mathcal{M}or_S(X, W)(S'') \arrow[r, "\mu_{f,S''}"'] & \mathcal{M}or_S(X, Y)(S''),
\end{tikzcd}$$

where the vertical arrows are restriction morphisms defined in 1.4.1. If both W and Y are projective, by Theorem 1.4.9, both functors $\mathcal{M}or_S(X, W)$ and $\mathcal{M}or_S(X, Y)$ are represented by schemes $\mathrm{Mor}_S(X, W)$ and $\mathrm{Mor}_S(X, Y)$ respectively. Therefore, it follows from Yoneda lemma (A.1.4) that the natural transformation μ_f corresponds uniquely to a morphism

$$\mathrm{Mor}_S(X, W) \to \mathrm{Mor}_S(X, Y).$$

$\square$

1.4.13 Alternative definition of $\mathcal{M}or$. Let X and Y be any schemes over S and $S' \to S$ be any base change morphism. Notice there is a bijection of sets

$$\alpha_{S'} : \mathrm{Hom}_{S'}(X_{S'}, Y_{S'}) \to \mathrm{Hom}_S(X_{S'}, Y)$$

taking any $f' : X_{S'} \to Y_{S'}$ to the composition of f with the first projection of $Y_{S'}$. The inverse map is given by taking any S-morphism $g : X_{S'} \to Y$ to the unique morphism defined by universal property of $Y_{S'}$. That is, we take g to the unique f making the following diagram commutative:

$$
\begin{tikzcd}
X_{S'} \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left] \arrow[ddr, bend right, "g"'] & & \\
& Y_{S'} \arrow[r] \arrow[d] & S' \arrow[d] \\
& Y \arrow[r] & S.
\end{tikzcd}
$$

Also, for any $S'' \to S'$ we have morphisms

$$
\begin{tikzcd}
\mathrm{Hom}_{S'}(X_{S'}, Y_{S'}) \arrow[r, "\alpha_{S'}"] \arrow[d] & \mathrm{Hom}_S(X_{S'}, Y) \arrow[d] \\
\mathrm{Hom}_{S''}(X_{S''}, Y_{S''}) \arrow[r, "\alpha_{S''}"] & \mathrm{Hom}_S(X_{S''}, Y)
\end{tikzcd}
$$

induced by pullback. In other words, these bijections are natural in S'. Thus, the functor $\mathcal{M}or_S(X, Y)$ is naturally isomorphic to the functor

$$\mathcal{M}or'_S(X, Y) : (\mathbf{Noe}/S)^{op} \to \mathbf{Set}$$

defined at the level of objects by $\mathcal{M}or'_S(X, Y)(S') := \mathrm{Hom}_S(X_{S'}, Y)$, and for any S-morphism $h : S'' \to S'$ by

$$
\begin{aligned}
\mathcal{M}or'_S(h) : \mathrm{Hom}_S(X_{S'}, Y) &\longrightarrow \mathrm{Hom}_S(X_{S''}, Y) \\
f' &\longmapsto f'',
\end{aligned}
$$

where f'' is the composition of the natural projection $X_{S''} \to X_{S'}$ with f'. Equivalently, f'' is the unique morphism making the following diagram commutative

$$
\begin{tikzcd}
X_{S''} \arrow[r] \arrow[d] \arrow[rrr, bend left, "f''"] & X_{S'} \arrow[r] \arrow[d] \arrow[rr, bend left, "f'"] & X \arrow[d] & Y \arrow[dl] \\
S'' \arrow[r] & S' \arrow[r] & S, &
\end{tikzcd}
$$

where all squares are cartesian.

1.4.14 Universal morphism as an evaluation morphism. Since we have the natural isomorphism of functors $\mathcal{M}or_S(X,Y) \cong \mathcal{M}or'_S(X,Y)$, if $\mathcal{M}or_S(X,Y)$ is representable by a scheme $\mathrm{Mor}_S(X,Y)$, then it also represents the functor $\mathcal{M}or'_S(X,Y)$. Hence, there exists a universal section

$$\mathrm{ev} \in \mathcal{M}or'_S(X,Y)(\mathrm{Mor}_S(X,Y)).$$

In other words, an S-morphism

$$\mathrm{ev} : X \times \mathrm{Mor}_S(X,Y) \to Y$$

satisfying the following property: for each S' in $\mathbf{Noe}/S$ and each S-morphism $f' : X_{S'} \to Y$, there exists a unique map $g : S' \to \mathrm{Mor}_S(X,Y)$ such that

$$\mathcal{M}or'_S(X,Y)(g)(\mathrm{ev}) = f',$$

i.e. such that the diagram

$$\begin{tikzcd}
X_{S'} \arrow[r] \arrow[d] \arrow[rr, bend left, "f'"] & X \times_S \mathrm{Mor}_S(X,Y) \arrow[r, "\mathrm{ev}"] \arrow[d] & Y \\
S' \arrow[r, "g"] & \mathrm{Mor}_S(X,Y) &
\end{tikzcd}$$

commutes, where the square is the base change by g. The morphism ev is said to be the *evaluation morphism* of $\mathrm{Mor}_S(X,Y)$.

1.5 Properties of the functor of morphisms

Let X be a projective and flat S-scheme. Recall that in Lemma 1.4.12 we have defined a functor

$$\mathrm{Mor}_S(X,-) : \mathbf{PrSch}/S \to \mathbf{Noe}/S \tag{1.14}$$

taking any projective S-scheme Y to the scheme $\mathrm{Mor}_S(X,Y)$. In this section we prove useful properties of this functor. For instance, we will see that it behaves well with fiber products and that it preserves closed and open embeddings. Many of these properties can actually be proven at the level of functors $\mathcal{M}or_S(X, Y)$. A good instance of that is the lemma below.

1.5.1 Lemma (Functorial properties). *Let S be a scheme. The following properties hold:*

1. *Let $\{X_i\}_{i\in I}$ be a finite family of schemes over S and Y be a scheme over S. Then*
$$\mathcal{M}or_S\left(\coprod_{i\in I} X_i, Y\right) \cong \prod_{i\in I} \mathcal{M}or_S(X_i, Y),$$
where the right hand side denotes the fiber product over h_S;

2. *Let $\mathbf{I}$ be a category and $\mathcal{F} : \mathbf{I} \to \mathbf{Sch}/S$ be a functor such that $\lim \mathcal{F}$ exists in $\mathbf{Sch}/S$. Then*
$$\mathcal{M}or_S(X, \lim \mathcal{F}) \cong \lim \mathcal{M}or_S(X, \mathcal{F}),$$
where $\mathcal{M}or_S(X, \mathcal{F}) : \mathbf{I} \to \mathbf{Psh}(\mathbf{Sch/S})$ is the functor $i \mapsto \mathcal{M}or_S(X, \mathcal{F}(i))$. In other words, $\mathcal{M}or_S(X, -)$ preserves with limits.

Proof. Both properties follow easily from the analogous properties of the bifunctor $\mathrm{Hom}_S(-,-)$. Indeed, for 1, notice that

$$\begin{aligned}\mathcal{M}or_S\left(\coprod_{i\in I} X_i, Y\right)(S') &= \mathrm{Hom}_{S'}\left(\left(\coprod_{i\in I} X_i\right) \times_S S', Y_{S'}\right) \cong \mathrm{Hom}_{S'}\left(\coprod_{i\in I}(X_i \times_S S'), Y_{S'}\right)\\ &\cong \prod_{i\in I} \mathrm{Hom}_{S'}\left(X_i \times_S S', Y_{S'}\right) \cong \prod_{i\in I} \mathcal{M}or_S\left(X_i, Y\right)(S').\end{aligned}$$

(see Comment 1.5.2 below). Similarly for 2, for any S-scheme S' consider the functors

$$\begin{aligned}\mathcal{F}_{S'} : \mathbf{I} &\longrightarrow \mathbf{Sch}/S'\\ i &\longmapsto \mathcal{F}(i) \times_S S'\end{aligned}$$

and for each S-scheme S' we define the functors

$$\begin{aligned}\mathrm{Hom}_{S'}(X_{S'}, \mathcal{F}_{S'}) : \mathbf{I} &\longrightarrow \mathbf{Set}\\ i &\longmapsto \mathrm{Hom}_{S'}(X_{S'}, \mathcal{F}(i) \times_S S').\end{aligned}$$

Then it follows that

$$\begin{aligned}\mathcal{M}or_S(X, \lim \mathcal{F})(S') &= \mathrm{Hom}_{S'}(X_{S'}, (\lim \mathcal{F}) \times_S S') \\ &\cong \mathrm{Hom}_{S'}(X_{S'}, \lim \mathcal{F}_{S'}) \cong \lim \mathrm{Hom}_{S'}(X_{S'}, \mathcal{F}_{S'}) \\ &\cong (\lim \mathcal{M}or_S(X, \mathcal{F}))\,(S').\end{aligned}$$

It is straightforward to check naturality on S' in both cases, since limits and colimits of presheaves are defined sectionwise. □

1.5.2 Comment. In general, if S is a scheme, $\{X_i\}_{i\in I}$ an arbitrary collection of schemes over S and $Y \to S$ a morphism, then

$$\left(\coprod_{i\in I} X_i\right) \times_S Y \cong \coprod_{i\in I}(X_i \times_S Y).$$

Indeed, recall that $X_i \hookrightarrow \coprod_{i\in I} X_i$ is an open immersion and $\{X_i\}_{i\in I}$ is an open cover of $\coprod_{i\in I} X_i$. Since open immersions are stable under base change we have

$$\iota_i : X_i \times_S Y \hookrightarrow \left(\coprod_{i\in I} X_i\right) \times_S Y$$

are also open immersions, and moreover, the collection $\{\iota_i(X_i \times_S Y)\}_{i\in I}$ is an open cover of $\left(\coprod_{i\in I} X_i\right) \times_S Y$. Therefore, the canonical morphism

$$\coprod_{i\in I}(X_i \times_S Y) \to \left(\coprod_{i\in I} X_i\right) \times_S Y$$

obtained by the universal property of coproducts (or equivalently by gluing all ι_i along empty intersections) is a surjective open immersion and hence an isomorphism.

1.5.3 Remark. We will use many times the following particular case of Lemma 1.5.1. Let X be an S-scheme and $Y \to Y''$, $Y' \to Y''$ be morphisms of schemes over S. We have a canonical isomorphism of functors

$$\mathcal{M}or_S(X, Y \times_{Y''} Y') \cong \mathcal{M}or_S(X, Y) \times_{\mathcal{M}or_S(X,Y'')} \mathcal{M}or_S(X, Y').$$

If X is projective and flat over S, and Y, Y' and Y'' are quasi-projective over S, we have the analogous isomorphism of schemes of morphisms. In particular, if

$Y'' = S$ we have $\mathrm{Mor}_S(X, S) \cong S$, therefore

$$\mathrm{Mor}_S(X, Y \times_S Y') \cong \mathrm{Mor}_S(X, Y) \times_S \mathrm{Mor}_S(X, Y').$$

1.5.4 Lemma. *Let X be a separated S-scheme, $\iota : W \hookrightarrow Y$ be an immersion of schemes over S and $\mu_\iota : \mathcal{M}or_S(X, W) \hookrightarrow \mathcal{M}or_S(X, Y)$ be the induced morphism of functors defined in 1.13. Consider $\iota_X : X \times_S W \hookrightarrow X \times_S Y$ to be the induced immersion on the base change and let $\eta_{\iota_X} : \mathcal{H}ilb_S(X \times_S W) \hookrightarrow \mathcal{H}ilb_S(X \times_S Y)$ be the induced morphism of functors defined in 1.3.3. Then the square*

$$\begin{tikzcd}
\mathcal{M}or_S(X, W) \arrow[r, hook, "\mu_\iota"] \arrow[d, hook] & \mathcal{M}or_S(X, Y) \arrow[d, hook] \\
\mathcal{H}ilb_S(X \times_S W) \arrow[r, hook, "\eta_{\iota_X}"] & \mathcal{H}ilb_S(X \times_S Y),
\end{tikzcd} \tag{1.15}$$

where vertical arrows are the natural inclusions in Theorem 1.4.9, is commutative and cartesian.

Proof. Recall that for each S-scheme S', the natural inclusions $\Theta_{S'}$ defined in the proof of Theorem 1.4.9 take each morphism $f \in \mathcal{M}or_S(X, Y)(S')$ to the closed subscheme defined by the graph morphism $\Gamma_f : X_{S'} \hookrightarrow (X \times_S Y)_{S'}$, which is a closed immersion as X is separated. Moreover, the natural morphism

$$\eta_{\iota_X, S'} : \mathcal{H}ilb_S(X \times_S W)(S') \hookrightarrow \mathcal{H}ilb_S(X \times_S Y)(S')$$

takes any closed subscheme of $(X \times_S W)_{S'}$ defined by an immersion

$$V \hookrightarrow (X \times_S W)_{S'}$$

to a subscheme defined by the composition

$$V \hookrightarrow (X \times_S W)_{S'} \overset{\iota_{X,S'}}{\hookrightarrow} (X \times_S Y)_{S'}.$$

Let $g \in \mathcal{M}or_S(X, W)(S')$ and denote

$$f := \mu_{\iota, S'}(g) = \iota_{S'} \circ g,$$

and consider the commutative diagram

$$\begin{tikzcd}
X_{S'} \arrow[d, hook, "\Gamma_g"'] \arrow[dr, "g"] & \\
(X \times_S W)_{S'} \arrow[r, "\pi"] \arrow[d, hook, "\iota_{X,S'}"] & W_{S'} \arrow[d, hook, "\iota_{S'}"'] \\
(X \times_S Y)_{S'} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & Y_{S'} \arrow[d] \\
X_{S'} \arrow[r] & S',
\end{tikzcd} \tag{1.16}$$

where all squares are cartesian. Note that the composition

$$X_{S'} \overset{\Gamma_g}{\hookrightarrow} (X \times_S W)_{S'} \overset{\iota_{X,S'}}{\hookrightarrow} (X \times_S Y)_{S'} \to X_{S'}$$

is the identity on $X_{S'}$. To check that (1.15) commutes, it suffices to check that $\Gamma_f = \Gamma_g \circ \iota_{X,S'}$, but this follows from the uniqueness of the graph morphism, since

$$\mathrm{pr}_1 \circ \iota_{X,S'} \circ \Gamma_g = \mathrm{id}_{X_{S'}} \quad \text{and} \quad \mathrm{pr}_2 \circ \iota_{X,S'} \circ \Gamma_g = \iota_{S'} \circ g = f.$$

Moreover, we claim that (1.15) is cartesian. By Definition A.2.2, in order to show this, it suffices to show that for each S-scheme S' and each pair

$$(V, f) \in \mathcal{H}ilb_S(X \times_S W)(S') \times_{\mathcal{H}ilb_S(X \times_S Y)(S')} \mathcal{M}or_S(X, Y)(S'),$$

there exists a unique morphism $g : X_{S'} \to W_{S'}$ satisfying two conditions:

1. the closed subscheme defined by the closed immersion

$$\eta_{\iota_X, S'}(\Gamma_g) = \iota_{X,S'} \circ \Gamma_g : X_{S'} \to (X \times_S Y)_{S'}$$

   is the same as the one defined by

$$V \hookrightarrow (X \times_S W)_{S'} \overset{\iota_{X,S'}}{\hookrightarrow} (X \times_S Y)_{S'};$$

2. $\mu_{\iota,S'}(g) = \iota_{S'} \circ g = f$.

Notice that by definition of the pair (V, f), we have that the closed subscheme defined by the composition

$$V \hookrightarrow (X \times_S W)'_S \overset{\iota_{X,S'}}{\hookrightarrow} (X \times_S Y)_{S'}$$

is the same as the one defined by $\Gamma_f : X_{S'} \hookrightarrow (X \times_S Y)_{S'}$. Therefore, we have an isomorphism $V \cong X_{S'}$. Define g as the composition

$$g : X_{S'} \xrightarrow{\cong} V \hookrightarrow (X \times_S W)_{S'} \xrightarrow{\pi} W_{S'}.$$

By the definition of graph morphism, we obtain a commutative diagram such as (1.16). In other words, g satisfies the conditions 1 and 2 above. To see that g is unique, just notice that if g' was another morphism satisfying condition 2, we would have

$$\iota_{S'} \circ g = \iota_{S'} \circ g'.$$

Therefore $g = g'$, since immersions are monomorphisms in the category of schemes (see [Stacks, 01L7]). □

1.5.5 Proposition. *Let X be a separated S-scheme, let $i : Z \hookrightarrow Y$ be a closed immersion and $j : U \hookrightarrow Y$ be an open immersion of schemes over S. Then the morphisms*

$$\mu_i : \mathcal{M}or_S(X, Z) \to \mathcal{M}or_S(X, Y),$$
$$\mu_j : \mathcal{M}or_S(X, U) \to \mathcal{M}or_S(X, Y)$$

make $\mathcal{M}or_S(X, Z)$ and $\mathcal{M}or_S(X, U)$ respectively to be a closed and an open subfunctor of $\mathcal{M}or_S(X, Y)$. In particular, if X is projective and flat over S and Y is quasi-projective over S, the morphisms

$$\begin{aligned} \mathrm{Mor}_S(X, i) : \mathrm{Mor}_S(X, Z) \to \mathrm{Mor}_S(X, Y) \\ \mathrm{Mor}_S(X, j) : \mathrm{Mor}_S(X, U) \to \mathrm{Mor}_S(X, Y) \end{aligned} \tag{1.17}$$

induced by applying the functor $\mathrm{Mor}_S(X, -)$ (1.14) to i and j, are a closed and an open immersion respectively.

Proof. Let $j_X : X \times_S U \hookrightarrow X \times_S Y$ be the open immersion induced by the base change of j. By Proposition 1.3.5, we have an open subfunctor

$$\eta_{j_X} : \mathcal{H}ilb_S(X \times_S U) \to \mathcal{H}ilb_S(X \times_S Y).$$

Thus, by Lemma 1.5.4, μ_j is also an open subfunctor.

Moreover, if X is projective and flat over S and Y is projective over S, all the functors involved are representable. By definition, μ_j induces the morphism (1.17), which is an open immersion by A.2.7.

The proof is exactly the same for the closed immersion $i : Z \hookrightarrow Y$, obtained by judiciously replacing j by i, U by Z, the words "open" by "closed" and applying Remark 1.3.6 instead of Proposition 1.3.5. $\square$

1.5.6 Remark. It follows from Proposition 1.5.5, that the functor (1.14) is defined on the category $\mathbf{QPrSch}/S$ of quasiprojective schemes over S. That is, we have the functor

$$\mathrm{Mor}_S(X, -) : \mathbf{QPrSch}/S \longrightarrow \mathbf{Sch}/S.$$

Chapter 2

Schemes of rational curves

We have defined schemes parametrizing morphisms of schemes over a base in the previous chapter. For this chapter we restrict our attention to $S = \operatorname{Spec} k$ where k is an algebraically closed field. Let us first fix the notation

$$\operatorname{Mor}(\mathbb{P}^1_k, X) \coloneqq \operatorname{Mor}_{\operatorname{Spec} k}(\mathbb{P}^1_k, X)$$

for the scheme parametrizing morphisms from $\mathbb{P}^1_k$ to a projective variety over k.

We have seen in Lemma 1.1.10 that $C \hookrightarrow X$ is a rational curve if and only if it is the image of a non-constant morphism $\mathbb{P}^1_k \to X$. Thus, we will abuse in language and say that $\operatorname{Mor}(\mathbb{P}^1_k, X)$ parametrizes rational curves on X.

By definition, any morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ corresponds to a k-point on $\operatorname{Mor}(\mathbb{P}^1_k, X)$. This k-point will be denoted

$$[f] \in \operatorname{Mor}(\mathbb{P}^1_k, X).$$

In section 2.1 we give a well known heuristic description of the scheme $\operatorname{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k)$ including its partition in terms of the degrees of the morphisms $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ following [Deb13].

In section 2.2 we consider the following question: let X be a projective variety over k and

$$\sigma : \operatorname{Bl}_Z(X) \to X$$

be the blow-up of Z at a closed subscheme. Can we find components of $\operatorname{Mor}(\mathbb{P}^1_k, \operatorname{Bl}_Z(X))$ using the induced morphism

$$\operatorname{Mor}(\mathbb{P}^1_k, \operatorname{Bl}_Z(X)) \to \operatorname{Mor}(\mathbb{P}^1_k, X)?$$

We use properties of blow-ups to describe a rough partition of $\operatorname{Mor}(\mathbb{P}^1_k, \operatorname{Bl}_Z(X))$. When $Z = \{p_1, \ldots, p_r\}$ is a finite collection of points in $\mathbb{P}^n_k$, we are able to expand the heuristic description of section 2.1. We determine a partition depending on the degrees of curves in $\mathbb{P}^n_k$ and their multiplicities at each point p_i, see Theorem 2.2.11. It is natural to look at the case of the blow-ups of $\mathbb{P}^2_k$ and in particular, Del Pezzo surfaces.

In section 2.3 we relate the scheme of morphisms and linear systems. More precisely: for each positive integer d and when $\operatorname{char} k = 0$ we define a morphism Ξ_d from $\operatorname{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)$ to the complete linear system of divisors of degree d. We prove that it is invariant with respect to an $\operatorname{Aut}(\mathbb{P}^1_k)$-action and its fiber is irreducible of dimension 3.

In section 2.4 we combine all of the above with results of Daigle and Melle-Hernández [DM12] on rational linear systems to describe the image of Ξ_d. This allows us to find components in $\operatorname{Mor}(\mathbb{P}^1_k, X)$ containing embedded resolutions of singularities of curves in the $\mathbb{P}^2_k$, see Theorem 2.3.5. To conclude the chapter we use the classification of Gimigliano, Harbourne and Idà [GHI13] of rational curves whose singularities are resolved by blowing-up points in general position. This classification allows us to find all the possible components given in Theorem 2.4.8 when $r \leq 7$. As an example, we provide a complete list of these components on a smooth cubic surface in Example 2.4.10.

2.1 Rational curves on projective spaces

Morphisms $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ can be described in a very elementary fashion. This elementary description will allow us to give a heuristic description of the scheme $\operatorname{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k)$ as a disjoint union of open subsets of projective spaces. This section is a slightly more detailed explanation of [Deb13, §2.1].

2.1.1 Regular morphisms from the projective line. Recall that a *rational map* $f : \mathbb{P}^1_k \dashrightarrow \mathbb{P}^n_k$ is an equivalence class of maps

$$(u : v) \mapsto (F_0(u, v) : \cdots : F_n(u, v))$$

where each F_i is a homogeneous polynomial in $k[u, v]$ and $\deg F_i = d$ for all i. We say two tuples $(F_0 : \cdots : F_n)$ and $(G_0 : \cdots G_n)$ are equivalent if

$$F_i G_j = F_j G_i \tag{2.1}$$

for all $0 \leq i, j \leq n$, see for instance [Sha13a, p. 51].

Claim. The relations (2.1) hold if and only if there exists a homogeneous polynomial $H \in k[u, v]$ such that $F_i = G_i \cdot H$.

Proof of claim. If the relations (2.1) hold, and since $k[u,v]$ is an UFD, we can assume that the collection $(F_0,\ldots,F_n)$ consists of polynomials which do not have any factor in common. In that case the relations (2.1) imply that F_i divides G_i, i.e., there exists an index i and a homogeneous polynomial $H \in k[u,v]$ such that $G_i = H \cdot F_i$. The relations also imply that $G_j = H \cdot F_j$ for the other indexes. The converse is clear. □

We say a rational map $f : \mathbb{P}^1_k \dashrightarrow \mathbb{P}^n_k$ is *regular* if for each $(a:b) \in \mathbb{P}^1_k$ there exists a representative $(F_0,\ldots,F_n)$ for f such that there exists at least one i such that $F_i(a,b) \neq 0$, see [Sha13a, p. 48]. In the proposition below we see that any rational map from $\mathbb{P}^1_k$ is in fact regular.

2.1.2 Lemma. *Let k be an algebraically closed field. Let $(F_0,\ldots,F_n)$ be a collection of homogeneous polynomials in $k[u,v]$. Then they have a factor in common if and only if $V(F_0,\ldots,F_n) \neq \emptyset$ in $\mathbb{P}^1_k$.*

Proof. Recall that each F_i splits into the product of linear factors over k. Therefore, if they have a factor in common it is clear that they have a nontrivial common root.

Conversely, if they have a common root $(\alpha_1 : \alpha_2)$, without loss of generality we can assume that $\alpha_2 \neq 0$ and therefore α_1/α_2 is a root of $f_i(u) := F_i(u,1)$ for each i.

Let $m(u)$ be the minimal polynomial of α_1/α_2 over k. Then $m(u)$ divides all of the $f_i(u)$, i.e. $f_i(u) = g_i(u)m(u)$. Once we homogenize the polynomials we obtain

$$F_i(u,v) = v^{\deg(f_i)} f_i(u/v) = v^{\deg(g_i)} g_i(u/v) \cdot v^{\deg(m)} m(u/v) = G_i(u,v) \cdot M(u,v)$$

for homogeneous polynomials G_i and M in $k[u,v]$. □

2.1.3 Remark. We conclude that for each regular morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ there exists a unique representative $(F_1,\ldots,F_n)$ for which the polynomials do not have factors in common.

Notice that the definition in 2.1.1 above is given in terms of classical language of varieties. It follows that, over an algebraically closed field k, the regular morphisms $\mathbb{P}^1_k \to \mathbb{P}^n_k$ correspond bijectively to morphisms of schemes between $\mathbb{P}^1_k$ and $\mathbb{P}^n_k$, see [Sha13b, Example 5.19, p.29]. More precisely there is a fully faithful functor between the category of varieties over k to the category of schemes over k (see also [Har77, Proposition II.2.1.6]). Hence from now on we will make no distinction between the terms regular morphism and morphism.

2.1.4 Lemma. *Let $F := (F_0, \dots, F_n)$ be a collection of homogeneous polynomials of degree d in $k[u,v]$. The collection does* not *have factors in common if and only if there exists an integer $m \geq d$ such that the k-linear map*

$$\begin{aligned}\theta_F : \bigoplus_{i=0}^{n} k[u,v]_{m-d} &\longrightarrow k[u,v]_m \\ \oplus_{i=0}^{n} G_i &\longmapsto \textstyle\sum_{i=0}^{n} F_i G_i\end{aligned} \tag{2.2}$$

is surjective, where $k[u,v]_m$ is the k-vector space generated by monomials of degree m.

Proof. By Lemma 2.1.2, the polynomials in the collection F_i do not have factors in common if and only $V(F_0, \dots, F_n) = \emptyset$ in $\mathbb{P}^1_k$. By the projective Nullstellensatz this means

$$(u,v) \subset \sqrt{(F_0, \dots, F_n)}.$$

Equivalently, it means that there exists an integer m such that

$$(u,v)^m = (u^m, u^{m-1}v, \dots, v^m) \subset (F_0, \dots, F_n). \tag{2.3}$$

If there exists an integer m such that (2.3) holds, then for each $0 \leq j \leq m$, there exists a collection of polynomials $\left\{G_i^{(j)}\right\}_{0 \leq 1 \leq n}$ in $k[u,v]$ such that

$$\sum_{i=0}^{n} G_i^{(j)} F_i = u^{m-j} v^j, \tag{2.4}$$

which are a fortiori homogeneous of degree $\deg G_i^{(j)} = m - d$. Thus the k-linear map (2.2) is surjective. Conversely if we have relations (2.4), then (2.3) holds. $\square$

2.1.5 Consider the morphism

$$\begin{aligned} f : \mathbb{P}^1_k &\longrightarrow \mathbb{P}^n_k \\ (u:v) &\longmapsto (F_0(u,v) : \dots : F_n(u,v)), \end{aligned} \tag{2.5}$$

where $(F_0(u,v) : \dots : F_n(u,v))$ is the unique representative of f for which the F_i are homogeneous polynomials of degree d with no common factor, as described in (2.1.1). Then we can write each polynomial in the collection as

$$F_i(u,v) = b_{i0}u^d + b_{i1}u^{d-1}v + \dots + b_{id}v^d,$$

and therefore, f corresponds to a unique point in $\mathbb{P}_k^{(n+1)(d+1)-1}$ given by the coordinates $\{b_{ij}\}$. By Lemma 2.1.4, the polynomials $F_0, \dots, F_n$ have no factor in common if and only if there exists an integer $m \geq d$ such that the map θ_F in (2.2) is surjective. We will choose a basis for the source and target of θ_F to find a convenient description in matrix form of this linear map.

Consider the ring of polynomials $k[x_{00}, x_{01}, \ldots, x_{nd}]$ and define $M_\theta(x_{00}, \ldots, x_{nd})$ to be the $(n+1)(m-d+1) \times (m+1)$ matrix:

$$\begin{pmatrix}
x_{00} & & & & x_{10} & & & & & x_{n0} & & & \\
x_{01} & x_{00} & & & x_{11} & x_{10} & & & & x_{n1} & x_{n0} & & \\
x_{02} & x_{01} & \ddots & & x_{12} & x_{11} & \ddots & & & x_{n2} & x_{n1} & \ddots & \\
\vdots & \vdots & \ddots & x_{00} & \vdots & \vdots & \ddots & x_{10} & \cdots & \vdots & \vdots & \ddots & x_{n0} \\
x_{0d} & x_{0(d-1)} & \ddots & \vdots & x_{1d} & x_{1(d-1)} & \ddots & \vdots & & x_{nd} & x_{n(d-1)} & \ddots & \vdots \\
 & x_{0d} & \ddots & \vdots & & x_{1d} & \ddots & \vdots & & & x_{nd} & \ddots & \vdots \\
 & & \ddots & \vdots & & & \ddots & \vdots & & & & \ddots & \vdots \\
 & & & x_{0d} & & & & x_{1d} & & & & & x_{nd}
\end{pmatrix} \tag{2.6}$$

whose entries consist of monomials in $k[x_{00}, \ldots, x_{nd}]$. Let both

$$k[u,v]_{m-d} \ \text{ and } \ k[u,v]_m$$

have the usual bases

$$\{u^{m-d}, u^{m-d-1}v, \ldots, v^{m-d}\} \ \text{ and } \ \{u^m, u^{m-1}v, \ldots, v^m\}$$

respectively. Then $\bigoplus_{i=0}^n k[u,v]_{m-d}$ has a basis consisting of $n+1$ copies of the monomial basis $\{u^{m-d}, u^{m-d-1}v, \ldots, v^{m-d}\}$. The corresponding matrix of θ_F under this choice of bases is given by the matrix $M_\theta(b_{00}, b_{01}, \ldots, b_{nd})$ with each entry monomial computed at the point

$$(b_{00} : \ldots : b_{nd}) \in \mathbb{P}_k^{(n+1)(d+1)-1} = \mathbb{P}_k^{nd+n+d}.$$

Notice that θ_F is surjective if and only if the rank of $M_\theta(b_{00}, \ldots, b_{nd})$ is $m+1$.

Now suppose that the collection $F = (F_1, \ldots, F_n)$ has factors in common. Then $\operatorname{rk} \theta_F < m+1$, that is, all $(m+1) \times (m+1)$ minors of $M_\theta(b_{00}, \ldots, b_{nd})$ vanish. The reasoning above shows that a collection $F = (F_1, \ldots, F_n)$ has factors in common if and only if the corresponding point in $\mathbb{P}_k^{nd+d+n}$ belongs to the variety cut out by the $(m+1) \times (m+1)$ minors of $M_\theta(x_{00}, \ldots, x_{nd})$. Let us denote this variety by V_d. Then we have just established a set theoretical bijection between regular morphisms $f : \mathbb{P}_k^1 \to \mathbb{P}_k^n$ given by collections $(F_1, \ldots, F_n)$ with no common factors, and the points of the quasi-projective variety

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k) := \mathbb{P}^{nd+n+d}_k \smallsetminus V_d.$$

In [Deb13] we find the analogous construction for fields which are not algebraically closed.

2.1.6 Proposition ([Deb13, p. 39])**.** *Let k be a field (not necessarily algebraically closed). Then we have a partition*

$$\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k) \cong \coprod_{d \in \mathbb{N}} \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k),$$

where each $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ is an open subset of $\mathbb{P}^{nd+n+d}_k$. In particular, each $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ is irreducible, nonsingular and has dimension $nd+n+d$.

2.1.7 Remark. Let $X \hookrightarrow \mathbb{P}^n_k$ be a closed immersion of schemes and define $\mathrm{Mor}_d(\mathbb{P}^1_k, X)$ to be the scheme fitting in the fibered diagram

$$\begin{tikzcd}
\mathrm{Mor}_d(\mathbb{P}^1_k, X) \arrow[r] \arrow[d, hook] & \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k) \arrow[d, hook] \\
\mathrm{Mor}(\mathbb{P}^1_k, X) \arrow[r] & \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k).
\end{tikzcd}$$

By Propositions 2.1.6 and 1.5.5 there is a partition

$$\mathrm{Mor}(\mathbb{P}^1_k, X) \cong \coprod_{d \in \mathbb{N}} \mathrm{Mor}_d(\mathbb{P}^1_k, X)$$

with $\mathrm{Mor}_d(\mathbb{P}^1_k, X)$ as a closed subscheme of $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$.

2.1.8 Remark. The heuristic description of $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n)$ above might be tempt us to try to use similar descriptions for $\mathrm{Mor}(\mathbb{P}^m_k, \mathbb{P}^n_k)$ for $m > 1$. Notice that the variety $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ is defined by the open condition induced by collections of $n+1$ polynomials in $k[u,v]_d$ not having factors in common. In this case we have by Lemma 2.1.2 that this condition coincides with these polynomials defining regular maps from $\mathbb{P}^1_k$. However the condition is not sufficient for regularity for $m > 1$. Indeed an easy counter example is given by the classical Cremona transformation

$$\begin{aligned} \mathbb{P}^2_k &\dashrightarrow \mathbb{P}^2_k \\ (x:y:z) &\mapsto (xy:xz:yz). \end{aligned}$$

2.1.9 Remark. Notice that if $[f] \in \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ is a k-point, then the morphism f is given by a collection of polynomials $F_i \in k[u,v]_d \cong H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d))$ with no factors in common. In particular, by the proof of [Har77, Theorem II.7.1(b)] we have that f is a morphism such that $f^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(d)$.

2.1.10 Degrees and field extensions. Let $[f] \in \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ be a k-point corresponding to a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ and let C be its scheme theoretic image. By Remark 2.1.9 we have $f^*\mathcal{O}_{\mathbb{P}^n_k}(1) \cong \mathcal{O}_{\mathbb{P}^1_k}(d)$, hence we have seen in Example 1.1.13 that if f is birational onto its image if and only if $d = \deg C$, that is, d is the leading coefficient of the Hilbert polynomial of C with respect to its embedding in $\mathbb{P}^n_k$. We say that f is *generically one-to-one* if f is birational onto its image.

Now suppose that f is not generically one-to-one, and let $\nu : \mathbb{P}^1_k \to C$ be the normalization of C. Then, there is a unique morphism g such that $f = \nu \circ g$. In particular, $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ corresponds uniquely to a pair of polynomials $G_0, G_1 \in k[u,v]$ of degree m_0 with no factors in common. It follows that m_0 divides d, and since ν is a birational morphism ([Stacks, 0BXC]), it follows that $d_0 \coloneqq d/m_0$ is the degree of C, in other words

$$d = m_0 \deg C. \tag{2.7}$$

For any $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ and $C = \overline{\mathrm{im}\,(f)}$ we will define the *degree* of f to be

$$\deg(f) = m_0 \deg C.$$

The morphism g induces a morphism between two copies of the function field of $\mathbb{P}^1_k$ denoted $g^* : k(t) \hookrightarrow k(t)$, and the integer m_0 is nothing but the degree of the field extension $[g^*(k(t)) : k(t)]$, see [Stacks, 02NY]. In particular, g consists of a composition between a cover of $\mathbb{P}^1_k$ and (if the characteristic is positive) a Frobenius endomorphism, see [Stacks, 0CCZ].

2.2 Rational curves on blowups at points

We have just seen in section 2.1 that for any projective variety X over a field k, the scheme $\mathrm{Mor}(\mathbb{P}^1_k, X)$ can be partitioned according to the degree of the morphisms. However, if $X \neq \mathbb{P}^n_k$, each partition can be composed of several irreducible components. A simple example is given when X is a smooth cubic surface in $\mathbb{P}^3_k$: we know that it contains exactly 27 lines and indeed we will see in 2.4.10 that the scheme $\mathrm{Mor}_1(\mathbb{P}^1_k, X)$ has 27 components.

With that in mind, the purpose of this section is to make use of the functor $\mathrm{Mor}(\mathbb{P}^1_k, -)$ to produce a finer partition of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ when X is the blow-up of a projective space at a finite number of points. First of all, we recall the definition of blow-up via its universal property.

2.2.1 Definition. Let Y be a scheme and $E \hookrightarrow Y$ be a closed subscheme. We say E is an *effective Cartier divisor* if for every point $p \in E$, there exists an affine neighbourhood $U = \operatorname{Spec} A \subset X$ such that

$$E \cap U \cong \operatorname{Spec}(A/(a))$$

where $a \in A$ is a non-zero divisor. In other words, E is locally cut out by one equation.

Let X be a scheme and let $Z \hookrightarrow X$ be a closed subscheme. Then the *blow-up* of X along Z is a pair consisting of a morphism

$$\sigma : \mathrm{Bl}_Z(X) \to X$$

and a closed subscheme

$$E \hookrightarrow \mathrm{Bl}_Z(X)$$

fitting on the fibered diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d, "\sigma"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

and satisfying the following property: the closed subscheme E is an effective Cartier divisor on $\mathrm{Bl}_Z(X)$ and moreover, for any morphism $X' \to X$ and a cartesian square

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

such that E' is an effective Cartier divisor on X', we have that this diagram factorizes as a fibered diagram

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

The subscheme E is said to be the *exceptional divisor* of $\mathrm{Bl}_Z(X)$.

Furthermore, for any closed subscheme $\iota : W \hookrightarrow X$, consider the fibered diagram

$$\begin{tikzcd} \sigma^{-1}(W \smallsetminus Z) \arrow[r, hook] \arrow[d] & \sigma^{-1}(W) \arrow[r, hook, "\iota'"] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d, "\sigma"] \\ W \smallsetminus Z \arrow[r, hook] & W \arrow[r, hook, "\iota"] & X. \end{tikzcd}$$

We define the *total transform* of W with respect to σ to be the scheme theoretic preimage $\sigma^{-1}(W)$. We define the *strict transform* of W with respect to σ to be the scheme theoretic image $\overline{\mathrm{im}\left(\iota'|_{\sigma^{-1}(W \smallsetminus Z)}\right)}$, that is the closure of $\sigma^{-1}(W \setminus Z)$ on $\mathrm{Bl}_Z(X)$.

2.2.2 Let $\sigma : \mathrm{Bl}_Z(X) \to X$ be the blow-up of X along a closed subscheme Z and consider the induced morphism

$$\sigma_M := \mathrm{Mor}(\mathbb{P}^1_k, \sigma) : \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \to \mathrm{Mor}(\mathbb{P}^1_k, X),$$

and the open subschemes $\sigma^{-1}(X \smallsetminus Z)$ and $X \smallsetminus Z$. Recall that

$$\sigma|_{\sigma^{-1}(X \smallsetminus Z)} : \sigma^{-1}(X \smallsetminus Z) \to X \smallsetminus Z$$

is an isomorphism, see [Stacks, 02OS]. Therefore, we have a commutative diagram

$$\begin{tikzcd} \sigma^{-1}(X \smallsetminus Z) \arrow[r, "\sim"] \arrow[d, hook] & X \smallsetminus Z \arrow[d, hook] \\ \mathrm{Bl}_Z(X) \arrow[r, "\sigma"] & X. \end{tikzcd}$$

If we apply the functor $\mathrm{Mor}(\mathbb{P}^1_k, -)$ we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}(\mathbb{P}^1_k, \sigma^{-1}(X \smallsetminus Z)) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Mor}(\mathbb{P}^1_k, X \smallsetminus Z) \arrow[d, hook] \\
\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \arrow[r, "\sigma_M"] & \mathrm{Mor}(\mathbb{P}^1_k, X),
\end{tikzcd}$$

where the top row is also an isomorphism and the vertical arrows are open immersions by Lemma 1.5.5. In other words, $\sigma_M|_{\mathrm{Mor}(\mathbb{P}^1_k, \sigma^{-1}(X \smallsetminus Z))}$ is an isomorphism between open subschemes of $\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X))$ and $\mathrm{Mor}(\mathbb{P}^1_k, X)$.

Of course, since the $\mathrm{Mor}(\mathbb{P}^1_k, X)$ usually has infinitely many components, this isomorphism can be between relatively small open subsets and the morphism σ_M is very far from being birational. However, we are already equipped to say something meaningful for the morphism σ_M when restricted to the complement $\mathrm{Mor}(\mathbb{P}^1_k, X) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, Z)$.

Notice that $\mathrm{Mor}(\mathbb{P}^1_k, Z) \hookrightarrow \mathrm{Mor}(\mathbb{P}^1_k, X)$ is a closed immersion by Lemma 1.5.5 and therefore $\mathrm{Mor}(\mathbb{P}^1_k, X) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, Z)$ is an open subscheme of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ parametrizing rational curves intersecting the open subset $X \smallsetminus Z$.

We will use this to prove a proposition regarding the morphism σ_M, but before stating it we recall one last definition.

2.2.3 Definition. A morphism of schemes $f : X \to Y$ is said to be *(locally) quasi-finite* if it is (locally) of finite type and for each point $p \in Y$, the fiber $f^{-1}(p)$ is a discrete topological space.

2.2.4 Proposition. *Let X be a projective scheme over an algebraically closed field k, Z be a closed subscheme of X and $\sigma : \mathrm{Bl}_Z(X) \to X$ be the blow-up of X along Z. Let*

$$\sigma_M : \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \to \mathrm{Mor}(\mathbb{P}^1_k, X)$$

be the induced morphism and let $N := \mathrm{Mor}(\mathbb{P}^1_k, X) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, Z)$ be the open subscheme parametrizing rational curves intersecting $X \smallsetminus Z$ and $N' := \sigma_M^{-1}(N)$. Then the restriction

$$\sigma_M|_{N'} : N' \to N$$

is locally quasi-finite. More specifically, it is a bijection on k-points.

Proof. Any k-point $[f] \in N$ corresponds to a morphism $f : \mathbb{P}^1_k \to X$ whose image intersects the open $X \smallsetminus Z$. Since $\sigma|_{\sigma^{-1}(X \smallsetminus Z)}$ is an isomorphism, there is a rational map $g : \mathbb{P}^1_k \dashrightarrow \mathrm{Bl}_Z(X)$ such that the diagram

$$\begin{tikzcd} & \mathrm{Bl}_Z(X) \arrow[d, "\sigma"] \\ \mathbb{P}^1_k \arrow[ur, dashed, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

is commutative. Since $\mathbb{P}^1_k$ is nonsingular, this rational map has domain of definition $\mathbb{P}^1_k$, that is, g is regular (see for instance [Sil09, Chapter II, Proposition 2.1.]). Furthermore, it is unique (see [Stacks, 0A1Y]). This is equivalent to say that the fiber of the morphism σ_M at the point $[f]$ has a unique k-point $[g]$.

Let $\sigma_M^{-1}([f])$ denote this fiber. By Theorem 1.4.9, the scheme $\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X))$ is locally of finite type over k, and thus, so is $\sigma_M^{-1}([f])$. Hence, its set of k-points is dense (see [GW10, Proposition 3.35]). We conclude that $\sigma_M^{-1}([f])$ consists of a single point $[g]$ and hence $\sigma_M|_{N'}$ is a bijection on k-points.

To see it is locally quasi-finite, recall that any irreducible component of N' and of N is of finite type. Let $N'_0 \subset N'$ be an irreducible component. The restriction $\sigma_M|_{N'_0}$ is bijective on k-points between schemes of finite type, therefore it is quasi-finite (see [GW10, Remark 12.16]). We conclude that $\sigma_M|_{N'}$ is locally quasi-finite. □

2.2.5 Schemes of rational curves on blow-ups. Let us consider the blow-up $\sigma\colon \mathrm{Bl}_Z(X) \to X$ with exceptional divisor E, and denote

$$N := \mathrm{Mor}(\mathbb{P}^1_k, X) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, Z) \ \text{ and } \ N' := \sigma_M^{-1}(N),$$

as in Proposition 2.2.4.

Claim. The open subscheme N' is isomorphic to $\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, E)$.

Proof of claim. Recall that the exceptional divisor of the blow-up fits in the cartesian square

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d, "\sigma"] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

By Remark 1.5.3, we have a cartesian square

$$\begin{tikzcd}
\mathrm{Mor}(\mathbb{P}^1_k, E) \arrow[r, hook] \arrow[d] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \arrow[d, "\sigma_M"] \\
\mathrm{Mor}(\mathbb{P}^1_k, Z) \arrow[r, hook] & \mathrm{Mor}(\mathbb{P}^1_k, X).
\end{tikzcd}$$

In other words, $\sigma_M^{-1}(\mathrm{Mor}(\mathbb{P}^1_k, Z)) \cong \mathrm{Mor}(\mathbb{P}^1_k, E)$. Therefore, we obtain

$$\begin{aligned} N' =& \sigma_M^{-1}(\mathrm{Mor}(\mathbb{P}^1_k, X) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, Z)) \cong \sigma_M^{-1}(\mathrm{Mor}(\mathbb{P}^1_k, X)) \smallsetminus \sigma_M^{-1}(\mathrm{Mor}(\mathbb{P}^1_k, Z)) \\ &\cong \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \smallsetminus \mathrm{Mor}(\mathbb{P}^1_k, E). \end{aligned}$$ □

It follows that N' parametrizes rational curves on $\mathrm{Bl}_Z(X)$ not contained in the exceptional divisor E. Furthermore, the open subscheme

$$U := \mathrm{Mor}(\mathbb{P}^1_k, \sigma^{-1}(X \smallsetminus Z))$$

is contained in N'. We conclude that there is a partition

$$\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) = \mathrm{Mor}(\mathbb{P}^1_k, \sigma^{-1}(X \smallsetminus Z)) \amalg N'' \amalg \mathrm{Mor}(\mathbb{P}^1_k, E), \tag{2.8}$$

where N'' is a closed subscheme whose underlying topological space is $N' \smallsetminus U$. By definition, the subscheme N'' parametrizes rational curves intersecting the exceptional divisor E properly, that is, curves intersecting E but not contained in E.

Proposition 2.2.4 gives us a hint for the behaviour of components in N''. In fact, it tells us that if N_0 is an irreducible component of N, then the preimage $N'_0 := \sigma_M^{-1}(N_0)$ can be roughly understood as a "splitting" of N_0. A variant of the theorem of dimension of fibers ([Mus17, Proposition 5.5.1]) tells us that there exists a unique component $N'_1 \subset N'_0$ dominating N_0 and such that $\dim N'_1 = \dim N_0$. However, since N'_0 might not be irreducible, it may split in many other components of dimension strictly smaller than the $\dim N_0$. Intuitively, the component N'_1 should be the closure of $U \cap N'_0$ in N'_0, since $U \cap N'_0$ is isomorphic to an open subset of N_0. Therefore, we expect that any irreducible component of

$$N''_0 := N'_0 \cap N'' \subset N''$$

has dimension strictly less than $\dim N_0$. The behaviour of components just described is illustrated in the fibered diagram

$$\begin{tikzcd}
\emptyset \arrow[r, dashed, "\cong"] \arrow[d] & N_0'' \cap N_1' \arrow[r, hook] \arrow[d, hook] & N_0'' \arrow[r, hook] \arrow[d, hook] & N' \cap N'' \arrow[r, hook] \arrow[d, hook] & N'' \arrow[d, hook] \\
U \cap N_0' \arrow[r, hook] & N_1' \arrow[r, hook] & N_0' \arrow[r, hook] \arrow[d, hook] & N' \arrow[r, hook] \arrow[d, hook] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X)) \arrow[d, "\sigma_M"] \\
 & & N_0 \arrow[r, hook] & N \arrow[r, hook] & \mathrm{Mor}(\mathbb{P}^1_k, X).
\end{tikzcd}$$

In the following subsections we investigate further this splitting when $X = \mathbb{P}^n_k$ and Z is a finite collection of points. We start by adopting an elementary description of morphisms from the projective line to the blow-up of a point in $\mathbb{P}^n_k$ based on the ones of Section 2.1.

2.2.6 Morphisms from $\mathbb{P}^1_k$ to a blow-up. Let k be an algebraically closed field, $\mathbb{P}^n_k$ be a projective space and $\sigma : \mathrm{Bl}_p(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ at the point $p = (1 : 0 : \cdots : 0)$. Recall that $\mathrm{Bl}_p(\mathbb{P}^n_k)$ can be defined as a closed subscheme of the fiber product $\mathbb{P}^n_k \times \mathbb{P}^{n-1}_k$. Explicitly, if we define coordinates $(x_0 : \cdots : x_n)$ for $\mathbb{P}^n_k$ and $(y_1 : \cdots : y_n)$ for $\mathbb{P}^{n-1}_k$, the closed subscheme $\mathrm{Bl}_p(\mathbb{P}^n_k)$ is given by the equations

$$\{x_i y_j = y_j x_i\} \text{ for } 1 \leq i, j \leq n$$

in $\mathbb{P}^n_k \times \mathbb{P}^{n-1}_k$ (see for instance [Sha13a, Chapter 2, §4.1.]). Therefore, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Bl}_p(\mathbb{P}^n_k) \arrow[dr, hook] \arrow[drr, bend left, "\tau"] \arrow[ddr, bend right, "\sigma"'] & & \\
 & \mathbb{P}^n_k \times \mathbb{P}^{n-1}_k \arrow[r] \arrow[d] & \mathbb{P}^{n-1}_k \arrow[d] \\
 & \mathbb{P}^n_k \arrow[r] & \operatorname{Spec} k.
\end{tikzcd} \tag{2.9}$$

Recall that

$$\mathrm{Hom}_k(\mathbb{P}^1_k, \mathbb{P}^n_k \times \mathbb{P}^{n-1}_k) \cong \mathrm{Hom}_k(\mathbb{P}^1_k, \mathbb{P}^n_k) \times \mathrm{Hom}_k(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k),$$

that is, any regular morphism $f:\mathbb{P}^1_k \to \mathbb{P}^n_k \times \mathbb{P}^{n-1}_k$ corresponds uniquely to a pair of regular morphisms $F:\mathbb{P}^1_k \to \mathbb{P}^n_k$ and $G:\mathbb{P}^1_k \to \mathbb{P}^{n-1}_k$. Furthermore, by 2.1.1, the pair (F,G) corresponds uniquely to a pair

$$((F_0:\cdots:F_n),(G_1:\cdots:G_n)) \tag{2.10}$$

of tuples of homogeneous polynomials in $k[u,v]$ with $\deg F_i = d$ and $\deg G_i = e$ and such that each tuple of polynomials have no factors in common. Finally, a regular morphism $f':\mathbb{P}^1_k \to \mathrm{Bl}_p(\mathbb{P}^n_k)$ is just a regular morphism $f:\mathbb{P}^1_k \to \mathbb{P}^n_k\times\mathbb{P}^{n-1}_k$ such that $f(\mathbb{P}^1_k) \subset \mathrm{Bl}_p(\mathbb{P}^n_k)$, in other words, it corresponds uniquely to tuples (2.10) such that

$$F_iG_j = F_jG_i \text{ for } 1 \le i,j \le n.$$

This description is useful even when we consider the case of blow-ups at multiple points. In fact, we will be able to describe morphisms from $\mathbb{P}^1_k$ to blow-ups at multiple points by reducing them to several of the cases above. To do this, we recall another general property of blow-ups.

2.2.7 Blow-up closure. Let X be a scheme, let $Z \hookrightarrow X$ be a closed subscheme and let $\sigma:\mathrm{Bl}_Z(X)\to X$ be the blow-up of X with exceptional divisor E. Let $f:X'\to X$ be a morphism and define $Z' := f^{-1}(Z)$ to be the scheme theoretic preimage of Z, that is, Z' fits the fibered diagram

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d, "f"] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

If $\sigma':\mathrm{Bl}_{Z'}(X')\to X'$ is the blow-up with exceptional divisor E', then the fibered diagram of the blow-up σ factors through a diagram

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d, hook] & \mathrm{Bl}_{Z'}(X') \arrow[d, hook] \arrow[dd, bend left, "\sigma'"] \\ E\times_X Z \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Z(X)\times_X X' \arrow[d] \\ Z' \arrow[r, hook] & X' \end{tikzcd}$$

such that the hooked arrows are closed immersions. See for instance [Ful98, Subsection B.6.9] or [Vak17, Lemma 22.2.6].

Notice that by the universal property of the blow-up, we also have a fibered diagram

$$\begin{tikzcd}
E' \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{Z'}(X') \arrow[d] \arrow[dd, bend left, "f\circ\sigma'"] \\
E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d, "\sigma"] \\
Z \arrow[r, hook] & X.
\end{tikzcd}$$

2.2.8 Proposition. *Let $\{p_1, \dots, p_r\}$ be a finite collection of distinct points in $\mathbb{P}^n_k$. Let $\sigma : X \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ at $\{p_1, \dots, p_r\}$ with exceptional divisor E and let $\sigma_i : \mathrm{Bl}_{p_i}(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ with exceptional divisor E_i for each i. Then, there exist closed immersions $X \hookrightarrow \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)$ and $E \hookrightarrow E_1 \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} E_r$ such that we have a factorization*

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & X \arrow[d, hook] \\
E_1 \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} E_r \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k) \arrow[d] \\
\{p_1, \dots, p_r\} \arrow[r, hook] & \mathbb{P}^n_k,
\end{tikzcd}$$

where the outer square is cartesian.

Proof. Let $\sigma_{1\cdots j} : \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ be the blow-ups of $\mathbb{P}^n_k$ at the points $p_1, \dots, p_j$ for $1 \le j < r$ and $E_{1\cdots j}$ be the respective exceptional divisors. Consider the fiber products

$$\begin{tikzcd}
\mathrm{Bl}_{p_{j+1}}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \arrow[r] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \arrow[d, "\sigma_{1\cdots j}"] \\
\mathrm{Bl}_{p_{j+1}}(\mathbb{P}^n_k) \arrow[r, "\sigma_{j+1}"] & \mathbb{P}^n_k.
\end{tikzcd}$$

Consider the point $q := \sigma_{1\cdots j}^{-1}(p_{j+1}) \in \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k)$ and let

$$\sigma_q : \mathrm{Bl}_{p_1,\dots,p_j,q}(\mathbb{P}^n_k) := \mathrm{Bl}_q(\mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k)) \to \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k)$$

be the blow-up of $\mathrm{Bl}_{p_1,\ldots,p_j}(\mathbb{P}^n_k)$ at q with exceptional divisor E_q. By 2.2.7, the morphism σ_q and the restriction $\sigma_q|_{E_q}$ factorize as the commutative diagram:

$$\begin{tikzcd}
E_q \arrow[r, hook] \arrow[d, hook] \arrow[dd, bend right=60, "\sigma_q|_{E_q}"'] & \mathrm{Bl}_{p_1,\ldots,p_j,q}(\mathbb{P}^n_k) \arrow[d, hook] \arrow[dd, bend left=60, "\sigma_q"] \\
E_{j+1} \times_{\mathbb{P}^n_k} \{q\} \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_{j+1}}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_1,\ldots,p_j}(\mathbb{P}^n_k) \arrow[d] \\
\{q\} \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\ldots,p_j}(\mathbb{P}^n_k) \arrow[d, "\sigma_{1\cdots j}"] \\
\{p_{j+1}\} \arrow[r, hook] & \mathbb{P}^n_k.
\end{tikzcd}$$

Claim. $\mathrm{Bl}_{p_1,\ldots,p_j,q}(\mathbb{P}^n_k) \cong \mathrm{Bl}_{p_1,\ldots,p_{j+1}}(\mathbb{P}^n_k)$.

Proof of claim. Consider the composition

$$\mathrm{Bl}_{p_1,\ldots,p_j,q}(\mathbb{P}^n_k) \xrightarrow{\sigma_q} \mathrm{Bl}_{p_1,\ldots,p_j}(\mathbb{P}^n_k) \xrightarrow{\sigma_{1\cdots j}} \mathbb{P}^n_k,$$

and the following fibered diagram

$$\begin{tikzcd}
E'_{1\cdots j} \arrow[r, hook] \arrow[d, "\cong"] & E'_{1\cdots j} \amalg E_q \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\ldots,p_j,q}(\mathbb{P}^n_k) \arrow[d, "\sigma_q"] \\
E_{1\cdots j} \arrow[r, hook] \arrow[d] & E_{1\cdots j} \amalg q \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\ldots,p_j}(\mathbb{P}^n_k) \arrow[d, "\sigma_{i\cdots j}"] \\
\{p_1,\ldots,p_j\} \arrow[r, hook] & \{p_1,\ldots,p_{j+1}\} \arrow[r, hook] & \mathbb{P}^n_k.
\end{tikzcd}$$

Clearly $E'_{1\cdots j}$ is an effective Cartier divisor (see [Stacks, 02OO]) and, by definition, so is E_q. Therefore, $E'_{1\cdots j} \amalg E_q$ is also an effective Cartier divisor. By the universal property of $\mathrm{Bl}_{p_1,\ldots,p_{j+1}}(\mathbb{P}^n_k)$, there exists

$$\alpha : \mathrm{Bl}_{p_1,\ldots,p_j,q}(\mathbb{P}^n_k) \to \mathrm{Bl}_{p_1,\ldots,p_{j+1}}(\mathbb{P}^n_k)$$

such that

$$\sigma_{1\cdots j} \circ \sigma_q = \sigma_{1\cdots j+1} \circ \alpha$$

and inducing a fibered diagram

$$\begin{tikzcd} E'_{1\cdots j} \amalg E_q \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_j,q}(\mathbb{P}^n_k) \arrow[d, "\alpha"] \arrow[dd, bend left, "\sigma_{1\cdots j}\circ\sigma_q"] \\ E_{1\cdots j+1} \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k) \arrow[d, "\sigma_{1\cdots j+1}"] \\ \{p_1,\dots,p_{j+1}\} \arrow[r, hook] & \mathbb{P}^n_k. \end{tikzcd}$$

On the other hand, since $E' := \sigma^{-1}_{i\cdots j+1}(\{p_1,\dots,p_j\})$ is clearly a disjoint union of effective Cartier divisors on $\mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k)$, the universal property of $\mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k)$ implies that there exists

$$\rho : \mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k) \to \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k)$$

such that

$$\sigma_{1\cdots j+1} = \sigma_{1\cdots j} \circ \rho$$

and induces a fibered diagram

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k) \arrow[d, "\rho"] \arrow[dd, bend left, "\sigma_{1\cdots j+1}"] \\ E_{1\cdots j} \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \arrow[d, "\sigma_{1\cdots j}"] \\ \{p_1,\dots,p_j\} \arrow[r, hook] & \mathbb{P}^n_k. \end{tikzcd}$$

Moreover, if we consider the fibered diagram

$$\begin{tikzcd} E'' \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k) \arrow[d, "\rho"] \\ q \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \arrow[d, "\sigma_{1\cdots j}"] \\ \{p_{j+1}\} \arrow[r, hook] & \mathbb{P}^n_k, \end{tikzcd}$$

then we see that E'' is a component of $E_{1\cdots j+1}$. Therefore, it is an effective Cartier divisor. The universal property of $\mathrm{Bl}_{p_1,\dots,p_j,q}(\mathbb{P}^n_k)$ implies that there exists a morphism

$$\beta : \mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k) \to \mathrm{Bl}_{p_1,\dots,p_r,q}(\mathbb{P}^n_k)$$

such that $\rho = \sigma_q \circ \beta$ and inducing the fibered diagram

$$\begin{tikzcd}
E'' \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_{j+1}}(\mathbb{P}^n_k) \arrow[d, "\beta"'] \arrow[dd, bend left, "\rho"] \\
E_q \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_j,q}(\mathbb{P}^n_k) \arrow[d, "\sigma_q"'] \\
q \arrow[r, hook] & \mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k).
\end{tikzcd}$$

It is straightforward to check that $E_{1\cdots j+1} = E' \amalg E''$. To prove the isomorphism in the claim, it suffices to check that α and β are mutually inverses. Notice that

$$\begin{aligned}
\sigma_{1\cdots j+1} \circ \alpha \circ \beta =& \sigma_{1\cdots j} \circ \sigma_q \circ \beta \\
=& \sigma_{1\cdots j} \circ \rho = \sigma_{i\cdots j+1}.
\end{aligned}$$

Universal property of $\sigma_{i\cdots j+1}$ implies $\alpha \circ \beta = \mathrm{id}$. On the other hand,

$$\begin{aligned}
\sigma_{1\cdots j} \circ \sigma_q \circ \beta \circ \alpha &= \sigma_{1\cdots j} \circ \rho \circ \alpha \\
&= \sigma_{1\cdots j+1} \circ \alpha = \sigma_{1\cdots j} \circ \sigma_q
\end{aligned}$$

Universal property of $\sigma_{1\dots j}$ implies that $\sigma_q \circ \beta \circ \alpha = \sigma_q$ and universal property of σ_q implies that $\beta \circ \alpha = \mathrm{id}$, and we obtain the desired isomorphism. $\square$

It follows that we have a closed immersion

$$\iota_r : \mathrm{Bl}_{p_1,\dots,p_r}(\mathbb{P}^n_k) \hookrightarrow \mathrm{Bl}_{p_1,\dots,p_{r-1}}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k).$$

For $1 < j < r$, we define recursively the closed immersions ι_j fitting in the fibered diagram

$$\begin{tikzcd}
\mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_{j+1}}(\mathbb{P}^n_k) \arrow[r, hook, "\iota_{j-1}"] \arrow[d] & \mathrm{Bl}_{p_1,\dots,p_{j-1}}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_j}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_{j+1}}(\mathbb{P}^n_k) \arrow[r] \arrow[d] & \mathrm{Bl}_{p_{j+1}}(\mathbb{P}^n_k) \arrow[d] \\
\mathrm{Bl}_{p_1,\dots,p_j}(\mathbb{P}^n_k) \arrow[r, hook, "\iota_j"] & \mathrm{Bl}_{p_1,\dots,p_{j-1}}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_j}(\mathbb{P}^n_k) \arrow[r] & \mathbb{P}^n_k
\end{tikzcd}$$

Thus, we can define the immersion of the statement as the composition

$$X = \mathrm{Bl}_{p_1,\dots,p_r}(\mathbb{P}^n_k) \overset{\iota_r}{\hookrightarrow} \mathrm{Bl}_{p_1,\dots,p_{r-1}}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k) \overset{\iota_{r-1}}{\hookrightarrow} \cdots$$
$$\cdots \overset{\iota_1}{\hookrightarrow} \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k). \qquad \square$$

2.2.9 Definition. Let $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be a non-constant morphism corresponding to a tuple $(F_0 : \ldots : F_n)$ of forms on $k[u, v]$. Let $p \in f(\mathbb{P}^1_k)$ be a point in the image. Up to change of coordinates on $\mathbb{P}^n_k$, we can assume $p = (1 : 0 : \cdots : 0)$. It follows that $p \in f(\mathbb{P}^1_k)$ if and only if $\{F_j\}_{1\leq j\leq n}$ have factors in common. By the claim in 2.1.1, we know that this is equivalent to saying that there exists a homogeneous $H \in k[u, v]$ such that H divides F_i for $1 \leq i \leq n$. We define

$$m_p(f) = \max\{\deg H \mid H \in k[u, v] \text{ and } H \text{ divides } F_j \text{ for } 1 \leq j \leq n\}$$

to be the *parametric multiplicity* of p on f.

2.2.10 Remark. Let $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be a morphism corresponding to a tuple $(F_0 : \cdots : F_n)$ of forms of degree d. If

$$p = (1 : 0 : \cdots : 0) \in f(\mathbb{P}^1_k),$$

then we can write $F_i = HF'_i$ for $1 \leq i \leq n$ with $\deg H = m_p(f)$, that is, we have the relation

$$m_p(f) = d - \deg F'_i.$$

If f is generically one-to-one and $\operatorname{char} k = 0$, then the parametric multiplicity $m_p(f)$ coincides with the multiplicity of the point p at the scheme theoretic image of f, see for instance [Pér07, Theorem 8] for $n = 2$.

When f is not generically one-to-one, it follows from (2.7) in 2.1.10 that if

$$\nu : \mathbb{P}^1_k \to C := \overline{\mathrm{im}\,(f)}$$

is the normalization of C, then there is morphism $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $f = \nu \circ g$ and we have the relation

$$m_p(f) = m_0 m_p(\nu), \tag{2.11}$$

where m_0 is the degree $[g^* k(t) : k(t)]$.

2.2.11 Theorem. *Let k be an algebraically closed field, $\{p_1, \ldots, p_r\} \subset \mathbb{P}^n_k$ be a finite collection of points in a projective space. Let $\sigma : X \to \mathbb{P}^n_k$ be the blow-up of $\mathbb{P}^n_k$ along $\{p_1, \ldots, p_r\}$ with exceptional divisor E. Let $\mathbf{m} := (m_1, \ldots, m_r)$ denote an r-tuple of non-negative integers. Then we have the partition in closed* *subschemes*

$$\mathrm{Mor}(\mathbb{P}^1_k, X) \cong \mathrm{Mor}_0(\mathbb{P}^1_k, X) \amalg \left(\coprod_{d>0} \coprod_{m_i \le d} M_{d,\mathbf{m}} \right) \amalg \mathrm{Mor}_{>0}(\mathbb{P}^1_k, E),$$

where

- $\mathrm{Mor}_0(\mathbb{P}^1_k, X)$ *parametrizes constant morphisms;*
- *a* k*-point* $[g]$ *belongs to* $M_{d,\mathbf{m}}$ *if and only if*

$$\begin{aligned} &\deg(\sigma \circ g) = d \ \ \textit{and} \\ &m_{p_i}(\sigma \circ g) = m_i \ \ \textit{for} \ \ 1 \le i \le r; \end{aligned}$$

- $\mathrm{Mor}_{>0}(\mathbb{P}^1_k, E) \cong \coprod_{i=1}^r \coprod_{e>0} \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)$.

In particular, for $\mathbf{0} := (0, \ldots, 0)$ *and each positive integer* d*, the subschemes* $M_{d,\mathbf{0}}$ *are nonsingular of dimension* $nd + d + n$ *and parametrize curves which do not intersect* E*.*

Proof. For this proof we will use the following notational convention: for any morphism of schemes $\alpha : X \to Y$ over k, we denote

$$\alpha_M : \mathrm{Mor}(\mathbb{P}^1_k, X) \to \mathrm{Mor}(\mathbb{P}^1_k, Y)$$

to be the corresponding morphism on schemes of morphisms.

Case $r = 1$:

We denote $p := p_1 = (1 : 0 : \cdots : 0)$ and $X = \mathrm{Bl}_p(\mathbb{P}^n_k)$. Let σ and τ be morphism fitting the diagram (2.9). We can describe the scheme $\mathrm{Mor}(\mathbb{P}^1_k, X)$ by looking at the fibers of σ_M, that is, we start by noticing the partition

$$\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_p(\mathbb{P}^n_k)) = \coprod_{d \in \mathbb{N}} \sigma_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)), \tag{2.12}$$

where $\sigma_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k))$ denotes the scheme theoretic inverse image of $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ under σ_M. That is, $\sigma_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k))$ fits in a cartesian square

$$\begin{tikzcd} \sigma_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)) \arrow[r, hook] \arrow[d] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_p(\mathbb{P}^n_k)) \arrow[d, "\sigma_M"] \\ \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k) \arrow[r, hook] & \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k). \end{tikzcd}$$

Claim 1. *If T is an irreducible component of* $\mathrm{Mor}(\mathbb{P}^1_k, X)$, *then*

$$T \subset \sigma_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k))$$

for a unique $d \geq 0$.

Proof of claim. Since T is irreducible, $\sigma_M(T)$ is irreducible, therefore we have $\sigma_M(T) \subset \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ for a unique $d \geq 0$. ☐

We can further partition (2.12) using τ_M. Indeed, for each $d \geq 0$, we have

$$\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) = \coprod_{e \geq 0} \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)), \tag{2.13}$$

where

$$\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k))$$

denotes the scheme theoretic intersection fitting on the fibered diagram:

$$\begin{tikzcd}
\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)) \arrow[r, hook] \arrow[d, hook] & \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)) \arrow[d, hook] \\
\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \arrow[r, hook] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_p(\mathbb{P}^n_k)).
\end{tikzcd}$$

Claim 2. *If T is an irreducible component of* $\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right)$, *then we have*

$$T \subset \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k))$$

for a unique $e \geq 0$.

Proof of claim. Since T is irreducible, $\tau_M(T)$ is irreducible, therefore we have $\tau_M(T) \subset \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)$ for a unique $e \geq 0$. ☐

Claim 3. *We have*

$$\begin{aligned}
\sigma_M^{-1}(\mathrm{Mor}_0(\mathbb{P}^1_k, \mathbb{P}^n_k)) &\cong \mathrm{Mor}_0(\mathbb{P}^1_k, \sigma^{-1}(\mathbb{P}^n_k \smallsetminus \{p\}) \amalg \mathrm{Mor}(\mathbb{P}^1_k, E) \\
&\cong \mathrm{Mor}_0(\mathbb{P}^1_k, X) \amalg \mathrm{Mor}_{>0}(\mathbb{P}^1_k, E)
\end{aligned}$$

Proof of claim. Let $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be the constant morphism sending $\mathbb{P}^1_k$ to p and let $[f]$ be the corresponding point in $\mathrm{Mor}_0(\mathbb{P}^1_k, \mathbb{P}^n_k)$. Notice that $\mathrm{Mor}(\mathbb{P}^1_k, \{p\}) = [f]$

and by Remark 1.5.3, we have a cartesian square

$$\begin{tikzcd}
\mathrm{Mor}(\mathbb{P}^1_k, E) \arrow[r, hook] \arrow[d] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_p(\mathbb{P}^n_k)) \arrow[d, "\sigma_M"] \\
{[f]} \arrow[r, hook] & \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k).
\end{tikzcd}$$

That is, the fiber σ_M at $[f]$ is precisely $\mathrm{Mor}(\mathbb{P}^1_k, E) \cong \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)$. It follows that

$$\sigma_M^{-1}([f]) \cong \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k) \cong \coprod_{e\geq 0} \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k).$$

In addition to that, we have that

$$\mathrm{Mor}_0(\mathbb{P}^1_k, \mathbb{P}^n_k) = \mathrm{Mor}_0(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\}) \amalg [f],$$

and by 2.2.2 and Remark 2.2.6, we have that

$$\sigma_M^{-1}(\mathrm{Mor}_0(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\})) \cong \mathrm{Mor}_0(\mathbb{P}^1_k, \sigma^{-1}(\mathbb{P}^n_k \smallsetminus \{p\})).$$

Moreover it is clear that $\mathrm{Mor}_0(\mathbb{P}^1_k, \sigma^{-1}(\mathbb{P}^n_k \smallsetminus \{p\}) \amalg \mathrm{Mor}_0(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)$ parametrize all the constant morphisms, that is this union is isomorphic to $\mathrm{Mor}_0(\mathbb{P}^1_k, X)$ and the claim follows. $\square$

For any point $[g]$ in $\sigma_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k))$ for some $d \geq 1$, the description in 2.2.6 implies that the morphism g corresponds uniquely to a pair

$$((F_0 : \cdots : F_n), (G_1 : \cdots : G_n)), \tag{2.14}$$

where:

- $(F_0 : \cdots : F_n)$ is a collection of homogeneous polynomials of degree d with no common factors;
- $(G_1 : \cdots : G_n)$ is also a collection of polynomials with no common factors;
- and $F_iG_j = F_jG_i$ for $1 \leq i, j \leq n$.

Claim 4. *Let d and e be positive integers. If $e > d$, then*

$$\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)) = \emptyset.$$

If $e \geq d$, a k-point $[g]$ belongs to

$$\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k))$$

if and only if

$$\deg(\sigma \circ g) = d \;\; \textit{and} \;\; m_p(\sigma \circ g) = d - e.$$

Moreover, if $d = e$, then this subscheme is irreducible of dimension $nd + n + d$.

Proof of claim. Consider a morphism $g : \mathbb{P}^1_k \to X$. We have two situations: either $p = (1 : 0 : \cdots : 0)$ belongs to the image $(\sigma \circ g)(\mathbb{P}^1_k)$ or it does not.

Suppose that $p \notin (\sigma \circ g)(\mathbb{P}^1_k)$. This is equivalent to saying that $F_1, \ldots, F_n$ in (2.14) have no roots in common and by Proposition 2.1.2 they have no factors in common. Since $G_1, \ldots, G_n$ also do not have factors in common, we have that $F_i G_j = F_j G_i$, for all $1 \leq i, j \leq n$ if and only if

$$G_i = F_i \;\; \text{for} \;\; 1 \leq i \leq n$$

(see Claim in 2.1.1). In particular, $\deg G_i = \deg F_i = d$, or equivalently, $\tau \circ g$ has degree d. In other words,

$$\tau_M([g]) \in \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k) \;\; \text{and} \;\; [g] \in \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)).$$

Recall that by 2.2.2, the morphism

$$\sigma_M|_{\mathrm{Mor}(\mathbb{P}^1_k, \sigma^{-1}(\mathbb{P}^n_k \smallsetminus \{p\}))} : \mathrm{Mor}(\mathbb{P}^1_k, \sigma^{-1}(\mathbb{P}^n_k \smallsetminus \{p\})) \to \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\})$$

is an isomorphism of open subsets of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ and $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k)$. Notice that

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\}) := \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k) \cap \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\})$$

is a non-empty open subscheme of $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ (indeed, since k is infinite we can always find morphisms $\mathbb{P}^1_k \to \mathbb{P}^n_k$ of degree d whose image avoids a point p). Therefore, $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\})$ is irreducible and nonsingular of dimension

$$\dim \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k) = nd + n + d,$$

by Proposition 2.1.6. Finally notice that, by definition,

$$[g] \in \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k))$$

if and only if $p \notin (\sigma \circ g)(\mathbb{P}^1_k)$, which is equivalent to say $[\sigma \circ g] \in \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\})$. Thus, we have

$$\sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)) \cong \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k \smallsetminus \{p\}).$$

Now suppose
$$p = (1 : 0 : \cdots : 0) \in (\sigma \circ g)(\mathbb{P}^1_k).$$

Since $d \geq 1$, we have that $p \in (\sigma \circ g)(\mathbb{P}^1_k)$ if and only if the polynomials $\{F_i\}_{1 \leq i \leq n}$ have roots in common. Thus, by Proposition 2.1.2, there exists a homogeneous polynomial H in $k[u, v]$ such that

$$F_i = H \cdot F_i' \ \text{ for } \ 1 \leq i \leq n$$

(notice that $H \nmid F_0$). We can assume without loss of generality that $m := \deg H$ is maximal. In such a situation we have equalities $F_i G_j = F_j G_i$ for all $1 \leq i, j \leq n$ if and only if
$$G_i = F_i' \ \text{ for } \ 1 \leq i \leq n.$$

In particular, all the polynomials $\{G_i\}_{1 \leq i \leq n}$ have degree $d - m$, that is, $\tau \circ g$ has degree $d - m$. In other words,

$$\begin{aligned} &\tau_M([g]) \in \mathrm{Mor}_{d-m}(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k) \ \text{ and} \\ &\qquad [g] \in \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_{d-m}(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k)) \end{aligned}$$

for some $1 < m < d$. $\square$

Define

$$M_{d,m} := \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)\right) \cap \tau_M^{-1}(\mathrm{Mor}_{d-m}(\mathbb{P}^1_k, \mathbb{P}^{n-1}_k))$$

for all $d \geq 1$ and $0 \leq m < d$. Notice that by Claim 4, $\deg(\tau \circ g) = d - m$ if and only if we have parametric multiplicity $m_p(\sigma \circ g) = m$. Thus, $M_{d,m}$ has k-points satisfying the conditions of the statement.

The partitions (2.13) and the one given by the Claim 3 yield

$$\mathrm{Mor}(\mathbb{P}^1_k, X) \cong \mathrm{Mor}_0(\mathbb{P}^1_k, X) \amalg \left(\coprod_{d>0} \coprod_{m \leq d} M_{d,m} \right) \amalg \mathrm{Mor}_{>0}(\mathbb{P}^1_k, E),$$

and concludes the proof for $r = 1$.

Case $r > 1$:

We fix the following notation:

- $\sigma : X = \mathrm{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ is the blow-up of $\mathbb{P}^n_k$ along $\{p_1, \ldots, p_r\}$ with exceptional divisor E;
- $\sigma_i : \mathrm{Bl}_{p_i}(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ is the blow-up of $\mathbb{P}^n_k$ at p_i with exceptional divisor E_i;

- $\iota : X \hookrightarrow \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)$ is the closed immersion defined in Proposition 2.2.8;

- $\mathrm{pr}_i : \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k) \to \mathrm{Bl}_{p_i}(\mathbb{P}^n_k)$ is the natural i-th projection;

- $\theta : \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ is the natural morphism to $\mathbb{P}^n_k$;

- N is $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k)$;

- The partition

$$\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_i}(\mathbb{P}^n_k)) \cong M^i_{\mathbb{P}^n_k} \amalg M_{E_i} \amalg \mathrm{Mor}(\mathbb{P}^1_k, E_i)$$

  induced by each σ_i by the case $r = 1$, where:

  - $M^i_{\mathbb{P}^n_k} := \coprod_{d_i \in \mathbb{N}} M^i_{d_i,0}$;
  - $M_{E_i} := \coprod_{d_i \in \mathbb{N}} \coprod_{m_i=1}^{d_i} M^i_{d_i,m_i}$;

- $f_i : \mathbb{P}^1_k \to \mathbb{P}^n_k$ is the constant morphism such that $f_i(\mathbb{P}^1_k) = p_i$.

Notice that we have the relations $\sigma_i \circ \mathrm{pr}_i = \theta$ for all i and $\theta \circ \iota = \sigma$.

Claim 5. *Let* $\mathbf{m} = (m_1, \ldots, m_r)$ *denote* r*-tuples of non-negative integers. We have an isomorphism*

$$\begin{aligned}&\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)) \\ &\qquad \cong \left(\coprod_{i=1}^r \theta_M^{-1}([f_i])\right) \amalg \left(\coprod_{d\in\mathbb{N}} \coprod_{m_i<d} M^1_{d,m_1} \times_N \cdots \times_N M^r_{d,m_r}\right).\end{aligned}$$

Proof of claim. By Remark 1.5.3, we have an isomorphism

$$\begin{aligned}&\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)) \\ &\qquad \cong \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k)) \times_N \cdots \times_N \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)).\end{aligned}$$

Let i and j be two distinct integers in $\{1, \ldots, r\}$ and notice that since coproducts commute with fibered products in the category of schemes, we have an isomorphism

$$\begin{aligned}
&\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_i}(\mathbb{P}^n_k)) \times_N \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_j}(\mathbb{P}^n_k)) \\
&\quad \cong \left(M^i_{\mathbb{P}^n_k} \times_N M^j_{\mathbb{P}^n_k}\right) \amalg \left(M^i_{\mathbb{P}^n_k} \times_N M_{E_j}\right) \amalg \left(M^i_{\mathbb{P}^n_k} \times_N \mathrm{Mor}(\mathbb{P}^1_k, E_j)\right) \\
&\quad\quad \amalg \left(M_{E_i} \times_N M^j_{\mathbb{P}^n_k}\right) \amalg \left(M_{E_i} \times_N M_{E_j}\right) \amalg \left(M_{E_i} \times_N \mathrm{Mor}(\mathbb{P}^1_k, E_j)\right) \\
&\quad\quad \amalg \left(\mathrm{Mor}(\mathbb{P}^1_k, E_i) \times_N M^j_{\mathbb{P}^n_k}\right) \amalg \left(\mathrm{Mor}(\mathbb{P}^1_k, E_i) \times_N M_{E_j}\right) \\
&\quad\quad \amalg \left(\mathrm{Mor}(\mathbb{P}^1_k, E_i) \times_N \mathrm{Mor}(\mathbb{P}^1_k, E_j)\right).
\end{aligned}$$

We can easily describe some of the products above. First, consider the morphisms π_i, π_j and θ_{ij} fitting in the fibered diagram

$$\begin{tikzcd}
\mathrm{Bl}_{p_i}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_j}(\mathbb{P}^n_k) \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] \arrow[dr, "\theta_{ij}"] & \mathbb{P}^1_k, \mathrm{Bl}_{p_i}(\mathbb{P}^n_k) \arrow[d, "\sigma_i"] \\
\mathrm{Bl}_{p_j}(\mathbb{P}^n_k) \arrow[r, "\sigma_j"] & N
\end{tikzcd}$$

and the natural projection

$$\mathrm{pr}_{ij} : \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k) \to \mathrm{Bl}_{p_i}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_j}(\mathbb{P}^n_k).$$

Notice that we have

$$\begin{aligned}
\mathrm{pr}_i &= \pi_i \circ \mathrm{pr}_{ij} \\
\mathrm{pr}_j &= \pi_j \circ \mathrm{pr}_{ij} \\
\theta &= \theta_{ij} \circ \mathrm{pr}_{ij}
\end{aligned} \tag{2.15}$$

Then, by definition of the partitions, we have

$$\begin{aligned}
M^i_{\mathbb{P}^n_k} \times_N \mathrm{Mor}(\mathbb{P}^1_k, E_j) &\cong \pi^{-1}_{i,M}(\sigma^{-1}_{i,M}(\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_i\})) \cap \pi^{-1}_{j,M}(\sigma^{-1}_{j,M}([f_j])) \\
&= \theta^{-1}_{ij,M}(\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_i\})) \cap \theta^{-1}_{ij,M}([f_j])) = \theta^{-1}_{ij,M}([f_j]).
\end{aligned} \tag{2.16}$$

Similarly, we obtain

$$\mathrm{Mor}(\mathbb{P}^1_k, E_i) \times_N M^j_{\mathbb{P}^n_k} \cong \theta^{-1}_{ij,M}([f_i]).$$

Furthermore, since

$$\sigma_{j,M}(\mathrm{Mor}(\mathbb{P}^1_k, E_j)) = [f_j] \notin \sigma_{i,M}(M_{E_i})$$
$$\text{and } \ [f_i] \neq [f_j],$$

we have

$$\begin{aligned} &M_{E_i} \times_N \mathrm{Mor}(\mathbb{P}^1_k, E_j) = \emptyset \\ &\mathrm{Mor}(\mathbb{P}^1_k, E_i) \times_N M_{E_j} = \emptyset \\ &\mathrm{Mor}(\mathbb{P}^1_k, E_i) \times_N \mathrm{Mor}(\mathbb{P}^1_k, E_j) = \emptyset. \end{aligned} \tag{2.17}$$

Since the relations (2.15), (2.16) and (2.17) hold for any pair of distinct indices i, j, we obtain:

$$\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)) \cong \left(\coprod_{i=1}^{r} \theta_M^{-1}([f_i]) \right) \amalg M',$$

where

$$M' \cong \coprod_{M'_i \in \{M^i_{\mathbb{P}^n_k}, M_{E_i}\}} M'_1 \times_N \cdots \times_N M'_r.$$

Let $[g] \in \mathrm{Mor}(\mathbb{P}^1_k, X)$ be a k-point and suppose that

$$\mathrm{pr}_{i,M}([g]) = [\mathrm{pr}_i \circ g] \in M^i_{d_i, m_i}.$$

Then, we have that

$$\sigma_{i,M} \circ \mathrm{pr}_{i,M}([g]) = \theta_M([g])$$

corresponds to a morphism

$$\theta \circ g : \mathbb{P}^1_k \to \mathbb{P}^n_k$$

of degree d_i. We conclude that

$$M^i_{d_i, m_i} \times_N M^j_{d_j, m_j} = \emptyset$$

whenever $i \neq j$. Thus, we obtain the partition

$$M' \cong \coprod_{d \in \mathbb{N}} \coprod_{m_i < d} M^1_{d, m_1} \times_N \cdots \times_N M^r_{d, m_r}. \qquad \square$$

Finally, consider ι_M to be the induced closed embedding of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ into $\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k)) \times_N \cdots \times_N \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_r}(\mathbb{P}^n_k))$. We can define the partition of the statement as follows.

Subschemes $M_{d,\mathbf{m}}$:

Define

$$M_{d,\mathbf{m}} := \iota_M^{-1}(M^1_{d,m_1} \times_N \cdots \times_N M^r_{d,m_r})$$

for all positive integers d and $1 \le m_1, \ldots, m_r \le d$. The universal property of fiber products yields

$$M^1_{d,m_1} \times_N \cdots \times_N M^r_{d,m_r} \cong \mathrm{pr}_{1,M}^{-1}\left(M^1_{d,m_1}\right) \cap \cdots \cap \mathrm{pr}_{r,M}^{-1}\left(M^r_{d,m_r}\right).$$

Hence, for any point k-point $[g] \in M_{d,\mathbf{m}}$ we have

$$\mathrm{pr}_{i,M} \circ \iota_M([g]) = [\mathrm{pr}_i \circ \iota \circ g] \in M^i_{d,m_i}.$$

On the other hand, by definition of M^i_{d,m_i} in the case $r = 1$, this is equivalent to

$$m_{p_i}(\sigma_i \circ \mathrm{pr}_i \circ \iota \circ g) = m_{p_i}(\sigma \circ g) = m_i.$$

In particular, if $\mathbf{m} = \mathbf{0} = (0, \ldots, 0)$, then since fibered products commute with fibered products and $\mathrm{Mor}(\mathbb{P}^1_k, -)$ commutes with fibered products, we have the isomorphism

$$\begin{aligned}
&M^1_{d,0} \times_N \cdots \times_N M^r_{d,0} \\
&\qquad \cong \theta_M^{-1}\left(\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_1\})\right) \cap \cdots \cap \theta_M^{-1}\left(\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_r\})\right) \\
&\qquad \cong \theta_M^{-1}\left(\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_1\}) \cap \cdots \cap \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_r\})\right) \\
&\qquad \cong \theta_M^{-1}\left(\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_1, \ldots p_r\})\right).
\end{aligned}$$

Furthermore, since $\sigma = \theta \circ \iota$ is an isomorphism when restricted to the preimage of $\mathbb{P}^n_k \setminus \{p_1, \ldots, p_r\}$, we have

$$M_{d,\mathbf{0}} \cong \sigma_M^{-1}\left(\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_1, \ldots p_r\})\right)$$

is irreducible of dimension $nd + n + d$, here we use that k is infinite to guarantee

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k \setminus \{p_1, \ldots, p_r\})$$

is always non-empty.

Subschemes of $\mathrm{Mor}(\mathbb{P}^1_k, E)$:

Just as the case $r = 1$, we have fibered diagram

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook, "\iota"] \\
\theta^{-1}(\{p_1, \dots, p_r\}) \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{p_1}(\mathbb{P}^n_k) \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} \mathrm{Bl}_{p_r}(\mathbb{P}^n_k) \arrow[d, "\theta"] \\
\{p_1, \dots, p_r\} \arrow[r, hook] & \mathbb{P}^n_k
\end{tikzcd}$$

inducing a corresponding fibered diagram

$$\begin{tikzcd}
\mathrm{Mor}(\mathbb{P}^1_k, E) \arrow[r, hook] \arrow[d, hook] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1,\dots,p_r}(\mathbb{P}^n_k)) \arrow[d, hook, "\iota_M"] \\
\theta_M^{-1}(\{[f_1], \dots, [f_r]\}) \arrow[r, hook] \arrow[d] & \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_1}(\mathbb{P}^n_k)) \times_N \cdots \times_N \mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_{p_r}(\mathbb{P}^n_k)) \arrow[d, "\theta_M"] \\
\{[f_1], \dots, [f_r]\} \arrow[r, hook] & N.
\end{tikzcd}$$

Since $E_i \cong \sigma^{-1}(p_i)$, we have $E = \coprod_{i=1}^r \sigma^{-1}(p_i) \cong \coprod_{i=1}^r E_i$. Since fibered products commute with coproducts, it follows that

$$\begin{aligned}
\mathrm{Mor}(\mathbb{P}^1_k, E) &\cong \iota_M^{-1}\left(\coprod_{i=1}^r \theta_M^{-1}([f_i])\right) \cong \coprod_{i=1}^r \sigma_M^{-1}([f_i]) \cong \coprod_{i=1}^r \mathrm{Mor}(\mathbb{P}^1_k, E_i) \\
&\cong \coprod_{i=1}^r \coprod_{e \geq 0} \mathrm{Mor}_e(\mathbb{P}^1_k, E_i)
\end{aligned}$$

As before, we have $\mathrm{Mor}_0(\mathbb{P}^1_k, X) \cong M_{0,\mathbf{0}} \amalg \mathrm{Mor}_0(\mathbb{P}^1_k, E)$, therefore we obtain the partition of the statement. □

2.2.12 Remark. Notice that one of the main features of the partition of Theorem 2.2.11 is that it reflects the intersection of curves with the exceptional divisor E. Indeed, by the definition of $M_{d,\mathbf{m}}$, if at least one of the m_i in $\mathbf{m} = (m_1, \dots, m_r)$ is positive then any k-point $[g] \in M_{d,\mathbf{m}}$ corresponds to a curve intersecting the exceptional divisor E properly, *i.e.* if $C := \overline{\mathrm{im}\,(g)}$, then $C \cap E \neq \emptyset$ and $C \not\subset E$.

Moreover, the subscheme $\mathrm{Mor}(\mathbb{P}^1_k, E)$ parametrizes curves strictly contained on the exceptional divisor E.

2.2.13 Warning. Although the partitions in Theorem 2.2.11 can be stated for any finite collection of points $\{p_1, \dots, p_r\}$ in the projective space, the behaviour of the components depends heavily on the configuration of these points. It follows that some of the components $M_{d,\mathbf{m}}$ in Theorem 2.2.11 can be empty. For example, let $\{p_1, p_2, p_3\}$ be three non-colinear points in $\mathbb{P}^2_k$. Then we have that $M_{1,(1,1,1)} = \emptyset$ on $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2_k)$.

2.3 Rational plane curves and linear systems

Let $L \subset \mathbb{P}^2_k$ be a line and recall that for each positive integer d, there is a projective space parametrizing effective divisors on $\mathbb{P}^2_k$ which are linearly equivalent to dL. This projective space is the *complete linear system* of dL, and is denoted $|dL|$, see also B.1.2. In fact, $|dL|$ can be defined as $\mathbb{P}(H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d))^\vee)$ and it has dimension $N = \binom{d+2}{d} - 1$.

The aim of this section is to prove that there is a regular morphism

$$\Xi_d : \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \to |dL|$$

and that its fibers are of dimension 3. Moreover, we prove that the image $\mathrm{im}(\Xi_d)$ contains locally closed subschemes of $|dL|$ for which we can compute the dimension. We start with a heuristic description of how this map should act on the k-points of these varieties. We have summarized some background on divisors and linear systems on surfaces used here on Appendix B.

2.3.1 Definition ([Ful98, §1.1 and §1.4])**.** Let X be a scheme of finite type over k. An *s-cycle* on X is a finite formal sum

$$\mathcal{Z} := \sum_i a_i Z_i,$$

where Z_i are integral subschemes of X of dimension s. The set of all s-cycles forms a group denoted $\mathrm{Cycl}_s(X)$ and called the *group of cycles* of dimension s on X.

If X is projective, *i.e.*, if it is embedded in some $\mathbb{P}^n_k$, the *degree* of the cycle $\mathcal{Z}$ is defined as

$$\deg Z := \sum a_i \deg Z_i,$$

where $\deg Z_i$ is the degree of the subscheme Z_i with respect to its embedding in $\mathbb{P}^n_k$. We say $\mathcal{Z}$ is *effective* if $a_i > 0$ for all i.

Let $f : X \to Y$ be a proper morphism between schemes of finite type over k. For any closed subvariety $Z \subset X$, we have that $W = \overline{\mathrm{im}\,(f|_Z)}$ is a closed subvariety. Let ζ and ξ be the generic points of Z and W respectively. The morphism $f|_Z$ defines a morphism between the function fields $(f|_Z)^* : \kappa(\xi) \to \kappa(\zeta)$. Define

$$\deg(Z/W) := \begin{cases} [\kappa(\xi) : \kappa(\zeta)], \text{ if } \dim W = \dim Z, \\ 0, \text{ if } \dim W < \dim Z. \end{cases}$$

For any s-cycle $\mathcal{Z} = \sum a_i Z_i$ on X, the cycle

$$f_* \mathcal{Z} = \sum_i a_i \deg(Z_i/W_i) W_i$$

is called the *proper pushforward* of $\mathcal{Z}$.

2.3.2 Let $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be a morphism corresponding to a tuple $(F_0, \ldots, F_n)$ of degree d and let $C = \overline{\operatorname{im}(f)}$ be its scheme theoretic image. Then, by definition, the proper pushforward of $\mathbb{P}^1_k$ as a cycle is the cycle on $\mathbb{P}^n_k$ given by

$$f_* \mathbb{P}^1_k = [\kappa(t) : k(C)] C.$$

This is also a 1-cycle in $\mathbb{P}^2_k$, *i.e.* a divisor in $\operatorname{Div} \mathbb{P}^2_k$.

Equivalently, if $f = \nu \circ g$ is the factorization through the normalization $\nu : \mathbb{P}^1_k \to C$ given in 2.1.10, then

$$f_* \mathbb{P}^1_k = [g^* k(t) : k(t)] C = m_0 C. \tag{2.18}$$

By definition, the degree of $f_* \mathbb{P}^1_k$ is $d = m_0 \deg C$. Let

$$\operatorname{Hom}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) := \{ f : \mathbb{P}^1_k \to \mathbb{P}^2_k \mid \deg F_i = d \text{ for all } i \}.$$

Let $\operatorname{Div}^{\mathit{eff}}_d \mathbb{P}^2_k$ be set of effective divisors in $\operatorname{Div} \mathbb{P}^2_k$ of degree d. We can define a set theoretical map

$$\begin{aligned} \operatorname{Hom}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) &\longrightarrow \operatorname{Div}^{\mathit{eff}}_d \mathbb{P}^2_k \\ f &\longmapsto f_*(\mathbb{P}^1_k). \end{aligned} \tag{2.19}$$

The main point of this section is to prove that when the characteristic of k is zero, the map (2.19) correponds to the map on the k-points of a regular morphism between $\operatorname{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$ and a complete linear system. This can be proven through the existence of the Chow scheme of 1-cycles in $\mathbb{P}^2_k$.

If the characteristic of k is positive, we cannot guarantee the existence of the

Chow scheme. However, we will point out a strategy to bypass the existence of the Chow scheme by using the functor of linear systems $\mathcal{LinSys}_{\mathcal{O}_{\mathbb{P}^n_k}(d)}$ defined in B.2.4.

2.3.3 Remark. It is easy to see that if $\operatorname{char} k = 0$, $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ is a non constant morphism and $p \in f(\mathbb{P}^1_k)$ then the parametric multiplicity $m_p(f)$ coincides with the multiplicity of the divisor $\mu_p(f_*\mathbb{P}^1_k)$ (see B.1.8 for the definition). Indeed, we already know this holds if f is generically one-to-one. In general, write $f = \nu \circ g$ where ν is the normalization of $C = \overline{\operatorname{im}(f)}$ and $m_0 = [g^*k(t) : k(t)]$. Therefore, by Remark 2.2.10 and equation (2.18) above

$$m_p(f) = m_0 m_p(\nu) = m_0 \mu_p(C) = \mu_p(f_*\mathbb{P}^1_k).$$

2.3.4 Two words about Chow schemes. Linear systems are examples of schemes parametrizing effective cycles. These schemes exist in more generality, at least when the characteristic of the ground field is zero. Let us briefly comment on this topic following [Kol96].

Let $X \hookrightarrow \mathbb{P}^n_k$ be an integral scheme and assume $\operatorname{char} k = 0$. For any scheme S over k we obtain a flat morphism $X \times S \to S$, and for each point $p \in S$ consider the fiber X_p. A natural question is for which closed subschemes $Z \subset X \times S$ the fiber Z_p is a a cycle in $\operatorname{Cycl}_s(X_p)$ for some s. That is, Z can be understood as a "family of cycles" on $X \times S$ over S. Another two reasonable expectations are that $Z \subset X \times S$ is proper over S and that $\deg Z_p = \deg Z_q$ for all points $p, q \in S$. These reasonable expectations motivate the definition of a *well defined family of proper cycles* of dimension s and degree d, see [Kol96, p. I.3.10]. We can define the set

$$\mathcal{Chow}_{s,d}(X)(S) = \left\{ \begin{array}{c} \text{Well defined families of effective,} \\ \text{proper, algebraic cycles on } X \times S \text{ over } S \\ \text{of dimension } s \text{ and degree } d \end{array} \right\}.$$

In fact, for any morphism $S' \to S$ between semi-normal varities (see [Kol96, Definition I.7.2.1]), Kóllar defines suitable morphisms

$$\mathcal{Chow}_{s,d}(X)(S) \to \mathcal{Chow}_{s,d}(X)(S')$$

so that the association

$$S \mapsto \mathcal{Chow}_{s,d}(X)(S)$$

is a functor from the category of semi-normal schemes to sets, see [Kol96, Proposition I.3.19].

Moreover, if $\operatorname{char} k = 0$, this functor is represented by a scheme called the *Chow scheme* and denoted $\mathrm{Chow}_{s,d}(X)$. In particular, we have

$$\mathrm{Hom}(\operatorname{Spec} k, \mathrm{Chow}_{s,d}(X)) \cong \{D \in \mathrm{Cycl}_s(X) \mid D \text{ effective and } \deg D = d\}.$$

The theory to prove the representability of $\mathcal{C}how_{s,d}(X)$ is deep and goes much beyond the scope of this thesis. The reader is referred to [Kol96, §1.3]. A theory of relative cycles has also been developed by Suslin-Voevodsky [SV00], although the Chow presheaves defined by them differ slightly from those defined by Kóllar. These sheaves are still not representable by a scheme in positive characteristic. However, the sheafification of these presheaves (on a sufficiently fine Grothendieck topology) is isomorphic to the sheafification of the functor of points of a scheme. This scheme is unique up to universal homeomorphism. The interested reader is referred to loc.cit or a more detailed exposition of both previous approaches given in [And19]. In particular, Anderson proves that the Chow scheme defined above following [Kol96] is the semi-normalization of the one obtained in [SV00], see [And19, Corollary 6.3.26].

2.3.5 Example. If $X = \mathbb{P}^n_k$ and $N = \binom{n+d}{d} - 1$, then we already know that $|dH| := \mathbb{P}(H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))^\vee)$ parametrizes the effective divisors of degree d on $\mathbb{P}^n_k$, where $H \subset \mathbb{P}^n_k$ is a hyperplane. If we assume that $\operatorname{char} k = 0$, then we have that the Chow scheme is isomorphic to the classical construction of Chow varieties constructed by Chow coordinates. In particular, we have that

$$\mathrm{Chow}_{n-1,d}(\mathbb{P}^n_k) \cong |dH|$$

see [Ryd03, Example 8.29] for an accessible proof using Chow coordinates.

2.3.6 Scheme of morphisms and cycles. Suppose $\operatorname{char} k = 0$. It follows from [Kol96, Corollary I.6.9] that there is a regular morphism[1]

$$\begin{aligned} \Xi : \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k) &\longrightarrow \mathrm{Chow}(\mathbb{P}^2_k) \\ [f] &\longmapsto [f_*\mathbb{P}^1_k]. \end{aligned}$$

In particular, let $d > 0$ and $L \subset \mathbb{P}^2_k$ be a line in $\mathbb{P}^2_k$. We can restrict this morphism to $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$ and define

$$\Xi_d := \Xi|_{\mathrm{Mor}_d(\mathbb{P}^1_k,\mathbb{P}^2_k)} : \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \longrightarrow \mathrm{Chow}_{1,d}(\mathbb{P}^2_k) \cong |dL|. \tag{2.20}$$

[1] Notice that in [Kol96, Corollary I.6.9] we need to take the semi-normalization of the scheme $\mathrm{Mor}(X,Y)$, however since $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)$ is nonsingular, seminormalization is not needed, see [Kol96, Proposition I.7.2.3].

2.3.7 Example. Consider the morphisms $f_1, f_2 : \mathbb{P}^1_k \to \mathbb{P}^2_k$ and given by the equations

$$f_1(u:v) = (u^2 : (u+v)^2 : v^2);$$
$$f_2(u:v) = (u^2 : u^2+v^2 : v^2).$$

The scheme theoretic image of f_1 is the irreducible conic

$$C = \{x_0^2 + x_1^2 + x_2^2 - 2(x_0x_1 + x_0x_2 + x_1x_2) = 0\},$$

while the scheme theoretic image of f_2 is the line

$$L = \{x_0 - x_1 + x_2 = 0\},$$

but notice that f_2 is a double cover of L. Thus, we have

$$\Xi_2([f_1]) = [C] \ \text{ and } \ \Xi_2([f_2]) = [2L].$$

2.3.8 A strategy for positive characteristic. To define the morphism Ξ_d we assumed the characteristic of k was zero and used that the Chow scheme $\mathrm{Chow}_{1,d}(\mathbb{P}^2_k) \cong |dL|$. We expect that this can be proven in a more direct way for k of arbitrary characteristic using the functor of linear systems $\mathcal{L}in\mathcal{S}ys_{\mathcal{O}_{\mathbb{P}^2_k}(d)}$, defined in B.2.4, in the following way. Since $\mathcal{M}or(\mathbb{P}^1_k, \mathbb{P}^2_k) \cong h_{\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)}$ we define the functor $\mathcal{M}or_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$ to be the fibered product in

$$\begin{tikzcd}
\mathcal{M}or_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \arrow[r, "\sim"] \arrow[d] & h_{\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)} \arrow[d] \\
\mathcal{M}or(\mathbb{P}^1_k, \mathbb{P}^2_k) \arrow[r, "\sim"] & h_{\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)}
\end{tikzcd}$$

and recall that if S is a scheme of finite type over k, we have

$$\mathcal{L}in\mathcal{S}ys_{\mathcal{O}_{\mathbb{P}^2_k}(d)}(S) = \left\{ \begin{array}{c} D \subset \mathbb{P}^2_k \times S \text{ relative effective Cartier divisor} \\ \text{such that } \mathcal{O}_{\mathbb{P}^2_k \times S}(D) \cong \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^2_k(d)}) \otimes_{\mathcal{O}_{\mathbb{P}^2_k \times S}} \mathrm{pr}_2^*(\mathcal{K}) \\ \text{for some } \mathcal{K} \in \mathrm{Pic}\, S \end{array} \right\}.$$

The key point of the argument would be to define a morphism

$$\begin{aligned} \Upsilon_d(S) : \mathcal{M}or_d(\mathbb{P}^1_k, \mathbb{P}^2_k)(S) &\longrightarrow \mathcal{L}in\mathcal{S}ys_{\mathcal{O}_{\mathbb{P}^2_k}(d)}(S) \\ [f : \mathbb{P}^1_k \times S \to \mathbb{P}^2_k \times S] &\longmapsto (\mathrm{pr}_1^*(\mathrm{pr}_1 \circ f)_*(\mathbb{P}^1_k \times S)) \end{aligned}$$

and check this defines a well defined morphism of functors. This would require checking the following:

- for every $f : \mathbb{P}^1_k \times S \to \mathbb{P}^2_k \times S$, the cycle $\mathrm{pr}_1^*(\mathrm{pr}_1 \circ f)_*(\mathbb{P}^1_k \times S)$ is a relative effective Cartier divisor;

- $\mathcal{O}_{\mathbb{P}^2_k \times S}(\mathrm{pr}_1^*(\mathrm{pr}_1 \circ f)_*(\mathbb{P}^1_k \times S)) \cong \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^2_k}(d)) \otimes_{\mathcal{O}_{\mathbb{P}^2_k \times S}} \mathrm{pr}_2^*(\mathcal{K})$ for some $\mathcal{K} \in \operatorname{Pic} S$;

- and for any morphism $g : S' \to S$, we have a well defined commutative diagram

$$\begin{tikzcd} \mathcal{M}or_d(\mathbb{P}^1_k, \mathbb{P}^2_k)(S) \arrow[r, "\Upsilon_d(S)"] \arrow[d] & \mathcal{L}in\mathcal{S}ys_{\mathcal{O}_{\mathbb{P}^2_k}(d)}(S) \arrow[d] \\ \mathcal{M}or_d(\mathbb{P}^1_k, \mathbb{P}^2_k)(S') \arrow[r, "\Upsilon_d(S')"] & \mathcal{L}in\mathcal{S}ys_{\mathcal{O}_{\mathbb{P}^2_k}(d)}(S'). \end{tikzcd}$$

To check all of the items, one might need a better description of $f : \mathbb{P}^1_k \times S \to \mathbb{P}^2_k \times S$, as it is does not simply correspond to polynomials of degree d. The third item, however, seems to be manageable by [SV00, Theorem 3.6.1].

If this is, in fact, a well defined morphism of functors, then the restriction of $\mathcal{M}or_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$ to the category **FinType**$/k$ is still representable by $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$, and therefore, the morphism Υ_d would induce a unique morphism of varieties $\Xi_d : \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \to |dL|$ by Yoneda Lemma. For now, whenever we use the morphisms Ξ_d we will assume that $\operatorname{char} k = 0$.

2.3.9 Two words about group actions. We can understand the fibers of the morphisms Ξ_d as the orbit of a group action. We recall basic definitions following [Mil17, Chapter 1].

We say that G is an *algebraic group* over k if it is a group object in the category **FinType**$/k$ of schemes of finite type over k. In other words, it is a pair (G, m) where G is a scheme of finite type over k, $m : G \times G \to G$ is a regular morphism and there exist morphisms $e : \operatorname{Spec} k \to G$ and $\mathrm{inv} : G \to G$ satisfying natural commutative diagrams:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "(\mathrm{inv}\,,\mathrm{id})"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(\mathrm{id}\,,\mathrm{inv})"'] \arrow[d] \\
\operatorname{Spec} k \arrow[r, "e"] & G & \operatorname{Spec} k \arrow[l, "e"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\operatorname{Spec} k \times G \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\simeq"'] & G \times G \arrow[d, "m"] & G \times \operatorname{Spec} k \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "\simeq"] \\
& G. &
\end{tikzcd}
$$

Let G be an algebraic group over k and X be a scheme of finite type over k. We say G *acts* on X if there is a regular morphism

$$\alpha : G \times X \longrightarrow X$$

satisfying commutative diagrams:

$$
\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{id} \times \alpha"] \arrow[d, "m \times \mathrm{id}"'] & G \times X \arrow[d, "\alpha"] \\
G \times X \arrow[r, "\alpha"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Spec} k \times X \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\simeq"'] & G \times X \arrow[d, "\alpha"] \\
& X
\end{tikzcd}
$$

(compare with [MFK94]).

Suppose that G is non-singular acting on X via α. Then the G-*orbit* of $p \in X$ is defined to be the scheme theoretic image of an orbit map

$$
\begin{aligned}
o_p : G \times p \cong G &\longrightarrow X \\
g &\longmapsto \alpha(g, p)
\end{aligned}
$$

and we will denote it by O_p. Moreover, we can define the *stabilizer* of a point $p \in X$ as the fiber $G_p := o_p^{-1}(p)$. It is an algebraic subgroup of G.

2.3.10 Lemma. *Let G be a nonsingular algebraic group over k acting over an integral scheme X over k. If there exists a point p such that $\dim G_p = 0$, then there exists an open subset $U_0 \subset X$ such that for any point $q \in U_0$ we have $\dim O_q = \dim G$.*

Proof. The morphism o_p is faithfully flat onto its image (see [Mil17, Proposition

7.4(b)]). Therefore, for any point in $q \in O_p$ we have

$$\dim o_p^{-1}(q) = \dim G - \dim O_p,$$

see [Har77, Corollary III.9.6]. By [MFK94, Definition 0.9], the set

$$Z_r = \{p \in X \mid \dim o_p^{-1}(p) \geq r\}$$

is closed in X. Thus, $U_0 \coloneqq X \setminus Z_1$ is an open subset consisting of points whose orbits under the action of G are 0-dimensional. It follows that if there exists a point p such that $\dim o_p^{-1}(p) = 0$, then $p \in U_0$. In other words, U_0 is non-empty and for any point $q \in U_0$, we have $\dim O_q = \dim G$. □

2.3.11 Lemma. *Let k be an algebraically closed field of characteristic 0 and*

$$\Xi_d : \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \to |dL|$$

be the morphism defined in (2.20). Then there exists an open subset

$$U \subset \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$$

such that for any k-point $[D] \in \mathrm{im}(\Xi_d|_U)$ we have that $\Xi_d^{-1}([D])$ is irreducible of dimension 3.

Proof. Recall that

$$\mathrm{Aut}(\mathbb{P}^1_k) \coloneqq \mathrm{Mor}_1(\mathbb{P}^1_k, \mathbb{P}^1_k)$$

is the variety parametrizing automorphisms of $\mathbb{P}^1_k$. It is well known that it is an algebraic group, and we have already seen it is nonsingular and $\dim \mathrm{Aut}(\mathbb{P}^1_k) = 3$. Furthermore, we have a natural action of $\mathrm{Aut}(\mathbb{P}^1_k)$ on each $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$ given by

$$\begin{aligned} \mathrm{Aut}(\mathbb{P}^1_k) \times \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) &\longrightarrow \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \\ ([g], [f]) &\longmapsto [f \circ g]. \end{aligned} \tag{2.21}$$

By definition, for every k-point in $\mathrm{im}(\Xi_d)$ corresponding to a divisor D, we have that $D = f_*\mathbb{P}^1$ for some $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$. On the other hand, for any automorphism $[g] \in \mathrm{Aut}(\mathbb{P}^1_k)$ we have that $D = (f \circ g)_*\mathbb{P}^1_k$. In other words, Ξ_d is $\mathrm{Aut}(\mathbb{P}^1_k)$-invariant.

Claim. There is an open subset $U \subset \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$ such that for each $[D] \in U$ fiber $\Xi_d^{-1}([D])$ is the $\mathrm{Aut}(\mathbb{P}^1_k)$-orbit of $[f]$ under the action (2.21).

Proof of claim. Let $\mathrm{Mor}_{bir}(\mathbb{P}^1_k, \mathbb{P}^2_k)$ denote the open subscheme of $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)$ parametrizing morphisms from $\mathbb{P}^1_k$ to $\mathbb{P}^2_k$ which are generically one-to-one (see

[Kol96, Definition II.2.6]) and let

$$U \coloneqq \mathrm{Mor}_{bir}(\mathbb{P}^1_k, \mathbb{P}^2_k) \cap \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k).$$

Consider two points $[f], [f'] \in U$ such that $f'_*\mathbb{P}^1_k = f_*\mathbb{P}^1_k$, then

$$\overline{\mathrm{im}\,(f)} \cong \overline{\mathrm{im}\,(f')} =: C.$$

Since both f and f' induce birational morphisms onto C there exists a morphism $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that the diagram

$$\begin{tikzcd}
\mathbb{P}^1_k \arrow[dr] \arrow[drr, bend left, "f"] \arrow[dd, "g"'] & & \\
& C \arrow[r, hook] & \mathbb{P}^2_k \\
\mathbb{P}^1_k \arrow[ur] \arrow[urr, bend right, "f'"'] & &
\end{tikzcd}$$

is commutative. Since

$$\deg(f) = \deg(f') = d \ \text{ and } \ \deg(g \circ f') = \deg(g)\deg(f') = \deg(f),$$

it follows that $\deg g = 1$, *i.e* $g \in \mathrm{Mor}_1(\mathbb{P}^1_k, \mathbb{P}^1_k)$ and $[f']$ is in the $\mathrm{Aut}(\mathbb{P}^1_k)$-orbit of $[f]$. $\square$

In particular, by the definition of the orbits in 2.3.9 and since $\mathrm{Aut}(\mathbb{P}^1_k)$ is irreducible, we have that $\varXi_d^{-1}([D])$ is irreducible for every $[D] \in U$.

Claim. There exists a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ such that the stabilizer $\mathrm{Aut}(\mathbb{P}^1_k)_{[f]}$ has dimension 0.

Proof of claim. Define $f(u : v) \coloneqq (u^d : v^d : u^{d-1}v)$ and notice that if $[g] \in \mathrm{Aut}(\mathbb{P}^1_k)$, then we can write $g(u : v) = (\alpha u + \beta v : \gamma u + \delta v)$ with $\alpha, \beta, \gamma, \delta \in k$ and $\det \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \neq 0$. Then it is easy to see that $f \circ g = f$ if and only if the system

$$\begin{cases} \beta = 0 \\ \gamma = 0 \\ \alpha^d = 1 \\ \delta^d = 1 \\ \alpha^{d-1}\delta = 1 \end{cases}$$

has a solution. The third and fourth equations imply that $\alpha = \epsilon_d^a$ and $\delta =$

ϵ_d^b, where ϵ_d is the primitive root of the unit. The fifth equation implies that $(d-1)a+b \equiv 0 \mod d$, or equivalently $a \equiv b \mod d$. It follows that $\alpha = \delta$ and hence $g = \mathrm{id}_{\mathbb{P}^1_k}$, as claimed. In particular, $\mathrm{Aut}(\mathbb{P}^1_k)_{[f]}$ consists of a single point and $\dim \mathrm{Aut}(\mathbb{P}^1_k)_{[f]} = 0$. □

By Lemma 2.3.10, there exists an open subset $U_0 \subset \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ such that for any $[f] \in U_0$ and $D := f_*\mathbb{P}^1_k$ we have

$$\dim \Xi_d^{-1}([D]) = \dim O_{[f]} = \dim \mathrm{Aut}(\mathbb{P}^1_k) = 3.$$

Finally, since $\Xi_d|_U$ is an $\mathrm{Aut}(\mathbb{P}^1_k)$-invariant morphism and $\mathrm{Aut}(\mathbb{P}^1_k)$ is defined over k we can can apply Remark [MFK94, (4) of p. 6 & 7] on $\Xi_d|_U$ and conclude that the dimensions of the fibers of $\Xi_d|_U$ are constant in the irreducible components of U. Since $U \cap U_0$ is non-empty and U is irreducible, it follows that $\dim \Xi_d^{-1}([D]) = 3$ for all $D \in \mathrm{im}\,(\Xi_d|_U)$. □

2.4 Rational curves on Del Pezzo surfaces

Let us denote $X = \mathrm{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^n_k)$ and recall that in Theorem 2.2.11 we have defined a partition of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ which is independent of a projective embedding of X. Once we fix a projective embedding $X \hookrightarrow \mathbb{P}^N_k$, we can use this partition to refine the partition of $\mathrm{Mor}(\mathbb{P}^1_k, X) = \coprod_{e\in\mathbb{N}} \mathrm{Mor}_e(\mathbb{P}^1_k, X)$ in degrees of Remark 2.1.7.

We say r points in $\mathbb{P}^2_k$ (with $1 \le r \le 8$) are in *general position* if no three of them lie on a line, no six of them lie on a conic and all of them do not lie on a cubic. If X is the blow-up of $\mathbb{P}^2_k$ at r points in general position then X is a Del Pezzo surface and there are well known projective embeddings for X. We use those projective embeddings to refine the partition of degrees on $\mathrm{Mor}(\mathbb{P}^1_k, X)$.

2.4.1 Corollary. *Let k be an algebraically closed field and let $r \le 8$ be a positive integer. Let $\sigma : X \to \mathbb{P}^2_k$ be the blow-up of $\mathbb{P}^2_k$ in r points in general position. Let $M_{d,\mathbf{m}}$ denote the closed subschemes defined in Theorem 2.2.11 and $\mathrm{Mor}_e(\mathbb{P}^1_k, E_i)$ denote the scheme parametrizing morphisms of degree e to the exceptional line E_i. Let a be a positive integer such that $-aK_X$ is very ample and let*

$$\iota_{-aK_X} : X \hookrightarrow \mathbb{P}^N_k$$

be the corresponding embedding. Then for each integer $e > 0$, the scheme parametriz-

ing morphisms of degree e from $\mathbb{P}^1_k$ to X with respect to ι_{-aK_X} is given by

$$\operatorname{Mor}_e(\mathbb{P}^1_k, X) = \left(\coprod_{e=a(3d-|\mathbf{m}|)} M_{d,\mathbf{m}} \right) \amalg \left(\coprod_{i=1}^{r} \operatorname{Mor}_e(\mathbb{P}^1_k, E_i) \right).$$

Proof. Let $L \in \operatorname{Pic} \mathbb{P}^2_k$ be the divisor class of a line in $\mathbb{P}^2_k$. The anticanonical class of X is given by

$$-K_X = 3\sigma^* L - \sum_{i=1}^{r} E_i.$$

Recall that it is ample (see [Man86, Theorem 24.4]), i.e. there is an integer $a > 0$ such that $-aK_X$ induces a closed immersion $\iota_{-aK_X} : X \hookrightarrow \mathbb{P}^N_k$ (if $r \leq 6$ we can take $a = 1$, see [Man86, Theorem 24.5]).

Let $e > 0$ and let $[f]$ be a closed point in $\operatorname{Mor}_e(\mathbb{P}^1_k, X)$ and consider the scheme theoretic image $C = \overline{\operatorname{im}(\sigma \circ f)}$.

Claim. Suppose C is not a point. Then, there exists a positive integer d such that

$$e = \left(3d - \textstyle\sum_{i=1}^{r} m_{p_i}(\sigma \circ f)\right) a,$$

where $m_{p_i}(\sigma \circ f)$ is the parametric multiplicity of $\sigma \circ f$ in p_i.

Proof of claim. Notice that $\widetilde{C} \coloneqq \overline{\operatorname{im}(f)}$ is the strict transform of C under σ. Let d be the degree of C in $\mathbb{P}^2_k$ and let $m_i \coloneqq \mu_{p_i}(C)$ be the multiplicity of C at each point p_i.

Suppose that f is generically one-to-one. Hence, so is $\sigma \circ f$. In particular, the parametric multiplicities coincide with the multiplicities of C at each p_i, that is, $m_{p_i}(\sigma \circ f) = \mu_{p_i}(C)$, see [Pér07, Theorem 8]. Notice that $[f] \in M_{d,\mathbf{m}}$ for some collection $\mathbf{m} = (m_1, \ldots, m_r)$.

Since $\sigma \circ f$ is generically one-to-one, it follows from Example 1.1.13 that e coincides with the degree of $\widetilde{C}$ in $\mathbb{P}^N_k$. Furthermore, the degree of $\widetilde{C}$ in $\mathbb{P}^N_k$ is the intersection number $\widetilde{C} \cdot (-aK_X)$ ([Har77, Remark V.1.6.2]).

Recall that we can write the total transform of C via σ as

$$\sigma^* C = \widetilde{C} + \sum_{i=1}^{r} m_i E_i,$$

see B.2. We can use B.1.3 and B.1.6 to compute

$$\begin{aligned}
e = \deg \widetilde{C} = \widetilde{C} \cdot (-aK_X) &= a \left(\sigma^* C \cdot (-K_X) - \left(\sum_{i=1}^{r} m_i E_i \right) \cdot (-K_X) \right) \\
&= a \left(\sigma^* C \cdot \sigma^* 3L - \sigma^* C \cdot \left(\sum_{i=1}^{r} E_i \right) - \left(\sum_{i=1}^{r} m_i E_i \right) \cdot \sigma^* 3L + \left(\sum_{i=1}^{r} m_i E_i \right) \cdot \left(\sum_{i=1}^{r} E_i \right) \right) \\
&= a \left(3 \deg C - \sum_{i=1}^{r} m_i \right) = a \left(3d - |\mathbf{m}| \right).
\end{aligned}$$

If f is not generically one-to-one, then let $\nu : \mathbb{P}^1_k \to \widetilde{C}$ be the normalization of $\widetilde{C}$. Then there exists a unique $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $f = \nu \circ g$. Let m_0 be the degree of the polynomials defining g, by (2.7), we have

$$e = m_0 \deg \widetilde{C}.$$

Moreover since σ is birational, so is $\sigma \circ \nu$ and since

$$\overline{\operatorname{im}(\sigma \circ \nu)} = \overline{\operatorname{im}(\sigma \circ f)} = C \quad \text{and} \quad \sigma \circ f = \sigma \circ \nu \circ g,$$

it follows that

$$d = m_0 \deg C.$$

By (2.11) in Remark 2.2.10 we have that

$$m_{p_i}(f) = m_0 m_{p_i}(\nu)$$

for all p_i. Since ν is generically one-to-one, it follows from the previous case that

$$\deg \widetilde{C} = a \left(3 \deg C - \sum_{i=1}^{r} m_{p_i}(\nu) \right).$$

Therefore $e = a(3d - |\mathbf{m}|)$ as before. □

Claim. Suppose C is a point. Then there exists i such that $[f] \in \mathrm{Mor}_e(\mathbb{P}^1_k, E_i)$.

Proof of claim. Clearly, if C is a point, then $\widetilde{C} = E_i$ for some i and $E_i \cong \mathbb{P}^1_k$ is a line. Therefore, $[f] \in \mathrm{Mor}_e(\mathbb{P}^1_k, X) \cap \mathrm{Mor}(\mathbb{P}^1_k, E_i)$ if and only if f is a morphism of degree e to E_i. □

The two claims yield the desired result. □

2.4.2 Remark. Notice that, under the hypotheses of Corollary 2.4.1, we have $\mathrm{Mor}_e(\mathbb{P}^1_k, E_i) \cong \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^1_k)$, therefore $\mathrm{Mor}_e(\mathbb{P}^1_k, E_i)$ is an irreducible nonsingular component of dimensions $2e+1$. Moreover, if $M_{d,\mathbf{0}} \subset \mathrm{Mor}_e(\mathbb{P}^1, X)$ for some $e > 0$ we can write the dimension of this component in terms of e as $\dim M_{d,\mathbf{0}} = e/a+2$.

2.4.3 Components in $M_{d,\mathbf{m}}$. Consider $\sigma : X \to \mathbb{P}^2_k$ to be the blow-up of $\mathbb{P}^2_k$ at r points in general position, with $r \leq 8$. That is, X is a Del Pezzo surface of degree $9-r$. Our objective is to find components in $M_{d,\mathbf{m}} \subset \mathrm{Mor}(\mathbb{P}^1_k, X)$ for given d and $\mathbf{m} = (m_1, \ldots, m_r)$.

The strategy is to look at the preimages of the composition of the maps

$$M_{d,\mathbf{m}} \xrightarrow{\sigma_M|_{M_{d,\mathbf{m}}}} \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \xrightarrow{\Xi_d} |dL|.$$

Notice that the image $\mathrm{im}\left(\Xi_d \circ \sigma_M|_{M_{d,\mathbf{m}}}\right)$ consists of effective divisors of degree d in $\mathbb{P}^2_k$ supported on rational curves passing through each blown-up point with specified multiplicities. Let us be more precise.

Let $f : \mathbb{P}^1_k \to X$ be a morphism and $C = \overline{\mathrm{im}\,(\sigma \circ f)}$. Suppose that $[f] \in M_{d,\mathbf{m}}$ and f is generically one-to-one. Then each m_i coincides with the multiplicity $\mu_{p_i}(C)$ at the point p_i. Hence, we need to describe the locus on $|dL|$ of divisors supported on irreducible rational curves passing through the blown-up points p_i with multiplicity m_i. We will see that under the assumption that $\widetilde{C} \coloneqq \overline{\mathrm{im}\,(f)}$ is smooth, the closure of $\Xi_d \circ \sigma_M(M_{d,\mathbf{m}})$ in $|dL|$ contains an irreducible locally closed subset of $|dL|$ dense in a linear system of $|dL|$ containing the divisor C, and we will determine its dimension.

Before proceeding to the proof of this, we introduce rational linear systems following Daigle and Melle-Hernández [DM12]. We follow their terminology and use the main result of their paper to compute the desired dimension.

2.4.4 Definition ([DM12, Definition 2.1])**.** Suppose $\mathrm{char}\, k = 0$ and let X be a rational surface over k. A linear system of divisors $\mathbb{L}$ on X is said to be *rational* if a general member is an irreducible rational curve, *i.e.*, if there exists a closed proper subset $Z \subset \mathbb{L}$ such that every point in the open $U \coloneqq \mathbb{L} \smallsetminus Z$ corresponds to an irreducible rational curve (see [Iit82, §7.9])

2.4.5 Remark. In the original definition [DM12, Definition 2.1], the authors assume $\dim \mathbb{L} \geq 1$. We will adopt the convention that if $\dim \mathbb{L} = 0$, then $\mathbb{L}$ consists of a single divisor supported on a rational curve on X.

2.4.6 Lemma. *Suppose* $\mathrm{char}\, k = 0$*. Let C be an irreducible rational curve of degree d in $\mathbb{P}^2_k$ passing through points $p_1, \ldots, p_r$ with multiplicities $m_1, \ldots, m_r$ and let $\sigma : X \to \mathbb{P}^2_k$ be the blow-up of $\mathbb{P}^2_k$ on the points p_i. Let $\widetilde{C}$ be the strict transform of C under σ. Suppose that $\widetilde{C}$ is non-singular. Then there exists a rational linear system $\mathbb{L}_C$ on $\mathbb{P}^2_k$ satisfying the following properties:*

1. $D \in \mathbb{L}_C \iff D$ *is an effective divisor of degree* d *in* $\mathbb{P}^2_k$ *such that the multiplicity of* D *at* p_i *is* m_i *for all* i*.*

2. $\dim \mathbb{L}_C = \max\{d^2 + 1 - \sum_{i=1}^r m_i^2, 0\}$*;*

3. $\mathbb{L}$ *is a rational linear system containing* C *if and only if* $\mathbb{L} \subseteq \mathbb{L}_C$*.*

Moreover, the intersection $\mathrm{im}(\Xi_d) \cap \mathbb{L}_C$ *is dense on* $\mathbb{L}_C$*.*

Proof. Let $\widetilde{C}$ be the strict transform of the curve C. Define

$$L_C = \left\{ \sigma_* D' \in \mathrm{Div}\, \mathbb{P}^2_k \mid D' \in |\widetilde{C}| \right\}. \tag{2.22}$$

Let L be the divisor class of a line $\mathbb{P}^2_k$ and $|dL|$ be the complete linear system of divisors of degree d in $\mathbb{P}^2_k$. For each divisor D in $\mathrm{Div}\, \mathbb{P}^2_k$ let $\mu_{p_i}(D)$ denote the multiplicity of D at the point p_i (see B.1.8).

Claim. We have $D \in L_C$ if and only if $D \in |dL|$ and $\mu_{p_i}(D) = m_i$ for each i.

Proof of claim. If $D \in |dL|$ and $\mu_{p_i}(D) = m_i$, by (B.2) we have that $\sigma^* D - \sum_{i=1}^r m_i E_i$ is effective and linearly equivalent to $\widetilde{C}$, therefore, by B.1.8,

$$D = \sigma_* \left(\sigma^* D - \sum_{i=1}^r m_i Ei \right) \in L_C.$$

Conversely, if $D \in L_C$, then D is linearly equivalent to $\sigma_* \widetilde{C} = C$, thus we have $\sigma^* D$ is linearly equivalent to $\sigma^* C$, by B.2, this happens if and only if

$$\mu_{p_i}(D) = \mu_{p_i}(C) = m_i \text{ for all } i. \qquad \square$$

Suppose $\widetilde{C}^2 \geq 0$. The set L_C is parametrized by a linear system on $\mathbb{P}^2_k$, which we will denote $\mathbb{L}_C$, of dimension $\dim |\widetilde{C}|$ (see [DM12, Definition 2.5]) and since $\widetilde{C}$ is nonsingular we have $\widetilde{C} \cong \mathbb{P}^1_k$, therefore by [DM12, Lemma 2.4.(a)] we obtain

$$\dim \mathbb{L}_C = \widetilde{C}^2 + 1 = \left(\sigma^* C - \sum_{i=1}^r m_i E_i \right)^2 + 1 = d^2 + 1 - \sum_{i=1}^r m_i^2.$$

therefore $\mathbb{L}_C$ has properties 1 and 2. It follows from [DM12, Theorem 2.8.] that $\mathbb{L}_C$ is a rational linear system satisfying property 3.

Claim. Suppose $\widetilde{C}^2 < 0$, then $|\widetilde{C}|$ is a point and consists of the effective divisor $\widetilde{C}$.

Proof of claim. Let $D \in |\widetilde{C}|$ be an effective divisor. Suppose that $D \neq \widetilde{C}$ is an irreducible curve, then since it is linearly equivalent to $\widetilde{C}$ it is numerically equivalent to $\widetilde{C}$, see Remark B.1.5, therefore

$$\widetilde{C} \cdot D = \widetilde{C}^2 < 0,$$

which is a contradiction since $\widetilde{C} \cdot D \geq 0$, see Remark B.1.4. Therefore $D = \widetilde{C}$.

Now suppose $D \in |\widetilde{C}|$ is just an effective divisor, since D is numerically equivalent to $\widetilde{C}$ we must have $\widetilde{C} \cdot D < 0$, therefore, by the case above one, of the curves on the support of D must be $\widetilde{C}$, in other words

$$D = a\widetilde{C} + D'$$

where $a \geq 1$, D' is an effective divisor and

$$D' \cdot \widetilde{C} < -a\widetilde{C}^2.$$

Moreover, we have that the class of $\sigma_* D$ is the same as the class of C in $\operatorname{Pic} \mathbb{P}^2_k$, in other words

$$\begin{aligned} aC + \sigma_* D' = C \text{ in } \operatorname{Pic} \mathbb{P}^2_k &\iff (a-1)C = -\sigma_* D' \text{ in } \operatorname{Pic} \mathbb{P}^2_k \\ &\iff (a-1)\deg(C) = -\deg(\sigma_* D'). \end{aligned}$$

In particular, since D' is effective, so is $\sigma_* D'$, thus $\deg(\sigma_* D') \geq 0$ and it follows that

$$0 \leq (a-1)\deg(C) = -\deg(\sigma_* D') \leq 0.$$

Hence $a = 1$ and $\deg(\sigma_* D') = 0$, that is $D' \in \ker \sigma_*$, or equivalently $D' = \sum_{i=1}^r m'_i E_i$ for $m'_i \in \mathbb{Z}$, see Remark B.1.7. Finally, since the classes of E_i are generators for $\operatorname{Pic} X \cong \mathbb{Z}^{r+1}$ we have that $D = \widetilde{C} + \sum_{i=1}^r m'_i E_i$ is linearly equivalent to $\widetilde{C}$ if and only if $m'_i = 0$ for all i, that is $D = \widetilde{C}$. □

The claim above tells us that if $\widetilde{C}^2 < 0$ then $L_C = \{C\}$, hence $\mathbb{L}_C$ is a 0-dimensional rational linear system for which 1, 2 are trivially satisfied. It follows from [DM12, Theorem 2.8. not (d) $\implies$ not (b)] that if we suppose $\mathbb{L}$ is a rational linear system containing C, then $\dim \mathbb{L} = 0$, that is $\mathbb{L}$ is the trivial linear system $\mathbb{L}_C$.

To prove the last assertion, notice that it follows from Remark 2.3.3 and property 1 that

$$\operatorname{im}(\Xi_d) \cap \mathbb{L}_C = \left\{ f_* \mathbb{P}^1_k \in \operatorname{Div}^{\mathit{eff}} \mathbb{P}^2_k \mid \deg(f) = d \text{ and } \mu_{p_i}(f_* \mathbb{P}^1_k) = m_i \text{ for all } i \right\}.$$

In other words, it is the set of divisors on $\mathbb{P}^2_k$ supported on rational curves passing through the points $p_1, \ldots, p_r$ with multiplicities m_i. Since $\mathbb{L}_C$ is a rational linear system, there is an open subset $U \subset \mathbb{L}_C$ whose points correspond to irreducible rational curves. For any irreducible rational curve $\iota : C \hookrightarrow \mathbb{P}^2_k$ in $\mathbb{L}_C$ we can take its normalization $\nu : \mathbb{P}^1_k \to C$ and define $f := \iota \circ \nu$ such that $C = f_*\mathbb{P}^1_k$ an by definition we have $\deg(f) = d$, see 2.1.10. It follows that $U \subset \operatorname{im}(\Xi_d) \cap \mathbb{L}_C$ and $\operatorname{im}(\Xi_d) \cap \mathbb{L}_C$ is dense on $\mathbb{L}_C$. □

2.4.7 Lemma (Dimension of fibers)**.** *Let $f : X \to Y$ be a morphism of algebraic varieties over k. Suppose that Y is irreducible and that all (closed) fibers of f are irreducible and of the same dimension m (in particular, f is surjective). Then:*

- *there is a unique irreducible component X^0 of X that dominates Y and;*
- *every irreducible component Z of X is a union of fibers of f with $\dim Z = \dim(\overline{f(Z)}) + m$. In particular, $\dim X^0 = \dim Y + m$.*

Proof. See [Mus17, Proof of Proposition 5.5.1]. □

2.4.8 Theorem. *Let k be an algebraically closed field of characteristic 0. Let*

$$\sigma : X \to \mathbb{P}^2_k$$

be the blow-up of $\mathbb{P}^2_k$ at r points. Suppose there exists a rational curve C of degree d in $\mathbb{P}^2_k$ passing through these points with multiplicities $\mathbf{m} = (m_1, \ldots, m_r)$ such that its strict transform under σ is nonsingular. Consider $M_{d,\mathbf{m}}$ to be the closed subscheme of $\operatorname{Mor}(\mathbb{P}^1_k, X)$ defined in Theorem 2.2.11. Then, there exists an irreducible component $M^0_{d,\mathbf{m}} \subset M_{d,\mathbf{m}}$ such that

$$\dim M^0_{d,\mathbf{m}} = \max\left\{d^2 + 1 - \textstyle\sum_{i=1}^r m_i^2, 0\right\} + 3.$$

Moreover, a general point of $M^0_{d,\mathbf{m}}$ corresponds to a generically one-to-one morphism.

Proof. Let $\sigma_M : \operatorname{Mor}(\mathbb{P}^1_k, X) \to \operatorname{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)$ be the induced morphism, Ξ_d be the morphism defined in (2.20), C be a curve satisfying the hypotheses of the statement and $\mathbb{L}_C \subset |dL|$ be the rational linear system divisors passing through the blown up points with multiplicities $m_1, \ldots, m_r$ defined on Lemma 2.4.6.

Claim. $\Xi_d \circ \sigma_M(M_{d,\mathbf{m}}) = \operatorname{im}(\Xi_d) \cap \mathbb{L}_C$.

Proof of claim. For each $[f] \in M_{d,\mathbf{m}}$ it follows from Remark 2.3.3 that

$$m_{p_i}(\sigma \circ f) = \mu_{p_i}((\sigma \circ f)_*\mathbb{P}^1_k) \tag{2.23}$$

for all i. In other words, we have that

$$\Xi_d(\sigma_M([f])) = (\sigma \circ f)_*\mathbb{P}^1_k \in \mathbb{L}_C.$$

Conversely, for any divisor $D \in \operatorname{im}(\Xi_d) \cap \mathbb{L}_C$, we have that there exists a morphism $g : \mathbb{P}^1_k \to \mathbb{P}^2_k$ such that

$$D = g_*\mathbb{P}^1_k \text{ and } \quad \deg(g) = \deg D = d.$$

Thus, by Proposition 2.2.4, there exists a unique $[f] \in \operatorname{Mor}(\mathbb{P}^1_k, X)$ such that $\sigma \circ f = g$, that is, $D = (\sigma \circ f)_*\mathbb{P}^1_k = \Xi_d(\sigma_M([f]))$. By Lemma 2.4.6 we have $\mu_{p_i}(D) = m_i$, thus, once again by Remark 2.3.3, we have (2.23). In other words, $[f] \in M_{d,\mathbf{m}}$. $\square$

Define

$$N := \overline{\operatorname{im}\left(\sigma_M|_{M_{d,\mathbf{m}}}\right)}_{red},$$

that is, N is the reduction of the scheme theoretical image of $\sigma_M|_{M_{d,\mathbf{m}}}$ in $\operatorname{Mor}(\mathbb{P}^1_k, \mathbb{P}^2_k)$. Then by Lemma 2.4.6, $\mathbb{L}_C$ has dimension $s := \dim \max\{d^2 + 1 - \sum_{i=1}^r m_i^2, 0\}$ and the image $\Xi_d(N)$ contains an open subset $U \subset \mathbb{L}_C$ such that each point in U corresponds to an irreducible rational curve. Moreover, we have seen in the proof of Lemma 2.3.11, that for each point $[f]$ of the open subset

$$V := \operatorname{Mor}_{bir}(\mathbb{P}^1_k, \mathbb{P}^2_k) \cap \operatorname{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$$

we have that the fiber over $\Xi_d([f])$ is irreducible of dimension 3.

Claim. $\Xi_d^{-1}(U) \subset V$.

Proof of claim. Suppose there exists a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$, such that

$$[f] \in \operatorname{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \smallsetminus V$$

such that $\Xi_d([f]) = f_*\mathbb{P}^1_k \in U$, that is $f_*\mathbb{P}^1_k$ is an irreducible rational curve, by definition of the proper pushforward we have $f_*\mathbb{P}^1_k = [k(t) : \kappa(C)]C$, but since f is not generically one-to-one, i.e. f is not birational to its image, we have that the induced degree of the field extension $[k(t) : \kappa(C)]$ is greater than 1. In other words, $f_*\mathbb{P}^1_k$ is not an irreducible rational curve, which is a contradiction. $\square$

Define $U' := \Xi|_N^{-1}(U)$. We have that $\Xi_d|_{U'}^{-1} : U' \to U$ is a surjective morphism of varieties such that every fiber is irreducible of dimension 3. Thus, by Lemma

2.4.7, there exists a unique irreducible component $U'' \subset U'$ such that

$$\Xi_d|_{U''} : U'' \to U$$

is dominant and U'' has pure dimension $s+3$.

Notice that by definition of the scheme theoretic image we have that $\sigma_M|_{M_{d,\mathbf{m}}}$ factors as

$$\sigma_M|_{M_{d,\mathbf{m}}} : M_{d,\mathbf{m}} \xrightarrow{\alpha} \overline{\operatorname{im}\left(\sigma_M|_{M_{d,\mathbf{m}}}\right)} \xhookrightarrow{\iota} \operatorname{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k)$$

where ι is a closed immersion and by the universal property of reduction we have a unique map α_{red} such that the diagram

$$\begin{tikzcd}
(M_{d,\mathbf{m}})_{red} \arrow[r, "\alpha_{red}"] \arrow[d] & N \arrow[d] \\
M_{d,\mathbf{m}} \arrow[r, "\alpha"] & \overline{\operatorname{im}\left(\sigma_M|_{M_{d,\mathbf{m}}}\right)}
\end{tikzcd}$$

is commutative. Consider the scheme theoretic preimage

$$W := \alpha_{red}^{-1}(U'').$$

By Proposition 2.2.4, $\sigma_M|_{M_{d,\mathbf{m}}}$ is bijective on closed points and since U' has underlying topological space isomorphic to $\Xi_d^{-1}(U) \subset \sigma_M(M_{d,\mathbf{m}})$ and reductions induce isomorphisms on underlying topological spaces, we have that $\alpha_{red}|_W$ is also bijective on closed points. This can be visualized in the following commutative diagram of sets

$$\begin{tikzcd}
W \arrow[r, "\sim"] \arrow[d, hook] & U'' \arrow[d, hook] & \\
\alpha_{red}^{-1}(U') \arrow[r, "\sim"] \arrow[d, hook] & U' \arrow[d, hook] & \Xi_d^{-1}(U) \arrow[l, "\sim"'] \arrow[d, hook] \\
(M_{d,\mathbf{m}})_{red} \arrow[r, "\alpha_{red}"] \arrow[rr, bend right, "\sigma_M|_{(M_{d,\mathbf{m}})_{red}}"'] & N & \sigma_M(M_{d,\mathbf{m}}). \arrow[l, hook']
\end{tikzcd}$$

In particular, every closed fiber of $\alpha_{red}|_W$ is irreducible of dimension 0. Again by Lemma 2.4.7, there exists a unique irreducible component $W' \subset W$ dominant-ing U'' and of pure dimension $s+3$. We illustrate the schemes defined here in the following commutative diagram

$$\begin{tikzcd}
W' \arrow[d, hook] \arrow[dr, "\text{dominant}"] & & & \\
W \arrow[r] \arrow[ddd] & U'' \arrow[d, hook] \arrow[drr, "\text{dominant}"] & & \\
& U' \arrow[rr] \arrow[dd, hook] & & U \arrow[dd, hook] \\
& & & \\
(M_{d,\mathbf{m}})_{red} \arrow[r, "\alpha_{red}"] & N \arrow[r, hook] & \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \arrow[r, "\Xi_d"] & \mathbb{L}_C
\end{tikzcd}$$

where the squares are cartesian. We take $M^0_{d,\mathbf{m}}$ to be the component of $M_{d,\mathbf{m}}$ whose underlying topological space is the closure $\overline{W'} \subset M_{d,\mathbf{m}}$.

It is easy to see that every point $[f] \in (\Xi_d \circ \sigma_M)^{-1}(U) \cap M^0_{d,\mathbf{m}}$ is generically one-to-one. Indeed just notice that, by definition of U, we have that $(\sigma \circ f)_*(\mathbb{P}^1_k) = D$ for an irreducible rational curve D, that is $\deg(\mathbb{P}^1_k / \overline{\mathrm{im}\,(\sigma \circ f)}) = 1$, see Definition 2.3.1. Recall that

$$\deg(\mathbb{P}^1_k / \overline{\mathrm{im}\,(\sigma \circ f)}) = \deg(\mathbb{P}^1_k / \overline{\mathrm{im}\,(f)}) \deg(X/\mathbb{P}^2_k)$$

see [Stacks, 02NZ]. Since σ is birational we have $\deg(X/\mathbb{P}^2_k) = 1$, therefore we obtain $\deg(\mathbb{P}^1_k / \overline{\mathrm{im}\,(f)}) = 1$, that is if f is birational onto its image. $\square$

2.4.9 We have now plenty of tools to find components of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ when X is a Del Pezzo surface given by the blow-up of at most 8 points $p_1, \ldots, p_r$ in general position.

Theorem 2.4.8 states that if a rational curve of d passing through the points $p_1, \ldots, p_r$ with multiplicities $m_1, \ldots, m_r$ is resolved by blowing-up the plane at these points, then we can find a component $M^0_{d,\mathbf{m}}$ in $\mathrm{Mor}(\mathbb{P}^1_k, X)$ and compute its dimension.

Curves which can be resolved by blowing-up points in general position have been completely classified when $r \leq 7$ by Gimigliano, Harbourne and Idà [GHI13].

In particular, their classification implies there is a complete list of the possible components $M^0_{d,\mathbf{m}}$ for Del Pezzo surfaces of degree ≥ 2, and allows us to compute their dimension explicitly.

As an example, we will give below the complete list of the components of $\mathrm{Mor}(\mathbb{P}^1_k, X)$ that we can obtain by combining Theorems 2.2.11, 2.4.8, Corollary 2.4.1 and [GHI13, Theorem 3.6.] when $r = 6$, that is, when X is a smooth cubic surface in $\mathbb{P}^3_k$.

2.4.10 Example. Let k be an algebraically closed field with $\operatorname{char} k = 0$. Let $p_1, \ldots, p_6$ be points in general position in $\mathbb{P}^2_k$, let $\sigma : X \to \mathbb{P}^2_k$ be the blow-up of those points with exceptional divisors $E_1, \ldots, E_6$, and consider the embedding $X \hookrightarrow \mathbb{P}^3_k$ given by the very ample linear system $|\sigma^* 3L - E_1 - \cdots - E_6|$.

For each integer e, Corollary 2.4.1 yields that we have 6 components of $\operatorname{Mor}_e(\mathbb{P}^1_k, X)$ given by $\operatorname{Mor}_e(\mathbb{P}^1_k, E_i) \cong \operatorname{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^1_k)$ which are nonsingular of dimension $2e+1$. These components parametrize covers of degree e of the exceptional lines.

Furthermore, for each e the classification in [GHI13, Theorem 3.6] allows us to determine each d and $\mathbf{m} = (m_1, \ldots, m_r)$ satisfying the following

- $e = 3d - |\mathbf{m}|$ and;
- there exists a rational curve C of degree d and $\mu_{p_i}(C) = m_i$ for each i which is resolved by blowing up the points p_i.

In particular, [GHI13, Theorem 3.6] yields that if C is resolved by blowing-up p_i, then for any permutation of τ of r elements we can find a rational curve C' of degree d such that $\mu_{p_i}(C) = m_{\tau(i)}$ which is also resolved by blowing up the points p_i. This follows since the classes of C and C' on $\operatorname{Pic} X$ are in the same orbit under the action the Weyl group of orthogonal transformations on $\operatorname{Pic} X$. We refer to [GHI13, Sections 2.5. and 3.2.] and for details.

It follows that we can compute the dimension and the number of components defined in Theorem 2.4.8 for each fixed d and to compute their number it suffices to compute the possible permutation types of $\mathbf{m} = (m_1, \ldots, m_r)$. We organize this data in the Table 2.1 below.

2.4.11 Remark. We compare the example above with the description of the components of $\operatorname{Mor}(\mathbb{P}^1_k, X)$ of degree $1 \leq e \leq 3$ given in [Kol08, Example 5.4.] when X is a smooth cubic surface. There, we have the following description:

- 27 components of degree $e = 1$ parametrizing lines in a cubic surface;
- 27 components of degree $e = 2$ parametrizing conics;
- 73 components of degree $e = 3$, 72 of those parametrizing twisted cubics on X and 1 component parametrizing plane cubics on X.

Notice that by the list obtained in Example 2.4.10, we have

- 27 components of degree $e = 1$ consisting of 21 on Table 2.1 and 6 corresponding to $\operatorname{Mor}_1(\mathbb{P}^1_k, \mathbb{P}^1_k)$;

e	d	$\mathbf{m}$	$\#M^0_{d,\mathbf{m}}$		$\dim M^0_{d,\mathbf{m}}$
			$t=0$	$t>0$	
1	1	$(1,1,0,0,0,0)$	15		3
	2	$(1,1,1,1,1,0)$	6		
2	1	$(1,0,0,0,0,0)$	6		4
	2	$(1,1,1,1,0,0)$	15		
	3	$(2,1,1,1,1,1)$	6		
$3+2t$	$1+t$	$(t,0,0,0,0,0)$	1	6	$5+2t$
	$2+t$	$(1+t,1,1,0,0,0)$	20	60	
	$2+2t$	$(1+t,1+t,1+t,1+t,0,0)$	20	60	
	$3+t$	$(2+t,1,1,1,1,0)$	30		
	$3+2t$	$(2+t,1+t,1+t,1+t,1,0)$	30	90	
	$3+3t$	$(2+2t,1+t,1+t,1+t,1+t,t)$	30		
	$4+2t$	$(2+t,2+t,2+t,1+t,1,1)$	20	60	
	$4+3t$	$(2+2t,2+t,2+t,1+t,1+t,1+t)$	20	60	
	$5+3t$	$(2+2t,2+t,2+t,2+t,2+t,2+t)$	1	6	
$4+2t$	$2+t$	$(1+t,1,0,0,0,0)$	15	30	$6+2t$
	$3+t$	$(2+t,1,1,1,0,0)$	60		
	$3+2t$	$(2+t,1+t,1+t,1+t,0,0)$	60		
	$4+2t$	$(2+t,2+t,2+t,1+t,1,0)$	60	120	
	$4+t$	$(3+t,1,1,1,1,1)$	6		
	$4+3t$	$(3+2t,1+t,1+t,1+t,1+t,1+t)$	6		
	$5+2t$	$(3+t,2+t,2+t,2+t,1,1)$	60		
	$5+3t$	$(3+2t,2+t,2+t,2+t,1+t,1+t)$	60		
	$6+3t$	$(3+2t,3+t,2+t,2+t,2+t,2+t)$	15	30	
6	4	$(2,2,2,0,0,0)$	20		4
	6	$(4,2,2,2,2,0)$	30		
	8	$(4,4,4,2,2,2)$	20		
	10	$(4,4,4,4,4,4)$	1		

Table 2.1: List of components $M^0_{d,\mathbf{m}}$ in a smooth cubic surface in $\mathbb{P}^3_k$.

- 33 components of degree $e=2$ consisting of 27 on Table 2.1 corresponding to the conics and 6 corresponding to double covers of the exceptional divisors $\mathrm{Mor}_2(\mathbb{P}^1_k,\mathbb{P}^1_k)$;
- 78 components of degree $e=3$ consisting of 72 on Table 2.1 (given by 1 for $d=1$, 20 for $d=2$, 30 for $d=3$, 20 for $d=4$ and 1 for $d=5$), and 6 corresponding to triple covers of the exceptional divisors $\mathrm{Mor}_3(\mathbb{P}^1_k,\mathbb{P}^1_k)$.

We clearly have some discrepancies on the descriptions. Let us shed some light on them. First of all, the description in [Kol08, Example 5.4.] only takes into account components containing morphisms which are generically one-to-one, while our description also takes into account some which do not. Let us see this case by case.

- For $e = 1$, we obtain the same components given by the classical description of lines on cubics which can be found, for instance, in [Har77, Theorem V.4.9.]. That is, we obtain the components which correspond to lines passing through 2 of the points p_i, conics passing through 5 points p_i and the exceptional divisors.

- For $e = 2$, we obtain the same 27 components corresponding to conics. In Example 2.4.10 we are also considering the double covers of the exceptional divisors E_i. However, we have that $M_{d,\mathbf{m}}$ also contains components corresponding to the double covers of the remaining 15 lines. We obtain those by taking $d = 2$ and $\mathbf{m}$ a permutation of $(2, 2, 0, 0, 0, 0)$, or taking $d = 4$ and $\mathbf{m}$ is a permutation of $(2, 2, 2, 2, 2, 0)$. We have in total 54 components of degree $e = 2$ when we count the double covers.

- For $e = 3$, we have counted in Example 2.4.10 the triple covers of exceptional divisors, but we are again missing the triple covers of the remaining 21 lines inside the components $M_{d,\mathbf{m}}$. These are components in $M_{d,\mathbf{m}}$ when $d = 3$ and $\mathbf{m}$ is a permutation of $(3, 3, 0, 0, 0, 0)$, or when $d = 6$ and $\mathbf{m}$ is a permutation of $(3, 3, 3, 3, 3, 0)$.

  We also have 72 components of table 2.1. It is easy to see that all these components correspond to the families of twisted cubics on X, that is, cubics not contained in a plane. Indeed for any cubic curve $C = \Pi \cap X$ where $\Pi \subset \mathbb{P}^3_k$ is a plane, we have that C is singular (since it is well known that a curve of degree ≥ 2 is singular on a projective plane) then C cannot be the image of any morphism $[f]$ in a component of type $M^0_{d,\mathbf{m}}$ since all such images are strict transforms of curves of degree d with multiplicity $\mathbf{m}$ which are resolved by the blow-up σ. Hence, we see that we are still missing the component of plane cubics altogether on the description of Example 2.4.10. Counting all of the components described above and triple covers, we will have in total $27 + 72 + 1 = 100$ components of degree $e = 3$.

This missing component of cubics shows that there are natural occurring components of $M_{d,\mathbf{m}}$ which correspond to curves on the plane and also are not resolved by the blow-up of the points p_i which are not just covers of a rational curve. As the degree e (or d) increases, we can expect that these consist of the majority of the components in the various $M_{d,\mathbf{m}}$. This expectation is natural and also evidenced by the Table 2.1, which shows that the number of components $M^0_{d,\mathbf{m}}$ is relatively small for arbitrarily high degree e.

2.4.12 Future work. The theorems in Chapter 2 have many directions for improvement and generalization. We point out at least three immediate directions regarding looking at other components, higher dimensions and characteristic issues. More precisely:

- Components: We can investigate them if we can still say something meaningful when we look at components of $M_{d,\mathbf{m}}$ which do not contain any curve satisfying conditions of Theorem 2.4.8.
- Higher dimensions: We can investigate them if a similar statement to Theorem 2.4.8 when X is the blow-up of $\mathbb{P}^n_k$ with $n > 2$. Also it is possible to proceed if an analogous statement to Theorem 2.2.11 holds when X is the blow-up of $\mathbb{P}^n_k$ in closed subschemes of higher dimensions. We expect to obtain a reasonable partition depending on the intersection number of rational curves with the exceptional divisor.
- Characteristic: we think that the analogous statements to Theorem 2.4.8 can be proven without the assumption of $\operatorname{char} k = 0$, since the classical descriptions of Del Pezzo surfaces and of linear systems hold for arbitrary characteristic.

Appendix A

Categorical remarks

A.1 Yoneda Lemma and Representability

In this section we briefly recall Yoneda Lemma and some of its corollaries used throughout the text. The discussion will follow closely the approach of [Lei14]. Recall that for every category $\mathbf{C}$, a presheaf $\mathcal{F}$ is a functor from $\mathbf{C}^{op}$ to $\mathbf{Set}$. For any two presheaves $\mathcal{F}$ and $\mathcal{G}$, a morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a collection of maps $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$ for each object X in $\mathbf{C}$ such that for any morphism $f : X \to Y$, the square

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "\alpha_Y"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(Y) \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}(X) \arrow[r, "\alpha_X"'] & \mathcal{G}(X) \end{tikzcd}$$

is commutative. Such a morphism α is said to be a *natural transformation*; equivalently, we say that the morphisms α_X are *natural* in X. The corresponding category is denoted $\mathbf{Psh}(\mathbf{C})$.

A.1.1 Example. Let $\mathbf{C}$ be any locally small category and let Y be an object in $\mathbf{C}$. We define a functor $h_Y : \mathbf{C}^{op} \to \mathbf{Set}$ defined on objects by $X \mapsto \mathrm{Hom}_{\mathbf{C}}(X, Y)$ and taking every morphism $f : X' \to X$ to the map

$$\begin{aligned} h_Y(f) : \mathrm{Hom}_{\mathbf{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathbf{C}}(X', Y) \\ g &\longmapsto g \circ f. \end{aligned}$$

In particular, the presheaf h_Y has an essential property: let $\mathrm{id}_Y : Y \to Y$ be the identity map in $h_Y(Y)$. Then for any object X in $\mathbf{C}$ and any morphism $g : X \to Y$ in $h_Y(X)$, we can write

$$g = h_Y(g)(\mathrm{id}_Y). \tag{A.1}$$

In other words, every section of this presheaf can be recovered from the morphisms of $\mathbf{C}$ and one section of $h_Y(Y)$, namely id_Y.

A.1.2 Theorem (Yoneda Lemma)**.** *Let* $\mathbf{C}$ *be a category and* $\mathbf{Psh}(\mathbf{C})$ *the category of presheaves. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$$

natural in X *and in* $\mathcal{F}$*.*

Proof. Let X be an object in $\mathbf{C}$ and $\mathcal{F}$ be a presheaf in $\mathbf{Psh}(\mathbf{C})$. We define the maps

$$\begin{aligned} \Theta_{X,\mathcal{F}} : \mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, \mathcal{F}) &\longrightarrow \mathcal{F}(X) \\ (\alpha : h_X \to \mathcal{F}) &\longmapsto \alpha_X(\mathrm{id}_X) \end{aligned}$$

and

$$\begin{aligned} \Xi_{X,\mathcal{F}} : \mathcal{F}(X) &\longrightarrow \mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, \mathcal{F}) \\ V &\longmapsto (\xi^V : h_X \to \mathcal{F}), \end{aligned}$$

where for each object Y in $\mathbf{C}$, the natural transformation ξ^V is given by

$$\begin{aligned} \xi^V_Y : h_X(Y) &\longrightarrow \mathcal{F}(Y) \\ f &\longmapsto \mathcal{F}(f)(V). \end{aligned} \tag{A.2}$$

We check that $\Theta_{X,\mathcal{F}}$ and $\Xi_{X,\mathcal{F}}$ are mutually inverses. Indeed, for any section $V \in \mathcal{F}(X)$, we have

$$\begin{aligned} \Theta_{X,\mathcal{F}} \circ \Xi_{X,\mathcal{F}}(V) &= \Theta_{X,\mathcal{F}}(\xi^V) = \xi^V_X(\mathrm{id}_X) \\ &= \mathcal{F}(\mathrm{id}_X)(V) = \mathrm{id}_{\mathcal{F}(X)}(V) = V; \end{aligned}$$

and conversely, for any natural transformation $\alpha : h_X \to \mathcal{F}$,

$$\Xi_{X,\mathcal{F}} \circ \Theta_{X,\mathcal{F}}(\alpha) = \Xi_{X,\mathcal{F}}(\alpha_X(\mathrm{id}_X)) = \xi^{\alpha_X(\mathrm{id}_X)}.$$

By (A.1), it follows that for each object Y in $\mathbf{C}$ we have $\xi^{\alpha_X(\mathrm{id}_X)}_Y : h_X(Y) \to \mathcal{F}(Y)$ such that

$$\xi^{\alpha_X(\mathrm{id}_X)}_Y(f) = \mathcal{F}(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Y(h_X(f)(\mathrm{id}_X)) = \alpha_Y(f),$$

in other words, $\xi^{\alpha_X(\mathrm{id}_X)} = \alpha$.

It is straightforward to check that these maps are natural in X and $\mathcal{F}$, that is, for any morphism $f : X \to Y$ in $\mathbf{C}$ and for any natural transformation $\alpha : \mathcal{F} \to \mathcal{G}$, we have commutative diagrams

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_Y, \mathcal{F}) \arrow[r, bend left, "\Theta_{Y,\mathcal{F}}"] \arrow[d, "H^{\mathcal{F}}(h_f)"'] & \mathcal{F}(Y) \arrow[l, bend left, "\Xi_{Y,\mathcal{F}}"] \arrow[d, "\mathcal{F}(f)"] \\
\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, \mathcal{F}) \arrow[r, bend left, "\Theta_{X,\mathcal{F}}"] & \mathcal{F}(X) \arrow[l, bend left, "\Xi_{X,\mathcal{F}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, \mathcal{F}) \arrow[r, bend left, "\Theta_{X,\mathcal{F}}"] \arrow[d, "H_{h_X}(\alpha)"'] & \mathcal{F}(X) \arrow[l, bend left, "\Xi_{X,\mathcal{F}}"] \arrow[d, "\alpha_X"] \\
\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, \mathcal{G}) \arrow[r, bend left, "\Theta_{X,\mathcal{G}}"] & \mathcal{G}(X), \arrow[l, bend left, "\Xi_{X,\mathcal{G}}"]
\end{tikzcd}
$$

where $H^{\mathcal{F}}$ and H_{h_X} denote the functors $\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(-, \mathcal{F})$ and $\mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, -)$ respectively. For further details see [Lei14, Theorem 4.2.1.]. □

A.1.3 As an immediate application of the Yoneda Lemma, we can prove that any category can be embedded in its corresponding category of presheaves. Namely, we can define a functor

$$h : \mathbf{C} \to \mathbf{Psh}(\mathbf{C})$$

defined on objects by $Y \mapsto h_Y$ and associating to each morphism $f : Y \to Y'$ the natural transformation $h_f : h_Y \to h_{Y'}$ given on each section X by

$$
\begin{aligned}
h_f(X) : \mathrm{Hom}_{\mathbf{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathbf{C}}(X, Y') \\
g &\longmapsto g \circ f.
\end{aligned}
$$

This functor is called *Yoneda embedding*.

A.1.4 Corollary. *Let* $\mathbf{C}$ *be a locally small category. The functor* $h : \mathbf{C} \to \mathbf{Psh}(\mathbf{C})$ *defined in A.1.3 is fully faithful.*

Proof. For any two objects X, Y in $\mathbf{C}$, it follows by Yoneda Lemma that

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = h_Y(X) \cong \mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(h_X, h_Y).$$

□

A.1.5 Definition. Let $\mathbf{C}$ be a locally small category and $\mathcal{F}$ a presheaf on $\mathbf{C}$. We say $\mathcal{F}$ is *representable* if there exists an object Y in $\mathbf{C}$ and a natural isomorphism $h_Y \cong \mathcal{F}$. In such a situation we say h_Y is a *representation* of $\mathcal{F}$.

A.1.6 Corollary. *Let* $\mathbf{C}$ *be a locally small category,* Y *an object of* $\mathbf{C}$ *and* $\mathcal{F}$ *an object in* $\mathbf{Psh}(\mathbf{C})$*. The functor* h_Y *is a representation of* $\mathcal{F}$ *if and only if there exists an element* $U \in \mathcal{F}(Y)$ *satisfying the following property:*

$$\begin{array}{l}\textit{for each object } X \textit{ in } \mathbf{C} \textit{ and each section } V \in \mathcal{F}(X), \textit{ there} \\ \textit{exists a unique map } f : X \to Y \textit{ such that } \mathcal{F}(f)(U) = V.\end{array} \tag{A.3}$$

Proof. Let $U \in \mathcal{F}(Y)$ and $\xi^U : h_Y \to \mathcal{F}$ be the natural transformation (A.2) defined in the proof of Yoneda Lemma. Then the statement of the proposition is equivalent to saying that h_Y is a representation of $\mathcal{F}$ if and only if there exists U such that each ξ^U_X is a bijection.

Indeed, if $\alpha : h_Y \to \mathcal{F}$ is a natural isomorphism, Yoneda Lemma implies that there exists $U \in \mathcal{F}(Y)$ such that $\alpha = \xi^U$. It follows that each ξ^U_X is a bijection. The converse is clear. □

A.1.7 Definition. Let $\mathbf{C}$ be a locally small category and $\mathcal{F}$ be an object of $\mathbf{Psh}(\mathbf{C})$. If h_Y is a representation of $\mathcal{F}$, then we say that a section $U \in \mathcal{F}(Y)$ satisfying (A.3) is the *universal section* of $\mathcal{F}$.

A.2 Fibered product of functors

A.2.1 Fiber product of presheaves. Recall that both limits and colimits in $\mathbf{Psh}(\mathbf{C})$ are computed sectionwise. In particular, if $\mathcal{F} \to \mathcal{G}$ and $\mathcal{H} \to \mathcal{G}$ are morphisms of presheaves, then the *fiber product* $\mathcal{F} \times_{\mathcal{G}} \mathcal{H}$ in $\mathbf{Psh}(\mathbf{C})$ is defined sectionwise by $(\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(X) := \mathcal{F}(X) \times_{\mathcal{G}(X)} \mathcal{H}(X)$ for every object X in $\mathbf{C}$ and for each morphism $f : X' \to X$ we have a morphism

$$(\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(f) : (\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(X) \to (\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(X')$$

given by the universal property of the fiber product, that is $(\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(f)$ is the unique morphism making the diagram

$$\begin{tikzcd}
(\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(X) \arrow[rr] \arrow[dr] \arrow[dd, "(\mathcal{F}\times_{\mathcal{G}}\mathcal{H})(f)"'] & & \mathcal{H}(X) \arrow[dr] \arrow[dd, "\mathcal{H}(f)"] & \\
& \mathcal{F}(X) \arrow[rr] \arrow[dd, "\mathcal{F}(f)"] & & \mathcal{G}(X) \arrow[dd, "\mathcal{G}(f)"] \\
(\mathcal{F} \times_{\mathcal{G}} \mathcal{H})(X') \arrow[rr] \arrow[dr] & & \mathcal{H}(X') \arrow[dr] & \\
& \mathcal{F}(X') \arrow[rr] & & \mathcal{G}(X')
\end{tikzcd}$$

commutative. We recall a few basic properties of this fiber product.

A.2.2 Definition. A commutative square of presheaves over $\mathbf{C}$

$$\begin{tikzcd}
\mathcal{K} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\
\mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

is said to be *cartesian* if there is a natural isomorphism $\mathcal{K} \cong \mathcal{F} \times_{\mathcal{G}} \mathcal{H}$.

This can be checked sectionwise, that is, the diagram is cartesian if for each object X in $\mathbf{C}$ and each element

$$(V, V') \in \mathcal{F}(X) \times_{\mathcal{G}(X)} \mathcal{H}(X),$$

there exists a unique section $V'' \in \mathcal{K}(X)$ such that $V'' \mapsto V'$ and $V'' \mapsto V'$ via the morphisms $\mathcal{K}(X) \to \mathcal{F}(X)$ and $\mathcal{K}(X) \to \mathcal{H}(X)$ respectively.

A.2.3 Proposition. *Let $f : X \to Z$ and $g : X \to Z$ be morphisms in $\mathbf{C}$, and suppose that the fiber product $X \times_Z Y$ exists in $\mathbf{C}$. Let $h_f : h_X \to h_Z$ and $h_g : h_Y \to h_Z$ be the corresponding natural transformations given in A.1.2, and $h_X \times_{h_Z} h_Y$ be the fiber product in $\mathbf{Psh}(\mathbf{C})$. Then $h_X \times_{h_Z} h_Y \cong h_{X \times_Z Y}$.*

Proof. By definition, for any object W we have

$$(h_X \times_{h_Z} h_Y)(W) = \{(\alpha, \beta) \mid f \circ \alpha = g \circ \beta\}.$$

By the universal property of $X \times_Z Y$, for every pair $(\alpha, \beta) \in (h_X \times_{h_Z} h_Y)(W)$

there exists a unique $\gamma : W \to X \times_Z Y$ such that the diagram

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!\gamma"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

is commutative. Therefore, there is a clear morphism

$$\begin{aligned}
(h_X \times_{h_Z} h_Y)\,(W) &\longrightarrow h_{X \times_Z Y}(W) \\
(\alpha, \beta) &\longmapsto \gamma
\end{aligned}$$

natural in W. The map is clearly surjective and the universal property of the cartesian square implies that it is injective. $\square$

A.2.4 Proposition. *A commutative square of presheaves over* $\mathbf{C}$

$$\begin{tikzcd}
\mathcal{K} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\
\mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

is cartesian if and only if for each pair of morphisms $h_X \to \mathcal{F}$ and $h_X \to \mathcal{H}$ from a representable *presheaf such that the solid diagram*

$$\begin{tikzcd}
h_X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{K} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\
& \mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

is commutative, there exists a unique dashed morphism such that the whole diagram is commutative.

Proof. The direct implication comes from the definition of a cartesian square. We prove the converse. Suppose there exists a presheaf $\mathcal{L}$ and commutative diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{K} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d, "\beta"] \\
& \mathcal{F} \arrow[r, "\alpha"'] & \mathcal{G}.
\end{tikzcd}$$

In particular, we have the analogous diagrams for each object X in the category $\mathbf{C}$. Moreover, Yoneda Lemma defines bijections

$$\Xi_{X,\mathcal{L}} : \mathcal{L}(X) \to \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{L}), \qquad \Theta_{X,\mathcal{K}} : \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{K}) \to \mathcal{K}(X)$$

natural in X. In addition, by assumption, for each morphism $h_X \to \mathcal{L}$ there is a commutative diagram

$$\begin{tikzcd}
h_X \arrow[r, "\alpha"] \arrow[dr, dashed, "\exists ! \Phi_X(\alpha)"'] & \mathcal{L} \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d, "\beta"] \\
& & \mathcal{F} \arrow[r, "\alpha"'] & \mathcal{G}
\end{tikzcd}$$

and therefore there is a map $\Phi_X : \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{L}) \to \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{K})$. The uniqueness of each $\Phi_X(\alpha)$ implies that for any morphism $f : X \to Y$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Psh(C)}}(h_Y, \mathcal{L}) \arrow[r, "\Phi_Y"] \arrow[d, "H^{\mathcal{L}}(h_f)"'] & \mathrm{Hom}_{\mathbf{Psh(C)}}(h_Y, \mathcal{K}) \arrow[d, "H^{\mathcal{K}}(h_f)"] \\
\mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{L}) \arrow[r, "\Phi_X"] & \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{K})
\end{tikzcd}$$

is commutative. Thus, we have maps

$$\mathcal{L}(X) \xrightarrow{\Xi_{X,\mathcal{L}}} \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{L}) \xrightarrow{\Phi_X} \mathrm{Hom}_{\mathbf{Psh(C)}}(h_X, \mathcal{K}) \xrightarrow{\Theta_{X,\mathcal{K}}} \mathcal{K}(X)$$

natural in X, and therefore a natural transformation $\mathcal{L} \to \mathcal{K}$ whose uniqueness follows straighforwardly from the definition and Yoneda Lemma. $\square$

A.2.5 From this point onwards, we will be interested in fiber products of functors when $\mathbf{C}$ is a full subcategory of the category $\mathbf{Sch}/S$ of schemes over a base scheme S. In particular, we want to define properties of functors in $\mathbf{Psh}(\mathbf{C})$ from properties of morphisms of schemes.

A.2.6 Definition. Let $\mathbf{C}$ be a full subcategory of $\mathbf{Sch}/S$ and $\mathcal{F}, \mathcal{G}$ be presheaves in $\mathbf{Psh}(\mathbf{C})$. We say $\mathcal{F}$ is a *subpresheaf* (or *subfunctor*) of $\mathcal{G}$ if there exists a morphism $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ which is injective sectionwise, that is, α is a monomorphism in $\mathbf{Psh}(\mathbf{C})$.

A subpresheaf $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ is said to be *open* (resp. *closed*) if for every scheme Y in $\mathbf{C}$ and morphism $\beta : h_Y \to \mathcal{G}$, there exists an open subscheme $\iota_\beta : U_\beta \hookrightarrow Y$ in $\mathbf{C}$ such that the fiber product $\mathcal{F} \times_{\mathcal{G}} h_Y$ is naturally isomorphic to h_{U_β}. In other words, there exists a morphism $h_{U_\beta} \to \mathcal{F}$ completing a cartesian diagram

$$\begin{tikzcd} h_{U_\beta} \arrow[r, "h_{\iota_\beta}"] \arrow[d] & h_Y \arrow[d, "\beta"] \\ \mathcal{F} \arrow[r, hook, "\alpha"'] & \mathcal{G}. \end{tikzcd} \tag{A.4}$$

A.2.7 Proposition. *Let $\mathbf{C}$ be a full subcategory of $\mathbf{Sch}/S$. Let X be an S-scheme. Any open (resp. closed) subpresheaf $\mathcal{H} \hookrightarrow h_X$ is representable by an open (resp. closed) subscheme of X.*

Proof. Let $h_{\mathrm{id}_X} : h_X \to h_X$ be the identity morphism. The induced fiber product $\mathcal{H} \times_{h_X} h_X$ is naturally isomorphic to $\mathcal{H}$. By the definition of open (resp. closed) subpresheaf, it follows that $\mathcal{H} \cong h_{U_{\mathrm{id}_X}}$, where U_{id_X} is an open (resp. closed) subscheme of X. □

A.2.8 It is useful to describe open subpresheaves in terms of sections. In that regard, we have the following proposition.

A.2.9 Proposition. *Let $\mathbf{C}$ be a full subcategory of $\mathbf{Sch}/S$. Then a monomorphism $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ is an open (resp. closed) subpresheaf if and only if for every S-scheme Y and section $V \in \mathcal{G}(Y)$, there exists an open (resp. closed) subscheme $U_V \hookrightarrow Y$ in $\mathbf{C}$ satisfying the following universal property:*

$$\begin{gathered} \textit{a morphism } f : X \to Y \textit{ in } \mathbf{C} \textit{ factors through } U_V \\ \textit{if and only if } \mathcal{G}(f)(V) \in \alpha_X(\mathcal{F}(X)). \end{gathered} \tag{A.5}$$

Proof. Let Y be any S-scheme. By Yoneda Lemma, a section $V \in \mathcal{G}(Y)$ corresponds uniquely to the morphism $\xi^V : h_Y \to \mathcal{G}$ given in A.2. The morphism

$\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ is an open (resp. closed) subpresheaf if for each morphism $\beta : h_Y \to \mathcal{G}$ there exists an open (resp. closed) immersion $\iota_\beta : U_\beta \hookrightarrow Y$ in $\mathbf{C}$ and a cartesian diagram (A.4). For each $V \in \mathcal{G}(Y)$ write $U_V := U_{\xi^V}$ and $\iota_V := \iota_{\xi^V}$ and consider the solid commutative diagrams

$$\begin{tikzcd}
h_X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "h_f"] \arrow[ddr, bend right] & & \\
& h_{U_V} \arrow[r, "h_{\iota_V}"] \arrow[d] & h_Y \arrow[d, "\xi^V"] \\
& \mathcal{F} \arrow[r, hook, "\alpha"'] & \mathcal{G}
\end{tikzcd} \tag{A.6}$$

for each X in $\mathbf{C}$. By Proposition A.2.4, the above square is cartesian if and only if for each X there exists a unique dashed arrow making (A.6) commute.

Claim. If the square in (A.6) is cartesian, then U_V satisfies (A.5).

Proof of claim. Suppose it is cartesian. Let $f : X \to Y$ be a morphism in $\mathbf{C}$ and suppose it factors through U_V, i.e. there exists a dashed morphism in (A.6) such that the upper triangle is commutative. Let V' be the image of id_X in $\mathcal{F}(X)$ via

$$h_X(X) \to h_{U_V}(X) \to \mathcal{F}(X).$$

Then by definition

$$\alpha_X(V') = \xi^V_X(h_f(X)(\mathrm{id}_X)) = \xi^V_X(f) = \mathcal{G}(f)(V),$$

that is, $\mathcal{G}(f)(V) \in \alpha_X(\mathcal{F}(X))$.

Conversely, suppose $\mathcal{G}(f)(V) \in \alpha_X(\mathcal{F}(X))$, then let $V' \in \mathcal{F}(X)$ be a section such that $\alpha_X(V') = \mathcal{G}(f)(V)$. Then $\xi^{V'} : h_X \to \mathcal{F}$ is a morphism making the outer diagram on (A.6) commutative. The universal property of the fibered product implies $f : X \to Y$ factors through U_V. □

Claim. If U_V satisfies (A.5), then the square in (A.6) is cartesian.

By assumption, we have that for any X and any morphism $f : X \to Y$, the existence of a morphism $h_X \to h_{U_V}$ making the upper triangle of the solid

diagram

$$\begin{tikzcd}
h_X \arrow[drr, bend left, "h_f"] \arrow[dr] \arrow[ddr, bend right] & & \\
& h_{U_V} \arrow[r, "h_{\iota_V}"] \arrow[d, dashed] & h_Y \arrow[d, "\xi^V"] \\
& \mathcal{F} \arrow[r, hook, "\alpha"'] & \mathcal{G}.
\end{tikzcd} \tag{A.7}$$

commutative is equivalent to the existence of a morphism $h_X \to \mathcal{F}$ such that the outer solid diagram is commutative.

In particular, if $X = U_V$ and $f = \iota_V$, then the morphism id_{U_V} yields a dashed morphism $h_{U_V} \dashrightarrow \mathcal{F}$ such that the square in (A.7) is commutative.

We claim that for each X in $\mathbf{C}$ and each $f : X \to Y$ factoring through U_V the induced solid diagram in (A.7) together with the dashed arrow defined in the previous paragraph also form a commutative diagram. Indeed, we have that $h_X \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ coincides with $h_X \to h_{U_V} \dashrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ by the commutativity of the upper triangle and the square. Since α is a monomorphism, we have that $h_X \to h_{U_V} \dashrightarrow \mathcal{F}$ coincides with $h_X \to \mathcal{F}$ and the whole diagram is commutative.

To prove that the commutative square in (A.7) is also cartesian, we use Proposition A.2.4. Hence, it suffices to prove that for each X in $\mathbf{C}$ and pair of morphisms $h_f : h_X \to h_Y$ and $h_X \to \mathcal{F}$ making the outer diagram in (A.7) commute, the induced morphism $h_X \to h_{U_V}$ (which exists by assumption) is unique, but this follows from open (resp. closed) immersions being monomorphisms in categories of schemes. □

Appendix B

Linear systems on surfaces

We very briefly recollect terminology on divisors on surfaces, their intersection pairing and how this pairing behaves under blow-up at points. This material is very well known, and thorough treatments can be found in [Har77, §II.6 and Chapter V] or [HS00, §A.2 and §A.3].

When X is a projective surface and D is a divisor on X, we define the functor of linear systems with respect to the divisor D and recall that this functor is representable. The representing scheme is a projective space parametrizing divisors linearly equivalent to D. This functorial view of linear systems will be useful to us to prove in Section 2.3 the existence of the regular morphism

$$\Xi_d : \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^2_k) \to |dL|$$

through morphisms between their functors of points.

B.1 Divisors on surfaces.

B.1.1 Definition. Let X be an irreducible nonsingular variety over k. A *divisor* on X is an element of the free group generated by irreducible and reduced subvarieties of codimension 1 on X, denoted by $\mathrm{Div}\, X$. That is, a divisor $D \in \mathrm{Div}\, X$ is a finite formal sum $\sum a_i C_i$ with C_i irreducible and reduced. We define its *degree* to be $\deg D = \sum a_i$ and we say a divisor D is *effective* if $a_i \geq 0$ for all i.

We say C is a *prime divisor* of X if C is a closed irreducible subvariety of X of codimenison 1. Let ζ be the generic point of a prime divisor C and $\kappa(X)$ be the function field of X. Recall that $\mathcal{O}_{X,\zeta}$ is a discrete valuation ring whose fraction field is isomorphic to $\kappa(X)$. In other words, for each such C we have a discrete valuation

$$v_C : \kappa(X) \to \mathbb{Z}.$$

For each non-zero $f \in \kappa(X)$, we have that $v_C(f) \neq 0$ only for finitely many prime divisors (see [Har77, Lemma II.6.1.]). Therefore, for each $f \in \kappa(X)$ we can define a divisor

$$\operatorname{div}(f) := \sum_{\substack{C \text{ is a prime} \\ \text{divisor}}} v_C(f)C.$$

We say a divisor D is *linearly equivalent* to a divisor D' if there exists $f \in \kappa(X)$ such that $D' - D = \operatorname{div}(f)$.

Linear equivalence defines a congruence relation on $\operatorname{Div} X$, that is, an equivalence relation preserving the group operation on $\operatorname{Div} X$. Therefore, the quotient of $\operatorname{Div} X$ with respect to this equivalence relation has a well defined group structure and is denoted $\operatorname{Cl} X$ called *group of divisor classes* on X. Since X is regular, this quotient group is isomorphic to the *Picard group* $\operatorname{Pic} X$, *i.e.*, the group of isomorphism classes of invertible sheaves on X, see [Har77, Corollary II.6.16.].

B.1.2 Linear systems. For each divisor $D \in \operatorname{Div} X$ we can define a k-vector space

$$L_D = \{f \in \kappa(X) \mid D + \operatorname{div}(f) \text{ is effective}\} \cup \{0\}.$$

Notice that the set of divisors linearly equivalent to D is in bijection with closed points of the projective space $\mathbb{P}(L_D)$ via the map

$$\begin{aligned} \mathbb{P}(L_D) &\longrightarrow \left\{ \begin{matrix} D' \in \operatorname{Div} X \mid D' \text{ is effective} \\ \text{and linearly equivalent to } D \end{matrix} \right\} \\ f \mod k^* &\longmapsto D + \operatorname{div}(f) \end{aligned}$$

We define $|D| := \mathbb{P}(L_D)$ and we will identify points in $|D|$ with effective divisors linearly equivalent to D in $\operatorname{Div} X$. We say $|D|$ is the *complete linear system* generated by D, and that a *linear system* on X is a linear subspace of $|D|$ for some $D \in \operatorname{Div} X$.

Each linear system $\mathbb{L}$ can be written as $\mathbb{P}(V)$ where $V \subset L_D$ is a vector subspace of dimension $m \leq \dim L_D$. Let $f_0, \dots, f_{m-1}$ be a basis for V, then we can define a rational map

$$\begin{aligned} \iota_{\mathbb{L}} : X &\dashrightarrow \mathbb{P}_k^{m-1} \\ p &\mapsto (f_0(p) : \cdots : f_{m-1}(p)). \end{aligned}$$

We say $\mathbb{L}$ is *very ample* if $\iota_{\mathbb{L}}$ is a closed immersion. A divisor D is said to be *very ample* if $|D|$ is very ample.

Lastly, the following terminology on linear systems is very commonly used: we say that *a general member of the linear system* $\mathbb{L}$ *has property* P if there exists a proper closed subset of $Z \subset \mathbb{L}$ such that every divisor corresponding to a point in $\mathbb{L} \setminus Z$ has property P, see for instance [Iit82, §7.9.]

B.1.3 Intersection pairing. From now on, let X be a nonsingular surface over k. Recall that two divisors $D, D' \in \operatorname{Div} X$ *intersect transversally* at a point $p \in X$ if the local equations of the curves on D and D' which pass through p generate the local ring $\mathcal{O}_{X,p}$. We define an *intersection pairing* for divisors on X to be a pairing

$$\begin{aligned} \operatorname{Div} X \times \operatorname{Div} X &\longrightarrow \mathbb{Z} \\ (D, D') &\longmapsto D \cdot D' \end{aligned}$$

satisfying the following conditions:

- If D and D' intersect transversally, then $D \cdot D' = \#(D \cap D')$;
- $D \cdot D' = D \cdot D'$;
- $(D_1 + D_2) \cdot D' = D_1 \cdot D' + D_2 \cdot D'$;
- if D_1 is linearly equivalent to D_2, then $D_1 \cdot D' = D_2 \cdot D'$.

An intersection pairing exists and is unique, see [Har77, Theorem 1.1.]. In fact, thanks to the last item, the pairing

$$\begin{aligned} \operatorname{Pic} X \times \operatorname{Pic} X &\longrightarrow \mathbb{Z} \\ ([D], [D']) &\longmapsto D \cdot D' \end{aligned}$$

is well defined. Hence, when there is no risk of confusion, we will use the notation D both for a divisor on $\operatorname{Div} X$ and its class on $\operatorname{Pic} X$.

B.1.4 Remark. Notice that if C and D are distinct non-singular irreducible curves we have $C \cdot D \geq 0$. In fact, this is holds for any pair of distinct irreducible curves, see [Har77, Proposition V.1.4].

B.1.5 Numerical equivalence. Let D be a divisor on $\operatorname{Div} X$. We say D is *numerically equivalent* to 0, and denote it $D \equiv 0$ if $D \cdot E = 0$ for all $E \in \operatorname{Div} X$. We say a divisor D' is numerically equivalent to D, and denote it $D \equiv D'$, if $(D - D') \equiv 0$. Notice that $\equiv$ defines an equivalence relation on $\operatorname{Div} X$ and, moreover, by the definition of the intersection pairing in B.1.3, if D is linearly equivalent to D' then $D \equiv D'$.

B.1.6 Intersections on blow-ups. Let $\sigma : \widetilde{X} \to X$ be the blow-up of X at the points $p_1, \ldots, p_r$, and let $E_1, \ldots, E_r$ be the corresponding exceptional divisors for each blown up point. We have that $\operatorname{Pic} \widetilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}^r$ and there are morphisms

$$\sigma^* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \quad \text{and} \quad \sigma_* : \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} X$$

such that

$$\sigma_* \circ \sigma^* = \mathrm{id}_{\operatorname{Pic} X}\,. \tag{B.1}$$

We can describe the intersection pairing of $\widetilde{X}$ using these morphisms. In fact, they satisfy the following properties:

1. For any $D, D' \in \operatorname{Pic} X$ we have $(\sigma^* D) \cdot (\sigma^* D') = D \cdot D'$;
2. For any $D \in \operatorname{Pic} X$ we have $(\sigma^* D) \cdot E_i = 0$ for all i;
3. $E_i^2 = E_i \cdot E_i = -1$, for all i;
4. $E_i \cdot E_j = 0$, for all $i \neq j$;
5. For any $D \in \operatorname{Pic} X$ and $D' \in \operatorname{Pic} \widetilde{X}$, we have $(\sigma^* D) \cdot D' = D \cdot (\sigma_* D')$.

See for instance [Har77, Proposition V.3.2].

B.1.7 Remark. The equality (B.1) tells us that we have a split exact sequence

$$0 \to \mathbb{Z}^r \rightleftarrows \operatorname{Pic} \widetilde{X} \underset{\sigma^*}{\overset{\sigma_*}{\rightleftarrows}} \operatorname{Pic} X \to 0.$$

In particular, we have $\ker(\sigma_*) = \mathbb{Z}^r$, that is $\sigma_*(E_i) = 0$ for all i.

B.1.8 Pullback and pushforwards of divisors under blow-ups. We can describe the morphisms σ^* and σ_* more explicitly. For any curve C on X, the strict transform of C is defined as

$$\widetilde{C} = \overline{\sigma^{-1}(C \smallsetminus p_1, \ldots, p_r)}.$$

By extension, for any divisor $D = \sum_{i=1}^s a_i C_i$ in $\operatorname{Div} X$, its *strict transform* is defined as $\widetilde{D} = \sum_{i=1}^s a_i \widetilde{C}_i$.

For any effective divisor D on X and a point $p \in X$, let g be a local equation for D in $\mathcal{O}_{X,p}$ and let $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$ be the maximal ideal of the local ring. We define the *multiplicity* $\mu_p(D)$ to be the largest integer m such that $g \in \mathfrak{m}_p^m$. When D consists of a curve, this coincides with the usual definition for the multiplicity of a curve at the point p. Thus, by [Har77, Proposition V.3.6], for any $D = \sum_{i=1}^{s} a_i C_i \in \operatorname{Pic} X$ we obtain

$$\sigma^*(D) = \sigma^*\left(\sum_{i=1}^{s} a_i C_i\right) = \widetilde{D} + \sum_{j=1}^{r}\sum_{i=1}^{s} a_i \mu_{p_j}(C_i) E_j. \tag{B.2}$$

B.2 Linear systems revisited

We have just seen in B.1.2 that for any nonsingular variety X, a divisor class $D \in \operatorname{Pic} X$ gives rise to a complete linear system $|D|$ isomorphic to a projective space. A more precise way to formulate this correspondence is by a representability result, *i.e.*, $|D|$ is actually a scheme representing a functor of linear systems. To define this functor we first recall the following definition.

B.2.1 Definition ([Stacks, 01WQ] and [Stacks, 01WX])**.** Let X be a scheme. A closed subscheme $D \hookrightarrow X$ is said to be an *effective Cartier divisor* if its ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible sheaf.

The *sheaf associated to* D, denoted $\mathcal{O}_X(D)$, is defined to be the dual of $\mathcal{I}_D$, that is,

$$\mathcal{O}_X(D) := \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X).$$

Let $X \to S$ be a morphism. Then D is a *relative effective Cartier divisor* of X over S if D is an effective Cartier divisor on X and $D \hookrightarrow X \to S$ is flat.

B.2.2 Remark. Definition B.2.1 is motivated by the following: if $X \to S$ is a morphism and D is a relative effective Cartier divisor, then flatness of $D \to S$ implies that for any morphism $f : S' \to S$ the fibered product $D_{S'}$ fitting in the fibered diagram

$$\begin{tikzcd}
D_{S'} \arrow[r, "f'|_{D_{S'}}"] \arrow[d, hook] & D \arrow[d, hook] \\
X_{S'} \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\
S' \arrow[r, "f"] & S
\end{tikzcd}$$

is a relative effective Cartier divisor, see [Stacks, 056Q]. Moreover, we have that $(f')^*\mathcal{O}_X(D) \cong \mathcal{O}_{X_{S'}}(D_{S'})$.

B.2.3 Remark. The definition of an effective Cartier divisor in Definition B.2.1 coincides with the one in Definition 2.2.1 by [Stacks, 01WS].

B.2.4 Definition. Let **FinType**$/k$ be the category of schemes of finite type over k, X be a projective surface over k and $\mathcal{L} \in \operatorname{Pic} X$ be an invertible sheaf. Then we define a *functor of linear systems* as

$$\mathscr{L}in\mathscr{S}ys_{\mathcal{L}} : \mathbf{FinType}/k \to \mathbf{Set}$$

defined for each finite type scheme S as

$$\mathscr{L}in\mathscr{S}ys_{\mathcal{L}}(S) = \left\{ \begin{array}{c} D \subset X \times S \text{ relative effective Cartier divisor} \\ \text{such that } \mathcal{O}_{X\times S}(D) \cong \operatorname{pr}_1^*(\mathcal{L}) \otimes_{\mathcal{O}_{X\times S}} \operatorname{pr}_2^*(\mathcal{K}) \\ \text{for some } \mathcal{K} \in \operatorname{Pic} S \end{array} \right\}$$

and defined for any morphism $S' \to S$ as

$$\begin{aligned} \mathscr{L}in\mathscr{S}ys_{\mathcal{L}}(S) &\longrightarrow \mathscr{L}in\mathscr{S}ys_{\mathcal{L}}(S') \\ D &\longmapsto D_{S'}. \end{aligned}$$

B.2.5 Proposition ([Mum66, Lecture 13, Proposition 2])**.** *Let $\mathcal{L}$ be an invertible sheaf on X. Then the functor $\mathscr{L}in\mathscr{S}ys_{\mathcal{L}}$ is representable by $\mathbb{P}(H^0(X,\mathcal{L})^\vee)$, where $H^0(X,\mathcal{L})^\vee$ is the dual of $H^0(X,\mathcal{L})$. In particular, if $\mathcal{L} \cong \mathcal{O}_X(D)$ for some effective Cartier divisor D on X, we denote*

$$|D| := \mathbb{P}(H^0(X,\mathcal{O}_X(D))^\vee) \cong \mathbb{P}(H^0(X,\mathcal{I}_D)).$$

B.2.6 Definition. Let X be a projective surface over k and $\mathcal{L} \in \operatorname{Pic} X$. The *complete linear system* on X with respect to $\mathcal{L}$ is the projective space representing the functor $\mathscr{L}in\mathscr{S}ys_{\mathcal{L}}$. A *linear system* is a linear subspace of this projective space.

B.2.7 Example. Let $X = \mathbb{P}^2_k$ be the projective plane. It is well known that every invertible sheaf is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(d)$ for some $d \in \mathbb{N}$, and that in addition, $H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d)) = k[x_0,x_1,x_2]_d$. Therefore, the complete linear systems on $\mathbb{P}^2_k$ are projective spaces isomorphic to $\mathbb{P}^N_k$, with $N = \binom{d+2}{2}$.

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Glossary of Notations

$[\kappa(\xi) : \kappa(\zeta)]$	Degree of the field extension between fraction fields of integral schemes, page 6
$[f]$	k-point in $\mathrm{Mor}(\mathbb{P}^1_k, X)$ corresponding to a morphism $f : \mathbb{P}^1_k \to X$, page 47
$\mathrm{Bl}_Z(X)$	Blow-up of X along a closed subscheme Z, page 54
$\mathcal{F} \times_{\mathcal{G}} \mathcal{H}$	Fibered product of functors, page 102
$\mathcal{F}$	Presheaf over a category $\mathbf{C}$, page 99
$\mathcal{O}_X(D)$	Sheaf associated to divisor D on X, page 113
$\mathbf{C}$	Locally small category, page 99
$\mathbf{C}^{op}$	Opposite category of $\mathbf{C}$, page 99
$\mathbf{FinType}/k$	Category of schemes of finite type over k, page 81
$\mathbf{I}$	Index category, page 41
$\mathbf{Noe}/S$	Category of locally noetherian schemes over S, page 9
$\mathbf{PrSch}/S$	Category of projective schemes over a base S, page xvi
$\mathbf{Psh}(\mathbf{C})$	Category of presheaves on category $\mathbf{C}$, page 99
$\mathbf{QPrSch}/S$	Category of quasi-projective schemes over S, page xvi
$\mathbf{Sch}/S$	Category of schemes over S, page 106
$\mathbf{Set}$	Category of sets, page 99
$\mathcal{C}how_{s,d}(X)$	Chow presheaf of well defined families of effective proper cycles on X, page 78
$\chi(X, \mathcal{F})$	Euler characteristic of $\mathcal{F}$ over X, page 2
$\mathrm{Chow}_{s,d}(X)$	Chow scheme of s-cycles of degree d over X, page 79
$\mathrm{Cl}\, X$	Group of divisor classes on X, page 110

$\mathrm{Cycl}_s(X)$	Group of cycles of dimension s on X, page 76
$\deg C$	Degree of a curve C, page 53
$\deg(f)$	Degree of a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^n_k$ or equivalently, $\deg(f_*\mathbb{P}^1_k)$, page 53
$\mathrm{Div}\, X$	Group of Weil divisors on X, page 109
$\mathrm{Div}^{\mathit{eff}}_d\, \mathbb{P}^2_k$	Effective divisors on $\mathbb{P}^2_k$ of degree d, page 77
η_ι	Natural transformation between Hilbert functors induced by an immersion ι, page 20
ev	Evaluation morphism of $\mathrm{Mor}_S(X, Y)$, page 40
$\mathit{\Gamma}_f$	Graph morphism of the morphism f, page 33
$\mathcal{H}\mathit{ilb}_S(X)$	Hilbert functor of X over S, page 9
$\mathcal{H}\mathit{ilb}^P_S(X)$	Hilbert functor of X over S with fixed polynomial P, page 11
$\mathrm{Hilb}^P_S(X)$	Hilbert scheme of X over S with respect to P, page 12
$\mathrm{Hilb}_S(X)$	Hilbert scheme of X over S, page 12
$\mathrm{Hom}_{\mathbf{C}}(X, Y)$	Set of arrows between X and Y in $\mathbf{C}$, page 99
$\mathrm{Hom}_S(X, Y$	Set of morphisms from X to Y over S, page 27
$\mathrm{im}(f)$	Set theoretic image of f, page xvi
$\kappa(p)$	Residue field of a point p on a scheme, page 10
$\mathcal{L}\mathit{in}\mathcal{S}\mathit{ys}_{\mathcal{L}}$	Functor of linear systems on a surface w.r.t line bundle $\mathcal{L}$, page 114
$\mathbb{L}$	Linear system on a variety, page 110
$\mathbb{L}_C$	Rational linear system containing C, page 88
$\mathcal{M}\mathit{or}_S(X, Y)$	Functor of morphisms from X to Y over a base S, page 27
$\mathrm{Mor}(\mathbb{P}^1_k, X)$	Scheme parametrizing rational curves, page 47
$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$	Scheme parametrizing rational curves of degree d, page 52
$\mathrm{Mor}_S(X, -)$	Covariant functor between schemes of morphisms, page 40

$\mathrm{Mor}_S(X,Y)$	Scheme of morphisms from X to Y over a base S, page 27
$\mu_p(D)$	Multiplicity of divisor at a point p, page 112
$\mathrm{Pic}\, X$	Picard group of X, page 110
$\mathcal{Q}uot_{\mathcal{E}/X/S}$	Functor of families of quotients of a coherent sheaf $\mathcal{E}$ on X which are flat and have proper support., page 26
$\overline{\mathrm{im}\,(f)}$	Scheme theoretic image of f, page xvi
σ	Blow up of X along a subscheme, page 47
σ^*	Pullback of σ w.r.t Picard groups, page 112
σ_*	Pushforward of σ w.r.t Picard groups, page 112
σ_M	Morphism between $\mathrm{Mor}(\mathbb{P}^1_k, \mathrm{Bl}_Z(X))$ and $\mathrm{Mor}(\mathbb{P}^1_k, X)$ induced by the blow-up σ, page 55
$\widetilde{C}$	Strict transform of a curve C under the blow-up σ, page 86
f^{un}	Universal morphism of $\mathrm{Mor}_S(X,Y)$, page 35
$f_*\mathcal{Z}$	Proper pushforward of the cycle $\mathcal{Z}$, page 77
G_p	Stabilizer of an algebraic group action at p, page 82
$H^i(X,\mathcal{F})$	i-th cohomology group of a coherent sheaf $\mathcal{F}$ over X., page 2
h_X	Representable presheaf, page 99
$H_{\mathcal{L},\mathcal{F}}$	Hilbert function, page 2
$k[x_0,\ldots,x_n]_t$	k-vector space of polynomials of degree t in $n+1$ variables, page 3
L	Line in $\mathbb{P}^2_k$, page 76
$m_p(f)$	Parametric multiplicity of f at a point p, page 65
O_p	Orbit of an algebraic group action at p, page 82
o_p	Orbit map of an algebraic group action at p, page 82
$p_a(X)$	Arithmetic genus of X, page 3

$p_g(X)$	Geometric genus of X, page 4
$P_{\mathcal{L},\mathcal{F}}(t)$	Hilbert polynomial of $\mathcal{F}$ with respect to $\mathcal{L}$ on a scheme X, page 2
$P_X(t)$	Hilbert polynomial of a projective scheme X over k, page 3
$W \cap Z$	Scheme theoretic intersection, page xvi
X_p	Fiber of a morphism $X \to S$ at a point $p \in S$, page 10
$X_{S'}, X \times_S S'$	Fibered product of X and S' over S, page 9

Index