Motivic Spaces with Proper Support

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Abstract

In this thesis we introduce the notion of a *cdp*-functor on the category of proper schemes over a Noetherian base, and we show that *cdp*-functors to Waldhausen categories extend to factors that satisfy the excision property. This allows us to associate with a *cdp*-functor an Euler-Poincaré characteristic that sends the class of a proper scheme to the class of its image. Applying this construction to the Yoneda embedding yields a monoidal proper-fibred Waldhausen category over Noetherian schemes, with canonical *cdp*-functors to its fibres. Also, we deduce a motivic measure to the Grothendieck ring of finitely presented simplicially stable motivic spaces with the *cdh*-topology.

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Introduction

A motivic measure is a map from the classes of varieties over a field to a ring that satisfies the scissors relations (19) and respects products. To each motivic measure one associates a zeta function, by applying the motivic measure to symmetric powers of algebraic varieties. For instance, counting points over a finite field gives rise to the Hasse-Weil zeta function through applying it to symmetric powers, as it was first shown by Kapranov in [**Kap00**]. Another example arises from Larsen-Lunts motivic measure that takes value in the monoid ring of stable birational classes of algebraic varieties over a field, which has important applications in birational algebraic geometry, see [**LL03**] and [**GS14**]. The map to the Grothendieck ring of varieties, which is generated by the isomorphism classes of varieties modulo the scissors relations, provides a universal motivic measure. There are other important questions in algebraic geometry tackled through the Grothendieck ring of varieties, see [**NS11**] and [**DL04**]. However, this ring is not fully understood; for instance, the class of the affine line was not shown to be a zero divisor for a field of characteristic zero until recently, see [**Bor15**].

More generally, for a category with a set of distinguished sequences (*e.g.* exact sequences, cofibre sequences, distinguished triangles), its Grothendieck group is the group generated by isomorphism classes of objects module splitting the sequences. It can be though of as a decategorification of the category, with respect to the considered sequences. For a category with an exact structure, Quillen introduced an algebraic $\mathtt{K}$-theory, that extends the Grothendieck group, see [**Qui73**]. That was generalised by Waldhausen in [**Wal85**], who defined what is now called a Waldhausen structure, to which he associated an algebraic $\mathtt{K}$-theory spectrum whose path components group coincides with its Grothendieck group. Functors that respect these structures induce maps of spectra, and hence homomorphisms between the Grothendieck groups.

Most Waldhausen categories one is familiar with arise from model structures. A notion first developed by Quillen in [**Qui67**], and recently it gained more attention due to its extensive use in Morel-Voevodsky motivic homotopy theory of schemes. For a field of characteristic zero, there exists a surjective motivic measure to the path components of a Waldhausen spectrum of a subcategory of motivic spaces, due to Röndigs, see [**Rön16**]. However, we are not aware of the existence of such measures in positive characteristic, unless one inverts the Tate sphere.

The motivation for this thesis is the idea to lift the Hasse-Weil zeta function to motivic spaces, suggested by Vladimir Guletskiĭ. This can be split into the following two questions.

(1) Is there a non-trivial motivic measure that takes value in a Waldhausen subcategory of motivic spaces, over a finite field?
(2) Does the motivic measure of counting points factorise through such a motivic measure, if it exists?

Regarding the first question, the Yoneda embedding and Kan extensions provide a functor form the category of schemes, over a Noetherian base, to pointed motivic spaces. However, sending a smooth scheme to its pointed motivic space does not induce a motivic measure, as it does not respect the scissors relations. Then, one may consider altering motivic spaces to induce a functor that gives rise to a motivic measure.

Some motivic measures, like the Hodge measure and the ℓ-adic measure arise from cohomology theories with proper[1] support, *i.e.* they satisfy the excision property, (**E**). Such a cohomology theory arises from a plain cohomology theory (that does not satisfy the excision property), and both versions coincide for proper schemes, over the base. Then, it becomes natural to ask if motivic spaces are a plain theory, that admits a properly supported counterpart. This question is not restricted to motivic spaces, and it can be asked in a greater generality. That is, for a scheme S, when does a weak monoidal functor $F : \mathsf{Prop}/S \to \mathscr{C}$, from proper S-schemes to a symmetric monoidal Waldhausen category, define a weak monoidal functor $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{prop}}_{\mathrm{ft}}/S \to \mathscr{C}$ that satisfies the excision property?

On the one hand, when $S = \mathtt{Spec}\, k$, for a field k of characteristic zero, the motivic measure to the simplicially stable motivic homotopy category, introduced in [**Rön16**], relies on a presentation of the Grothendieck group of varieties, in which the generators are classes of smooth projective varieties and the relations are induced by blow up squares, recalled in Theorem 3.1.2. On the other hand, the aforementioned cohomology theories send *cdp*-squares[2] of proper schemes to (homotopy) pushout squares. That led us to distinguish functors to a Waldhausen category that satisfy the properties (**PS1**)-(**PS3**), the most relevant of which is sending *cdp*-squares of proper schemes to pushout squares, which accounts to independence of compactifications. We call a functor that satisfies these properties a *cdp*-functor, and we use Nagata's Compactification Theorem to show that such functors give rise to motivic measures. The below theorem is our main result.

[1] They are usually called cohomology theories with compact support. However, the term 'compact' referees to a notion of smallness that we use, and we prefer to use 'proper support' to avoid confusion.

[2] A generalisation of blow up squares, see Definition A.4.28.

Theorem (4.1.32). *Let S be a Noetherian scheme of finite Krull dimension, and let $F : (\mathsf{Prop}/S, \times, S) \to (\mathscr{C}, \wedge, \mathbb{1})$ be a weak monoidal cdp-functor to a symmetric monoidal Waldhausen category. Then, there exists a functor*

$$F^{\mathrm{c}} : (\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S, \times, S) \to (\mathscr{C}, \wedge, \mathbb{1}),$$

where $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S$ is the category of separated schemes of finite type over S whose morphisms are finite compositions of proper morphisms and formal inverses of open immersions, such that

- *there exists a natural isomorphism $\varphi : F \Rightarrow F^{\mathrm{c}}_{|\mathsf{Prop}/S}$;*
- *F^{c} satisfies the excision property, i.e. for every closed immersion $i : v \hookrightarrow x$ in $\mathsf{Sch}^{\mathrm{ft}}/S$ with complementary open immersion $j : u \hookrightarrow x$, the sequence*

$$F^{\mathrm{c}}(v) \xrightarrow{i_!} F^{\mathrm{c}}(x) \xrightarrow{j^!} F^{\mathrm{c}}(u).$$

  is a cofibre sequence in $\mathscr{C}$; and
- *F^{c} is weak monoidal, i.e. there exist natural transformations*

$$\phi^{\mathrm{c}} : F^{\mathrm{c}} \wedge F^{\mathrm{c}} \to F^{\mathrm{c}}(\times) \qquad \text{and} \qquad \phi^{\mathrm{c}}_S : \mathbb{1} \to F(S)$$

  that satisfy the associativity and unitality axioms, whose components are weak equivalences in $\mathscr{C}$.

Therefore, there exists a motivic measure

$$\mu_F : \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0(\mathscr{C}),$$

that sends the class of a proper S-scheme x to the class of $F(x)$.

The (pointed) Yoneda embedding is not a *cdp*-functor. Therefore, we provide a brief account of how to associate a motivic measures to functors that are not a *cdp*-functor. In particular, in §.4.2.2, we apply the above theorem to a properly supported version of the Yoneda embedding, and we obtain motivic spaces with proper support, with the *cdh*-topology.

Regarding question 2, we distinguish a Quillen adjunction that counts points for $\mathbb{A}^1$-rigid schemes, which we expect to factorise the classical motivic measure of counting points.

It became expected that a Grothendieck group of a category is a shadow of a richer structure, a $\mathtt{K}$-theory, that encodes deeper information about the category one started with. However, the category of varieties does not admit a Waldhausen structure, due to the lack of enough cokernels. Recently, Zakharevich introduced, in [**Zak17**], the notion of an assembler, and used it to define a spectrum whose path components coincide with the Grothendieck group of varieties. Then, Campbell defined a variation of a Waldhausen structure, called a semi-Waldhausen structure, on the category varieties, in which closed immersions play the role of cofibrations, resulting in an E_∞-ring spectrum with the same property, see [**Cam17**]. Applying Theorem 4.1.32 to the properly supported Yoneda embedding in §.4.2, we recover a spectrum that we expect its path components to be isomorphic to the modified Grothendieck ring of varieties. In fact, such spectrum arises from a fibre of a monoidal proper-fibred Waldhausen category over Noetherian schemes.

0.1. Thesis Outline

The thesis consists of an introduction, four chapters, and an appendix. The first three chapters review known materials that are needed for our constructions; whereas, in Chapter 4, we present our constructions and results.

The development of motivic homotopy theory depends on the well-established theory of model categories and their localisations. Therefore, we devote Chapter 1 to review the main notions of homotopy categories, needed to work in the realm of motivic homotopy theory. It starts with the notion of localisation of categories and the general theory of model categories. In particular, we focus on certain types of model structures that are particularly relevant to motivic homotopy theory, namely proper, cellular, simplicial and monoidal model structures. Then, we move to the central notion of localisation of model categories, especially Bousfield localisation. Since we need to consider stable homotopy categories, we recall stabilisation using symmetric spectra, followed by a brief account of triangulated categories. Finally, we conclude the chapter with a review of algebraic $\mathtt{K}$-theory.

In Chapter 2, we review motivic homotopy theory and geometric motives. We began by recalling the standard model structures of simplicial (pre)sheaves. Then, we review the motivic spaces, motivic spectra, and motivic complexes. we recall the main constructions of motivic spaces (spectra) and complexes, and recall some of the relations between them.

Chapter 3 is concerned mainly with the motivic measure of counting points over a finite field. We also recall with some details how this motivic measure lifts to effective Chow motives.

Chapter 4 begins with a section on compactifications, needed to extend *cdp*-functors. Afterwards, we prove the existence of properly supported extensions for *cdp*-functors. We provide a brief outline how to compactify functors that are not *cdp*-functors, which is applied to the Yoneda embedding to obtain a monoidal proper-fibred Waldhausen category over Noetherian schemes, with canonical *cdp*-functors to its fibres. Then, we apply this constriction to obtain properly supported motivic spaces, with the *cdh*-topology.

This is followed by calculations to examine a candidate for a functor to realise the motivic measure of counting points on the motivic homotopy categories.

The thesis assumes the reader's familiarity with basics of category theory, as in [**ML98**]. Yet, in the Appendix A, we briefly recall the main categorical notions used in the thesis.

0.2. Conventions and Notations

Throughout this thesis, all *schemes* are assumed to be separated over the ring of integers, and hence all morphisms of schemes in this thesis are separated. We denote the category of schemes and their morphisms by Sch. For a scheme S, let

- $\mathsf{Sch}^{\mathrm{ft}}/S$ denote the category of schemes of finite type over S;
- Var/S denote the full subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ of reduced S-schemes;
- Sm/S denote the full subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ of smooth S-schemes;
- Prop/S denote the full subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ of proper S-schemes;
- Proj/S denote the full subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ of projective S-schemes;
- SmProp/S denote the intersection of Sm/S and Prop/S; and
- SmProj/S denote the intersection of Sm/S and Proj/S.

Since we do not use the category Sch/S of all schemes over S, we abuse notation and refer to an object in $\mathsf{Sch}^{\mathrm{ft}}/S$ by an *S-scheme*. Also, an object in Var/S is called an *S-variety*.

For a subcategory $\mathscr{C}$ in $\mathsf{Sch}^{\mathrm{ft}}/S$ and for sets $\mathscr{P}$ and $\mathscr{I}$ of morphisms in $\mathsf{Sch}^{\mathrm{ft}}/S$ that are closed under composition and contain isomorphisms of S-schemes, we denote the subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ whose objects belong to $\mathtt{Ob}(\mathscr{C})$ and whose morphisms belong to $\mathscr{P}$ by $\mathscr{C}^{\mathscr{P}}$, whereas the category $(\mathscr{C}^{\mathscr{I}})^{\mathrm{op}}$ is denoted by $\mathscr{C}_{\mathscr{I}}$. Also, we denote the subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ whose objects belong to $\mathtt{Ob}(\mathscr{C})$ and whose morphisms are finite compositions of morphisms in $\mathscr{P}$ and formal inverses of morphisms in $\mathscr{I}$ by $\mathscr{C}^{\mathscr{P}}_{\mathscr{I}}$.

Also, we let $\mathsf{Noe}^{\mathrm{fd}}$ denote the full subcategory in Sch of Noetherian schemes of finite Krull dimensions.

CHAPTER 1

Homotopy Theory

Many notions in mathematics are invariant with respect to a set of morphisms between the studied objects. In such situations, the homotopy category with respect to those morphisms becomes the category of main interest, as a natural framework to consider such notions. For instance, most invariants of algebraic topology are invariant with respect to homotopy equivalences, which makes topological homotopy types natural objects to study.

Usually it is difficult to study a homotopy category $\mathcal{H}$ directly, and one uses a presentation of $\mathcal{H}$ by a pair of a category $\mathscr{C}$ and a set S of its morphisms. That is $\mathcal{H}$ is a localisation of $\mathscr{C}$ with respect to S, which presents $\mathcal{H}$ as a 'minimal' category under $\mathscr{C}$ in which morphisms of S are inverted. However, it is important to emphasise that a homotopy theory is concerned with a homotopy category rather than its presentations. In some occasions, different presentations of a homotopy category may possess technical advantages over the others, and one may consider those more suitable for the given occasion. Also, one usually favours presentations with additional technical sets of morphisms, *e.g.* fibrations or cofibrations, which allow for a simpler description of the homotopy category, and provide tools to work with homotopy (co)limits.

We commence this chapter with a review of localisation of categories, explaining the difficulties one may encounter with localisation. Then, following Quillen, we review model categories, which provide a convenient framework to do homotopy theory, avoiding the technical difficulties that arise with localisation in general.

Some homotopy categories admit a rich structure, providing technical advantages to work with. To that end, we provide a brief account on stable homotopy categories of symmetric spectra, in §.1.3, and on triangulated categories, in §.1.4. Then, we conclude this chapter with algebraic $\mathtt{K}$-theory.

1.1. Localisation of Categories

The notion of localisation of categories generalises localisation of rings, modules, and topological spaces, in that it 'universally' inverts a set of morphisms in a given category.

Definition 1.1.1. Let $\mathscr{C}$ be a category, and let S be a set of morphisms in $\mathscr{C}$. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be S-*local* if it sends morphisms in S to isomorphisms in $\mathscr{D}$. An S-local functor $\mathsf{L}_S : \mathscr{C} \to \mathcal{H}_S\mathscr{C}$ is a *localisation* of $\mathscr{C}$ with respect to S if

(1) for every S-local functor $F : \mathscr{C} \to \mathscr{D}$ there exists a functor $G_F : \mathcal{H}_S\mathscr{C} \to \mathscr{D}$ and a natural isomorphism $\phi : G_F \circ \mathsf{L}_S \Rightarrow F$; and
(2) the functor
$$\mathsf{L}_S^* : \mathsf{Fun}(\mathcal{H}_S\mathscr{C}, \mathscr{D}) \to \mathsf{Fun}(\mathscr{C}, \mathscr{D}),$$
given by precomposition with L_S, is fully faithful for every category $\mathscr{D}$.

At first encounter, the definition above may not reflect its intended universality. It is formulated so that a localisation is unique up to equivalences of categories, if it exists, see [**KS06**, Prop.7.1.2]. Hence, by a homotopy category, one may referee to such a category up to equivalences of categories. However, in our view, a notion deserves to be called universal if it can be viewed as a universal morphism. To that end, we devote the following paragraphs, where we use subdivision categories to recognise the property of being S-local in terms of the existence of certain strong 2-commutative squares, as in Lemma 1.1.2, which are used to realise a localisation of a large category as a 2-universal 1-morphism, as in Lemma 1.1.3. Readers comfortable with the definition above, and not interested in such formalities, may skip to §.1.1.1.

We begin by recalling the notion of a subdivision category, as in [**ML98**, p.224]. For a category $\mathscr{C}$, its *subdivision category* $\mathscr{C}^\S$ is given by

- the set $\mathtt{Ob}(\mathscr{C}^\S) \coloneqq \mathtt{Ob}(\mathscr{C}) \bigsqcup \mathsf{Mor}(\mathscr{C})$; and
- for each $X, Y \in \mathtt{Ob}(\mathscr{C}^\S)$, the set of morphisms

$$\mathscr{C}^\S(X,Y) \coloneqq \begin{cases} * & \text{if } X = Y; \\ * & \text{if } Y \in \mathsf{Mor}(\mathscr{C}), X \in \mathtt{Ob}(\mathscr{C}) \text{ and either } \mathtt{dom}\, Y = X \text{ or } \mathtt{codom}\, Y = X; \\ \varnothing & \text{otherwise;} \end{cases}$$

with the canonical composition and identity morphisms, where $*$ and $\varnothing$ are a singleton and an empty set in the fixed universe, respectively, see §.A.1. When $\mathscr{C}$ is a (locally) small category, so is $\mathscr{C}^\S$. For every morphism $f \in \mathsf{Mor}(\mathscr{C})$, denote the unique morphisms $\mathtt{dom}\, f \to f$ and $\mathtt{codom}\, f \to f$ in $\mathscr{C}^\S$ by α_f and β_f, respectively. There exists canonical functors $\vec{\pi} : \mathscr{C}^\S \to \mathscr{C}$ and $\overleftarrow{\pi} : \mathscr{C}^\S \to \mathscr{C}^{\mathrm{op}}$ given on objects of $\mathscr{C}^\S$ by

$$\vec{\pi}(C) = C \quad , \quad \vec{\pi}(f) = \mathtt{codom}(f) \qquad \text{and} \qquad \overleftarrow{\pi}(C) = C^{\mathrm{op}} \quad , \quad \overleftarrow{\pi}(f) = \mathtt{dom}(f)^{\mathrm{op}},$$

and on the non-identity morphisms of $\mathscr{C}^\S$ by

$$\vec{\pi}(\alpha_f) = f \quad , \quad \vec{\pi}(\beta_f) = \mathrm{id}_{\mathtt{codom}_f} \qquad \text{and} \qquad \overleftarrow{\pi}(\alpha_f) = \mathrm{id}_{\mathtt{dom}_f^{\mathrm{op}}} \quad , \quad \overleftarrow{\pi}(\beta_f) = f^{\mathrm{op}},$$

for every $C \in \mathtt{Ob}(\mathscr{C})$ and $f \in \mathsf{Mor}(\mathscr{C})$.

Lemma 1.1.2. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between large categories, let S be a set of morphisms in $\mathscr{C}$, and let $i_S : \mathscr{S} \hookrightarrow \mathscr{C}$ be the subcategory in $\mathscr{C}$ generated by S. Then, F is S-local if and only if there exists a strong 2-commutative square

$$Sq_F: \quad \begin{tikzcd} \mathscr{S}^{\S} \arrow[r, hook, "i_S \circ \vec{\pi}"] \arrow[d, "\overleftarrow{\pi}"'] & \mathscr{C} \arrow[d, "F"] \arrow[dl, Rightarrow, "e", "\sim"'] \\ \mathscr{S}^{\mathrm{op}} \arrow[r, "j"'] & \mathscr{D} \end{tikzcd}$$

in CAT_2, *i.e.* if there exists a functor $j : \mathscr{S}^{\mathrm{op}} \to \mathscr{D}$ and a natural isomorphism $e : F \circ i_S \circ \vec{\pi} \Rightarrow j \circ \overleftarrow{\pi}$.

Proof. Assuming that F is S-local, the existence of the strong 2-commutative square Sq_F is evident, where the functor j is given by

$$j(X^{\mathrm{op}}) = F(X) \qquad \text{and} \qquad j(s^{\mathrm{op}}) = F(s)^{-1}$$

for $X^{\mathrm{op}} \in \mathsf{Ob}(\mathscr{S}^{\mathrm{op}})$ and s^{op} in $\mathscr{S}^{\mathrm{op}}$, whereas the natural isomorphism e is given by

$$e_X = \mathrm{id}_{F(X)} \qquad \text{and} \qquad e_s = F(s)^{-1}$$

for every $X \in \mathsf{Ob}(\mathscr{S})$ and $s \in \mathsf{Mor}(\mathscr{S})$.

On the other hand, assume that the strong 2-commutative square Sq_F exists. Then, the natural isomorphism e induces a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "e_X"] \arrow[d, "F(s)"'] & j(X^{\mathrm{op}}) \arrow[r, "e_X^{-1}"] \arrow[d, equal] & F(X) \arrow[d, "F(s)"] \\ F(Y) \arrow[r, "e_s"] \arrow[d, equal] & j(X^{\mathrm{op}}) \arrow[r, "e_s^{-1}"] & F(Y) \arrow[d, equal] \\ F(Y) \arrow[r, "e_Y"'] & j(Y^{\mathrm{op}}) \arrow[r, "e_Y^{-1}"'] \arrow[u, "j(s^{\mathrm{op}})"'] & F(Y) \end{tikzcd}$$

in $\mathscr{D}$ whose composite horizontal morphisms are identities, for every morphism $s : X \to Y$ in $\mathscr{S}$. Which implies that $F(s)$ is an isomorphism with an inverse $e_X^{-1} \circ j(s^{\mathrm{op}}) \circ e_Y$, for every morphism $s : X \to Y$ in $\mathscr{S}$. Hence, F is S-local. □

Let $\mathsf{CAT}_2^{\bullet\leftarrow\bullet\rightarrow\bullet}$ denote the strict 2-category of strict 2-functors from the span category $\bullet\leftarrow\bullet\rightarrow\bullet$ to CAT_2 (*i.e.* spans of large categories), their pseudo-natural transformations, and modifications of the latter, see §.A.2.1.2, and let $\Delta : \mathsf{CAT}_2 \longrightarrow \mathsf{CAT}_2^{\bullet\leftarrow\bullet\rightarrow\bullet}$ denote the evident constant strict 2-functor. Then, a functor $F : \mathscr{C} \to \mathscr{D}$ between large categories is S-local, *i.e.* fits into a strong 2-commutative square Sq_F, if and only if it fits into a 1-morphism $((H, G, F), (\phi, \psi))$ from the span $\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \vec{\pi}} \mathscr{C}$ to $\Delta(\mathscr{D})$ in $\mathsf{CAT}_2^{\bullet\leftarrow\bullet\rightarrow\bullet}$. To ease the notation, when $\psi = \phi = \mathrm{id}_{F \circ i_S \circ \vec{\pi}}$, we denote such a 1-morphism by $\underline{F}$.

LEMMA 1.1.3. Let $\mathscr{C}$ be a large category, and let S be a set of morphisms in $\mathscr{C}$. Then, an S-local functor $\mathtt{L}_S : \mathscr{C} \to \mathcal{H}_S\mathscr{C}$ between large categories is a localisation of $\mathscr{C}$ with respect to S if and only if $\underline{\mathtt{L}_S}$ is a strict 2-universal 1-morphism from the span $\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \overrightarrow{\pi}} \mathscr{C}$ to the constant strict 2-functor $\Delta : \mathsf{CAT}_2 \to \mathsf{CAT}_2^{\bullet\leftarrow\bullet\to\bullet}$, where $i_S : \mathscr{S} \hookrightarrow \mathscr{C}$ is the subcategory in $\mathscr{C}$ generated by S.

PROOF. For a large category $\mathscr{D}$, one has $\mathtt{Map}_{\mathsf{CAT}_2}(\mathcal{H}_S\mathscr{C}, \mathscr{D}) = \mathsf{Fun}(\mathcal{H}_S\mathscr{C}, \mathscr{D})$. Then, Definition A.2.7 implies that $\underline{\mathtt{L}_S}$ is a strict 2-universal 1-morphism from $\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \overrightarrow{\pi}} \mathscr{C}$ to the strict 2-functor $\Delta : \mathsf{CAT}_2 \to \mathsf{CAT}_2^{\bullet\leftarrow\bullet\to\bullet}$ if and only if the induced functor

$$\mathtt{L}_S^* : \mathsf{Fun}(\mathcal{H}_S\mathscr{C}, \mathscr{D}) \longrightarrow \mathtt{Map}_{\mathsf{CAT}_2^{\bullet\leftarrow\bullet\to\bullet}}\left(\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \overrightarrow{\pi}} \mathscr{C}, \Delta(\mathscr{D})\right),$$

given by $\mathtt{L}_S^*(G) = \underline{G \circ \mathtt{L}_S}$, is an equivalence of categories for every large category $\mathscr{D}$. Given the axiom of choice, that is equivalent to $\mathtt{L}_S^*$ being essentially surjective and fully faithful.

Since a functor $\mathscr{C} \to \mathscr{D}$ is S-local if and only if it fits into a 1-morphism from the span $\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \overrightarrow{\pi}} \mathscr{C}$ to $\Delta(\mathscr{D})$ in $\mathsf{CAT}_2^{\bullet\leftarrow\bullet\to\bullet}$, the essential surjectivity of $\mathtt{L}_S^*$ for every large category $\mathscr{D}$ is equivalent to Definition 1.1.1.(1).

On the other hand, $\mathtt{L}_S^*$ is fully faithful if and only if the precomposition with the natural isomorphism $\mathrm{id}_{\mathtt{L}_S}$ induces a bijection of sets

$$\begin{aligned}(\mathtt{L}_S^*)_{G,H} : \mathsf{Fun}(\mathcal{H}_S\mathscr{C}, \mathscr{D})(G,H) &\cong \mathtt{Map}_{\mathsf{CAT}_2^{\bullet\leftarrow\bullet\to\bullet}}\left(\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \overrightarrow{\pi}} \mathscr{C}, \Delta(\mathscr{D})\right)\left(\underline{G\,\mathtt{L}_S}, \underline{H\,\mathtt{L}_S}\right)\\ &\cong \mathsf{Fun}(\mathscr{C}, \mathscr{D})(G\,\mathtt{L}_S, H\,\mathtt{L}_S)\end{aligned}$$

for every large category $\mathscr{D}$ and for every pair of functors $G, H : \mathcal{H}_S\mathscr{C} \to \mathscr{D}$. That, in turn, is equivalent to Definition 1.1.1.(2). □

The lemma above means in particular that a localisation $\mathtt{L}_S$ fits into a strict 2-pushout square of the span $\mathscr{S}^{\mathrm{op}} \xleftarrow{\overleftarrow{\pi}} \mathscr{S}^{\S} \xrightarrow{i_S \circ \overrightarrow{\pi}} \mathscr{C}$ in the strict 2-category CAT_2.

1.1.1. Properties of Localisations. For every category $\mathscr{C}$ and a set S of its morphisms, there exists a localisation $\mathtt{L}_S : \mathscr{C} \to \mathcal{H}_S\mathscr{C}$, in which $\mathcal{H}_S\mathscr{C}$ is the (a priori big) category $\mathscr{C}[S^{-1}]$ of fractions of $\mathscr{C}$ with respect to S. The set of objects of $\mathscr{C}[S^{-1}]$ equals the set of objects of $\mathscr{C}$, whereas its morphisms are equivalence classes of zigzags of morphisms in $\mathscr{C}$ with the components directed backwards being elements in S, modulo the evident equivalence relations, see [**GZ67**, §.I.1.1].

The construction of the category of fractions has some disadvantages, mainly due to the 'size' of its $\mathtt{hom}$-sets and to the nature of its morphisms. First, the category $\mathscr{C}[S^{-1}]$ is not necessarily locally small even when $\mathscr{C}$ is, which restricts possible constructions on $\mathscr{C}[S^{-1}]$, including the $\mathtt{hom}$-bifunctor. Also, morphisms of the category of fractions are rather formal and hard to work with, compared to those of $\mathscr{C}$.

Some of the difficulties one encounters with the category of fractions can be remedied when the localisation is reflective, as in Definition 1.1.4, or when the presenting category admits an additional structure making the homotopy category more accessible, like left or right calculus of fractions, or a model structure. In the rest of this section, we briefly recall reflective localisations, and we devote the next section for the study of model structures and their localisations.

DEFINITION 1.1.4. Let $\mathscr{C}$ be a category, let S be a set of morphisms in $\mathscr{C}$, and let $\mathtt{L}_S : \mathscr{C} \longrightarrow \mathcal{H}_S\mathscr{C}$ be a localisation of $\mathscr{C}$ with respect to S. If $\mathtt{L}_S$ admits a fully faithful right adjoint U_S, the localisation $\mathtt{L}_S$ is said to be *reflective*.

DEFINITION 1.1.5. Let $\mathscr{C}$ be a category, let S be a set of morphisms in $\mathscr{C}$, and let $\mathtt{L}_S : \mathscr{C} \longrightarrow \mathcal{H}_S\mathscr{C}$ be a localisation of $\mathscr{C}$ with respect to S. An object $Z \in \mathscr{C}$ is said to be S-*local* if the induced map

$$f^* : \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$$

is a bijection of sets for every morphism $f : X \to Y$ in S, *i.e.* if the representable functor $\mathtt{h}_Z$ factorises through $\mathtt{L}_S^{\mathrm{op}}$.

REMARK 1.1.6. When the localisation $\mathtt{L}_S$ is reflective, with a reflector U_S, the adjunction $\mathtt{L}_S \dashv U_S$ implies that $U_S(X)$ is an S-local for every $X \in \mathcal{H}_S\mathscr{C}$. Also, the Yoneda lemma implies that a morphism $\mathtt{L}_S(f)$ is an isomorphism in $\mathcal{H}_S\mathscr{C}$ if and only if the induced map

$$f^* : \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$$

is a bijection for every S-local object Z in $\mathscr{C}$. Therefore, a reflective homotopy category $\mathcal{H}_S\mathscr{C}$ is equivalent to the full subcategory of S-local objects in $\mathscr{C}$, which makes a reflective homotopy category more accessible, compared to a general homotopy category.

1.2. Model Categories

Model structures were first developed by Quillen in [**Qui67**] as a framework to study homotopy theories. The existence of a model structure on a presentation of a homotopy category addresses some of the issues arising in localisation in general, and makes the homotopy category more accessible, through realising it using a better understood quotient category, as in Theorem 1.2.15.

In this section, we recall the basic notions and properties of model categories, distinguishing special types of model structures that are of a special importance in motivic homotopy theory, namely left proper, cellular, and simplicial model structures. Then, we follow by a brief account on left Bousfield localisation.

1.2.1. Basics of Model Categories. The development of model structures is motivated by the homotopy theory of (topological and simplicial) spaces, and hence it relays on generalisations of familiar techniques in topology, which are recalled below.

Definition 1.2.1. Let $\mathscr{C}$ be a category, and let f and g be morphisms in $\mathscr{C}$. The morphism f is said to be a *retract* of g if there exist commutative squares $D : f \to g$, and $R : g \to f$ in $\mathscr{C}$ such that $R \circ D = \mathrm{id}_f$ in $\mathsf{Mor}(\mathscr{C})$, *i.e.*

$$\begin{tikzcd} X \arrow[r, "d_0"] \arrow[d, "f"'] \arrow[dr, phantom, "D"] & X' \arrow[r, "r_0"] \arrow[d, "g"] \arrow[dr, phantom, "R"] & X \arrow[d, "f"] \\ Y \arrow[r, "d_1"'] & Y' \arrow[r, "r_1"'] & Y \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{id}_f"] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y. \end{tikzcd}$$

Definition 1.2.2. Let $\mathscr{C}$ be a category, and let $i : U \to V$ and $p : X \to Y$ be morphisms in $\mathscr{C}$. The morphism i is said to have the *left lifting property* (*LLP*) with respect to p, and p is said to have the *right lifting property* (*RLP*) with respect to i, if for every solid commutative square

$$\begin{tikzcd} U \arrow[r, "e_0"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r, "e_1"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

in $\mathscr{C}$, there exists a dotted lift $h : V \to X$, not necessarily unique, that makes the whole diagram commute. A morphism f is said to have the LLP (resp. RLP) with respect to a set I of morphisms in $\mathscr{C}$ if it has the LLP (resp. RLP) with respect to every morphism in I.

Example 1.2.3. Let $\mathscr{C}$ be a category. Every morphism in $\mathscr{C}$ have both the RLP and LLP with respect to the set of isomorphisms in $\mathscr{C}$.

Sets of morphisms defined using the left and right lifting properties are fundamental in the study of model structures. For a set of morphisms I in $\mathscr{C}$, a morphism in $\mathscr{C}$ is called an *I-projective* (resp. *I-injective*) if it has the LLP (resp. RLP) with respect to I. The set of I-projective (resp. I-injective) morphisms in $\mathscr{C}$ is denoted by $I\text{-}\mathtt{proj}$ (resp. $I\text{-}\mathtt{inj}$). Then, a morphism in $\mathscr{C}$ is called an *I-cofibration* (resp. *I-fibration*) if it has the LLP (resp. RLP) with respect to $I\text{-}\mathtt{inj}$ (resp. $I\text{-}\mathtt{proj}$). The set of I-cofibrations (resp. I-fibrations) is denoted by $I\text{-}\mathtt{cof}$ (resp. $I\text{-}\mathtt{fib}$).

The sets $I\text{-}\mathtt{proj}$ and $I\text{-}\mathtt{inj}$ are closed under retracts and compositions, and they contain all isomorphisms of $\mathscr{C}$. Also, the set $I\text{-}\mathtt{proj}$ is closed under pushouts, whereas $I\text{-}\mathtt{inj}$ is closed under pullbacks.

Hovey's definition of model categories, presented in [**Hov99**], is more restrictive than the original definition due to Quillen [**Qui67**], as it requires the existence of functorial factorisations, recalled below.

Definition 1.2.4. Let $\mathscr{C}$ be a category. A *functorial factorisation* (α,β) in $\mathscr{C}$ is a pair of functors

$$\alpha,\beta : \mathsf{Mor}(\mathscr{C}) \to \mathsf{Mor}(\mathscr{C})$$

that form a factorisation system, *i.e.* for every morphism $f : X \to Y$ in $\mathscr{C}$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha(f)"'] & & Y \\ & Z \arrow[ur, "\beta(f)"'] & \end{tikzcd}$$

in $\mathscr{C}$, for $Z = \mathtt{dom}\,\beta(f) = \mathtt{codom}\,\alpha(f)$. Alternatively, the functorial factorisation (α,β) can be given by a functor

$$(\alpha,\beta) : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}^{\mathbf{3}}.$$

Remark 1.2.5. The three notions of factorisation, lifting, and retract are interactively connected, and this might be best shown through the retract argument and its consequences.

Lemma 1.2.6 (The Retract Argument). Let $\mathscr{C}$ be a category, and assume that a morphism $f : X \to Z$ factorises in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Z \\ & Y \arrow[ur, "p"'] & \end{tikzcd} .$$

If f has the RLP with respect to i, then it is a retract of p. Dually, if f has the LLP with respect to p, then it is a retract of i.

Proof. See [**Hov99**, Lem.1.1.9]. □

Definition 1.2.7. Let $\mathscr{C}$ be a category, let C, F and W be sets of morphisms in $\mathscr{C}$, and let (α,β) and (γ,δ) be functorial factorisations in $\mathscr{C}$. The quintuple $\mathscr{M} = \big(C, F, W, (\alpha,\beta), (\gamma,\delta)\big)$ is called a *model structure* on $\mathscr{C}$ if

CM1 $\mathscr{C}$ is bicomplete;

CM2 (Two-out-of-three) morphisms in W satisfy the two-out-of-three property, *i.e.* for composable morphisms f and g in $\mathscr{C}$, if two of the morphisms f, g, and $g \circ f$ belong to W, then so does the third;

CM3 (Stability under retract) the sets C, F, and W are closed under retracts;

CM4 (Lifting) morphisms in C have the LLP with respect to $F \bigcap W$, and morphisms in $C \bigcap W$ have the LLP with respect to F; and

CM5 (Factorisation) for every morphism $f : X \to Y$ in $\mathscr{C}$, one has

a) $\beta(f) \in F \bigcap W$, and $\alpha(f) \in C$; and

b) $\delta(f) \in F$, and $\gamma(f) \in C \bigcap W$.

Then, the pair $(\mathscr{C}, \mathscr{M})$ is called a *model category*, and the sets $C, F, W, C \bigcap W$ and $F \bigcap W$ are called the sets of *cofibrations, fibrations, weak equivalences, weak cofibrations, and weak fibrations*, respectively, in $\mathscr{M}$. More generally, a presentation $(\mathscr{C}, S)$

of a homotopy category is called a *category with weak equivalences* if S contains all isomorphisms of $\mathscr{C}$ and satisfies the two-out-of-three property.

A bicomplete category may admit different model structures, see §.2.1.2 for a discussion on different model structures on the category of simplicial presheaves on an essentially small site.

Model structures are used to be called *closed model structures*, that is to refer to the relations among the axioms in Definition 1.2.7. For instance, the set of cofibrations (resp. weak cofibrations) is precisely the set of morphisms with the LLP with respect to weak fibrations (resp. fibrations); whereas the set of fibrations (resp. weak fibrations) is precisely the set of morphisms with the RLP with respect to weak cofibrations (resp. cofibrations), see [**Hov99**, Lem.1.1.10]. In the presence of the two-out-of-three and factorisation axioms, we see that any two of the sets C, F, and W, in a model structure, determine the third. In fact, the retract axiom can be replaced by the requirement that any two of the sets C, F, and W determine the third, as in [**GM03**, Ch.V, §1.4, p293].

For a model category $(\mathscr{C}, \mathscr{M})$, cofibrations, weak cofibrations, fibrations, and weak fibrations form subcategories in $\mathscr{C}$, each of which contains all isomorphisms of $\mathscr{C}$ and is closed under retracts. Also, cofibrations and weak cofibrations are closed under pushouts. Dually, fibrations and weak fibrations are closed under pullbacks, see [**Hov99**, Cor.1.1.11].

Example 1.2.8. There exist model structures on the category of topological spaces Top, with weak equivalences, cofibrations, and fibrations given by

- weak homotopy equivalences, LLP with respect to weak Serre fibrations, and Serre fibrations, respectively; it is called the *classical model structure* or the *Quillen-Serre model structure* on Top, see [**Qui67**, §.II.3.Th.1]; and
- homotopy equivalences, closed Hurewicz cofibrations, and Hurewicz fibrations, respectively, called the *Hurewicz-Strøm model structure*, see [**Str72**, Th.3].

Example 1.2.9. Let $(\mathscr{C}, \mathscr{M})$ be a model category, and let $U, Y \in \mathscr{C}$. Then, the faithful (but not full) forgetful functors $U \downarrow \mathscr{C} \to \mathscr{C}$, $\mathscr{C} \downarrow Y \to \mathscr{C}$, and $U \downarrow \mathscr{C} \downarrow Y \to \mathscr{C}$ induce canonical model structures $U \downarrow \mathscr{M}$, $\mathscr{M} \downarrow Y$, and $U \downarrow \mathscr{M} \downarrow Y$ on the bicomplete categories $U \downarrow \mathscr{C}$, $\mathscr{C} \downarrow Y$, and $U \downarrow \mathscr{C} \downarrow Y$, respectively. The cofibrations, fibrations, and weak equivalences in theses canonical model structures are morphisms whose images are cofibrations, fibrations, and weak equivalences, respectively. The forgetful functor $U \downarrow \mathscr{C} \to \mathscr{C}$ admits a left adjoint given by coproduct with U, whereas $\mathscr{C} \downarrow Y \to \mathscr{C}$ admits a right adjoint given by product with Y.

The category $* \downarrow \mathscr{C}$ is pointed, and the model structure $* \downarrow \mathscr{M}$ satisfies several desired properties that do not always hold for $\mathscr{M}$. The category $* \downarrow \mathscr{C}$ is usually denoted by $\mathscr{C}_\bullet$ and the left adjoint $(- \coprod *, *) : \mathscr{C} \to \mathscr{C}_\bullet$ is denoted by $-_+$.

1.2.1.1. *The Homotopy Category of a Model Category.* A model category is a particular presentation of a homotopy category, with an extra structure that enables us to realise the homotopy category through a better understood quotient category with respect to a congruence relation induced from the model structure. Hence, model categories provide useful tools to deal with and understand homotopy theories. In fact, when Quillen introduced the notion of a model category in [**Qui67**], it was called "a category of model for homotopy theory".

The homotopy category of a model category $(\mathscr{C}, \mathscr{M})$ is defined to be a localisation $\mathtt{L}_{\mathscr{M}} : \mathscr{C} \to \mathcal{H}_{\mathscr{M}}\mathscr{C}$ of $\mathscr{C}$ with respect to the set of weak equivalences in $\mathscr{M}$. Hence, different model structures with the same set of weak equivalences present equivalent homotopy categories.

One defines homotopy relations in a model category in an analogous manner to topological spaces. One starts by axiomatising the cylinder and path spaces, resulting in the cylinder and path objects, and use the latter to define left and right homotopies, respectively.

For a space V, since id_V is left homotopic to itself, the canonical maps $i_0, i_1 : V \to V \times I$ and the universal property of coproducts produce a factorisation $\mathrm{pr}_1 \circ (i_0 \coprod i_1)$ of the fold map $\nabla_V : V \coprod V \to V$, illustrated by the commutative diagram

$$\begin{tikzcd}
& V \coprod V \arrow[d, dotted, "i_0 \coprod i_1"] & \\
V \arrow[ur] \arrow[r, "i_0"] \arrow[dr, bend right, "\mathrm{id}_V"'] & V \times I \arrow[d, "pr_1"] & V. \arrow[ul] \arrow[l, "i_1"'] \arrow[dl, bend left, "\mathrm{id}_V"] \\
& V &
\end{tikzcd}$$

Recall that the projection $V \times I \overset{Pr_1}{\to} V$ is a weak equivalence. Moreover, when we restrict ourselves to the category **CGHaus** of compactly generated Hausdorff spaces, we find that the map $i_0 \coprod i_1$ is a Hurewicz cofibration, which gives rise to the following definition.

Definition 1.2.10. Let $(\mathscr{C}, \mathscr{M})$ be a model category, and let $V, X \in \mathscr{C}$.

- A *cylinder object* for V is a factorisation of the fold morphism $\nabla_V : V \coprod V \to V$

$$C: \qquad V \coprod V \xrightarrow{i_0 \coprod i_1} \mathtt{Cyl}(V) \xrightarrow{p_C} V\ ,$$

  where p_C is a weak equivalence in $\mathscr{M}$, and $i_0, i_1 : V \to \mathtt{Cyl}(V)$ are morphisms in $\mathscr{C}$ for which $i_0 \coprod i_1$ is a cofibration in $\mathscr{M}$.
- A *path object* for X is a factorisation of the diagonal morphism $\Delta_X : X \to X \times X$

$$P: \qquad X \xrightarrow{i_P} \mathtt{Path}(X) \xrightarrow{p_0 \times p_1} X \times X\ ,$$

where i_P is a weak equivalence in $\mathscr{M}$, and $p_0, p_1 : \mathtt{Path}(X) \to X$ are morphisms in $\mathscr{C}$ for which $p_0 \times p_1$ is a fibration in $\mathscr{M}$.

REMARK 1.2.11. The functorial factorisations in a model category provide canonical functorial cylinder and path objects. In fact, it is sufficient to consider homotopies defined using these canonical cylinder and path objects, see [**Hir03**, Prop.7.3.4]. Denote the canonical cylinder object for $V \in \mathscr{C}$ that is induced by the functorial factorisation (α, β), applied to the fold map ∇_V, by

$$C_V : \qquad V \coprod V \xrightarrow{i_0 \coprod i_1} \mathtt{Cyl}_{\mathscr{M}}(V) \xrightarrow{p_C} V \ ,$$

and the canonical path object for $X \in \mathscr{C}$ that is induced by the functorial factorisation (γ, δ), applied to the diagonal map Δ_V, by

$$P_X : \qquad X \xrightarrow{i_P} \mathtt{Path}_{\mathscr{M}}(X) \xrightarrow{p_0 \times p_1} X \times X \ .$$

DEFINITION 1.2.12. Let $(\mathscr{C}, \mathscr{M})$ be a model category, and let $f_0, f_1 : V \to X$ be morphisms in $\mathscr{C}$.

- A *left homotopy* from f_0 to f_1 is a pair (C, H_l), where C is a cylinder object for V and $H_l : \mathtt{Cyl}(V) \to X$ is a morphism in $\mathscr{C}$ for which $f_0 = H_l \circ i_0$ and $f_1 = H_l \circ i_1$, *i.e.* that makes the following diagram

$$\begin{tikzcd} V \arrow[r, "i_0"] \arrow[dr, bend right, "f_0"'] & \mathtt{Cyl}(V) \arrow[d, "H_l"] & V \arrow[l, "i_1"'] \arrow[dl, bend left, "f_1"] \\ & X & \end{tikzcd}$$

  commute; if there exists a left homotopy (C, H_l) from f_0 to f_1, we say that f_0 is *left homotopic* to f_1, and we write $f_0 \overset{l}{\simeq} f_1$.
- A *right homotopy* from f_0 to f_1 is a pair (P, H_r), where P is a path object for X and $H_r : V \to \mathtt{Path}(X)$ is a morphism in $\mathscr{C}$ for which $f_0 = p_0 \circ H_r$ and $f_1 = p_1 \circ H_r$, *i.e.* that makes the following diagram

$$\begin{tikzcd} & V \arrow[dl, bend right, "f_1"'] \arrow[d, "H_r"] \arrow[dr, bend left, "f_0"] & \\ X & \mathtt{Path}(X) \arrow[l, "p_1"] \arrow[r, "p_0"'] & X \end{tikzcd}$$

  commute; if there exists a right homotopy (P, H_r) from f_0 to f_1, we say that f_0 is *right homotopic* to f_1, and we write $f_0 \overset{r}{\simeq} f_1$;
- if f_0 is both left and right homotopic to f_1, then we say that f_0 is *homotopic* to f_1, and we write $f_0 \simeq f_1$; and
- a morphism $f : V \to X$ in $\mathscr{C}$ is called a *homotopy equivalence* if there exists a morphism $g : X \to V$ such that $fg \simeq \mathrm{id}_X$ and $gf \simeq \mathrm{id}_V$.

For topological spaces, the cylinder functor $- \times I$ is a left adjoint to the path functor $-^I$, which allows for the interchange between right and left topological homotopies.

Also, it explains the terminology of left and right homotopies, where the left homotopy is the one defined by the left adjoint cylinder functor. However, even the 'nicest' cylinder functor $\mathtt{Cyl}_{\mathscr{M}}(-)$ and path functor $\mathtt{Path}_{\mathscr{M}}(-)$, induced by the functorial factorisations of a model structure $\mathscr{M}$, do not have to be adjoint, and one needs to distinguish between left and right homotopies.

In a general model category, neither the left nor the right homotopy defines equivalence relations on $\mathtt{hom}$-sets. In particular, such relations are not necessarily symmetric, as in Kan-Quillen's model structure on simplicial sets, see [**GJ09**, §.I.6]. However, that can be remedied through restricting attention to the subcategory of fibrant and cofibrant objects. For a model category $(\mathscr{C}, \mathscr{M})$, an object $V \in \mathscr{C}$ is said to be *cofibrant* if the unique morphism $\varnothing \to V$ is a cofibration in $\mathscr{M}$, and an object $X \in \mathscr{C}$ is said to be *fibrant* if the unique morphism $X \to *$ is a fibration in $\mathscr{M}$. The full subcategories $\mathscr{C}_c, \mathscr{C}_f$, and $\mathscr{C}_{cf}$ of cofibrant, fibrant, and cofibrant-fibrant objects, respectively, play an essential role in realising the homotopy category of $(\mathscr{C}, \mathscr{M})$.

A model structure $\mathscr{M}$ on a category $\mathscr{C}$ induces model structures $\mathscr{M}_c, \mathscr{M}_f$ and $\mathscr{M}_{cf}$ on the bicomplete categories $\mathscr{C}_c, \mathscr{C}_f$, and $\mathscr{C}_{cf}$, respectively, in which a morphism is a cofibration, a fibration, or a weak equivalence, if and only if it is mapped by the inclusion functor to a cofibration, a fibration, or a weak equivalence, respectively. Whereas, the functorial factorisations are given by the restriction of the functorial factorisations of $\mathscr{M}$.

Proposition 1.2.13. Let $(\mathscr{C}, \mathscr{M})$ be a model category. Then, left and right homotopies between cofibrant-fibrant objects in $(\mathscr{C}, \mathscr{M})$ coincide. Moreover, the homotopy relation is a congruence relation on the category $\mathscr{C}_{cf}$ of cofibrant-fibrant objects.

Proof. See [**Hov99**, Cor.1.2.6 and Cor.1.2.7]. □

Proposition 1.2.14. Let $(\mathscr{C}, \mathscr{M})$ be a model category, and let f be a morphism between cofibrant-fibrant objects in $\mathscr{C}$. Then, f is a weak equivalence in $\mathscr{M}$ if and only if it is a homotopy equivalence.

Proof. See [**Hov99**, Prop.1.2.8]. □

Theorem 1.2.15. Let $(\mathscr{C}, \mathscr{M})$ be a model category and let $Q : \mathscr{C}_{cf} \to \pi\mathscr{C}_{cf}$ be the quotient functor of $\mathscr{C}_{cf}$ with respect to the homotopy congruence relation. Then, there exists an equivalence of categories $\pi\mathscr{C}_{cf} \to \mathcal{H}\mathscr{C}_{cf}$.

Proof. See [**Hov99**, Cor.1.2.9]. □

Since the inclusion functors induce equivalences of categories

$$\mathcal{H}\mathscr{C}_{cf} \to \mathcal{H}\mathscr{C}_c \to \mathcal{H}\mathscr{C} \qquad \text{and} \qquad \mathcal{H}\mathscr{C}_{cf} \to \mathcal{H}\mathscr{C}_f \to \mathcal{H}\mathscr{C},$$

by [**Hov99**, Prop.1.2.3], the categories $\mathcal{H}\mathscr{C}$ and $\pi\mathscr{C}_{cf}$ are equivalent. In particular, when $\mathscr{C}$ is locally small there exists a locally small homotopy category of $\mathscr{C}$ with respect to the weak equivalences of $\mathscr{M}$, namely $\pi\mathscr{C}_{cf}$.

Cofibrant and Fibrant Replacements. Proposition 1.2.14 and Theorem 1.2.15 illustrate the importance of the cofibrant and fibrant objects in a model category. The functorial factorisations of a model category provide a machinery to functorially 'approximate' its objects by cofibrant or fibrant objects, where 'approximate' means replacing objects by weakly equivalent ones, and hence by isomorphic objects in the homotopy category.

Definition 1.2.16. Let $(\mathscr{C},\mathscr{M})$ be a model category. Define a functor $Q:\mathscr{C}\to\mathscr{C}_c$ that sends each morphism $i:U\to V$ in $\mathscr{C}$ to the morphism

$$Q(i) \coloneqq \mathsf{codom}_{\mathsf{Mor}\mathscr{C}}\ \alpha\left(\begin{tikzcd} U \arrow[r, "i"] & V \\ \varnothing \arrow[u] \arrow[r] & \varnothing \arrow[u] \end{tikzcd}\right) = \mathsf{dom}_{\mathsf{Mor}\mathscr{C}}\ \beta\left(\begin{tikzcd} U \arrow[r, "i"] & V \\ \varnothing \arrow[u] \arrow[r] & \varnothing \arrow[u] \end{tikzcd}\right)$$

in $\mathscr{C}_c$. Since (α,β) is a functorial factorisation in $\mathscr{M}$, the assignment above gives a well-defined functor Q. It is called the *cofibrant replacement functor* of $\mathscr{M}$.

For every object $V\in\mathscr{C}$, the morphism $\beta(\varnothing\to V):Q(V)\to V$ is a weak fibration.

Definition 1.2.17. Let $(\mathscr{C},\mathscr{M})$ be a model category. Define a functor $R:\mathscr{C}\to\mathscr{C}_f$ that sends each morphism $p:X\to Y$ in $\mathscr{C}$ to the morphism

$$R(p) \coloneqq \mathsf{codom}_{\mathsf{Mor}\mathscr{C}}\ \gamma\left(\begin{tikzcd} X \arrow[r, "p"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & * \end{tikzcd}\right) = \mathsf{dom}_{\mathsf{Mor}\mathscr{C}}\ \delta\left(\begin{tikzcd} X \arrow[r, "p"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & * \end{tikzcd}\right)$$

in $\mathscr{C}_f$. The functor R is called the *fibrant replacement functor* of $\mathscr{M}$.

For every object $X\in\mathscr{C}$, the morphism $\gamma(X\to *):X\to R(X)$ is a weak cofibration.

The two-out-of-three property shows that both the cofibrant and fibrant replacement functors preserve weak equivalences, which is essential for Definition 1.2.24 of total derived functors.

Remark 1.2.18. A model category may admit different cofibrant and fibrant approximations, see [**Hir03**, §.14.6]. For instance, in left localisations of model categories, it is desired to have a cofibrant approximation that maps arbitrary morphisms to cofibrations between cofibrant objects, which may be called *cofibration cofibrant approximation.* The cofibrant replacement functor Q given in Definition 1.2.16 does not satisfy this property. However, an evident iteration of Q does.

1.2.1.2. *Quillen Functors.* Although morphisms of a mathematical structure are usually defined to be those preserving that structure, considering only functors preserving the whole model structure is too restrictive, because it excludes motivating examples of particular interests in the classical homotopy theory, like the identity functor from the Quillen-Serre model category $\mathsf{Top}_{\mathrm{QS}}$ to the Hurewicz-Strøm model category $\mathsf{Top}_{\mathrm{HS}}$. Since one is interested in the homotopy categories rather than their presentations, one considers functors between the presenting model categories that canonically induce total derived functors between the homotopy categories. The widely-adopted notion of a morphism of model categories is what is now called a Quillen adjunction. It preserves enough aspects of the model structures so that it both induces canonical adjunction between the homotopy categories and cover the functors one usually is interested in.

Definition 1.2.19. Let $(\mathscr{C},\mathscr{M})$ and $(\mathscr{D},\mathscr{N})$ be model categories.

- A functor $F:\mathscr{C}\to\mathscr{D}$ is called a *left Quillen functor* if it is a left adjoint and preserves cofibrations and weak cofibrations.
- A functor $G:\mathscr{D}\to\mathscr{C}$ is called a *right Quillen functor* if it is a right adjoint and preserves fibrations and weak fibrations.

Given an adjunction $F:\mathscr{C}\rightleftarrows\mathscr{D}:G$, the functor F is a left Quillen functor if and only if G is a right Quillen functor, see [**Hov99**, Lem.1.2.3]. Such an adjunction is called a *Quillen adjunction.*

Example 1.2.20. The adjunction

$$|-|:\mathsf{sSet}_{\mathrm{KQ}}\rightleftarrows\mathsf{Top}_{\mathrm{QS}}:\mathtt{Sing}$$

of the geometric realisation and the singular functor, recalled in §1.2.4.1, is a Quillen adjunction between Kan-Quillen's model structure on simplicial sets and Quillen-Serre's model structure on topological spaces, see [**Hov99**, Th.3.6.7 and Th.2.4.23].

Example 1.2.21. For a model category $(\mathscr{C},\mathscr{M})$ and for an object $U\in\mathscr{C}$, the left adjoint functor $-\coprod U:\mathscr{C}\to U\downarrow\mathscr{C}$ is a left Quillen functor with respect to the model structure $U\downarrow\mathscr{M}$, as in Example 1.2.9. Particularly, the adjoining base point functor $-_+$ is a left Quillen functor. Moreover, a Quillen adjunction $F:(\mathscr{C},\mathscr{M})\rightleftarrows(\mathscr{D},\mathscr{N}):G$ induces a Quillen adjunction $F_\bullet:(\mathscr{C}_\bullet,\mathscr{M}_\bullet)\rightleftarrows(\mathscr{D}_\bullet,\mathscr{N}_\bullet):G_\bullet$, with $F_\bullet(X_+)$ being canonically isomorphic to $F(X)_+$ for every $X\in\mathscr{C}$, see [**Hov99**, Prop.1.3.5].

Although Quillen functors are not required to preserve all weak equivalences, Ken Brown's Lemma 1.2.22 implies that they preserve just enough weak equivalences to induce adjoint functors between the homotopy categories.

Lemma 1.2.22 (Ken Brown's Lemma). Let $(\mathscr{C},\mathscr{M})$ be a model category, let $(\mathscr{D},W_{\mathscr{D}})$ be a category with weak equivalences, and let $F:\mathscr{C}\to\mathscr{D}$ be a functor. If F sends weak cofibrations between cofibrant objects to weak equivalences, then it sends all weak equivalences between cofibrant objects to weak equivalences. Dually, if F sends weak fibrations between fibrant objects to weak equivalences, then it sends all weak equivalences between fibrant objects to weak equivalences.

PROOF. See [**Hov99**, Lem.1.1.12]. □

COROLLARY 1.2.23. Let $F : (\mathscr{C}, \mathscr{M}) \rightleftarrows (\mathscr{D}, \mathscr{N}) : G$ be a Quillen adjunction. Then, F preserves all weak equivalences between cofibrant objects, and G preserve all weak equivalences between fibrant objects.

Therefore, the restrictions

$$F|_{\mathscr{C}_c} : \mathscr{C}_c \to \mathscr{D} \qquad \text{and} \qquad G|_{\mathscr{D}_f} : \mathscr{D}_f \to \mathscr{C}$$

preserve all weak equivalences. Hence, they induce well-defined functors between the homotopy categories

$$\mathcal{H}F|_{\mathscr{C}_c} : \mathcal{H}\mathscr{C}_c \to \mathcal{H}\mathscr{D} \qquad \text{and} \qquad \mathcal{H}G|_{\mathscr{D}_f} : \mathcal{H}\mathscr{D}_f \to \mathcal{H}\mathscr{C}.$$

These functors, in addition to the cofibrant and fibrant replacements, give rise to an adjunction $\mathcal{H}\mathscr{C} \leftrightarrows \mathcal{H}\mathscr{D}$ between the homotopy categories.

DEFINITION 1.2.24. Let $F : (\mathscr{C}, \mathscr{M}) \leftrightarrows (\mathscr{D}, \mathscr{N}) : G$ be a Quillen adjunction.

- The *total left derived functor* $\mathbf{L}F : \mathcal{H}\mathscr{C} \to \mathcal{H}\mathscr{D}$ is the composition

$$\mathcal{H}\mathscr{C} \xrightarrow{\mathcal{H}Q} \mathcal{H}\mathscr{C}_c \xrightarrow{\mathcal{H}F} \mathcal{H}\mathscr{D} ,$$

  where Q is the cofibrant replacement of $\mathscr{M}$, as in Definition 1.2.16.
- The *total right derived functor* $\mathbf{R}G : \mathcal{H}\mathscr{D} \to \mathcal{H}\mathscr{C}$ is the composition

$$\mathcal{H}\mathscr{D} \xrightarrow{\mathcal{H}R} \mathcal{H}\mathscr{D}_f \xrightarrow{\mathcal{H}G} \mathcal{H}\mathscr{C} ,$$

  where R is the fibrant replacement of $\mathscr{N}$, as in Definition 1.2.17.

This definition is the main reason to require the factorisation to be fixed for a model structure, and for it to be functorial, see [**Hov99**, §.1.3.2]. In fact, one obtains derived functors for every Quillen adjunction and for every choice of cofibrant and fibrant approximation functors. That is particularly useful when the functorial factorisations are given by the small object argument for some large cardinality, as it is the case of the local model structures of simplicial presheaves, see §.2.1. In which cases, one looks for more nicely behaved cofibrant and fibrant approximation functors.

EXAMPLE 1.2.25. The Quillen adjunction of the geometric realisation and the singular functor, recalled in §.1.2.4.1, induces an equivalence of homotopy categories

$$\mathbf{L}|-| : \mathcal{H}\mathsf{sSet}_{\mathrm{KQ}} \rightleftarrows \mathcal{H}\mathsf{Top}_{\mathrm{QS}} : \mathbf{R}\,\mathtt{Sing},$$

see [**Hov99**, Th.3.6.7 and Th.2.4.23].

More generally, a Quillen adjunction $F : (\mathscr{C}, \mathscr{M}) \rightleftarrows (\mathscr{D}, \mathscr{N}) : G$ is called a *Quillen equivalence* if the total derived functors $\mathbf{L}F$ and $\mathbf{R}G$ are adjoint equivalences of categories, see [**Hov99**, §.1.3.3].

1.2.2. Cellular Model Categories. The localisation of a model structure, as in Definition 1.2.62, possesses some technical challenges, mainly in terms of the existence of the functorial factorisations for the localised structure. Such challenges may be overcome when the original model structure is cellular, which is roughly a model structure that both satisfies some relative smallness conditions and contains a large enough set of cofibrations that behave like inclusions of sets, see Definition 1.2.40. The Bousfield-Smith cardinality argument, recalled in Theorem 1.2.67, uses a bounded version of the small object argument which relies on the aforementioned properties of a cellular model structure to establish the functorial factorisations for the localised model structure.

Before recalling cellular model structures, one needs to be familiar with the notion of (presented) relative cell complexes, some relative smallness notions, and the small object argument. In fact, relative cell complexes and relative smallness are formulated to express the small object argument. Hence, readers looking for motivations for the following constructions are encouraged to skim the small object argument, in §.1.2.2.3, before proceeding from here.

1.2.2.1. *Relative I-$\mathtt{cell}$ Complexes.* Let $\mathscr{C}$ be a cocomplete category, and let λ be an ordinal. A *λ-sequence* in $\mathscr{C}$ is a colimit-preserving functor $Z : \lambda \to \mathscr{C}$. Denote the image of the unique morphism $\upsilon \to \xi$ along Z by z_υ^ξ for ordinals $\upsilon < \xi < \lambda$, denote $z_\xi^{\xi+1}$ by z_ξ, and denote the morphism $Z_\xi \to \mathtt{colim}\, Z$, induced by the universal property of colimits, by z_ξ^λ, for every ordinal $\xi < \lambda$.

Definition 1.2.26. Let $\mathscr{C}$ be a cocomplete category, and let Z a λ-sequence in $\mathscr{C}$. The *transfinite composition* of Z is the colimit injection $z_0^\lambda : Z_0 \to \mathtt{colim}\, Z$. Let I be a set of morphisms in $\mathscr{C}$, a transfinite composition $Z_0 \to \mathtt{colim}\, Z$ is called a *transfinite composition of morphisms in I* if z_ξ belongs to I, for every $\xi + 1 < \lambda$. A set of morphisms I in $\mathscr{C}$ is said to be *closed under transfinite compositions* if it contains all transfinite compositions of morphisms in I.

Example 1.2.27. Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$. Then, the set I-$\mathtt{proj}$ is closed under transfinite compositions, and so is I-$\mathtt{cof}$. In particular, (weak) cofibrations in a model category are closed under transfinite compositions.

Definition 1.2.28. Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$, and let $f : A \to B$ be a morphism in $\mathscr{C}$. We say that f is a *relative I-$\mathtt{cell}$ complex* if it is a transfinite composition of pushouts of morphisms in I. That is, a morphism $f : A \to B$ in $\mathscr{C}$ is a relative I-$\mathtt{cell}$ complex if there exist an ordinal λ and a λ-sequence

Z in $A{\downarrow}\mathscr{C}$, where z_ξ is a cobase change in $\mathscr{C}$ of a morphism g_ξ in I, for every ordinal ξ with $\xi + 1 < \lambda$, such that f is the transfinite composite of Z, visualised in the diagram

$$\begin{tikzcd}
 & X_0 \arrow[r, "g_0"] \arrow[dl] & Y_0 \arrow[d] & & X_\xi \arrow[r, "g_\xi"] \arrow[dl] & Y_\xi \arrow[d] & X_{\xi+1} \arrow[r, "g_{\xi+1}"] \arrow[dl] & Y_{\xi+1} \arrow[d] & \\
A = Z_0 \arrow[rr, "z_0"'] \arrow[rrrrrrrr, bend right, "f"'] & & Z_1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed] & Z_\xi \arrow[rr, "z_\xi"'] & & Z_{\xi+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rr, "z_{\xi+1}"'] & & Z_{\xi+2} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed] & \mathtt{colim}\, Z = B
\end{tikzcd}$$

The set of relative I-$\mathtt{cell}$ complexes in $\mathscr{C}$ is denoted by I-$\mathtt{cell}$. A relative I-$\mathtt{cell}$ subcomplex of $f : A \to B$ is a relative I-$\mathtt{cell}$ complex $f' : A \to B'$ with a monomorphism $f' \to f$ in $A \downarrow \mathscr{C}$. An object $B \in \mathscr{C}$ is said to be an I-$\mathtt{cell}$ *complex* if the initial morphism $\varnothing \to B$ is a relative I-$\mathtt{cell}$ complex. A monomorphism $f : A \to B$ in $\mathscr{C}$ that is a relative I-$\mathtt{cell}$ complex between I-$\mathtt{cell}$ complexes A and B is called an *inclusion of I-$\mathtt{cell}$-complexes.*

The notion of relative I-$\mathtt{cell}$ complexes is an abstraction of gluing of cells in topology, and hence the name.

Relative I-$\mathtt{cell}$ complexes may be expressed in different ways as transfinite compositions of pushouts of morphisms in I. However, when I-$\mathtt{cell}$ consists of effective monomorphisms, fixing the presentation for relative I-$\mathtt{cell}$ complexes makes them behave like inclusions of sets, as in Proposition 1.2.31, which is essential for the Bousfield-Smith cardinality argument, recalled in Theorem 1.2.67.

Definition 1.2.29. Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$, and let $f : A \to B$ be a relative I-$\mathtt{cell}$ complex. A *presentation* P of f is a pair $(Z, (S_\xi, g_\xi, i_\xi)_{\xi<\lambda})$, where Z is a λ-sequence in $A{\downarrow}\mathscr{C}$ for some ordinal λ, with a transfinite composition isomorphic to f, such that for every ordinal $\xi + 1 < \lambda$,

- S_ξ is a set (indexing cells);
- g_ξ is a function $g_\xi : S_\xi \to I$ (choosing cells); and
- i_ξ is a function $i_\xi : S_\xi \to \mathtt{Ob}(\mathscr{C}{\downarrow}Z_\xi)$ (gluing cells);

with $\mathtt{dom}\, g_\xi(s_\xi) = \mathtt{dom}\, i_\xi(s_\xi)$, for every $s_\xi \in S_\xi$, for which there exists the pushout square (1) in $\mathscr{C}$, on the next page, where $X(s_\xi) \coloneqq \mathtt{dom}\, i_\xi(s_\xi)$ and $Y(s_\xi) \coloneqq \mathtt{codom}\, i_\xi(s_\xi)$. The set $\bigsqcup_{\xi<\lambda} S_\xi$ is called the *set of cells* of the presentation, and it is cardinality is called the *size* of the presentation. Moreover, the pair (f, P) is called a *presented relative I-$\mathtt{cell}$ complex.* A *presented relative I-$\mathtt{cell}$ subcomplex* of (f, P) is a pair (f', P'), where f' is a relative I-$\mathtt{cell}$ subcomplex of f and P' is a presentation of f' whose set of cells is a subset of cells of P, and whose choice and gluing maps are the restrictions of those of P.

$$\begin{tikzcd} \coprod_{s_\xi \in S_\xi} X(s_\xi) \arrow[r, "\coprod_{s_\xi \in S_\xi} g_\xi(s_\xi)"] \arrow[d, "\coprod_{s_\xi \in S_\xi} i_\xi(s_\xi)"'] & \coprod_{s_\xi \in S_\xi} Y(s_\xi) \arrow[d] \\ Z_\xi \arrow[r, "z_\xi"'] & Z_{\xi+1} \end{tikzcd} \tag{1}$$

Remark 1.2.30. Inclusions of I-`cell` complexes allow set-theoretic arguments in abstract cocomplete categories, when I-`cell` consists of effective monomorphisms, *i.e.* equalisers, as it is the case in cellular model categories. In which case, they become analogous to inclusions of sets, admitting operations similar to the intersection and union of sets, see Proposition 1.2.31.

Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$ such that I-`cell` consists of monomorphisms. Assume that B is an I-`cell` complex, and let S be the set of cells of a presentation of the initial morphism $\varnothing \to B$ in $\mathscr{C}$. Then, every inclusion of I-`cell` complexes $f : A \hookrightarrow B$ is determined up to isomorphisms by a subset of cells, that is a subset of S, see [**Hir03**, Prop.10.6.10]. Moreover, every subset of S, that is compatible with the choice and gluing maps, determines uniquely up to isomorphisms an inclusion of I-`cell` complexes $f : A \hookrightarrow B$, see [**Hir03**, Prop.10.6.11].

Proposition 1.2.31. Let C be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$ such that I-`cell` consists of effective monomorphisms, let B be an I-`cell` complex, and let S be the set of cells of a presentation of the initial morphism $\varnothing \to B$ in $\mathscr{C}$. Assume that $f_1 : B_1 \hookrightarrow B$ and $f_2 : B_2 \hookrightarrow B$ are inclusions of I-`cell` complexes, with subsets of cells $S_1 \subset S$ and $S_2 \subset S$, respectively. Then, the subsets $S_1 \bigcap S_2 \subset S_1 \bigcup S_2 \subset S$ of cells determine up to isomorphisms inclusions of I-`cell` complexes

$$i_1 : B_1 \hookrightarrow B_1 \bigcup B_2 \quad , \quad i_2 : B_2 \hookrightarrow B_1 \bigcup B_2 \quad , \quad i : B_1 \bigcup B_2 \hookrightarrow B,$$

$$j_1 : B_1 \bigcap B_2 \hookrightarrow B_1 \quad \text{and} \quad j_2 : B_1 \bigcap B_2 \hookrightarrow B_2,$$

for objects $B_1 \bigcap B_2, B_1 \bigcup B_2 \in \mathscr{C}$. Moreover, the square

$$\begin{tikzcd} B_1 \bigcap B_2 \arrow[r, "j_2"] \arrow[d, "j_1"'] & B_2 \arrow[d, "i_2"] \\ B_1 \arrow[r, "i_1"'] & B_1 \bigcup B_2 \end{tikzcd}$$

is bicartesian in $\mathscr{C}$, and the pullback of the span $B_1 \hookrightarrow B \hookleftarrow B_2$ exists in $\mathscr{C}$ and is isomorphic to $B_1 \bigcap B_2$.

Proof. See [**Hir03**, Prop.12.2.3 and Th.12.2.6]. $\square$

1.2.2.2. *Relative Smallness.* Obtaining a functorial factorisation usually comes down to some objects being 'small' with respect to certain colimits, that is morphisms from those objects to the colimits factorise through the colimits cocones. There are several such smallness notions, which are influenced by the considered colimits and the desired properties of the factorisation. We recall briefly the smallness notions needed for the small object argument and the Bousfield-Smith cardinality argument.

The notion of κ-small relative objects, as in Definition 1.2.33, is modelled over factorisations in the category Set of small sets and their maps, and it captures smallness with respect to transfinite compositions, which are the colimits that arise in the small object argument, see §.1.2.2.3.

Let Z be a λ-sequence in Set, and let $f : A \to \mathtt{colim}\, Z$ be a map. Then, being able to factorise f through Z_ξ, for some ordinal $\xi < \lambda$, depends on the relation between the cardinality of A and the ordinal λ, provided the axiom of choice. A sufficient and necessary condition for such a factorisation to occur, for any such map f, is axiomatised in the following definition.

Definition 1.2.32. Let κ be a cardinal. An ordinal λ is said to be *κ-filtered* if it is a limit ordinal and for every set $A \subseteq \lambda$ with $|\, A \,| \leq \kappa$ one has $\mathtt{sup}\, A < \lambda$.

For a finite cardinal κ, one has $\mathtt{sup}\, A \in A$, and hence all limit ordinals are finitely filtered. However, when κ is an infinite cardinal, κ-filtered ordinals are limit ordinals that are greater than or equal to κ^+.

Definition 1.2.33. Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$, and let κ be a cardinal. An object $K \in \mathscr{C}$ is said to be *κ-small relative* to I, if for every κ-filtered ordinals λ, the induce map

$$\mathtt{colim}\, \mathscr{C}(K, Z) \to \mathscr{C}(K, \mathtt{colim}\, Z)$$

is an isomorphism, for every λ-sequence $Z : \lambda \to \mathscr{C}$ of morphisms in I. Also. K is said to be *small relative* to I if it is κ-small relative to I for some cardinal κ.

Small objects relative to I are closed under retracts and small colimits, see [**Hir03**, Prop.10.4.7 and Prop.10.4.8].

Example 1.2.34. A set A is $|A|$-small relative to any set of morphisms in Set.

Example 1.2.35. Finite CW-complexes are $\aleph_0$-small relative to the set of inclusions of CW-complexes, see [**Hir03**, Ex.10.4.3]. On the other hand, topological spaces that are small relative to $\mathsf{Mor}(\mathsf{Top})$ are precisely the discrete topological spaces, see [**Bou77**, Ex.4.4].

A stronger variation of relative κ-smallness, recalled in the following definition, constitutes the main technical ingredient for the Bousfield-Smith cardinality argument.

Definition 1.2.36. Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$, and let κ be a cardinal. An object $K \in \mathscr{C}$ is said to be *κ-compact relative* to I, if for every presented relative I-$\mathtt{cell}$ complex $(f : A \to B, P)$, every morphism $K \to B$ factorises through a presented relative I-$\mathtt{cell}$ subcomplex of $(f : A \to B, P)$ of size at most κ. K is said to be *compact relative* to I if it is κ-compact relative to I for some cardinal κ.

When I-$\mathtt{cell}$ consists of monomorphisms, compact objects relative to I are small relative to I, see [**Hir03**, Prop.10.8.7]. Also, they are closed under retracts and small colimits, see [**Hir03**, Prop.10.8.4 and Prop.10.8.8].

Example 1.2.37. Let I be the set of the canonical inclusions

$$\left\{|\partial^n| : |\partial\Delta^n| \hookrightarrow |\Delta^n| \mid n \geq 0\right\}$$

of the boundaries of the standard topological simplices, see §.1.2.4.1. Then, finite CW-complexes are $\aleph_0$-compact relative to I. Also, for an infinite cardinal κ, every CW-complex of size κ is κ-compact relative to I, see [**Hir03**, Ex.10.8.3].

1.2.2.3. *The Small Object Argument.* The small object argument became the standard technique to obtain functorial factorisations, since a countable version of which was first used by Quillen in [**Qui67**, §.II.3.Lem.3] to show that any continuous map of topological spaces admits a factorisation as a cofibration followed by a weak Serre fibration.

We believe that it is more profitable for a non-specialised reader if we present an explanation of the main idea behind the small object argument, which links the factorisation with small relative objects, before stating the argument in Definition 1.2.38.

Since fibrations (resp. weak fibrations) have the RLP with respect to weak cofibrations (resp. cofibration), the question of finding a functorial factorisation for a model structure on a category $\mathscr{C}$ follows from being able to find a functorial factorisation (σ, τ) with $\sigma(f) \in I$ and $\tau(f) \in I$-$\mathtt{inj}$ for every morphism f in $\mathscr{C}$, for a suitable set of morphisms I.

Let $\mathscr{C}$ be a cocomplete category, let I be a set of morphisms in $\mathscr{C}$ and let f be a morphism in $\mathscr{C}$. When f does not belong to I-$\mathtt{inj}$, there exists a commutative square

$$\begin{tikzcd} K \arrow[r, "e_0"] \arrow[d, "g"'] \arrow[dr, phantom, "E"] & X \arrow[d, "f"] \\ L \arrow[r, "e_1"'] & Y, \end{tikzcd}$$

with $g \in I$, called an *I-lifting problem* for f, that does not admit a lift. When the desired factorisation exists, the square admits a *partial lifting*

$$\begin{tikzcd}
K \arrow[r, "e_0"] \arrow[dd, "g"'] & X_0 \arrow[d, "\sigma(f)"] \arrow[dd, bend left, "f"] \\
& Z \arrow[d, "\tau(f)"] \\
L \arrow[r, "e_1"'] \arrow[ur, dotted, "h"] & Y,
\end{tikzcd}$$

for having $\tau(f) \in I\text{-}\mathtt{inj}$. Hence, given a morphism f, as a first step in finding the desired factorisation, one may look for a factorisation of f satisfying the necessary condition of providing partial liftings for all I-lifting problems for f. Let I_f be the set of I-lifting problems for f, and consider the solid commutative square

$$\begin{tikzcd}
\coprod\limits_{E \in I_f} K_E \arrow[r, "\coprod\limits_{E \in I_f} e_{0,E}"] \arrow[dd, "\coprod\limits_{E \in I_f} g_E"'] & X \arrow[d, dotted, "i"] \arrow[dd, bend left, "f"] \\
& Z \arrow[d, dotted, "p"] \\
\coprod\limits_{E \in I_f} L_E \arrow[r, "\coprod\limits_{E \in I_f} e_{1,E}"'] \arrow[ur, dotted, "h"] & Y
\end{tikzcd} \tag{2}$$

in $\mathscr{C}$. Notice that all I-lifting problems for f admit partial liftings when the dotted morphisms in (2) exist, and make the whole diagram commute. In particular, such dotted morphisms exist for (Z, i, h) being the pushout of the span

$$\begin{tikzcd}
\coprod\limits_{E \in I_f} L_E & \coprod\limits_{E \in I_f} K_E \arrow[l, "\coprod\limits_{E \in I_f} g_E"'] \arrow[r, "\coprod\limits_{E \in I_f} e_{0,E}"] & X,
\end{tikzcd}$$

and p being induced by the universal property of pushouts, in which case $i \in I\text{-}\mathtt{cell}$.

Since not all lifting problems for p arise from those of f, the morphism p does not necessarily belong to $I\text{-}\mathtt{inj}$. Let $X_0 \coloneqq X$, $X_1 \coloneqq Z$, $f_0 \coloneqq f$, $f_1 \coloneqq p$, and let $x_0 \coloneqq i$. Iterating the preceding argument yields a sequence $X_\bullet$ in $X \downarrow \mathscr{C}$ and a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "x_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "x_1"] \arrow[dl, "f_1"] & \cdots \arrow[r] & X_k \arrow[r, "x_k"] \arrow[lll, bend left=10, "f_k"] & X_{k+1}, \arrow[llll, bend left=15, "f_{k+1}"] \\
Y & & & &
\end{tikzcd}$$

for every integer $k \geq 0$. For every non-negative integer $k' \leq k$, one has $x_{k'} \in I\text{-}\mathtt{cell}$. Also, all I-lifting problems for $f_{k'}$ are solved for f_k for every $k > k'$, *i.e.* they admit partial liftings at f_k. Taking the colimit of the sequence $X_\bullet$ yields a factorisation $f_k = f_\omega \circ x_k^\omega$, where $x_k^\omega : X_k \to X_\omega$ is the colimit injection for every integer $k \geq 0$, and $f_\omega : X_\omega \to Y$ is the morphism induced by the universal property of colimits. Similarly, for an integer $k \geq 0$, all I-lifting problems for f_k are solved for f_ω, and $x_k^\omega \in I\text{-}\mathtt{cell}$ as $I\text{-}\mathtt{cell}$ is closed under transfinite compositions. Having established the initial, successor[1], and the limit cases, one may iterate the construction for any ordinal λ to obtain a factorisation $f_\xi = f_\lambda \circ x_\xi^\lambda$ in which $x_\xi^\lambda \in I\text{-}\mathtt{cell}$ and all I-lifting problems for f_ξ are solved for f_λ, for every ordinal $\xi < \lambda$.

The argument above shows that, for ordinals $\xi < \lambda$, the factorisation $f = f_\xi \circ x_\xi \circ \cdots \circ x_0$ can be refined into a factorisation $f = f_\lambda \circ x_\lambda \circ \cdots \circ x_0$ in which f_λ admits lifts for more I-lifting problems than f_ξ. The transfinite composition $x_\lambda \circ \cdots \circ x_{\xi+1} \circ x_\xi \circ \cdots \circ x_0$ always belongs to $I\text{-}\mathtt{cell}$, by it very construction. The main point of the small object argument is to show that, under smallness conditions on I, halting the aforementioned iterative process at a big enough ordinal λ guarantees that f_λ belongs to $I\text{-}\mathtt{inj}$.

When there exists some cardinal κ for which domains of morphisms in I are κ-small relative to $I\text{-}\mathtt{cell}$, it suffice to choose λ to be a κ-filtered ordinal for f_λ to belong to $I\text{-}\mathtt{inj}$. That is, for an I-lifting problem

$$\begin{tikzcd}
K \arrow[r, "e_0"] \arrow[d, "g"'] & X_\lambda \arrow[d, "f_\lambda"] \\
L \arrow[r, "e_1"'] & Y
\end{tikzcd}$$

for f_λ, since domains of morphisms in I are κ-small relative to $I\text{-}\mathtt{cell}$, e_0 factorise at X_ξ for an ordinal $\xi < \lambda$, inducing the solid I-lifting problem

$$\begin{tikzcd}
K \arrow[r, "e_{0,\xi}"] \arrow[ddd, "g"'] & X_\xi \arrow[d, "x_\xi"'] \arrow[ddd, bend left, "f_\xi"] \\
 & X_{\xi+1} \arrow[d, "x_{\xi+1}^\lambda"] \\
 & X_\lambda \arrow[d, "f_\lambda"'] \\
L \arrow[r, "e_1"'] \arrow[uur, dotted, "h_\xi"] & Y
\end{tikzcd}$$

for f_ξ, where $e_0 = x_\xi^\lambda \circ e_{0,\xi}$. Such I-lifting problem admits the dotted partial lifting by the very construction of x_ξ and $f_{\xi+1}$. Then, $h \coloneqq x_{\xi+1}^\lambda \circ h_\xi$ is a lift for E, and $f_\lambda \in I\text{-}\mathtt{inj}$.

Definition 1.2.38. Let $\mathscr{C}$ be a cocomplete category. A set I of morphisms in $\mathscr{C}$ is said to *admit the small object argument* if the domains of morphisms in I are small relative to $I\text{-}\mathtt{cell}$.

[1]The same argument used for integers greater than zero applies for any successor ordinal.

THEOREM 1.2.39 (The Transfinite Small Object Argument). Let $\mathscr{C}$ be a cocomplete category, and let I be a small set of morphisms in $\mathscr{C}$ that admits the small object argument. Then, there exists a functorial factorisation (σ, τ) on $\mathscr{C}$ such that every morphism f in $\mathscr{C}$ factorises as $f = \tau(f) \circ \sigma(f)$, where $\sigma(f) \in I\text{-}\mathtt{cell}$ and $\tau(f) \in I\text{-}\mathtt{inj}$.

PROOF. See [**Hir03**, Prop.10.5.16]. □

The sets $I\text{-}\mathtt{cof}$ and $I\text{-}\mathtt{cell}$ are defined through different concepts, the former is given by lifting properties whereas the latter is given by transfinite compositions and pushouts. Yet, when I admits the small object argument, I-cofibrations coincide with retracts of relative $I\text{-}\mathtt{cell}$ complexes, and I-cofibrant objects coincide with retracts of $I\text{-}\mathtt{cell}$ complexes, see [**Hir03**, Lem.10.5.25].

1.2.2.4. *Cellular Model Structures.*

DEFINITION 1.2.40. A *cofibrantly generated* model structure is a triple $(\mathscr{M}, I, J)$, where $\mathscr{M}$ is a model structure on a category $\mathscr{C}$, and I and J are small sets of morphisms in $\mathscr{C}$ that admit the small object argument, such that

(1) the set of fibrations in $\mathscr{M}$ coincides with the set $J\text{-}\mathtt{inj}$; and
(2) the set of weak fibrations in $\mathscr{M}$ coincides with the set $I\text{-}\mathtt{inj}$.

Then, I and J are called the sets of *generating cofibrations* and *generating weak cofibrations*, respectively. Moreover, a cofibrantly generated model structure $(\mathscr{M}, I, J)$ is said to be *cellular* if

(3) the domains and codomains of morphisms in I are compact relative to I;
(4) the domains of morphisms in J are small relative to $I\text{-}\mathtt{cell}$; and
(5) cofibrations in $\mathscr{M}$ are effective monomorphisms.

Since I and J admit the small object argument, cofibrations and weak cofibrations in $\mathscr{M}$ coincide with retracts of relative $I\text{-}\mathtt{cell}$ complexes and retracts of relative $J\text{-}\mathtt{cell}$ complexes, respectively, see [**Hir03**, Prop.11.2.1].

THEOREM 1.2.41 (Recognising Cellular Model Structures). Let $(\mathscr{C}, \mathscr{M})$ be a model category, and let I and J be small sets of morphisms in $\mathscr{C}$. Then, $(\mathscr{M}, I, J)$ is a cellular model structure on $\mathscr{C}$ if and only if

(1) the set of weak fibrations in $\mathscr{M}$ coincides with $I\text{-}\mathtt{inj}$;
(2) the set of fibrations in $\mathscr{M}$ coincides with $J\text{-}\mathtt{inj}$;
(3) the domains and codomains of morphisms in I are compact relative to I;
(4) the domains of morphisms in J are small relative to $I\text{-}\mathtt{cell}$; and
(5) relative $I\text{-}\mathtt{cell}$ complexes are effective monomorphisms.

PROOF. See [**Hir03**, Th.12.1.8]. □

The following theorem is the main result about cellular model structures that is particularly useful for Bousfield localisation, §.1.2.6, in addition the boundedness technical result presented in [**Hir03**, Prop.12.5.3].

Theorem 1.2.42. Let $(\mathscr{M}, I, J)$ be a cellular model structure on $\mathscr{C}$. Then, cofibrant objects in $\mathscr{M}$ are small relative to the set of all cofibrations.

Proof. See [**Hir03**, Th.12.4.3]. $\square$

1.2.3. Proper Model Categories. Gluing of (pointed) topological spaces, *i.e.* pushing-out along cofibrations, is invariant under weak equivalences, and hence it presents homotopy pushouts, see [**Hir03**, Th.13.1.10 and Th.13.3.10]. The same is not true for a general model category. Proper model structures guarantee that pushouts (resp. pullbacks) along cofibrations (resp. fibrations) present homotopy pushouts (resp. homotopy pullbacks).

Definition 1.2.43. A model structure $\mathscr{M}$ is said to be

(1) *left proper* if weak equivalences are closed under pushouts along cofibrations;
(2) *right proper* if weak equivalences are closed under pullbacks along fibrations; and
(3) *proper* if it is both left and right proper.

Left proper model categories admit homotopy pushouts given by pushouts of cofibrant factors of the spans in question. In particular, pushouts along cofibrations present homotopy pushouts, as seen below.

Lemma 1.2.44. Let $(\mathscr{C}, \mathscr{M})$ a model category. Assume that $\mathscr{M}$ is

- left proper, then for every solid commutative diagram

$$\begin{tikzcd}
U \arrow[rr, "i"] \arrow[dr, "f", "\sim"'] \arrow[dd, "j"'] & & V \arrow[dr, "g", "\sim"'] \arrow[dd, dashed, "i_V" near end] & \\
& U' \arrow[rr, "i'" near start] \arrow[dd, "j'" near start] & & V' \arrow[dd, dashed, "i_{V'}"] \\
W \arrow[rr, dashed, "i_W" near end] \arrow[dr, "h", "\sim"'] & & Z \arrow[dr, dotted, "h", "\sim"'] & \\
& W' \arrow[rr, dashed, "i_{W'}"] & & Z',
\end{tikzcd}$$

in which i and i' are cofibrations and the solid diagonal morphisms are weak equivalences, the induced morphism $Z \to Z'$ is a weak equivalence, for $Z := W \coprod_U V$ and $Z' := W' \coprod_{U'} V'$; and

- right proper, then for every solid commutative diagram

$$\begin{tikzcd}
Y \arrow[dr, "f", "\sim"'] & & X \arrow[ll, "p"'] \arrow[dr, "g", "\sim"'] & \\
& Y' & & X' \arrow[ll, "p'"'] \\
Z \arrow[uu, "q"] \arrow[dr, "h", "\sim"'] & & W \arrow[ll, dashed, "\pi_W"'] \arrow[uu, dashed, "\pi_X"'] \arrow[dr, dotted, "h", "\sim"'] \arrow[ul, phantom, "\lrcorner" very near start] & \\
& Z' \arrow[uu, "q'"] & & W' \arrow[ll, dashed, "\pi_{W'}"'] \arrow[uu, dashed, "\pi_{X'}"'] \arrow[ul, phantom, "\lrcorner" very near start]
\end{tikzcd},$$

  in which p and p' are fibrations and the solid diagonal morphisms are weak equivalences, the induced morphism $W \to W'$ is a weak equivalence, for $W := Z \times_Y X$ and $W' := Z' \times_{Y'} X'$.

Proof. See [**Hir03**, Prop.13.3.10]. □

1.2.4. Simplicial Model Categories. Homotopy categories of model structures are enriched over Kan-Quillen's homotopy category of simplicial sets $\mathcal{H}\mathsf{sSet}_{\mathrm{KQ}}$, and hence the latter influences the considered homotopy theory, see [**Hov99**, §.6]. Simplicial model structures allow one to take advantage of the well-studied simplicial methods to do homotopy in abstract categories. Moreover, simplicial methods have proven fruitful in other areas, for instance see Deligne's influential paper [**Del74**].

1.2.4.1. *The Simplex Category.* Let $\varDelta$ denote the *simplex category*, *i.e.* the skeleton of the category of non-empty finite ordered sets and order-preserving maps between them. The morphisms in $\varDelta$ are generated by the sets of coface and codegeneracy maps, recalled below.

For integers $n \geq 1$, $0 \leq i \leq n$, the i^{th}-*coface map* $\partial_n^i : [n-1] \to [n]$ is the unique injective such map in $\varDelta$ skipping the value i, *i.e.*

$$\partial_n^i(j) = \begin{cases} j & \text{for } j < i; \\ j+1 & \text{for } j \geq i. \end{cases}$$

Whereas, for integers $0 \leq i \leq n$, the i^{th}-*codegeneracy map* $\sigma_n^i : [n+1] \to [n]$ is the unique surjective such map in $\varDelta$ repeating the value i, *i.e.*

$$\sigma_n^i(j) = \begin{cases} j & \text{for } j \leq i; \\ j-1 & \text{for } j > i. \end{cases}$$

Any morphism $\mu : [m] \to [n]$ in $\varDelta$ is a composition of faces and degeneracies, and can be expressed uniquely as a composition

$$\mu = \partial_n^{i_1} \circ \partial_{n-1}^{i_2} \circ \ldots \circ \partial_{n-s+1}^{i_s} \circ \sigma_{m-t}^{j_t} \circ \sigma_{m-t+1}^{j_{t-1}} \circ \ldots \circ \sigma_{m-1}^{j_1},$$

where $n-m = s-t$, $n \geq i_1 > i_2 > \ldots > i_s \geq 0$, and $m-1 \geq j_1 > j_2 > \ldots > j_t \geq 0$, see [**May92**].

Let $\mathscr{C}$ be a category, the category of *simplicial objects* (resp. *cosimplicial objects*) in $\mathscr{C}$ is the functor category $\mathit{\Delta}^{\mathrm{op}}\mathscr{C} \coloneqq \mathsf{Fun}(\mathit{\Delta}^{\mathrm{op}}, \mathscr{C})$ (resp. $\mathit{\Delta}\mathscr{C} \coloneqq \mathsf{Fun}(\mathit{\Delta}, \mathscr{C})$). For a simplicial object $X : \mathit{\Delta}^{\mathrm{op}} \to \mathscr{C}$ and a cosimplicial object $R : \mathit{\Delta} \to \mathscr{C}$, it is conventional to denote $X([n])$ and $R([n])$ by X_n and R^n, respectively, and one usually writes

$$d_i^n \coloneqq X(\partial_n^i) \qquad , \qquad s_i^n \coloneqq X(\sigma_n^i) \qquad , \qquad d_n^i \coloneqq R(\partial_n^i) \qquad \text{and} \qquad s_n^i \coloneqq R(\sigma_n^i).$$

Simplicial Set. The category sSet of *simplicial* (*small*) *sets* is the category of simplicial objects in Set. It is common to denote the Yoneda embedding $\mathsf{h}_{-} : \mathit{\Delta} \to \mathsf{sSet}$ by Δ^{-}. Then, the Yoneda lemma implies the existence of a canonical isomorphism $X_{-} \cong \mathsf{sSet}(\Delta^{-}, X)$, for every simplicial set X. For an integer $n \geq 0$, the simplicial set Δ^n, represented by $[n]$, is called the *standard n-simplex*.

Since the category Set is bicomplete, so is sSet. In fact, sSet is Cartesian closed, as it admits an internal $\underline{\mathtt{Hom}}$-functor

$$\underline{\mathtt{Hom}} : \mathsf{sSet}^{\mathrm{op}} \times \mathsf{sSet} \to \mathsf{sSet},$$

called the *function complex*, which is given by $\underline{\mathtt{Hom}}(-, --)_{\bullet} \coloneqq \mathsf{sSet}(- \times \Delta^{\bullet}, --)$.

Let $\Delta_{\mathrm{top}}^{\bullet} : \mathit{\Delta} \to \mathsf{Top}$ be the *standard cosimplicial topological space*, given on an object $[n] \in \mathit{\Delta}$ by the *standard topological n-simplex*

$$\Delta_{\mathrm{top}}^n = \left\{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0\right\},$$

and on a morphism $\mu : [m] \to [n]$ in $\mathit{\Delta}$ by the map

$$\begin{array}{rccc} \Delta(\mu) : & \Delta_{\mathrm{top}}^m & \to & \Delta_{\mathrm{top}}^n \\ & u & \mapsto & t, \end{array} \quad \text{with } t_i = \sum_{j \in \mu^{-1}(i)} u_j \text{ for } 0 \leq i \leq n.$$

There exists a tensor-$\underline{\mathtt{Hom}}$ adjunction

$$|-| : \mathsf{sSet} \rightleftarrows \mathsf{Top} : \mathtt{Sing},$$

associated with the functor $\Delta_{\mathrm{top}}^{\bullet}$, as in Example A.3.8. The functor $\mathtt{Sing}$ is the *singular simplicial functor* given by $\mathtt{Sing}(-)_{\bullet} \coloneqq \mathsf{Top}(\Delta_{\mathrm{top}}^{\bullet}, -)$, whereas $|-|$ is the *geometric realisation functor* given by the left Kan extension of $\Delta_{\mathrm{top}}^{\bullet}$ along the Yoneda embedding $\Delta^{\bullet} : \mathit{\Delta} \hookrightarrow \mathsf{sSet}$, see [**Kan58**, §.2-3]. The geometric realisation functor can be given for a simplicial set X by the space

$$|\, X \,| = \left(\coprod_{n=0}^{\infty} (X_n \times \Delta_{\mathrm{top}}^n)\right) / \sim, \tag{3}$$

where $\sim$ is the smallest equivalence relation that identifies $(x_m, u) \in X_m \times \Delta_{\mathrm{top}}^m$ and $(x_n, t) \in X_n \times \Delta_{\mathrm{top}}^n$ whenever there exists a morphism $\mu : [m] \to [n]$ in $\mathit{\Delta}$ for which

$$x_m = X_\mu(x_n) \qquad \text{and} \qquad t = \Delta_{\mathrm{top}}^{\mu}(u),$$

and for a morphism $f : X \to Y$ of simplicial sets by $|\, f \,|([(x_n, t)]) = [(f_n(x_n), t)]$.

In addition to the standard simplices and the adjunction $|-| \dashv \mathtt{Sing}$, the boundary simplices, horns, and simplicial circle play an essential role in the homotopy theories of simplicial sets, and hence of simplicial (pre)sheaves. For an integer n, the *boundary of the standard n-simplex* $\partial\Delta^n$ is a simplicial subset in Δ^n given on an object $[p]$ by

$$\partial\Delta^n_p = \{\alpha : [p] \to [n] \mid \alpha \text{ is not surjective}\},$$

and the quotient $S^n \coloneqq {}^{\Delta^n}/_{\partial\Delta^n}$ is called the *n-simplicial circle*. For integers $0 \leq r \leq n$, the *r-horn* Λ^n_r is the smallest simplicial subset in Δ^n that contains $\partial^i_n \in \Delta^n_{n-1}$ for $0 \leq i \neq r \leq n$. Moreover, the geometric realisation of $\Delta^n, \partial\Delta^n$, and S^n is $\Delta^n_{\mathrm{top}}, \partial\Delta^n_{\mathrm{top}}$, and S^n_{top}, respectively.

Kan-Quillen's Model Structure on Simplicial Sets. The category of simplicial sets admits a left proper cellular model structure, cofibrantly generated by sets

$$\mathbf{I} \coloneqq \{\partial_{[n]} : \partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{Z}_{\geq 0}\} \qquad \text{and} \qquad \mathbf{J} \coloneqq \{\lambda^n_r : \Lambda^n_r \hookrightarrow \Delta^n \mid n \in \mathbb{Z}_{\geq 0}, 0 \leq r \leq n\}$$

of generating cofibrations and generating weak cofibrations, respectably, called *Kan-Quillen's model structure* on simplicial sets, see [**Hov99**, §.3]. In this model structure, a morphism of simplicial sets is a weak equivalence if and only if its geometric realisation is a weak equivalence of topological spaces, and is a cofibration if and only if it is an injection, see [**Qui67**, §.II.3.Prop.2]. Recall Example 1.2.25, the geometric realisation-singular simplicial adjunction is a Quillen equivalence between Kan-Quillen's model structure on simplicial sets and Quillen-Serre's model structure on topological spaces.

The fibrations in Kan-Quillen's model structure are morphisms of simplicial sets with the RLP with respect to $\mathbf{J}$, they are called *Kan fibrations*. A simplicial set K is said to be a *Kan complex* if the terminal morphism $K \to *$ is a Kan fibration.

Theorem 1.2.45 (Quillen). The geometric realisation of a Kan fibration is a Serre fibration.

Proof. See [**Hov99**, Cor.3.6.2]. □

However, there are far more Serre fibrations than realisations of Kan fibrations. For instance, every topological space is a Hurewicz fibrant object, and hence a Serre fibrant object.

On the other hand, a map of topological spaces is a Serre fibration if and only if its mapped to a Kan fibration by the singular simplicial functor, see [**May92**, §.III]. In particular, for every topological space X, the singular simplicial set $S(X)$ is a Kan complex. Hence, the endofunctor $S(|-|)$ is a fibrant approximation functor. However, the cardinality $S(|X|)$ is much bigger than the cardinality of X, and $S(|-|)$ is difficult to use in model structures induced by $\mathsf{sSet}_{\mathrm{KQ}}$.

Kan introduced a better behaved fibrant replacement functor in [**Kan57**], known as *Kan's* $\mathtt{Ex}^\infty$ *functor*, prior to Quillen's introduction of the notion of model structures. Although the construction of the functor $\mathtt{Ex}^\infty$ bears a resemblance to the small object argument, it relays heavily on the specificity of the simplex category $\mathit{\Delta}$, particularly on the barycentric subdivision functor, recalled below.

Let $\mathtt{Path} : \mathsf{Cat} \to \mathsf{Cat}$ be the path functor, that sends a small category to the poset of its nonempty (finite) paths, partially ordered by inclusion, and let $\mathtt{N} : \mathsf{Cat} \hookrightarrow \mathsf{sSet}$ be the fully faithful nerve functor, as in Example A.3.8.(2). The tensor-$\underline{\mathtt{Hom}}$ adjoint functors associated with the composite functor $\mathtt{N} \circ \mathtt{Path} \circ [-] : \mathit{\Delta} \to \mathsf{sSet}$, as in Example A.3.8, are called the *barycentric subdivision* $\mathtt{sd}$ and *Kan's extension functor* $\mathtt{Ex}$,

$$\begin{tikzcd}
\mathit{\Delta} \arrow[r, "{[-]}"] \arrow[d, "\Delta^-"'] & \mathsf{Cat} \arrow[r, shift left, "\mathtt{Path}"] \arrow[r, shift right, dotted, "\mathrm{id}_{\mathsf{Cat}}"'] & \mathsf{Cat} \arrow[r, "\mathtt{N}"] & \mathsf{sSet}, \arrow[llld, bend left, shift left, "\mathtt{Ex}"] \\
\mathsf{sSet} \arrow[urrr, bend right, shift left, "\mathtt{sd}"] & & &
\end{tikzcd}$$

see [**Kan57**]. Then, one has

$$\mathtt{Ex}(X)_\bullet \cong \mathsf{sSet}(\Delta^\bullet, \mathtt{Ex}(X)) \cong \mathsf{sSet}(\mathtt{sd}\,\Delta^\bullet, X).$$

In particular, the functor $\mathtt{Ex}$ preserves 0-simplices. On the other hand, the composite functor $\mathtt{N} \circ [-] : \mathit{\Delta} \to \mathsf{sSet}$ coincides with the dense Yoneda embedding Δ^-, and hence its tensor-$\underline{\mathtt{Hom}}$ adjoint functors may be given by the identity functors, see §.A.3.3. There exist natural transformations $i : [-] \rightleftarrows \mathtt{Path} \circ [-] : p$ given component-wise, for $[n] \in \mathit{\Delta}$, by

$$i_n(k) = [k] \in \mathtt{Path}([n]) \qquad \text{and} \qquad p_n(P) = \mathtt{codom}\, P \in [n],$$

for every $k \in [n]$ and $P \in \mathtt{Path}([n])$. One has $p \circ i = \mathrm{id}_{[-]}$, and hence i (resp. p) is a natural split monomorphism (resp. epimorphism). The natural transformation $j \coloneqq \mathtt{N}\,p$ lifts to a natural split epimorphism and a natural split monomorphism

$$j_* : \mathtt{sd} \to \mathrm{id}_{\mathsf{sSet}} \qquad \text{and} \qquad j^* : \mathrm{id}_{\mathsf{sSet}} \to \mathtt{Ex},$$

respectively. Moreover, the components of j_* and j^* are weak equivalences of simplicial sets, see [**Kan57**, Lem.7.4 and Lem.7.5]. Then, the functor $\mathtt{Ex}^\infty : \mathsf{sSet} \to \mathsf{sSet}$ is defined to be the colimit of the injective system

$$\mathrm{id}_{\mathsf{sSet}} \xrightarrow{j^*} \mathtt{Ex} \xrightarrow{j^*_{Ex}} \mathtt{Ex}^2 \xrightarrow{j^*_{Ex^2}} \mathtt{Ex}^3 \xrightarrow{j^*_{Ex^3}} \dots .$$

The induced natural monomorphism $R_{\mathrm{Kan}} : \mathrm{id}_{\mathsf{sSet}} \to \mathtt{Ex}^\infty$ is a fibrant replacement for the model category $\mathsf{sSet}_{\mathrm{KQ}}$, see [**Kan57**, Th.4.2].

1.2.4.2. *Simplicial Categories.* *Simplicial categories*[2] are sSet-enriched categories that are bitensored over sSet, *i.e.*

- for every object $X \in \mathscr{C}$, the functor $\mathtt{Map}(X,-) : \mathscr{C} \to \mathsf{sSet}$ has a sSet-enriched left adjoint
$$X \otimes - : \mathsf{sSet} \to \mathscr{C}; \text{ and}$$
- for every object $Y \in \mathscr{C}$, the functor $\mathtt{Map}(-,Y) : \mathscr{C}^{op} \to \mathsf{sSet}$ has a sSet-enriched left adjoint
$$Y^{-} : \mathsf{sSet} \to \mathscr{C}^{op}.$$

Thus, there exist isomorphisms

$$\mathtt{Map}_{\mathscr{C}}(X \otimes K, Y) \cong \mathtt{Map}_{\mathsf{sSet}}(K, \mathtt{Map}_{\mathscr{C}}(X,Y)) \cong \mathtt{Map}_{\mathscr{C}^{\mathrm{op}}}(Y^K, X) \cong \mathtt{Map}_{\mathscr{C}}(X, Y^K) \tag{4}$$

of simplicial sets, natural in $X, Y \in \mathscr{C}$ and $K \in \mathsf{sSet}$, giving rise to a sSet-adjunction

$$- \otimes K : \mathscr{C} \rightleftarrows \mathscr{C} : -^K,$$

for every simplicial set K, see [**GJ09**, §.II.Lem.2.2].

Remark 1.2.46. Every simplicial category $\mathscr{C}$ defines a category whose $\mathtt{hom}$-sets are given for every $X, Y \in \mathscr{C}$ by $\mathscr{C}(X,Y) = \mathtt{Map}(X,Y)_0$. It is called the *underlying category* of $\mathscr{C}$, and it is denoted by $\mathscr{C}_0$. The category $\mathscr{C}_0$ is said to admit the *simplicial structure* $(\mathtt{Map}, \otimes, -^{-})$. Moreover, the Yoneda lemma implies the existence of canonical isomorphisms

$$\mathtt{Map}(X,Y)_n \cong \mathtt{Map}(\Delta^n, \mathtt{Map}(X,Y))_0 \cong \mathscr{C}(X \otimes \Delta^n, Y).$$

for every $X, Y \in \mathscr{C}$ and $n \geq 0$.

The Category of Simplicial Objects. The prototypical example of simplicial categories is the category of simplicial objects in a bicomplete category, as it admits a canonical simplicial structure, called the *standard simplicial structure*, which is recalled below, see also [**GS07**, §.4.2].

Let $\mathscr{C}$ be a bicomplete category, let $\varphi : A \to B$ be a map of small sets, and let $f : X \to Y$ be a morphism in $\mathscr{C}$. The universal properties of coproducts and products induce canonical morphisms

$$\coprod_{\varphi} f : \coprod_{A} X \to \coprod_{B} Y \qquad \text{and} \qquad \prod_{\varphi} f : \prod_{B} X \to \prod_{A} Y,$$

which give rise to functors

$$\boxtimes : \Delta^{\mathrm{op}}\mathscr{C} \times \mathsf{sSet} \to \mathsf{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{C}) \quad \text{and} \quad -^{\boxtimes} : \mathsf{sSet}^{\mathrm{op}} \times \Delta^{\mathrm{op}}\mathscr{C} \to \mathsf{Fun}(\Delta^{\mathrm{op}} \times \Delta, \mathscr{C}),$$

given on objects, for a simplicial object X in $\mathscr{C}$ and a simplicial set K, by

$$(X \boxtimes K)([-],[--]) = \coprod_{K_{--}} X_{-} \qquad \text{and} \qquad X^{\boxtimes K}([-],[--]) = \prod_{K_{-}} X_{--},$$

[2] Some authors call a simplicial object in the category of small categories a simplicial category. To avoid confusion, we call a simplicial object in Cat a *simplicial small category*.

and similarly on morphisms. Then, precomposing with the diagonal functor $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ and taking ends, as in [**ML98**, §.IX.5], yield functors

$$\otimes : \Delta^{\mathrm{op}}\mathscr{C} \times \mathsf{sSet} \to \Delta^{\mathrm{op}}\mathscr{C} \qquad \text{and} \qquad \underline{\mathtt{Hom}}_{\Delta} : \mathsf{sSet}^{\mathrm{op}} \times \Delta^{\mathrm{op}}\mathscr{C} \to \mathscr{C}. \tag{5}$$

In particular, for a simplicial object X in $\mathscr{C}$ and a simplicial set K, ons has

$$(X \otimes K)_n = \coprod_{K_n} X_n \qquad \text{and} \qquad \underline{\mathtt{Hom}}_{\Delta}(K, X) = \int_{[n]\in\Delta} \prod_{K_n} X_n,$$

see [**GS09**, §.2]. In fact, $\underline{\mathtt{Hom}}_{\Delta}(-, X)$ is the right X-$\underline{\mathtt{Hom}}$ functor, as in Example A.3.8, *i.e.* it is the right Kan extension of X along the functor $\Delta^{-\mathrm{op}} : \Delta^{\mathrm{op}} \to \mathsf{sSet}^{\mathrm{op}}$, and hence $\underline{\mathtt{Hom}}_{\Delta}(\Delta^n, X) \cong X_n$ for every integer $n \geq 0$.

The tensor bifunctor of the standard simplicial structure on the category of simplicial objects $\Delta^{\mathrm{op}}\mathscr{C}$ is the bifunctor $\otimes$ in (5); the $\mathtt{Map}$-simplicial sets bifunctor

$$\mathtt{Map}_{\Delta^{\mathrm{op}}\mathscr{C}} : (\Delta^{\mathrm{op}}\mathscr{C})^{\mathrm{op}} \times \Delta^{\mathrm{op}}\mathscr{C} \to \mathsf{sSet}$$

is given by

$$\mathtt{Map}_{\Delta^{\mathrm{op}}\mathscr{C}}(-, --)_\bullet \coloneqq \Delta^{\mathrm{op}}\mathscr{C}(- \otimes \Delta^\bullet, --);$$

whereas the cotensor bifunctor $--^- : \mathsf{sSet}^{\mathrm{op}} \times \Delta^{\mathrm{op}}\mathscr{C} \to \Delta^{\mathrm{op}}\mathscr{C}$ is given by

$$(--^-)_\bullet = \underline{\mathtt{Hom}}_{\Delta}(- \times \Delta^\bullet, --).$$

Example 1.2.47. For $\mathscr{C} = \mathsf{Set}$, the bifunctors $\otimes$ and $\underline{\mathtt{Hom}}_{\Delta}$ coincide with the Cartesian product and the $\mathtt{hom}$-set bifunctors of simplicial sets, respectively. Also, the functor $\mathtt{Map}$ coincides with the cotensor bifunctor.

Simplicial Model Categories. Simplicial model categories are categories endued with a simplicial structure and a model structure, that are compatible in the sense of the following theorem.

Theorem 1.2.48 (Homotopy Lifting–Extension Theorem). Let $\mathscr{C}$ be a simplicial category, and let $\mathscr{M}$ be a model structure on the category $\mathscr{C}_0$. Then, the following statements are equivalent

- for every cofibration $i : U \to V$ and every fibration $p : X \to Y$ in $\mathscr{M}$, the induced morphism of simplicial sets

$$\mathtt{Map}(i, p) : \mathtt{Map}(V, X) \to \mathtt{Map}(U, X) \times_{\mathtt{Map}(U,Y)} \mathtt{Map}(V, Y)$$

  is a Kan fibration; moreover, if either of i or p is a weak equivalence in $\mathscr{M}$, then so is $\mathtt{Map}(i, p)$;
- for every cofibration $i : U \to V$ in $\mathscr{M}$ and every cofibration $j : K \to L$ in $\mathsf{sSet}_{\mathrm{KQ}}$, the induced morphism

$$i \,\square\, j : U \otimes L \coprod_{U \otimes K} V \otimes K \to V \otimes L$$

is a cofibration in $\mathscr{M}$; moreover, if either of i or j is a weak equivalence in $\mathscr{M}$ or $\mathsf{sSet}_{\mathrm{KQ}}$, respectively, then so is $i \,\square\, j$; and

- for every fibration $p : X \to Y$ in $\mathscr{M}$ and every cofibration $j : K \to L$ in $\mathsf{sSet}_{\mathrm{KQ}}$, the induced morphism

$$p^j : X^L \to X^K \times_{Y^K} Y^L$$

is a fibration in $\mathscr{M}$; moreover, if either of p or j is a weak equivalence in $\mathscr{M}$ or $\mathsf{sSet}_{\mathrm{KQ}}$, respectively, then so is p^j.

Proof. See [**Hir03**, Prop.9.3.7]. □

Definition 1.2.49. A *simplicial model category* is a pair $(\mathscr{C}, \mathscr{M})$, where $\mathscr{C}$ is a simplicial category and $\mathscr{M}$ is a model structure on the category $\mathscr{C}_0$, that satisfies any of the equivalent statements in Theorem 1.2.48.

Example 1.2.50. The pair of the Kan-Quillen's model structure and the standard simplicial structure endues the category of simplicial sets with a simplicial model structure, see [**Hir03**, Ex.9.1.13].

In a simplicial model categories $(\mathscr{C}, \mathscr{M})$, the model structure can be determined on the level of simplicial sets. That is, a morphism i is a cofibration (resp. weak cofibration) in $\mathscr{M}$ if and only if the morphism $\mathtt{Map}(i,p)$ is a weak Kan fibration for every weak fibration (resp. fibration) p in $\mathscr{M}$, see [**Hir03**, Prop.9.4.4]. The dual statement holds for (weak) fibrations.

Proposition 1.2.51 (Detecting Weak Equivalences). Let $(\mathscr{C}, \mathscr{M})$ be a simplicial model category, and let $f : X \to Y$ be a morphism in the category $\mathscr{C}_0$. Then, the following statements are equivalent

- the morphism f is a weak equivalence in $\mathscr{M}$;
- for every fibrant object Z in $\mathscr{M}$, the induced morphism

$$Q(f)^* : \mathtt{Map}\big(Q(Y), Z\big) \to \mathtt{Map}\big(Q(X), Z\big)$$

is a weak equivalence in $\mathsf{sSet}_{\mathrm{KQ}}$, where Q is a cofibrant approximation functor for $\mathscr{M}$; and

- for every cofibrant object W in $\mathscr{M}$, the induced morphism

$$R(f)_* : \mathtt{Map}\big(W, R(X)\big) \to \mathtt{Map}\big(W, R(Y)\big)$$

is a weak equivalence in $\mathsf{sSet}_{\mathrm{KQ}}$, where R is a fibrant approximation functor for $\mathscr{M}$.

Proof. See [**Hir03**, Th.9.7.4]. □

1.2.5. Monoidal Model Categories. Most model categories one is interested in admit monoidal structures that are compatible with the model structures, in an analogous manner to Theorem 1.2.48. Such monoidal structures descent to monoidal structures on the homotopy categories.

DEFINITION 1.2.52. A *symmetric monoidal model structure* on a category $\mathscr{C}$ is a pair $(\mathscr{M}, \mathscr{S})$ of a model structure $\mathscr{M}$ and a closed symmetric monoidal structure $\mathscr{S} = (\otimes, \mathbb{1}, \underline{\mathtt{Hom}}, \psi, \alpha, \lambda, \rho)$ such that

(1) for cofibrations $i : U \to V$ and $j : K \to L$ in $\mathscr{M}$, the pushout product

$$i \,\square\, j : U \otimes L \coprod_{U \otimes K} V \otimes K \to V \otimes L$$

is a cofibration in $\mathscr{M}$; moreover, if either of i or j is a weak equivalence in $\mathscr{M}$, then so is $i \,\square\, j$; and

(2) for the cofibrant replacement morphism $Q(\mathbb{1}) \to \mathbb{1}$, the morphism

$$X \otimes Q(\mathbb{1}) \to X \otimes \mathbb{1} \cong_{\rho} X$$

is a weak equivalence in $\mathscr{M}$, for every cofibrant object X in $\mathscr{M}$.

Then, the triple $(\mathscr{C}, \mathscr{M}, \mathscr{S})$ is called a *symmetric monoidal model category.*

In a symmetric monoidal model category $(\mathscr{C}, \mathscr{M}, \mathscr{S})$, the adjunction $- \otimes X \dashv \underline{\mathtt{Hom}}(X, -)$ is a Quillen adjunction, for every cofibrant object X in $\mathscr{M}$. Hence, the homotopy category $\mathcal{H}\mathscr{C}$ admits a canonical closed symmetric monoidal structure, with a monoidal product and internal $\underline{\mathtt{Hom}}$ bifunctors given by the derived functors $\otimes^{\mathbf{L}}$ and $\mathbf{R}\underline{\mathtt{Hom}}$, respectively, see [**Hov99**, Th.4.3.2]. Axiom (2), in Definition 1.2.52, is needed for the derived monoidal structure to exist. However, when $\mathbb{1}$ is cofibrant in $\mathscr{M}$, (2) follows from (1), see [**Hov99**, §.4.2] and [**SS03a**, §.3.1].

EXAMPLE 1.2.53. The pair of the Kan-Quillen's model structure and the closed Cartesian monoidal structure endues the category of simplicial sets with a symmetric monoidal model structure, see [**Hov99**, Prop.4.2.8].

A monoidal Quillen adjunction between symmetric monoidal model categories is defined so that it induces a strong monoidal functor between the homotopy categories.

DEFINITION 1.2.54. A *weak monoidal Quillen adjunction* $F : (\mathscr{C}, \mathscr{M}, \mathscr{S}) \rightleftarrows (\mathscr{D}, \mathscr{N}, \mathscr{T}) : G$ between symmetric monoidal model categories is a Quillen adjunction $F \dashv G$, in which F is an oplax monoidal functor, such that

(1) for cofibrant objects $X, Y \in \mathscr{C}$, the oplax morphism

$$F(X \otimes Y) \to F(X) \otimes F(Y)$$

is a weak equivalence in $\mathscr{N}$; and

(2) the composition of the morphism $F(Q(\mathbb{1}_{\mathscr{C}})) \to F(\mathbb{1}_{\mathscr{C}})$, induced by the cofibrant replacement morphism, with the oplax morphism $F(\mathbb{1}_{\mathscr{C}}) \to \mathbb{1}_{\mathscr{D}}$ is a weak equivalence in $\mathscr{N}$, see [**SS03a**, §.3.2].

Moreover, a *strong monoidal Quillen adjunction* is a Quillen adjunction $F \dashv G$, in which F is strong monoidal, and the morphism $F(Q(\mathbb{1}_{\mathscr{C}})) \to F(\mathbb{1}_{\mathscr{C}})$, induced by the cofibrant replacement morphism, is a weak equivalence in $\mathscr{N}$, see [**Hov99**, Def.4 .2.16].

For a weak monoidal Quillen adjunction $F \dashv G$, since F is oplax monoidal, one finds that G is lax monoidal.

THEOREM 1.2.55. Let $F : (\mathscr{C}, \mathscr{M}, \mathscr{S}) \rightleftarrows (\mathscr{D}, \mathscr{N}, \mathscr{T}) : G$ be a weak monoidal Quillen adjunction between symmetric monoidal model categories. Then, the left derived functor $\mathbf{L}F$ is strong monoidal.

PROOF. See the proof of [**Hov99**, Th.4.3.3]. □

PROPOSITION 1.2.56. Let $(\mathscr{C}, \mathscr{M}, \mathscr{S})$ be a symmetric monoidal model category, whose unit $\mathbb{1}$ is a cofibrant object in $\mathscr{M}$ and coincides with the terminal object $*$ of $\mathscr{C}$. Then, the pointed category $\mathscr{C}_\bullet$ admits a closed symmetric monoidal structure $\mathscr{S}_\bullet$, making $(\mathscr{C}_\bullet, \mathscr{M}_\bullet, \mathscr{S}_\bullet)$ into a symmetric monoidal model category, with a unit $\mathbb{1}_\bullet = \mathbb{1}_+ = (* \coprod *, *)$, a smash product $\wedge$ given for pointed objects $(X, x), (Y, y) \in \mathscr{C}_\bullet$ by the pushout of the span

$$\begin{tikzcd} X \coprod Y \arrow[r, "(\mathrm{id}_X \otimes y) \coprod (x \otimes \mathrm{id}_Y)"] \arrow[d] & X \otimes Y \\ * & \end{tikzcd}$$

in $\mathscr{C}$, with the canonical base-point, and an internal $\underline{\mathtt{Hom}}_\bullet$ given for pointed objects $(X, x), (Y, y) \in \mathscr{C}_\bullet$ by the pullback of the cospan

$$\begin{tikzcd} \underline{\mathtt{Hom}}(*, Y) & \underline{\mathtt{Hom}}(X, Y) \arrow[l, "\underline{\mathtt{Hom}}(x, \mathrm{id}_Y)"'] \\ * \arrow[u, "\underline{\mathtt{Hom}}(*, y)"] & \end{tikzcd}$$

in $\mathscr{C}$, with the canonical base-point induced by the point y and the morphism $X \to *$.

PROOF. See [**Hov99**, Prop.4.2.9]. □

EXAMPLE 1.2.57. The smash product endues the pointed model category $\mathsf{sSet}_{\bullet,\mathrm{KQ}}$ with a symmetric monoidal model structure, with pointed internal $\underline{\mathtt{Hom}}_\bullet$ given for pointed simplicial sets (X, x_0) and (Y, y_0) by

$$\underline{\mathtt{Hom}}_\bullet\big((X, x_0), (Y, y_0)\big)_n = \Big(\mathsf{sSet}_\bullet\big((X, x_0) \wedge \Delta^n_+, (Y, y_0)\big), y_0 \circ p_X\Big),$$

for every $[n] \in \mathit{\Delta}$, where $p_X : X \to \Delta^0$ is the terminal such morphism in sSet, see [**Hov99**, Cor.4.2.10].

1.2.5.1. *Modules over Monoidal Model Categories.* A module over a symmetric monoidal model category $(\mathscr{C}, \mathscr{M}, \mathscr{S})$ is a model category whose homotopy category admits an action of the monoidal homotopy category $\mathcal{H}\mathscr{C}$.

DEFINITION 1.2.58. Let $(\mathscr{C}, \mathscr{M}, \mathscr{S})$ be a symmetric monoidal model category. A $\mathscr{C}$-(*right*) *model structure* is a pair of a $(\mathscr{C}, \mathscr{S})$-enriched category $\mathscr{D}$ that is bitensored over $(\mathscr{C}, \mathscr{S})$ and a model structure $\mathscr{N}$ on $\mathscr{D}$, such that

(1) for every cofibration $i : U \to V$ in $\mathscr{N}$ and every cofibration $j : K \to L$ in $\mathscr{M}$, the induced morphism
$$i \,\square\, j : U \otimes L \coprod_{U \otimes K} V \otimes K \to V \otimes L$$
is a cofibration in $\mathscr{N}$; moreover, if either of i or j is a weak equivalence in $\mathscr{N}$ or $\mathscr{M}$, respectively, then $i \,\square\, j$ is a weak equivalence in $\mathscr{N}$; and

(2) for the cofibrant replacement morphism $Q(\mathbb{1}_{\mathscr{C}}) \to \mathbb{1}_{\mathscr{C}}$, the morphism $X \otimes Q(\mathbb{1}_{\mathscr{C}}) \to X \otimes \mathbb{1}_{\mathscr{C}}$ is a weak equivalence in $\mathscr{N}$, for every cofibrant object X in $\mathscr{N}$.

Then, the pair $(\mathscr{D}, \mathscr{N})$ is called a $(\mathscr{C}, \mathscr{M}, \mathscr{S})$-(*right*) *model category.*

When no confusion arises, we may abuse notations and refer to $(\mathscr{C}, \mathscr{M}, \mathscr{S})$-model categories by $\mathscr{C}$-model categories.

PROPOSITION 1.2.59. Let $(\mathscr{C}, \mathscr{M}, \mathscr{S})$ be a symmetric monoidal model category, and let $(\mathscr{D}, \mathscr{N})$ be a $\mathscr{C}$-model category. Then, the homotopy category $\mathcal{H}\mathscr{D}$ is $\mathcal{H}\mathscr{C}$-enriched and bitensored over $\mathcal{H}\mathscr{C}$.

PROOF. See [**Hov99**, Th.4.3.4]. □

Every symmetric monoidal model category $(\mathscr{C}, \mathscr{M}, \mathscr{S})$ is a right model category over itself. Moreover, if $(\mathscr{D}, \mathscr{N})$ is a $\mathscr{C}$-model category, then $(\mathscr{D}_\bullet, \mathscr{N}_\bullet)$ is a $\mathscr{C}_\bullet$-model category, with respect to the symmetric monoidal model structure $(\mathscr{M}_\bullet, \mathscr{S}_\bullet)$, see [**Hov99**, Prop.4.2.19].

EXAMPLE 1.2.60. A simplicial model category is a $\mathsf{sSet}_{\mathrm{KQ}}$-model category, and hence a pointed simplicial model category is a $\mathsf{sSet}_{\bullet,\mathrm{KQ}}$-model category, see [**Hov99**, p.114].

EXAMPLE 1.2.61. Let $\mathscr{C}$ be a pointed simplicial model category, and hence a $\mathsf{sSet}_\bullet$-model category, the derived adjoint functors
$$- \wedge^{\mathbf{L}} (S^1, 0) : \mathcal{H}\mathscr{C} \rightleftarrows \mathcal{H}\mathscr{C} : \mathbf{R}\underline{\mathrm{Hom}}\big((S^1, 0), -\big)$$
are called the *suspension* and *loop* functors, and usually denoted by $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$, respectively.

For every object $X \in \mathscr{C}$, the object $\boldsymbol{\Sigma} X$ (resp. $\boldsymbol{\Omega} X$) admits a canonical cogroup (resp. group) structure, and it can be used to compute the homotopy groups of the

$\texttt{Map}$-simplicial sets, and hence detect weak equivalences in $\mathscr{C}$, similar to pointed topological spaces, see [**Hov99**, §.6.1].

1.2.6. Left Bousfield Localisation of Model Structures. A localisation of a model category presents a localisation of its homotopy category. Let $\mathcal{H}$ be a homotopy category presented by a model category $(\mathscr{C}, \mathscr{M})$. The localisation of the model structure $\mathscr{M}$ with respect to a set S of morphisms in $\mathscr{C}$, if it exists, is a 'minimal' model structure $\mathscr{M}_S$ on $\mathscr{C}$ whose weak equivalences contain weak equivalences of $\mathscr{M}$ and morphisms of S. Then, the homotopy category $\mathcal{H}_S$ of $(\mathscr{C}, \mathscr{M}_S)$ is a localisation of the homotopy category $\mathcal{H}$ with respect to the image of S in $\mathcal{H}$, *i.e.* $\mathcal{H}_S \cong \mathtt{L}_S \mathcal{H}$.

Definition 1.2.62. Let $(\mathscr{C}, \mathscr{M})$ be a model category, and let S be a set of morphisms in $\mathscr{C}$.

(1) A *left localisation* of $\mathscr{M}$ with respect to S, if it exists, is a pair $(\mathbf{L}_S \mathscr{M}, \eta_S)$, where $\mathbf{L}_S \mathscr{M}$ is a model structure on $\mathscr{C}$, and $\eta_S : (\mathscr{C}, \mathscr{M}) \to (\mathscr{C}, \mathbf{L}_S \mathscr{M})$ is a left Quillen functor such that
   (a) the total left derived functor $\mathtt{L}\,\eta_S$ takes morphisms in $\mathtt{L}_{\mathscr{M}}(S)$ to isomorphisms; and
   (b) any left Quillen functor $\theta : (\mathscr{C}, \mathscr{M}) \to (\mathscr{D}, \mathscr{N})$, for which the total left derived functor $\mathtt{L}\,\theta : \mathcal{H}\mathscr{C} \to \mathcal{H}\mathscr{D}$ takes morphisms in $\mathtt{L}_{\mathscr{M}}(S)$ to isomorphisms in $\mathcal{H}\mathscr{D}$, factorises uniquely through η_S.
(2) A *right localisation* of $\mathscr{M}$ with respect to S, if it exists, is a pair $(\mathbf{R}_S \mathscr{M}, \epsilon_S)$, where $\mathbf{R}_S \mathscr{M}$ is a model structure on $\mathscr{C}$, and $\epsilon_S : \mathscr{C} \to \mathbf{R}_S \mathscr{C}$ is a right Quillen functor such that
   (a) the total right derived functor $\mathtt{R}\,\epsilon_S$ takes morphisms in $\mathtt{L}_{\mathscr{M}}(S)$ to isomorphisms; and
   (b) any right Quillen functor $\theta : (\mathscr{C}, \mathscr{M}) \to (\mathscr{D}, \mathscr{N})$, for which the total right derived functor $\mathtt{R}\theta : \mathcal{H}\mathscr{C} \to \mathcal{H}\mathscr{D}$ takes morphisms in $\mathtt{L}_{\mathscr{M}}(S)$ to isomorphisms in $\mathcal{H}\mathscr{D}$, factorises uniquely through ϵ_S.

In the sequel, we restrict ourself to left localisations, yet most of what comes next can be easily dualised for right localisations, see [**Hir03**, Ch.3 and Ch.5]. Also, we find it convenient to restrict ourselves to simplicial model categories. Readers interested in the general argument are encouraged to consult [**Hir03**].

Proposition 1.2.51 imposes restrictions on the fibrant objects and the weak equivalences of the localised model structure, when it exists, giving rise to the following definition.

Definition 1.2.63. Let $(\mathscr{C}, \mathscr{M})$ be a simplicial model category, let Q be a cofibrant approximation functor for $\mathscr{M}$, and let S be a set of morphisms in $\mathscr{C}$. An object $Z \in \mathscr{C}$

is called an *S-local object* if Z is a fibrant object in $\mathscr{M}$ for which the induced morphism

$$Q(i)^* : \mathtt{Map}(Q(V), Z) \to \mathtt{Map}(Q(U), Z).$$

is a weak equivalence in $\mathsf{sSet}_{\mathrm{KQ}}$, for every morphism $i : U \to V$ in S. Whereas, a morphism $i : U \to V$ in $\mathscr{C}$ is called an *S-local equivalence* if the induced morphism

$$Q(i)^* : \mathtt{Map}(Q(V), Z) \to \mathtt{Map}(Q(U), Z)$$

is a weak equivalence in $\mathsf{sSet}_{\mathrm{KQ}}$, for every S-local object $Z \in \mathscr{C}$.

Both weak equivalences in $\mathscr{M}$ and morphisms in S are *S-weak equivalences*. Moreover, S-weak equivalences satisfy the two-out-of-three property and are closed under retracts, see [**Hir03**, Prop.3.2.3 and Prop.3.2.4].

S-local objects (resp. S-weak equivalences) are intended to form the fibrant objects (resp. weak equivalences) in the localised model structure, if it exists. Their role may be better illustrated through the following theorem.

Theorem 1.2.64. Let $F : (\mathscr{C}, \mathscr{M}) \rightleftarrows (\mathscr{D}, \mathscr{N}) : G$ be a Quillen adjunction between simplicial model categories, and let S be a set of morphisms in $\mathscr{C}$. Then, the following statements are equivalent

(1) the total left derived functor $\mathbf{L}F : \mathcal{H}\mathscr{C} \to \mathcal{H}\mathscr{D}$ takes morphisms in $\mathtt{L}_{\mathscr{M}}(S)$ to isomorphisms in $\mathcal{H}\mathscr{D}$;
(2) the functor F takes the cofibrant replacements of morphisms in S to weak equivalences in $\mathscr{N}$;
(3) the functor G takes fibrant objects in $\mathscr{N}$ to S-local objects in $\mathscr{C}$; and
(4) the functor F takes S-local equivalences between cofibrant objects to weak equivalences in $\mathscr{N}$.

Proof. See [**Hir03**, Th.3.1.6]. □

Definition 1.2.65. Let $(\mathscr{C}, \mathscr{M})$ be a simplicial model category, and let S be a set of morphisms in $\mathscr{C}$. A *left Bousfield localisation* of $\mathscr{M}$ with respect to S, if it exists, is a model structure $\mathbf{L}_S\mathscr{M}$ on $\mathscr{C}$ whose weak equivalences, cofibrations, and fibrations are S-weak equivalences, cofibrations in $\mathscr{M}$, and S-local fibrations, respectively; where *S-local fibrations* are the morphisms in $\mathscr{C}$ with the RLP with respect to S-weak cofibrations.

When a left Bousfield localisation $\mathbf{L}_S\mathscr{M}$ exists, the pair $(\mathbf{L}_S\mathscr{M}, \mathrm{id}_{\mathscr{C}})$ forms a left localisation of $\mathscr{M}$ with respect to S, see [**Hir03**, Th.3.3.19]. In which case, the fibrant objects in $\mathbf{L}_S\mathscr{M}$ are S-local objects, but the converse does not always hold. However, when in addition $\mathscr{M}$ is left proper, fibrant objects in $\mathbf{L}_S\mathscr{M}$ coincide with S-local objects, see [**Hir03**, Prop.3.4.1].

In general, the main obstacle to the existence of a left Bousfield localisation is the existence of its functorial factorisations, particularly the factorisation as S-weak cofibrations and S-local fibrations. Since the main tool used to produce functorial factorisations is the small object argument, it is natural to look for a small set J_S of S-weak cofibrations that both admits the small object argument and for which the set J_S-$\mathtt{inj}$ coincides with the set of S-local fibrations. In cellular model categories, inclusions of I-$\mathtt{cell}$ complexes play an essential role in obtaining such a small set J_S, and that is mainly due to their set-theoretic-like nature, seen in Proposition 1.2.31.

Lemma 1.2.66. Let $(\mathscr{M}, I, J)$ be a left proper cofibrantly generated simplicial model structure on a category $\mathscr{C}$, let S be a set of morphisms in $\mathscr{C}$, and let $p : X \to Y$ be a fibration in $\mathscr{M}$. Then, the morphism p is an S-local fibration if and only if it has the RLP with respect to the set of all S-weak inclusions of I-$\mathtt{cell}$ complexes.

Proof. See [**Hir03**, Prop.4.5.1 and Lem.4.5.2]. □

Since the small object argument applies for small sets, the set of all S-weak inclusions of I-$\mathtt{cell}$ complexes needs to be further refined to a small set. The set of all isomorphism classes of S-weak inclusions of I-$\mathtt{cell}$ complexes of size at most κ is a small set, let $J_{S,\kappa}$ be a set of its representatives, and hence a small set. In general, the set $J_{S,\kappa}$-$\mathtt{inj}$ does not coincide with the set of S-local fibrations. However, the Bousfield-Smith cardinality argument shows that there exists an accessible cardinal for which the two sets coincide, see [**Hir03**, §.4.5].

Theorem 1.2.67. Let $(\mathscr{M}, I, J)$ be a left proper cellular simplicial model structure on a category $\mathscr{C}$, and let S be a small set of morphisms in $\mathscr{C}$. Then, a left Bousfield localisation $\mathbf{L}_S\mathscr{M}$ exists and fibrant objects in $\mathbf{L}_S\mathscr{M}$ coincide with S-local objects. Moreover, $\mathbf{L}_S\mathscr{M}$ is a left proper cellular simplicial model structure with generating cofibrations I and generating weak cofibrations $J_S \coloneqq J_{S,\kappa}$, for a large enough accessible cardinal κ.

Proof. See [**Hir03**, Th.4.1.1 and §.4.5-6]. □

1.3. Stable Homotopy Categories

Stable homotopy theories, in which the suspension functor is quasi-inverted, are better behaved and admit richer structures, allowing for more invariants compared to unstable homotopy theories. Stabilising a homotopy category can be obtained in different ways, some of which have advantages over others. The universal stabilisation is given by the Spanier-Whitehead construction, and its main advantage is being applied on the level of homotopy categories. However, in general, it does not produce cocomplete categories. On the other hand, the categories of spectra and symmetric spectra are constructed on the level of model categories, with the latter inheriting symmetric monoidal structures. Since monoidal structures are particularly interesting for us, we restrict our attention to symmetric spectra.

1.3.1. Symmetric Spectra. Let $(\mathscr{C},\mathscr{S})$ be a closed symmetric monoidal category, and let $T\in\mathscr{C}$. A *T-spectrum* is a pair (X,e) of a sequence $X=\{X_n \mid n\in\mathbb{Z}_{\geq 0}\}$ of *terms* in $\mathscr{C}$ and a sequence $e=\{e_n : X_n\otimes T\to X_{n+1}\mid n\in\mathbb{Z}_{\geq 0}\}$ of *assembly morphisms* in $\mathscr{C}$. A *morphism of T-spectra* $f:(X,e)\to(Y,c)$ is a sequence $f=\{f_n : X_n\to Y_n \mid n\in\mathbb{Z}_{\geq 0}\}$ of morphisms in $\mathscr{C}$ that commute with the assembly morphisms.

A *T-symmetric spectrum* (X,τ,e) is a T-spectrum (X,e) with a left action τ of the symmetric group Σ_n on X_n, for every integer $n\geq 0$, such that the composition

$$e_{n+p-1}\circ\cdots\circ\left(e_{n+1}\otimes\mathrm{id}_{T^{\otimes(p-2)}}\right)\circ\left(e_n\otimes\mathrm{id}_{T^{\otimes(p-1)}}\right): X_n\otimes T^{\otimes p}\to X_{n+1}\otimes T^{\otimes(p-1)}\to\cdots\to X_{n+p-1}\otimes T\to X_{n+p}$$

is $\Sigma_{n+p}\supset(\Sigma_n\times\Sigma_p)$-equivariant for every pair of integers $p,n\geq 0$, where Σ_p acts on $T^{\otimes p}$ by permutation of factors. A *morphism of T-symmetric spectra* $f:(X,\tau,e)\to(Y,\varsigma,d)$ is a morphism of T-spectra $f:(X,e)\to(Y,d)$ whose n^{th}-term is Σ_n-equivariant. Denote the category of T-symmetric spectra in $\mathscr{C}$ by $\mathsf{Spt}^{\Sigma}(\mathscr{C},T)$. There exists a full embedding

$$\boldsymbol{\Sigma}_T^{\infty}:\mathscr{C}\hookrightarrow\mathsf{Spt}^{\Sigma}(\mathscr{C},T),$$

sending an object X in $\mathscr{C}$ to its *T-symmetric suspension spectrum* $\boldsymbol{\Sigma}_T^{\infty}X=(X,X\otimes T,X\otimes T^{\otimes 2},X\otimes T^{\otimes 3},\cdots)$, with the canonical left action and the identity assemble morphisms.

Alternatively, the category of T-symmetric spectra in $\mathscr{C}$ can by given as a subcategory of symmetric sequences in $\mathscr{C}$. This makes it easier to endue it with a monoidal structure. Let Σ denote the skeleton of the groupoid of finite sets with isomorphisms between them, and denote objects of Σ by the cardinality of their representative. The category Σ is symmetric monoidal with a canonical monoidal product

$$\oplus:\Sigma\times\Sigma\to\Sigma,$$

induced by the canonical injection $\Sigma_p\times\Sigma_q\subseteq\Sigma_{p+q}$, whose unit is 0. Let the category of *symmetric sequences* in $\mathscr{C}$ be the functor category $\mathscr{C}^{\Sigma}$. When $\mathscr{C}$ is (co)complete, the category $\mathscr{C}^{\Sigma}$ is (co)complete with level-wise (co)limits. Moreover, since $\mathscr{C}$ is a closed symmetric monoidal category, so is the category $\mathscr{C}^{\Sigma}$. To see that, one may rerun a variant of the argument used to established the standard simplicial structure on the category of simplicial objects. The monoidal product $\otimes$ and internal $\underline{\mathtt{Hom}}$ in $\mathscr{C}$ define functors

$$\boxtimes:\mathscr{C}^{\Sigma}\times\mathscr{C}^{\Sigma}\to\mathscr{C}^{\Sigma\times\Sigma}\qquad\text{and}\qquad\mathtt{hom}:\left(\mathscr{C}^{\Sigma}\right)^{\mathrm{op}}\times\mathscr{C}^{\Sigma\times\Sigma}\to\mathscr{C}^{\Sigma^{\mathrm{op}}\times\Sigma\times\Sigma},$$

given for $X,Y\in\mathscr{C}^{\Sigma}$ and $Z\in\mathscr{C}^{\Sigma\times\Sigma}$ by

$$(X\boxtimes Y)_{-,--}=X_{-}\otimes Y_{--}\qquad\text{and}\qquad\mathtt{hom}(X,Z)_{-,--,\bullet}=\underline{\mathtt{Hom}}(X_{-},Z_{\bullet,--}),$$

and similarly on morphisms. One may be tempted to define the monoidal product on $\mathscr{C}^{\Sigma}$ through a precomposition with the diagonal functor $\Sigma\to\Sigma\times\Sigma$, *i.e.* $(X\otimes Y)_n=$

$X_n \times Y_n$ for every $X, Y \in \mathscr{C}^\Sigma$. Although such definition yields a monoidal product on $\mathscr{C}^\Sigma$, it does not give rise to a closed monoidal structure in general, due to the lack of a dense canonical functor $\Sigma^{\mathrm{op}} \to \mathscr{C}^\Sigma$, compared to $\Delta^\bullet : \varDelta \to \mathsf{sSet}$. Let $\underline{\mathtt{Hom}}_\boxtimes : \mathscr{C}^{\Sigma^{\mathrm{op}}} \times \mathscr{C}^{\Sigma\times\Sigma} \to \mathscr{C}^\Sigma$ be the functor induced by taking the end of $\mathtt{hom}$, *i.e.*

$$\underline{\mathtt{Hom}}_\boxtimes(X, Z)_n = \int_{p\in\Sigma} \underline{\mathtt{Hom}}(X_p, Z_{n,p}),$$

for every $X \in \mathscr{C}^\Sigma$ and $Z \in \mathscr{C}^{\Sigma\times\Sigma}$. Then, the monoidal product $\otimes$ and internal $\underline{\mathtt{Hom}}_\otimes$ on $\mathscr{C}^\Sigma$ are given by

$$- \otimes -- \coloneqq \mathtt{Lan}_\oplus - \boxtimes -- \qquad \text{and} \qquad \underline{\mathtt{Hom}}_\otimes(-, --) \coloneqq \underline{\mathtt{Hom}}_\boxtimes(-, -- \circ \oplus),$$

i.e.

$$(X \otimes Y)_n = \int^{(p,q)\in\oplus\downarrow n} X_p \otimes Y_q \qquad \text{and} \qquad \underline{\mathtt{Hom}}_\otimes(X, Y)_n = \int_{p\in\Sigma} \underline{\mathtt{Hom}}(X_p, Y_{n+p}),$$

for every $X, Y \in \mathscr{C}^\Sigma$, with a unit $(\mathbb{1}, \varnothing, \varnothing, \cdots) \in \mathscr{C}^\Sigma$, see [**May04**, §.6]. Expanding the monoidal product coend formula yields

$$(X \otimes Y)_n = \coprod_{p+q=n} \Big(\big(\coprod_{\Sigma_n} X_p \otimes Y_q \big) / \Sigma_p \times \Sigma_q \Big),$$

Hence, a morphism of symmetric sequences $X \otimes Y \to Z$ can be realised by $(\Sigma_p \times \Sigma_q)$-equivariant morphisms $X_p \otimes Y_q \to Z_{p+q}$, for every $p, q \in \Sigma$, see [**Jar00**, p.506]. The symmetric suspension spectrum $\mathbf{\Sigma}_T^\infty \mathbb{1} = (\mathbb{1}, T, T^{\otimes 2}, T^{\otimes 3}, \cdots)$ is a commutative monoid in $\mathscr{C}^\Sigma$, and hence the *category of T-symmetric spectra* $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$ can be defined equivalently as the category of right $\mathbf{\Sigma}_T^\infty \mathbb{1}$-module in the category of symmetric sequences $\mathscr{C}^\Sigma$, see [**HSS99**, Prop.2.2.1]. Thus, the category of T-symmetric spectra in $\mathscr{C}$ is a bicomplete category, with level-wise (co)limits. Also, it forms a closed monoidal category with a monoidal product

$$X \wedge Y \coloneqq \mathtt{coeq}\big(X \otimes \mathbf{\Sigma}_T^\infty \mathbb{1} \otimes Y \rightrightarrows X \otimes Y \big),$$

where the horizontal morphisms are given by the action of $\mathbf{\Sigma}_T^\infty \mathbb{1}$ on X and Y, see [**Hov01**, §.7].

For every integer $n \in \Sigma$, there exists an *n-evaluation functor*

$$\mathtt{Ev}_n : \mathsf{Spt}^\Sigma(\mathscr{C}, T) \to \mathscr{C}$$

that sends a T-spectrum to its n^{th}-term, and a natural transformation

$$\sigma_n : \mathtt{Ev}_n \otimes T \to \mathtt{Ev}_{n+1},$$

given by the assembly morphisms, *i.e.* $\sigma_{n,(X,\tau,e)} = e_n$. The functor $\mathtt{Ev}_n$ admits a left adjoint

$$F_n : \mathscr{C} \to \mathsf{Spt}^\Sigma(\mathscr{C}, T),$$

given on an object $X \in \mathscr{C}$ by the monoidal product

$$F_n(X) \coloneqq (\varnothing, \varnothing, \cdots, \coprod_{\Sigma_n} X, \varnothing, \cdots) \otimes \mathbf{\Sigma}_T^\infty \mathbb{1}$$

in $\mathscr{C}^\Sigma$, where the nontrivial term in $(\varnothing, \varnothing, \cdots, \coprod_{\Sigma_n} X, \varnothing, \cdots)$ is centred in level n. Then, in particular, one has $F_0 = \mathbf{\Sigma}_T^\infty$. Also, $\mathtt{Ev}_n$ has a right adjoint

$$M_n : \mathscr{C} \to \mathsf{Spt}^\Sigma(\mathscr{C}, T),$$

given on an object $X \in \mathscr{C}$ by the internal $\underline{\mathtt{Hom}}_\otimes$

$$M_n(X) \coloneqq \underline{\mathtt{Hom}}_\otimes \big(\mathbf{\Sigma}_T(\mathbb{1}), (*, *, \cdots, \prod_{\Sigma_n} X, *, \cdots) \big),$$

in $\mathscr{C}^\Sigma$, where the nontrivial term in $(*, *, \cdots, \prod_{\Sigma_n} X, *, \cdots)$ is centred in level n.

1.3.1.1. *Model Structures on Symmetric Spectra.* Throughout this section, fix a left proper cellular symmetric monoidal model category $(\mathscr{C}, \mathscr{M}, \mathscr{S})$, with sets I and J of generating cofibration and weak cofibration, respectively. The model structure $(\mathscr{M}, I, J)$ induces several symmetric monoidal model structures on the category of T-symmetric spectra in $\mathscr{C}$.

The Projective Model Structure. The functor category $\mathscr{C}^\Sigma$ admits canonical symmetric monoidal model structures induced level-wise from $(\mathscr{C}, \mathscr{M})$. These structures are inhabited by the category $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$.

DEFINITION 1.3.1. A morphism f of T-symmetric spectra is called a *level* (*weak*) *equivalence*, a *level cofibration*, or a *level fibration*, if the morphism f_n is a weak equivalence, a cofibration, or a fibration for every $n \in \Sigma$. Moreover, f is said to be an *injective fibration* (resp. *projective cofibration*) if it has the RLP with respect to level weak cofibrations (resp. LLP with respect to level weak fibrations).

THEOREM 1.3.2. There exists a left proper model cellular structure on $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$ whose weak equivalence, cofibrations, and fibrations are level equivalences, projective cofibrations, and level fibrations, respectively; it is called the *projective model structure* on $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$. Moreover, the sets $I_T \coloneqq \bigcup_{n \in \Sigma} F_n(I)$ and $J_T \coloneqq \bigcup_{n \in \Sigma} F_n(J)$ are the generating cofibration and weak cofibration, respectively, for the *projective model structure.*

PROOF. See [**Hov01**, Th.8.2]. $\square$

Denote the resulting model category by $\mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathtt{proj}}$. Since the functor $\mathtt{Ev}_n$ takes level equivalences and level fibrations to equivalences and fibrations in $\mathscr{M}$, it is a right Quillen functor, and hence F_n is a left Quillen functor, for every $n \in \Sigma$.

THEOREM 1.3.3. The pair of the projective model structure and the closed symmetric monoidal structure endues the category $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$ with a symmetric monoidal model structure. Moreover, the functor $- \wedge \mathbf{\Sigma}_T^\infty T : \mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathtt{proj}} \to \mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathtt{proj}}$ is a left Quillen functor.

PROOF. See [**Hov01**, Th.8.3]. □

The Stable Model Structure. The left Quillen functor $- \wedge \mathbf{\Sigma}_T^\infty T$, in Theorem 1.3.3, is not necessarily a left Quillen equivalence. Since the model category $\mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathtt{proj}}$ is left proper and cellular, it admits Bousfield localisations with respect to small subsets of its morphisms; and the functor $- \wedge \mathbf{\Sigma}_T^\infty T$ becomes a left Quillen equivalence for an adequate such localisation, see Theorem 1.3.5.

For every $X \in \mathscr{C}$ and $n \in \Sigma$, let

$$\hat{\zeta}_n^{QX} : QX \otimes T = \coprod_{\Sigma_1} (QX \otimes T) \longrightarrow \coprod_{\Sigma_{n+1}} (QX \otimes T) = \mathtt{Ev}_{n+1} F_n QX$$

be the canonical such morphism induced by the inclusion $\Sigma_1 \subset \Sigma_{n+1}$, where Q is the cofibrant replacement functor of $(\mathscr{C}, \mathscr{M})$, and let

$$\zeta_n^{QX} : F_{n+1}(QX \otimes T) \longrightarrow F_n QX$$

be the preimage of $\hat{\zeta}_n^{QX}$ along the adjunction $F_{n+1} \dashv \mathtt{Ev}_{n+1}$.

DEFINITION 1.3.4. The *stable model structure* on $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$ is a left Bousfield localisation of the projective model structure on $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$ with respect to the small set

$$\zeta_T(I) \coloneqq \left\{ \zeta_n^{QX} : F_{n+1}(QX \otimes T) \longrightarrow F_n QX \mid X \in \mathtt{dom}(I) \bigcup \mathtt{codom}(I) \right\}.$$

Denote the resulting model category by $\mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathrm{stab}}$. The $\zeta_T(I)$-weak equivalences and $\zeta_T(I)$-fibrations are called *T-stable (weak) equivalences* and *T-stable fibrations*, respectively.

THEOREM 1.3.5. Assume that the domains of morphisms of I are cofibrant objects in $\mathscr{M}$. Then, the pair of the stable model structure and the closed symmetric monoidal structure endues $\mathsf{Spt}^\Sigma(\mathscr{C}, T)$ with a symmetric monoidal model structure. Moreover, the functor $- \wedge \mathbf{\Sigma}_T^\infty T : \mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathrm{stab}} \to \mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathrm{stab}}$ is a let Quillen equivalence, *i.e.* the total left derived functor $- \wedge^{\mathbf{L}} \mathbf{\Sigma}_T^\infty T$ is an autoequivalence of categories on $\mathcal{H}\mathsf{Spt}^\Sigma(\mathscr{C}, T)_{\mathrm{stab}}$, with a quasi-inverse $\mathbf{R}\underline{\mathtt{Hom}}_\wedge(\mathbf{\Sigma}_T^\infty T, -)$.

PROOF. See [**Hov01**, Th.8.10 and Th.8.11]. □

The total derived functors $- \wedge^{\mathbf{L}} \mathbf{\Sigma}_T^\infty T$ and $\mathbf{R}\underline{\mathtt{Hom}}_\wedge(\mathbf{\Sigma}_T^\infty T, -)$ are called the *T-suspension functor* and the *T-loop functor*, respectively, and usually denoted by $\mathbf{\Sigma}_T$ and $\mathbf{\Omega}_T$, respectively.

The T-suspension functor has a simple description on the stable homotopy category. Define the *left shift* functor $s_l : \mathsf{Spt}^\Sigma(\mathscr{C}, T) \to \mathsf{Spt}^\Sigma(\mathscr{C}, T)$ by

$$s_l(-) \coloneqq - \wedge F_1(\mathbb{1}),$$

i.e. for $X \in \mathsf{Spt}^{\Sigma}(\mathscr{C}, T)$, one has $s_l(X)_0 = \varnothing$ and $s_l(X)_n = \coprod_{\Sigma_n} X_{n-1}/\Sigma_{n-1}$, for every integer $n \geq 1$, with the canonical Σ_n-action. Let the *right shift* functor $s_r : \mathsf{Spt}^{\Sigma}(\mathscr{C}, T) \to \mathsf{Spt}^{\Sigma}(\mathscr{C}, T)$ be the right adjoint of s_l given by

$$s_r(-) \coloneqq \underline{\mathtt{Hom}}_{\wedge}(F_1(\mathbb{1}), -),$$

i.e. for $X \in \mathsf{Spt}^{\Sigma}(\mathscr{C}, T)$, one has $s_r(X)_n = X_{n+1}$, for every integer $n \geq 0$, with the Σ_n-action induced by the canonical inclusion $\Sigma_n \hookrightarrow \Sigma_{n+1}$.

Theorem 1.3.6. There exists a natural isomorphism between the total derived functors $- \wedge^{\mathbf{L}} \mathbf{\Sigma}_T^{\infty} T$ and $\mathbf{R}\, s_r$. Also, there exists a natural isomorphism between the total derived functors $\mathbf{R}\underline{\mathtt{Hom}}_{\wedge}(\mathbf{\Sigma}_T^{\infty} T, -)$ and $\mathbf{L}\, s_l$.

Proof. See [**Hov01**, Th.8.10]. □

The Additive Structure of the Stable Homotopy Category. The stable homotopy category $\mathcal{H} \coloneqq \mathcal{H}\mathsf{Spt}^{\Sigma}(\mathscr{C}, T)_{\mathrm{stab}}$ is an additive category. The abelian group structure on the set $\mathcal{H}(X, Y)$ is induced formally by the abelian cogroup structure on $X \cong \mathbf{\Sigma}^2(\mathbf{\Omega}^2 X)$, for every $X, Y \in \mathcal{H}$, see [**Hel68**, Cor.7.2]. The finite coproduct on $\mathsf{Spt}^{\Sigma}(\mathscr{C}, T)$ induces a finite coproduct on $\mathcal{H}$, making $\mathcal{H}$ into an additive category, by [**ML98**, §.VIII.2]. Moreover, the suspension functor $\mathbf{\Sigma}_T$ is an additive endofunctor.

The pair $(\mathcal{H}, \mathbf{\Sigma}_T)$ admits a canonical triangulated structure. Therefore, we devote the next section to the study of triangulated categories, and we recall such canonical triangulated structure in Example 1.4.2.

1.4. Triangulated Categories

The study of derived categories led to the formulation of the notion of triangulated categories, which is usually credited to Verdier (1963). The central notion in triangulated categories is that of distinguished triangles, which play the role of short exact sequences in abelian categories. Also, distinguished triangles induce long exact sequences of abelian groups allowing the development of the conventional diagram machinery that is frequently used in homological algebra on abelian categories.

1.4.1. Preliminaries of Triangulated Categories. Fix an additive category with a suspension $(\mathscr{A}, \mathbf{\Sigma})$, *i.e.* $\mathbf{\Sigma} : \mathscr{A} \to \mathscr{A}$ is an additive autoequivalence of categories, and let $\phi_{\mathbf{\Sigma}}$ be the adjunction induced by the natural isomorphism $\mathrm{id}_{\mathscr{A}} \Rightarrow \mathbf{\Sigma}^{-1} \circ \mathbf{\Sigma}$.

A *triangle* in $(\mathscr{A}, \mathbf{\Sigma})$ is a diagram

$$T: \qquad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathbf{\Sigma} X$$

in $\mathscr{A}$. When convenient, we refer to the triangle T by the triangle (u, v, w) on (X, Y, Z), and when no confusion may arise that is shortened to the triangle (u, v, w). A triangle (u, v, w) is called a *candidate triangle* if $v \circ u = 0$, $w \circ v = 0$, and $\mathbf{\Sigma}(u) \circ w = 0$. A *morphism of triangles* is a natural transformation between them.

Definition 1.4.1. A *triangulated structure* on $(\mathscr{A},\mathbf{\Sigma})$ is a set $\mathscr{T}$ of triangles in $(\mathscr{A},\mathbf{\Sigma})$, called *distinguished triangles* of $\mathscr{T}$, subject to the following axioms

TR1 (a) every morphism $u : X \to Y$ in $\mathscr{A}$ can be completed into a triangle (u, v, w) on $\big(X, Y, \mathsf{Cone}(u)\big)$ in $\mathscr{T}$, for some object $\mathsf{Cone}(u) \in \mathscr{A}$, which is called a *cone* of u in $\mathscr{T}$;

(b) for every object $A \in \mathscr{A}$ the triangle $(\mathrm{id}_A, 0, 0)$ on $(A, A, \mathbf{0})$ belongs to $\mathscr{T}$;

(c) every triangle in $(\mathscr{A},\mathbf{\Sigma})$ that is isomorphic to a triangle in $\mathscr{T}$ is in $\mathscr{T}$;

TR2 a triangle (u, v, w) is in $\mathscr{T}$ if and only if the triangle $(v, w, -\mathbf{\Sigma}u)$ belongs to $\mathscr{T}$;

TR3 given two triangles T and T' in $\mathscr{T}$ on (X, Y, Z) and (X', Y', Z'), respectively, and two compatible morphisms $f : X \to X'$ and $g : Y \to Y'$, *i.e.* morphisms that make the solid diagram

$$\begin{tikzcd}
T: \arrow[d, dotted] & X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dotted, "h"'] & \mathbf{\Sigma}X \arrow[d, dotted, "\mathbf{\Sigma}f"'] \\
T': & X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \mathbf{\Sigma}X',
\end{tikzcd}$$

commute, the pair (f, g) can be completed into a morphism of triangles $(f, g, h) : T \to T'$; and

TR4 (Verdier's axiom) for triangles (u_1, v_1, w_1), (u_2, v_2, w_2), and (u_3, v_3, w_3) in $\mathscr{T}$, in which $u_2 = u_3 \circ u_1$, there exists a triangle (u_4, v_4, w_4) in $\mathscr{T}$ that makes the diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "u_2"] \arrow[dr, "u_1"'] & & Z \arrow[rr, bend left, "v_3"] \arrow[dr, "v_2"] & & X' \arrow[rr, bend left, "w_4"] \arrow[dr, "w_3"] & & \mathbf{\Sigma}Z' \\
& Y \arrow[ur, "u_3"] \arrow[dr, "v_1"'] & & Y' \arrow[ur, dotted, "v_4"] \arrow[dr, "w_2"] & & \mathbf{\Sigma}Y \arrow[ur, "\mathbf{\Sigma}v_1"'] & \\
& & Z' \arrow[ur, dotted, "u_4"] \arrow[rr, bend right, "w_1"'] & & \mathbf{\Sigma}X \arrow[ur, "\mathbf{\Sigma}u_1"'] & &
\end{tikzcd} \tag{6}$$

commute.

The triple $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ is said to be a *triangulated category.* A triangulated functor $F : (\mathscr{B},\mathbf{\Omega},\mathscr{S}) \to (\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ between triangulated categories is an additive functor that preserves the triangulated structure, *i.e.* it is an additive functor $F : \mathscr{B} \to \mathscr{A}$ with a natural isomorphism $\Phi : F \circ \mathbf{\Omega} \to \mathbf{\Sigma} \circ F$ which sends a triangle (u, v, w) in $\mathscr{S}$ to a triangle $(F(u), F(v), \Phi_X \circ F(w))$ in $\mathscr{T}$.

Verdier's axiom asserts the existence of the dotted morphisms making the whole diagram commute not knowing a priori that the solid diagram is commutative, which imposes a strong restriction on the set of distinguished triangles. Verdier's axiom is usually called the *octahedron axiom* based on a possible rearrangement of the diagram (6) into a three-dimensional octahedron, see [**Wei94**, p.375]; whereas the diagram (6) is due to May, given in [**May05**].

EXAMPLE 1.4.2. Let $(\mathscr{C},\mathscr{M})$ be a left proper cellular monoidal model category, let $T \in \mathscr{C}$, and let $\mathcal{H} \coloneqq \mathcal{H}\mathsf{Spt}^{\Sigma}(\mathscr{C},T)_{\mathrm{stab}}$ be the stable homotopy category of T-symmetric spectra, as in §.1.3.1.1. Then, the additive category with suspension $(\mathcal{H},\mathbf{\Sigma}_T)$ admits a canonical triangulated structure, called the *cofibration-triangulation*, whose distinguished triangles are T-symmetric suspension spectra of cofibre sequences in $\mathscr{C}$, see [**Hel68**, §.9].

EXAMPLE 1.4.3. Let $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ be a triangulated category, and let $\mathscr{A} \to \tilde{\mathscr{A}}$ be its Karoubian envelope, see [**BS01**, Def.1.2]. Then, the additive structure, the suspension functor, and the triangulated structure descend to $\tilde{\mathscr{A}}$, making it into a triangulated category, see [**BS01**, Th.1.7].

DEFINITION 1.4.4. Let $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ be a triangulated category, let $\mathscr{B}$ be an abelian category, and let $H : \mathscr{A} \to \mathscr{B}$ be a functor. The functor H is said to be *homological* on $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ if it maps every triangle (u,v,w) in $\mathscr{T}$ to a half exact sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

in $\mathscr{A}$. A *cohomological* functors is defined dually.

REMARK 1.4.5. The corepresentable and representable functors $\mathtt{h}^A$ and $\mathtt{h}_A$ are homological and cohomological functors, respectively, for every object $A \in \mathscr{A}$. This, in particular, shows that $\mathtt{Cone}(u)$ is both a weak cokernel for u and a weak kernel for $\mathbf{\Sigma} u$, for every morphism u in $\mathscr{A}$.

A candidate triangle is said to be *exact* if it is mapped to a half exact sequence by every corepresentable and representable functor.

1.4.1.1. *The Diagram Lemmas.* A triangulated structure encodes enough information to provide the main diagram machinery one usually uses in homological algebra.

LEMMA 1.4.6 (5-Lemma). Let $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ be a triangulated category, and let $(f,g,h) : T \to T'$ be a morphism of exact triangles, such that both f and g are isomorphisms. Then, h is an isomorphism.

PROOF. A direct result of the Yoneda lemma and the 5-Lemma for abelian groups, see [**Nee01**, Ex.1.1.15 and Prop.1.1.20]. □

As a result of the 5-Lemma, the completion of a morphism into a distinguish triangle in a triangulated structure is unique up to non-canonical isomorphisms. That implies that a morphism is an isomorphism if and only if its cone is isomorphic to the zero object, see [**Nee01**, Cor.1.2.6].

Direct consequences of the 5-Lemma also include:

- a direct sum of triangles is distinguished if and only if its summands are, see [**Nee01**, Prop.1.2.1 and Prop.1.2.3]; and
- a direct sum of objects is a cone of the zero morphism from the desuspension of one of the object to the other, see [**Nee01**, Cor.1.2.7 and Lem.1.2.8].

Lemma 1.4.7 (3×3-Lemma). Let $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$ be a triangulated category. Then, every commutative square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

in $\mathscr{A}$ can be completed into the diagram below

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, dashed, "v"] \arrow[d, "g"'] & Z \arrow[r, dashed, "w"] \arrow[d, dotted, "h"'] & \mathbf{\Sigma} X \arrow[d, "\mathbf{\Sigma} f"'] \\ X' \arrow[r, "u'"] \arrow[d, dashed, "f'"'] & Y \arrow[r, dashed, "v'"] \arrow[d, dashed, "g'"'] & Z \arrow[r, dashed, "w'"] \arrow[d, dashed, "h'"'] & \mathbf{\Sigma} X' \arrow[d, dashed, "\mathbf{\Sigma} f'"'] \\ X'' \arrow[r, dotted, "u''"] \arrow[d, "f''"'] & Y \arrow[r, dashed, "v''"] \arrow[d, "g''"'] & Z \arrow[r, dashed, "w''"] \arrow[d, dashed, "h''"'] & \mathbf{\Sigma} X'' \arrow[d, dashed, "\mathbf{\Sigma} f''"'] \\ \mathbf{\Sigma} X \arrow[r, "\mathbf{\Sigma} u"] & \mathbf{\Sigma} Y \arrow[r, dashed, "\mathbf{\Sigma} v"] & \mathbf{\Sigma} Z \arrow[r, dashed, "\mathbf{\Sigma} w"] & \mathbf{\Sigma}^2 X, \end{tikzcd}$$

that is commutative everywhere apart from the bottom right square, which is anticommutative, with all horizontal and vertical triangles being distinguished in $\mathscr{T}$.

Proof. See [**May05**, Lem.1.7]. $\square$

1.4.2. Homotopy (Co)limits in Triangulated Categories. In general homotopy theory, one relays on the extra machinery of the presenting category to present homotopy (co)limits, which are usually difficult to realise directly on a homotopy category. However, triangulated categories, and hence stable homotopy categories, possess intrinsic and easy-to-express homotopy pullbacks and pushouts. However, that comes at the expense of not having well-behaved homotopy (co)limits, which means that such homotopy (co)limits are not functorial.

1.4.2.1. *Homotopy Cartesian Squares.* Recall that in an abelian category, a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ X' \arrow[r, "f'"'] & Z' \end{tikzcd} \tag{7}$$

is Cartesian or cocartesian if and only if the sequence

$$\mathbf{0} \longrightarrow X \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} X' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z' \longrightarrow 0$$

is left exact or right exact, respectively. Since distinguished triangles play the role of exact sequences in a triangulated category, one distinguishes a commutative square (7) that gives rise to a distinguished triangle

$$X \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} X' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z \xrightarrow{\delta} \mathbf{\Sigma} X,$$

for some morphism $\delta : Z \to \mathbf{\Sigma} X$, and calls it a *homotopy Cartesian square*. This notion is self-dual, due to axiom **TR2** in Definition 1.4.1. In a homotopy Cartesian square (7), the pair (f, g) is called a *homotopy pullback* of (f', g'), and (f', g') is called a *homotopy pushout* of (f, g).

Both the homotopy pullbacks and homotopy pushouts always exist in any triangulated category, and they are unique up to non-canonical isomorphisms. This allows the construction of Verdier's Quotient of a triangulated category with respect to a triangulated subcategory, as in §.1.4.3.

When a triangulated category admits certain (co)limits, it provide more general homotopy (co)limits, see [**Nee01**, §.1.6].

1.4.3. Thick Subcategories and Verdier's Quotient. Verdier's quotient provides a universal machinery to contract triangulated subcategories, through localising the ambient triangulated category with respect to an associated set of morphisms.

Definition 1.4.8. Let $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$ be a triangulated category. A *full triangulated subcategory* in $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$ is a full subcategory $i : \mathscr{B} \hookrightarrow \mathscr{A}$ that is closed with respect to the suspension and desuspension, such that $(\mathscr{B}, \mathbf{\Sigma}_{\mathscr{B}})$ is endued with a triangulated structure with respect to which the functor i is triangulated.

The restriction of the definition to full subcategories guarantees the uniqueness of the triangulated substructure on $(\mathscr{B}, \mathbf{\Sigma}_{\mathscr{B}})$ in $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$, denoted by $\mathscr{T}_{\mathscr{B}}$, which given by the set of all triangles in $\mathscr{B}$ that are distinguished in $\mathscr{T}$. Hence, we may abuse the notation and refer to $(\mathscr{B}, \mathbf{\Sigma}_{\mathscr{B}}, \mathscr{T}_{\mathscr{B}})$ by $\mathscr{B}$.

In practice, one is interested in full triangulated subcategories that are closed under isomorphisms in the ambient category, called *strict full triangulated subcategories*, see [**Tho97**] and [**Nee01**, Def.1.5.1]. In such categories, stability with respect to the suspension becomes a consequence of the other conditions.

Example 1.4.9. Let $F : (\mathscr{B}, \Omega, \mathscr{S}) \to (\mathscr{A}, \Sigma, \mathscr{T})$ be a triangulated functor, and let $\mathtt{ker}\, F$ be the strict kernel of the additive functor F, *i.e.* $\mathtt{ker}\, F$ is the full subcategory in $\mathscr{B}$ with a set of objects

$$\mathtt{Ob}(\mathtt{ker}\, F) = \left\{ B \in \mathscr{B} \mid F(B) \cong \mathbf{0}_{\mathscr{A}} \right\}.$$

Then, $\mathtt{ker}\, F$ is a strict full triangulated subcategory in $(\mathscr{B}, \Omega, \mathscr{S})$, see [**Nee01**, Lem.2.1.4].

In fact, $\mathtt{ker}\, F$ is not only additive, but also contains all summands of its objects, see [**Nee01**, Lem.2.1.5]. A strict full triangulated subcategory that contains all summands of its objects is called a *thick triangulated subcategory*, see [**Ric89**].

Definition 1.4.10. Let $\mathscr{B}$ be a strict full triangulated subcategory in a triangulated category $(\mathscr{A}, \Sigma, \mathscr{T})$. A *Verdier's quotient* of $(\mathscr{A}, \Sigma, \mathscr{T})$ with respect to $\mathscr{B}$, if it exists, is an initial universal triangulated functor

$$Q_{\mathscr{B}} : (\mathscr{A}, \Sigma, \mathscr{T}) \to ({}^{\mathscr{A}}/_{\mathscr{B}}, \Sigma_{\mathscr{A}/\mathscr{B}}, \mathscr{T}_{\mathscr{A}/\mathscr{B}})$$

for which $\mathscr{B} \subset \mathtt{ker}\, Q_{\mathscr{B}}$.

Remark 1.4.11. Using Lemma 1.4.6, one shows that a triangulated functor F from $\mathscr{A}$ contracts objects of $\mathscr{B}$ if and only if it inverts morphisms in the set

$$W_{\mathscr{B}} \coloneqq \left\{ u \in \mathsf{Mor}(\mathscr{A}) \mid \mathsf{Cone}(u) \in \mathscr{B} \right\}.$$

Morphisms of $W_{\mathscr{B}}$ are called *$\mathscr{B}$-weak equivalences*, and they admit several desired properties that are essential for the construction of Verdier's quotient. In particular, $W_{\mathscr{B}}$ satisfies the two-out-of-three property and is closed with respect to homotopy pushouts and homotopy pullbacks, see [**Nee01**, Lem.1.5.5-8]. These properties, in addition to cones being weak cokernels, as in Remark 1.4.5, imply that $\mathscr{A}$ admits left and right calculus of fractions with respect to $W_{\mathscr{B}}$, see [**GZ67**, §.I].

Let $\mathcal{H}_{\mathscr{B}}$ denote a homotopy category of $(\mathscr{A}, W_{\mathscr{B}})$. The category $\mathcal{H}_{\mathscr{B}}$ admits a natural triangulated structure induced from $(\mathscr{A}, \Sigma, \mathscr{T})$, with respect to which the localisation functor $\mathtt{L}_{W_{\mathscr{B}}}$ is triangulated. Since $\mathtt{L}_{W_{\mathscr{B}}}$ is triangulated and inverts the morphisms of $W_{\mathscr{B}}$, it contracts objects of $\mathscr{B}$, by Remark 1.4.11, and hence one has $\mathscr{B} \subseteq \mathtt{ker}\, \mathtt{L}_{W_{\mathscr{B}}}$.

Theorem 1.4.12. Let $\mathscr{B}$ be a strict full triangulated subcategory in a triangulated category $(\mathscr{A}, \Sigma, \mathscr{T})$. Then, a Verdier's quotient of $(\mathscr{A}, \Sigma, \mathscr{T})$ with respect to $\mathscr{B}$ always exists, and it is given by the localisation of $\mathscr{A}$ with respect to the set of morphisms $W_{\mathscr{B}}$. Moreover, $\mathtt{ker}\, Q_{\mathscr{B}}$ is the full subcategory of summands of objects in $\mathscr{B}$. In particular, when $\mathscr{B}$ is a thick triangulated subcategory, one has $\mathtt{ker}\, Q_{\mathscr{B}} = \mathscr{B}$.

Proof. See [**Nee01**, Lem 2.1.26-33]. □

Example 1.4.13. The derived category $\mathcal{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ is a Verdier's quotient of the classical homotopy category of complexes $\mathcal{K}(\mathscr{A})$ with respect to acyclic complexes.

1.4.4. Brown's Representability. Brown's Representability for triangulated categories, recalled in Theorem 1.4.15, provides a criterion for cohomological functors on 'nice' triangulated categories to be representable.

Let $(\mathscr{A}, \Sigma, \mathscr{T})$ be a triangulated category admitting small coproducts, and let B be a set of objects in $\mathscr{A}$. The triangulated category $(\mathscr{A}, \Sigma, \mathscr{T})$ is said to be *generated by* B if it is the smallest thick subcategory in $(\mathscr{A}, \Sigma, \mathscr{T})$ that is closed with respect to coproducts in $\mathscr{A}$ and contains B. Moreover, $(\mathscr{A}, \Sigma, \mathscr{T})$ is said to be *compactly generated* if it is generated by a small set B of *compact objects* in $\mathscr{A}$, *i.e.* for every $K \in B$ and for every small set $\{X_\lambda \in \mathscr{A} \mid \lambda \in \Lambda\}$ of objects in $\mathscr{A}$, the canonical morphism

$$\bigoplus_{\lambda\in\Lambda} \mathscr{A}(K, X_\lambda) \to \mathscr{A}(K, \bigoplus_{\lambda\in\Lambda} X_\lambda)$$

is an isomorphism of abelian groups, *cf.* Definition 1.2.33.

Definition 1.4.14. Let $(\mathscr{A}, \Sigma, \mathscr{T})$ be a triangulated category admitting small coproducts. The triangulated category $(\mathscr{A}, \Sigma, \mathscr{T})$ is said to satisfy the Brown's Representability Theorem if for every cohomological functor $F : \mathscr{A}^{\mathrm{op}} \to \mathsf{Ab}$ the following statements are equivalent

- F is representable; and
- F commutes with small products.

Theorem 1.4.15. Let $(\mathscr{A}, \Sigma, \mathscr{T})$ be a compactly generated triangulated category. Then, $(\mathscr{A}, \Sigma, \mathscr{T})$ satisfies the Brown's Representability Theorem.

Proof. See [**Nee96**, Th.3.1] and [**SS03b**, Lem.2.2.1]. □

In fact, the statement of this theorem holds in a greater generality, see [**Nee01**, Ch.8].

1.4.5. *t*-Structures and Weight Structures. The axioms of a triangulated category capture core properties of the derived category of an abelian category; yet, they do not provide all the tools available in homological algebra, in particular there is no canonical choice of cones, compared to the mapping cones for complexes.

1.4.5.1. *t-Structures.* A t-structure on a triangulated category, first introduced in [**BBD82**, §.1.3], guarantees that the triangulated category is equivalent to the derived category of an abelian category, which can be retrieved by means of the t-structure, see Theorem 1.4.20.

Definition 1.4.16. Let $(\mathscr{A}, \Sigma, \mathscr{T})$ be a triangulated category. A *t-structure* on $(\mathscr{A}, \Sigma, \mathscr{T})$ is a pair $\mathsf{t} = (\mathsf{t}^{\leq 0}, \mathsf{t}^{\geq 0})$ of subsets of objects of $\mathscr{A}$ such that

t1 $\mathsf{t}^{\leq 0}$ and $\mathsf{t}^{\geq 0}$ are strict, *i.e.* they contain all objects isomorphic to their elements;
t2 $\mathsf{t}^{\leq 0} \subset \Sigma^{-1}\mathsf{t}^{\leq 0}$ and $\Sigma^{-1}\mathsf{t}^{\geq 0} \subset \mathsf{t}^{\geq 0}$;
t3 $\mathscr{A}(\mathsf{t}^{\leq 0}, \Sigma^{-1}\mathsf{t}^{\geq 0}) = \mathbf{0}$, *i.e.* for every $X \in \mathsf{t}^{\leq 0}$ and $Y \in \mathsf{t}^{\geq 0}$, $\mathscr{A}(X, \Sigma^{-1} Y) = \mathbf{0}$; and

t4 (t-decomposition) for every object $X \in \mathscr{A}$ there exists a distinguished triangle

$$X^{\leq 0} \to X \to \mathbf{\Sigma}^{-1} X^{\geq 0} \to \mathbf{\Sigma} X^{\leq 0},$$

with $X^{\leq 0} \in \mathsf{t}^{\leq 0}$ and $X^{\geq 0} \in \mathsf{t}^{\geq 0}$.

The full subcategory $\heartsuit\mathsf{t} \subset \mathscr{A}$ with objects $\mathtt{Ob}(\heartsuit\mathsf{t}) \coloneqq \mathsf{t}^{\leq 0} \bigcap \mathsf{t}^{\geq 0} \subset \mathtt{Ob}(\mathscr{A})$ is called the *heart of the t-structure* $\mathsf{t} = (\mathsf{t}^{\leq 0}, \mathsf{t}^{\geq 0})$.

Example 1.4.17. Let $\mathscr{B}$ be an abelian category. The derived category $\mathcal{D}(\mathscr{B})$ admits a t-structure, given by the pair $\left(\mathtt{Ob}(\mathcal{D}^{\leq 0}(\mathscr{B})), \mathtt{Ob}(\mathcal{D}^{\geq 0}(\mathscr{B}))\right)$ of the full subcategories of complexes whose cohomology vanish in positive degrees and negative degrees, respectively; it is called the *standard t-structure* on the derived category $\mathcal{D}(\mathscr{B})$, see [**GM03**, Prop.IV.4.3].

For a t-structure t on a triangulated category $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$, let $\mathsf{t}^{\leq n}$ and $\mathsf{t}^{\geq n}$ denote full additive subcategories $\mathbf{\Sigma}^{-n}\,\mathsf{t}^{\leq 0}$ and $\mathbf{\Sigma}^{-n}\,\mathsf{t}^{\geq 0}$, respectively.

Although the t-decomposition property does not require the functoriality or uniqueness of the decomposition, the other axioms imply its uniqueness up to canonical isomorphisms.

Lemma 1.4.18. Let $\mathsf{t} = (\mathsf{t}^{\leq 0}, \mathsf{t}^{\geq 0})$ be a t-structure on a triangulated category $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$. Then,

- the inclusion $\iota_{\mathsf{t}}^{\leq n} : \mathsf{t}^{\leq n} \hookrightarrow \mathscr{A}$ admits an additive right adjoint $\tau_{\mathsf{t}}^{\leq n} : \mathscr{A} \to \mathsf{t}^{\leq n}$; and
- the inclusion $\iota_{\mathsf{t}}^{\geq n} : \mathsf{t}^{\geq n} \hookrightarrow \mathscr{A}$ admits an additive left adjoint $\tau_{\mathsf{t}}^{\geq n} : \mathscr{A} \to \mathsf{t}^{\geq n}$.

Moreover, there exists a t-decomposition

$$\tau_{\mathsf{t}}^{\leq 0} X \to X \to \mathbf{\Sigma}^{-1}\, \tau_{\mathsf{t}}^{\geq 0} X \to \mathbf{\Sigma}\, \tau_{\mathsf{t}}^{\leq 0} X \tag{8}$$

for every $X \in \mathscr{A}$. Also, every t-decomposition for X is canonically isomorphic to (8).

Proof. See [**GM03**, Lem.IV.4.5]. □

The functors $\tau_{\mathsf{t}}^{\leq n}$ and $\tau_{\mathsf{t}}^{\geq n}$ are called the *t-structure truncation functors*.

Example 1.4.19. Let $\mathscr{B}$ be an abelian category, and let t be the standard t-structure on the derived category $\mathcal{D}(\mathscr{B})$, with the standard triangulated structure. Then, the functors $\tau_{\mathsf{t}}^{\leq n}$ and $\tau_{\mathsf{t}}^{\geq n}$ coincide with the canonical truncation functors for complexes in $\mathscr{B}$.

The functors $\tau_{\mathsf{t}}^{\leq 0}$ and $\tau_{\mathsf{t}}^{\geq 0}$ are used to construct kernels and cokernels for morphisms in $\heartsuit\mathsf{t}$, which is the main component of the proof of the following theorem, see [**GM03**, §.IV.4.7].

Theorem 1.4.20. Let t be a t-structure on a triangulated category $(\mathscr{A}, \mathbf{\Sigma}, \mathscr{T})$. Then, $\heartsuit\mathsf{t}$ is an admissible abelian subcategory in $\mathscr{A}$.

Proof. See [**BBD82**, Th.1.3.6]. □

1.4.5.2. *Co-t-structures or Weight structures.* A co-t-structure or weight structure is a dual notion to a t-structure. It was introduced independently in [**Pau08**] under the name of a co-t-structure and in [**Bon10**] under the name of a weight structure. When it exists, it is used to realise a triangulated category as the classical homotopy category of complexes in an additive category. Bondarko uses the Chow weight structure, constructed in [**Bon10**, §.6.5], to establish an equivalence of categories $\mathcal{K}^b(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Q}}(k)) \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})$ for a perfect field k, and hence an isomorphism between the Grothendieck rings $\mathtt{K}_{\oplus}(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Q}}(k))$ and $\mathtt{K}_{\triangle}\left(\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})\right)$, see [**Bon11**].

Definition 1.4.21. Let $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ be a triangulated category. A *weight structure* on $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ is a pair $\mathsf{w}=(\mathsf{w}^{\le 0},\mathsf{w}^{\ge 0})$ of subsets of objects of $\mathscr{A}$ such that

- **w1** $\mathsf{w}^{\le 0},\mathsf{w}^{\ge 0}$ are additive and Karoubi-closed in $\mathscr{A}$, *i.e.* they contain all retracts of their objects;
- **w2** $\mathsf{w}^{\le 0}\subset \mathbf{\Sigma}^{-1}\,\mathsf{w}^{\le 0}$ and $\mathbf{\Sigma}^{-1}\,\mathsf{w}^{\ge 0}\subset \mathsf{w}^{\ge 0}$;
- **w3** $\mathscr{A}(\mathbf{\Sigma}^{-1}\,\mathsf{w}^{\ge 0},\mathsf{w}^{\le 0})=\mathbf{0}$, *i.e.* for every $X\in\mathsf{w}^{\ge 0}$ and $Y\in\mathsf{w}^{\le 0}$, $\mathscr{A}(\mathbf{\Sigma}^{-1}\,X,Y)=\mathbf{0}$; and
- **w4** (weight decomposition) for every object $X\in\mathscr{A}$ there exists a distinguished triangle
$$\mathbf{\Sigma}^{-1}\,X^{\ge 0}\to X\to X^{\le 0}\to \mathbf{\Sigma}\mathbf{\Sigma}^{-1}\,X^{\ge 0},$$
with $X^{\le 0}\in\mathsf{w}^{\le 0}$ and $X^{\ge 0}\in\mathsf{w}^{\ge 0}$.

The full subcategory $\heartsuit\mathsf{w}\subset\mathscr{A}$ with objects $\mathtt{Ob}(\heartsuit\mathsf{w}) \coloneqq \mathsf{w}^{\le 0}\bigcap\mathsf{w}^{\ge 0}\subset\mathtt{Ob}(\mathscr{A})$ is called the *heart of the weight structure* $\mathsf{w}=(\mathsf{w}^{\le 0},\mathsf{w}^{\ge 0})$.

Example 1.4.22. Let $\mathscr{B}$ be a Karoubian additive category, and let $\mathcal{K}^b(\mathscr{B})$ be the bounded classical homotopy category of complexes in $\mathscr{B}$. Consider the standard triangulated structure on $\mathcal{K}^b(\mathscr{B})$, in which cones are isomorphic to the mapping cones. Then, the sets $\mathsf{w}^{\le 0}$ and $\mathsf{w}^{\ge 0}$ of complexes that are homotopy equivalent to bounded complexes concentrated in non-positive and non-negative degrees, respectively, define a weight structure w on $\mathcal{K}^b(\mathscr{B})$. Moreover, $\mathscr{B}$ is equivalent to $\heartsuit\mathsf{w}$, and the weight decomposition is given by the naive truncation of complexes, see [**Bon10**].

Theorem 1.4.23. Let w be a weight structure on a triangulated category $(\mathscr{A},\mathbf{\Sigma},\mathscr{T})$ such that
$$\bigcup_{n\in\mathbb{Z}_{\ge 0}}\mathbf{\Sigma}^{-n}\left(\mathsf{w}^{\le 0}\right)=\mathtt{Ob}(\mathscr{A})\qquad\text{and}\qquad\bigcup_{n\in\mathbb{Z}_{\ge 0}}\mathbf{\Sigma}^{-n}\left(\mathsf{w}^{\ge 0}\right)=\mathtt{Ob}(\mathscr{A}).$$
If $\heartsuit\mathsf{w}$ is Karoubian then $\heartsuit\mathsf{w}$ generates $\mathscr{A}$.

Proof. See [**Bon10**, Prop.5.2.2]. □

1.5. Algebraic K-Theories

Algebraic $\mathtt{K}$-theory of a category with a structure is a decategorification that generalises its Grothendieck group and emphasises the properties encoded in the structure.

1.5.1. Quillen Exact Categories. Let $\mathscr{A}$ be a category, for a short sequence

$$M' \xrightarrow{i} M \xrightarrow{q} M'' \tag{9}$$

in $\mathscr{A}$, i (resp. q) is called the *inflation* (resp. *deflation*) of the sequence. Let $\mathscr{S}$ be a set of short sequences in $\mathscr{A}$, and let $M', M'' \in \mathscr{A}$. An $\mathscr{S}$*-extension* of M' by M'', if it exists, is an object $M \in \mathscr{A}$ that fits into a short sequence (9) in $\mathscr{S}$. A subcategory $\mathscr{B} \subset \mathscr{A}$ is said to be (essentially) closed under $\mathscr{S}$-extensions in $\mathscr{A}$ if for every short sequence (9) in $\mathscr{S}$ in which M', M'' are (isomorphic to) objects in $\mathscr{B}$, then M is (isomorphic to) an object in $\mathscr{B}$.

Quillen exact categories are modelled over a pair $(\mathscr{B}, \mathscr{E})$ of a full additive subcategory $\mathscr{B}$ of an abelian category $\mathscr{A}$, and the set $\mathscr{E}$ of all short exact sequences in $\mathscr{B}$ which are also exact in $\mathscr{A}$, where $\mathscr{B}$ is (essentially) closed under $\mathscr{E}$-extensions in $\mathscr{A}$, see [**Qui73**, §.2]. Although every monomorphism in $\mathscr{B}$ fits as an inflation in a short exact sequence in $\mathscr{A}$, it might not be an inflation in a short exact sequence in $\mathscr{B}$ that is exact in $\mathscr{A}$. Inflations (resp. deflations) in $\mathscr{E}$ are called *admissible monomorphisms* (resp. *admissible epimorphisms*). Admissible monomorphisms (resp. admissible epimorphisms) will be distinguished by arrows $\rightarrowtail$ (resp. $\twoheadrightarrow$).

Lemma 1.5.1. Let $\mathscr{B}$ be a full additive subcategory in an abelian category $\mathscr{A}$, which is (essentially) closed under extensions in $\mathscr{A}$, and let $\mathscr{E}$ be the set of all short exact sequences in $\mathscr{B}$ which are also exact in $\mathscr{A}$. Then,

QE1
- the set $\mathscr{E}$ is closed under isomorphisms in $\mathscr{B}$;
- the set $\mathscr{E}$ contains all split extensions in $\mathscr{B}$, *i.e.* the short sequence
$$X_1 \overset{i_1}{\rightarrowtail} X_1 \bigoplus X_2 \overset{p_2}{\twoheadrightarrow} X_2$$
in $\mathscr{B}$ belongs to $\mathscr{E}$ for every $X_1, X_2 \in \mathscr{B}$; and
- the inflation (resp. deflation) of every sequence in $\mathscr{E}$ is the kernel of its deflation (resp. cokernel of its inflation);

QE2 the set of inflations (resp. deflations) in $\mathscr{E}$ is closed under compositions and pushouts (resp. pullbacks) in $\mathscr{B}$; and

QE3 every morphism $M' \to M$ that has a cokernel in $\mathscr{B}$ and factors an inflation $M' \to N$ in $\mathscr{E}$ is an inflation in $\mathscr{E}$. Dually, every morphism $M \to M''$ that has a kernel in $\mathscr{B}$ and factors a deflation $N \to M''$ in $\mathscr{E}$ is a deflation in $\mathscr{E}$.

Proof. See [**Qui73**] and [**Büh10**]. ☐

Definition 1.5.2. Let $\mathscr{B}$ be an additive category, and let $\mathscr{E}$ be a set of short sequences in $\mathscr{B}$. The set $\mathscr{E}$ is called a *Quillen exact structure* on $\mathscr{B}$ if it satisfies the statements**QE1-QE3** in Lemma 1.5.1. Then, the pair $(\mathscr{B}, \mathscr{E})$ is called a *Quillen exact category*, and sequences in $\mathscr{E}$ are called *exact sequences*. A *Quillen exact functor* $F : (\mathscr{B}, \mathscr{E}) \to (\mathscr{C}, \mathscr{F})$ between Quillen exact categories is an additive functor $F : \mathscr{B} \to \mathscr{C}$ that maps sequences in $\mathscr{E}$ to sequences in $\mathscr{F}$.

Example 1.5.3. Every additive category admits the *split Quillen exact structure*, given by the split exact sequences, which is the smallest Quillen exact structure on an additive category. On the other hand, an abelian category admits another Quillen exact structure, given by all short exact sequences.

Theorem 1.5.4 (Gabriel-Quillen Embedding Theorem). Every small Quillen exact category $(\mathscr{B}, \mathscr{E})$ can be realised as a full additive subcategory in an abelian category $\mathscr{A}$, such that $\mathscr{B}$ is (essentially) closed under extensions in $\mathscr{A}$.

Proof. See [**Büh10**, App.A]. $\square$

Definition 1.5.5 (Quillen's Q Construction). Let $(\mathscr{B}, \mathscr{E})$ be an essentially small Quillen exact category. Define the category $Q_{\mathscr{E}}\mathscr{B}$ to be the category with

- the set of objects $\mathtt{Ob}(Q_{\mathscr{E}}\mathscr{B}) \coloneqq \mathtt{Ob}(\mathscr{B})$; and
- for every $X, Y \in \mathtt{Ob}(Q_{\mathscr{E}}\mathscr{B})$, the set $\mathtt{hom}_{Q\mathscr{B}}(X, Y)$ is the isomorphism classes of roofs $X \overset{i}{\to} Z \overset{p}{\leftarrow} Y$ with i and p being admissible monomorphism and admissible epimorphism, respectively;

while the composition is given by pushouts in $\mathscr{B}$. Then, let

$$\mathbf{B}^{+}_{\mathscr{E}}\mathscr{B} \coloneqq \mathbf{\Omega}\,\mathbf{B}|\mathtt{N}\,Q_{\mathscr{E}}\mathscr{B}|,$$

where $\mathtt{N}$ is the nerve functor, $|-|$ is the geometric realisation functor, $\mathbf{B}$ is the classifying space functor, and $\mathbf{\Omega}$ is the loop functor, see [**Nee97**, §.0].

In fact, $\mathbf{B}^{+}_{\mathscr{E}}\mathscr{B}$ is a pointed space, whose point is induced from the zero object in $\mathscr{B}$.

Quillen's $\mathtt{K}$*-groups* for $(\mathscr{B}, \mathscr{E})$ are the homotopy groups of $\mathbf{B}^{+}_{\mathscr{E}}\mathscr{B}$, *i.e.* for an integer $n \geq 0$, let

$$\mathtt{K}^{\mathscr{E}}_{n}(\mathscr{B}) \coloneqq \pi_n(\mathbf{B}^{+}_{\mathscr{E}}\mathscr{B}).$$

Example 1.5.6. Let $(\mathscr{B}, \mathscr{E})$ be a Quillen exact category. Then, it's Quillen's $\mathtt{K}^{\mathscr{E}}_{0}$-group is given by the abelian group generated by isomorphism classes of objects in $\mathscr{B}$ and relations $\{[Y] = [X] + [Z] \mid \text{there exists an exact sequence } X \to Y \to Z \text{ in } \mathscr{E}\}$, where $[U]$ denotes the isomorphism class of an object U in $\mathscr{B}$. Particularly,

- for an additive category $\mathscr{B}$, with the split Quillen exact structure $\mathscr{E}$, the abelian group $\mathtt{K}^{\mathscr{E}}_{0}(\mathscr{B})$ is generated by isomorphism classes of objects in $\mathscr{B}$ and relations $\{[X \oplus Y] = [X] + [Y] \mid X, Y \in \mathtt{Ob}(\mathscr{B})\}$, it is denoted by $\mathtt{K}_{\oplus}(\mathscr{B})$; and
- for an abelian category $\mathscr{A}$, with the short exact sequences structure $\mathscr{E}$, the abelian group $\mathtt{K}^{\mathscr{E}}_{0}(\mathscr{A})$ is generated by isomorphism classes of objects in $\mathscr{A}$ and relations $\{[Y] = [X] + [Z] \mid \text{there exists a short exact sequence } \mathbf{0} \to X \to Y \to Z \to \mathbf{0} \text{ in } \mathscr{A}\}$;

i.e. in both cases, the $\mathtt{K}^{\mathscr{E}}_{0}$-group coincides with the Grothendieck group.

Since a Quillen exact category $(\mathscr{B},\mathscr{E})$ is additive and all split extensions in $\mathscr{B}$ belongs to $\mathscr{E}$, there exists a canonical quotient group homomorphism

$$\mathtt{K}_{\oplus}(\mathscr{B}) \to \mathtt{K}_0^{\mathscr{E}}(\mathscr{B}),$$

which is not injective in general.

1.5.2. Waldhausen K-Theory.

Definition 1.5.7. Let $\mathscr{C}$ be a category with a zero object, let $\mathbf{c}\mathscr{C}$ be a subcategory in $\mathscr{C}$ that contains all isomorphisms in $\mathscr{C}$. The pair $(\mathscr{C},\mathbf{c}\mathscr{C})$ is called a *category with cofibrations*, and $\mathbf{c}\mathscr{C}$ is called a *subcategory of cofibrations* in $\mathscr{C}$, if

W1 the initial morphism $\mathbf{0} \to U$ belongs to $\mathbf{c}\mathscr{C}$ for every object $U \in \mathscr{C}$; and

W2 the pushout of all morphisms of $\mathbf{c}\mathscr{C}$ exist in $\mathscr{C}$, and $\mathbf{c}\mathscr{C}$ is closed under pushouts.

A cofibration in a category with cofibrations is denoted by a feathered arrow $\rightarrowtail$.

A category with cofibrations is closed under finite coproducts. That is because it has an initial object $\mathbf{0} \in \mathscr{C}$, and for objects $U, V \in \mathscr{C}$, the initial morphisms $\mathbf{0} \rightarrowtail U$ and $\mathbf{0} \rightarrowtail V$ are cofibrations, and hence the coproduct $U \coprod V = U \coprod_{\mathbf{0}} U$ exists in $\mathscr{C}$. Moreover, all cofibrations has cokernels in $\mathscr{C}$, given by pushouts along the terminal morphisms. Let $i : U \rightarrowtail V$ be a cofibration, and denote by V/U its cokernel $V \coprod_U \mathbf{0}$. For a cofibration $i : U \rightarrowtail V$, a cokernel sequence $U \overset{i}{\rightarrowtail} V \twoheadrightarrow V/U$ is called a *cofibre sequence* of i.

Definition 1.5.8. Let $\left(\mathscr{C},\mathbf{c}\mathscr{C}\right)$ be a category with cofibrations, and let $\mathbf{w}\mathscr{C}$ be a subcategory in $\mathscr{C}$ that contains all isomorphisms in $\mathscr{C}$. The triple $\left(\mathscr{C},\mathbf{c}\mathscr{C},\mathbf{w}\mathscr{C}\right)$ is called a *Waldhausen category*, and $\left(\mathbf{c}\mathscr{C},\mathbf{w}\mathscr{C}\right)$ is called a *Waldhausen structure* on $\mathscr{C}$, if they satisfy the *glueing axiom*, *i.e.*

W3 for every solid commutative diagram

$$\begin{tikzcd}
U \arrow[rr, tail, "i"] \arrow[dr, "f"] \arrow[dd, "j"'] & & V \arrow[dr, "g"] \arrow[dd, dashed, "i_V"] & \\
& U' \arrow[rr, tail, "i'"] \arrow[dd, "j'"'] & & V' \arrow[dd, dashed, "i_{V'}"] \\
W \arrow[rr, tail, dashed, "i_W"] \arrow[dr, "h"] & & Z \arrow[dr, dotted, "h"] & \\
& W' \arrow[rr, tail, dashed, "i_{W'}"] & & Z',
\end{tikzcd}$$

in $\mathscr{C}$, such that the morphisms i, i' in $\mathbf{c}\mathscr{C}$ and the diagonal solid morphisms belong to $\mathbf{w}\mathscr{C}$, then the induced morphism $Z \to Z'$ belongs to $\mathbf{w}\mathscr{C}$, for $Z := W \coprod_U V$ and $Z' := W' \coprod_{U'} V'$.

Then, morphisms in $\mathbf{w}\mathscr{C}$ are called the *weak equivalences* of the Waldhausen structure. If $\mathbf{w}\mathscr{C}$ also satisfies the two-out-of-three property then the Waldhausen category is said to be *saturated*. Moreover, the Waldhausen structure is said to satisfy the *extension axiom* if for a morphism of cofibre sequences

$$\begin{tikzcd} U \arrow[r] \arrow[d, "u"'] & V \arrow[r] \arrow[d, "v"] & V/U \arrow[d, "c"] \\ U' \arrow[r] & V' \arrow[r] & V'/U', \end{tikzcd}$$

having u and c in $\mathbf{w}\mathscr{C}$ implies that v is in $\mathbf{w}\mathscr{C}$.

When no confusion arise, we refer to the Waldhausen category $(\mathscr{C}, \mathbf{c}\mathscr{C}, \mathbf{w}\mathscr{C})$ by $\mathscr{C}$.

For Waldhausen categories $\mathscr{C}$ and $\mathscr{D}$, a functor $F : \mathscr{C} \to \mathscr{D}$ is said to be *exact* with respect to the Waldhausen structures, if it preserves cofibrations, weak equivalences, and pushouts along cofibrations, *i.e.* $F\big(\mathbf{c}\mathscr{C}\big) \subset \mathbf{c}\mathscr{D}$, $F\big(\mathbf{w}\mathscr{C}\big) \subset \mathbf{w}\mathscr{D}$, and for every cofibration $i : U \rightarrowtail V$ in $\mathbf{c}\mathscr{C}$ the canonical morphism

$$F(V) \coprod_{F(U)} F(X) \to F(V \coprod_{U} X)$$

is an isomorphism in $\mathscr{D}$, for every morphism $f : U \to X$ in $\mathscr{C}$. On the other hand, a natural transformation $\alpha : F \Rightarrow G : \mathscr{C} \to \mathscr{D}$ between exact functors is said to be a *weak equivalence* if the components of α belong to $\mathbf{w}\mathscr{D}$, see [**Wal85**, p.330].

Example 1.5.9. Let $\mathscr{C}_\tau$ be an essentially small site. Then, the category of (finitely presented objects of) pointed τ-sheaves of sets on $\mathscr{C}$ is a Waldhausen category, in which the cofibrations are the monomorphisms and the weak equivalence are the isomorphisms.

Example 1.5.10. The full subcategory of cofibrant objects in a pointed model category is a category with cofibrations. Moreover, if the model category is left proper, then the full subcategory of cofibrant objects is a saturated Waldhausen category whose set of cofibrations is the set of cofibration of the model category, and whose set of weak equivalences is the set of weak equivalences of the model category. That includes Kan-Quillen's model category of (finitely presented objects[3] of) pointed simplicial sets; and the local injective model category of (finitely presented objects of) pointed simplicial (pre)sheaves on an essentially small site. A left Quillen functor between such model categories is not necessarily an exact functor with respect to the corresponding Waldhausen structures. Although a left Quillen functor preserves all cofibrations and all pushouts (the latter for being left adjoint), it does not necessarily preserve all weak equivalences. However, since the geometric realisation $|-| : \mathsf{sSet}_\bullet \to \mathsf{Top}_\bullet$ preserves all weak equivalences, it is an exact functor with respect to the Waldhausen structure

[3]An object $X \in \mathscr{C}$ is said to be *finitely presented* in $\mathscr{C}$ if the corepresentable functor $\mathtt{h}^X$ preserves filtered colimits.

associated to Kan-Quillen's and Quillen-Serre's model structures of pointed simplicial sets and pointed topological spaces, respectively.

EXAMPLE 1.5.11. Let $(\mathscr{A}, \mathscr{E})$ be a Quillen exact category, let $\mathbf{c}\mathscr{A}$ be the set of admissible monomorphisms in $\mathscr{E}$, and let $\mathbf{w}\mathscr{A}$ be the set of all isomorphisms in $\mathscr{A}$. Then, $\big(\mathscr{A}, \mathbf{c}\mathscr{A}, \mathbf{w}\mathscr{A}\big)$ is a Waldhausen category, see [**TT90**, §.1.2.9].

DEFINITION 1.5.12 (Waldhausen S-construction). Let $\mathscr{C}$ be an essentially small Waldhausen category. The simplicial essentially small category $S_\bullet\mathscr{C} : \Delta^{\mathrm{op}} \to \mathsf{CAT}$ is the largest simplicial subcategory $S_\bullet\mathscr{C}$ of the simplicial functor category $\mathsf{Fun}(\mathsf{Mor}([\bullet]), \mathscr{C})$ such that, for every integer $n \geq 0$,

- for every functor $F : \mathsf{Mor}([n]) \to \mathscr{C}$ in $\mathtt{Ob}(S_n\mathscr{C})$ and for every pair of composable morphisms $\varphi : i \leq j$ and $\vartheta : j \leq k$ in $[n]$, the sequence
$$F(\varphi) \xrightarrow{F(\mathrm{id}_i, \vartheta)} F(\vartheta \circ \varphi) \xrightarrow{F(\varphi, \mathrm{id}_k)} F(\vartheta)$$
is a cofibre sequence; and
- for every natural transformation $\tau : F \to G$ in the category $S_n\mathscr{C}$, the component morphism $\tau_\varphi : F(\varphi) \to G(\varphi)$ is a weak equivalence, for every morphism φ in $[n]$.

DEFINITION 1.5.13. Let $\mathscr{C}$ be an essentially small Waldhausen category. The *Waldhausen* $\mathtt{K}$*-theory* of $\mathscr{C}$ is the spectra
$$\mathtt{K}(\mathscr{C}) \coloneqq \Omega^\infty \left|\mathtt{N}\left(S_\bullet\mathscr{C}\right)\right|.$$
For every integer $n \geq 0$, *Waldhausen* $\mathtt{K}$*-group* $\mathtt{K}_n(\mathscr{C})$ is the homotopy group
$$\mathtt{K}_n(\mathscr{C}) \coloneqq \pi_n \mathtt{K}(\mathscr{C}) = \pi_{n+1}\left|\mathtt{N}\left(S_\bullet\mathscr{C}\right)\right|.$$

An exact functor $F : \mathscr{C} \to \mathscr{D}$ between essentially small Waldhausen categories induces a map of spectra $\mathtt{K}(F) : \mathtt{K}(\mathscr{C}) \to \mathtt{K}(\mathscr{D})$, and a weak equivalence $\alpha : F \Rightarrow G$ between exact functors induces a homotopy $\mathtt{K}(F) \Rightarrow \mathtt{K}(G)$, see [**Wal85**, Prop.1.3.1].

THEOREM 1.5.14. Let $(\mathscr{A}, \mathscr{E})$ be an essentially small Quillen exact category. Then, there exists a natural homotopy equivalence between Quillen's $\mathtt{K}$-theory spectra of $(\mathscr{A}, \mathscr{E})$ and Waldhausen's $\mathtt{K}$-theory spectra of the associated Waldhausen category, as in Example 1.5.11.

PROOF. See [**Wal85**, §.1.9]. □

The Waldhausen K_0 group is known to have a simpler expression, given as follows: $K_0(\mathscr{C})$ is the abelian group generated by isomorphism classes of objects in $\mathscr{C}$ modulo the relations

(1) $[U] = [V]$ if there exists a weak equivalence $U \to V$; and
(2) $[V] = [U] + [V/U]$ for every cofibre sequence $U \rightarrowtail V \twoheadrightarrow V/U$;

where $[X]$ denotes the isomorphism class of an object X in $\mathscr{C}$, see [**TT90**, §.1.5.6]. Thus, an exact functor between essentially small Waldhausen categories induces a canonical group homomorphism between their Waldhausen K_0 groups.

For an essentially small Waldhausen category $\mathscr{C}$, there exist cofibre sequences

$$\mathbf{0} \rightarrowtail \mathbf{0} \twoheadrightarrow \mathbf{0} \qquad \text{and} \qquad U \rightarrowtail U \coprod V \twoheadrightarrow V,$$

and hence $[0]$ is the identity in $K_0(\mathscr{C})$ and $[U \coprod V] = [U] + [V]$, for every $U, V \in \mathscr{C}$. Also, for cofibrations $i : U \rightarrowtail V$ and $j : U \rightarrowtail W$, there exist a cofibre sequence

$$U \rightarrowtail V \coprod_U W \twoheadrightarrow V/U \coprod W/U,$$

which implies $[V \coprod_U W] = [V] + [W] - [U]$.

Lemma 1.5.15 (Eilenberg Swindle). Let $\mathscr{C}$ be an essentially small Waldhausen category that is closed under countable coproducts. Then, the spectrum $\mathtt{K}(\mathscr{C})$ is connected, *i.e.* the group $\mathtt{K}_0(\mathscr{C})$ vanishes.

Proof. Let U be an object in $\mathscr{C}$, then there exists $\coprod_{\mathbb{N}} U \in \mathscr{C}$. Since the initial morphisms $\mathbf{0} \to U$ and $\mathbf{0} \to \coprod_{\mathbb{N}} U$ are cofibrations, the morphism $U \to \coprod_{\mathbb{N}} U$ induced by the pushout square

$$\begin{tikzcd} \mathbf{0} \arrow[r] \arrow[d] & \coprod_{\mathbb{N}} U \arrow[d] \\ U \arrow[r] & \coprod_{\mathbb{N}} U \end{tikzcd}$$

in $\mathscr{C}$ is a cofibration. Since the composition $\mathbf{0} \to U \to \mathbf{0}$ coincides with the unique isomorphism $\mathrm{id}_{\mathbf{0}}$ and a cobase change along a composition is given by the composition of cobase changes, there exists a cofibre sequence

$$U \rightarrowtail \coprod_{\mathbb{N}} U \twoheadrightarrow \coprod_{\mathbb{N}} U$$

in $\mathscr{C}$. Thus, $[U] = [\bigvee_{\mathbb{N}} U] - [\bigvee_{\mathbb{N}} U] = 0 \in \mathtt{K}_0(\mathscr{C})$, and hence $\mathtt{K}_0(\mathscr{C}) = \mathbf{0}$. □

In particular, the Waldhausen $\mathtt{K}$-theory spectra of a model category, when defined, is connected. One is usually interested in Waldhausen categories that satisfy some finiteness conditions, which do not admit Eilenberg Swindle. For example, since finitely presented objects are closed under finite colimits, given a Waldhausen category $(\mathscr{C}, \mathbf{c}\mathscr{C}, \mathbf{w}\mathscr{C})$, the full subcategory of its finitely presented objects $\mathscr{C}^c$ admits a Waldhausen structure $(\mathbf{c}\mathscr{C}^c, \mathbf{w}\mathscr{C}^c)$, given by the restriction of the structure $(\mathbf{c}\mathscr{C}, \mathbf{w}\mathscr{C})$ to $\mathscr{C}^c$.

Example 1.5.16. Let $\mathscr{C}$ be the category of spectra of pointed simplicial sets. Then, $\mathtt{K}_0(\mathscr{C}) = \mathbf{0}$ and $\mathtt{K}_0(\mathscr{C}^c) = \mathbb{Z}$, see [**Bon10**, Prop.5.5.1].

Definition 1.5.17. Let $\mathscr{C}$ be an essentially small Waldhausen category. A *cylinder functor* on $\mathscr{C}$ is a quadruple $(\mathtt{Cyl}, p, i, j)$ of a functor $\mathtt{Cyl} : \mathsf{Mor}(\mathscr{C}) \to \mathscr{C}$ and natural transformations $p : \mathtt{Cyl} \Rightarrow \mathtt{codom}$, $i : \mathtt{codom} \Rightarrow \mathtt{Cyl}$, and $j : \mathtt{dom} \Rightarrow \mathtt{Cyl}$, such that for every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in $\mathscr{C}$, one has

- $p_f \circ j_f = f$ and $p_f \circ i_f = \mathrm{id}_Y$;
- the induced diagram

$$\begin{tikzcd} X \coprod Y \arrow[r, "j_f \coprod i_f"] \arrow[d, "x \coprod y"'] & \mathtt{Cyl}\, f \arrow[r, "p_f"] \arrow[d, "\mathtt{Cyl}(x,y)"'] & Y \arrow[d, "y"] \\ X' \coprod Y' \arrow[r, "j_{f'} \coprod i_{f'}"'] & \mathtt{Cyl}\, f' \arrow[r, "p_{f'}"'] & Y' \end{tikzcd}$$

  commutes;
- $j_f \coprod i_f$ belongs to $\mathbf{c}\mathscr{C}$;
- if both x and y are in $\mathbf{w}\mathscr{C}$, then so is $\mathtt{Cyl}(x, y)$; and
- if both x and y are in $\mathbf{c}\mathscr{C}$, then so are $\mathtt{Cyl}(x, y)$ and the morphism

$$\mathtt{Cyl}\, f \coprod_{X \coprod Y} X' \coprod Y' \longrightarrow \mathtt{Cyl}\, f'.$$

Moreover, the cylinder functor $\mathtt{Cyl}$ is said to satisfy the *cylinder axiom* if p_f belongs to $\mathbf{w}\mathscr{C}$ for every morphism f in $\mathscr{C}$.

When a cylinder functor exists, following the notation in the classical homotopy theory of topological spaces, let

$$\mathtt{Cyl}(U) \coloneqq \mathtt{Cyl}(\mathrm{id}_U) \quad , \quad \mathsf{Cone}_{\mathtt{Cyl}}(U) \coloneqq \mathtt{Cyl}(U \to \mathbf{0}) \quad \text{and} \quad \mathbf{\Sigma}_{\mathtt{Cyl}}\, U \coloneqq \mathsf{Cone}_{\mathtt{Cyl}}(U)/U,$$

for every object $U \in \mathscr{C}$. Then, one has a cofibre sequence

$$U \to \mathsf{Cone}_{\mathtt{Cyl}}(U) \to \mathbf{\Sigma}_{\mathtt{Cyl}}(U).$$

The morphism $\mathsf{Cone}_{\mathtt{Cyl}}(U) \to 0$ belongs to $\mathbf{w}\mathscr{C}$, and hence $[\mathsf{Cone}_{\mathtt{Cyl}}(U)] = 0 \in \mathtt{K}_0(\mathscr{C})$ and $[\mathbf{\Sigma}_{\mathtt{Cyl}}\, U] = -[U] \in \mathtt{K}_0(\mathscr{C})$.

Example 1.5.18. The mapping cylinder a morphism of pointed simplicial sets, that is the pushout of the morphism along the cylinder object of its domain, defines a cylinder functor for the Waldhausen category of (finitely presented objects of) pointed simplicial sets, recalled in Example 1.5.10, see [**TT90**, §.1.3.3]. Also, for a pointed simplicial set K, the pointed simplicial sets $\mathsf{Cone}_{\mathtt{Cyl}}(K)$ and $\mathbf{\Sigma}_{\mathtt{Cyl}}\, K$ coincide with usual cone and suspension, respectively. In particular, one has $\mathbf{\Sigma}_{\mathtt{Cyl}}\, K \cong K \wedge (S^1, 0)$.

THEOREM 1.5.19 (Localisation Theorem). Let $(\mathscr{C}, \mathbf{c}\mathscr{C})$ be an essentially small category with cofibrations that admits two subcategories $\mathbf{v}\mathscr{C} \subset \mathbf{w}\mathscr{C}$ of weak equivalences making it into Waldhausen categories $(\mathscr{C}, \mathbf{c}\mathscr{C}, \mathbf{v}\mathscr{C})$ and $(\mathscr{C}, \mathbf{c}\mathscr{C}, \mathbf{w}\mathscr{C})$, respectively. Let $\mathscr{C}^{\mathbf{w}}$ be the full subcategory in $\mathscr{C}$ of $\mathbf{w}\mathscr{C}$-acyclic objects in $\mathscr{C}$, *i.e.* objects whose initial morphisms are in $\mathbf{w}\mathscr{C}$. Assume that $\mathbf{w}\mathscr{C}$ is saturated and that every morphism f in $\mathscr{C}$ factorises as $f = p \circ j$ where $j \in \mathbf{c}\mathscr{C}$ and $p \in \mathbf{v}(\mathscr{C})$. Then, the exact inclusions

$$(\mathscr{C}^{\mathbf{w}}, \mathbf{c}\mathscr{C}^{\mathbf{w}}, \mathbf{v}\mathscr{C}^{\mathbf{w}}) \hookrightarrow (\mathscr{C}, \mathbf{c}\mathscr{C}, \mathbf{v}\mathscr{C}) \overset{\mathrm{id}_{\mathscr{C}}}{\to} (\mathscr{C}, \mathbf{c}\mathscr{C}, \mathbf{w}\mathscr{C})$$

induce an exact sequence of abelian groups

$$\mathtt{K}_0(\mathbf{v}\mathscr{C}^{\mathbf{w}}) \to \mathtt{K}_0(\mathbf{v}\mathscr{C}) \to \mathtt{K}_0(\mathbf{w}\mathscr{C}) \to \mathbf{0}.$$

PROOF. See [**Wei13**, Th.9.6]. □

The factorisation in the hypothesis of the Localisation Theorem is usually obtained via a cylinder functor that satisfies the cylinder axiom, *cf.* [**Wal85**, Th.1.6.4] and [**TT90**, §.1.8.1-2].

1.5.2.1. *Symmetric Monoidal Waldhausen Categories.* Similar to the case of model categories, a symmetric monoidal Waldhausen category admits a symmetric monoidal structure and a Waldhausen structure that are compatible. That allows one to endue the Waldhausen $\mathtt{K}$-theory with a homotopy commutative monoid structure.

DEFINITION 1.5.20. A *symmetric monoidal Waldhausen category* $(\mathscr{C}, \mathscr{S})$ is a pair of a Waldhausen category $\mathscr{C}$ and a symmetric monoidal structure $\mathscr{S} = (\wedge, \mathbb{1}_\wedge, \psi, \alpha, \lambda, \rho)$ on $\mathscr{C}$ such that

(1) the endofunctors $X \wedge -$ and $- \wedge X$ are exact for every $X \in \mathscr{C}$; and
(2) for cofibrations $i : U \to V$ and $i' : U' \to V'$ in $\mathscr{C}$, the pushout product

$$i \,\square\, i' : U \wedge V' \coprod_{U \wedge U'} V \wedge U' \to V \wedge V'$$

is a cofibration in $\mathscr{C}$.

EXAMPLE 1.5.21. Let $\mathsf{FSet}_\bullet$ be the Waldhausen category of pointed finite sets whose cofibrations (resp. weak equivalences) are pointed monomorphisms (resp. isomorphisms). Recall that Barratt-Priddy-Quillen Theorem implies

$$\mathtt{K}(\mathsf{FSet}_\bullet) \simeq \mathbb{S},$$

where $\mathbb{S} = (S^0, S^1, S^2, \ldots)$ is the sphere spectrum, see [**Rog10**, Th.8.9.3]. The smash product endues the category $\mathsf{FSet}_\bullet$ with a symmetric monoidal structure, with a unite $*_+ = (* \coprod *, *)$, making it into a symmetric monoidal Waldhausen category. For a pointed finite set (X, x), one has $[(X, x)] = |X \smallsetminus \{x\}| \cdot [*_+] \in \mathtt{K}_0(\mathsf{FSet}_\bullet)$, and there exists an isomorphism

$$(X, x) \cong \coprod_{X \smallsetminus \{x\}} *_+.$$

For an essentially small symmetric monoidal Waldhausen category $(\mathscr{C}, \mathscr{S})$, provided the axiom of choice, there exists an exact functor of Waldhausen categories $\upsilon_{\mathscr{C}} : \mathsf{Set}_\bullet \to \mathscr{C}$, for which

$$\upsilon_{\mathscr{C}}\big((X,x)\big) \cong \coprod_{X \smallsetminus \{x\}} 1_\wedge,$$

for every pointed finite set (X, x). Therefore, there exists a map of spectra

$$\mathtt{K}(\upsilon_{\mathscr{C}}) : \mathbb{S} \cong \mathtt{K}(\mathsf{FSet}_\bullet) \to \mathtt{K}(\mathscr{C}).$$

On the other hand, the monoidal product defines a paring

$$\mathtt{K}(\wedge) : \mathtt{K}(\mathscr{C}) \wedge \mathtt{K}(\mathscr{C}) \to \mathtt{K}(\mathscr{C}),$$

see [**Wal85**, p.342]. Then, the coherence natural isomorphisms of the monoidal structure induce a homotopy commutative monoid structure on the Waldhausen $\mathtt{K}$-theory spectrum $\mathtt{K}(\mathscr{C})$, *i.e.* it makes $\mathtt{K}(\mathscr{C})$ into a ring spectrum, see [**BM11**, Cor.2.8]. In particular, $\mathtt{K}_0(\mathscr{C})$ is a ring and its ring characteristic $\mathbb{Z} \to \mathtt{K}_0(\mathscr{C})$ is given by the ring homomorphism $\mathtt{K}_0(\upsilon_S)$.

Example 1.5.22. Recall Example 1.5.9, for an essentially small site $\mathscr{C}_\tau$, the Waldhausen category of pointed τ-sheaves of sets on $\mathscr{C}$ is a symmetric monoidal Waldhausen category, whose monoidal product is given by the smash product. Similar to Proposition 1.2.56, the unit of the symmetric monoidal structure is given by $*_+ = (* \coprod *, *)$, whereas the smash product of pointed τ-sheaves $(\mathscr{X}, x)$ and $(\mathscr{Y}, y)$ is given by the pushout of the span

$$\begin{tikzcd} \mathscr{X} \coprod \mathscr{Y} \arrow[r, "(\mathrm{id}_{\mathscr{X}} \times y) \coprod (x \times \mathrm{id}_{\mathscr{Y}})"] \arrow[d] & \mathscr{X} \times \mathscr{Y} \\ * & \end{tikzcd}$$

in $\mathsf{Shv}_\tau(\mathscr{C})$, with the canonical base-point.

An exact functor $F : \mathscr{C} \to \mathscr{D}$ between symmetric monoidal Waldhausen categories is said to be *weak monoidal* if F is lax monoidal, such that the coherence morphism

$$F(X) \wedge_{\mathscr{D}} F(Y) \to F(X \wedge_{\mathscr{C}} Y)$$

belongs to $\mathbf{w}\mathscr{D}$ for every $X, Y \in \mathscr{C}$, and so is the coherence morphism $\mathbb{1}_{\mathscr{D}} \to G(\mathbb{1}_{\mathscr{C}})$. For a weak monoidal exact functor $F : \mathscr{C} \to \mathscr{D}$ between essentially small symmetric monoidal Waldhausen categories, the map of spectra

$$\mathtt{K}(F) : \mathtt{K}(\mathscr{C}) \to \mathtt{K}(\mathscr{D})$$

is a morphism of ring spectra, with respect to the induced structures.

1.5.3. Grothendieck Group of Triangulated Categories. Let $(\mathscr{C}, \boldsymbol{\Sigma}, \mathscr{T})$ be a triangulated category. The *Grothendieck group* of $(\mathscr{C}, \boldsymbol{\Sigma}, \mathscr{T})$, denoted by $\mathtt{K}_{\triangle}(\mathscr{C})$, is the abelian group generated by isomorphism classes of objects in $\mathscr{C}$ and relations

$$\left\{[Y] = [X] + [Z] \mid \text{there exists a distinguished triangle } X \to Y \to Z \to \boldsymbol{\Sigma} X \text{ in } \mathscr{T}\right\},$$

where $[U]$ denotes the isomorphism class of an object U in $\mathscr{C}$. Then, $[\mathbf{0}] = 0$ and $[\boldsymbol{\Sigma} X] = -[X]$, for every $X \in \mathscr{C}$.

Example 1.5.23. Let $\mathscr{B}$ be an essentially small additive category, and let $\mathcal{K}^b(\mathscr{B})$ be the bounded homotopy category of complexes in $\mathscr{B}$. Then, the Euler characteristic

$$\begin{array}{rccc} \chi: & \mathtt{K}_{\triangle}\left(\mathcal{K}^b(\mathscr{B})\right) & \to & \mathtt{K}_{\oplus}\left(\mathscr{B}\right) \\ & [C_\bullet] & \mapsto & \sum_{n\in\mathbb{Z}}(-1)^n[C_n], \end{array}$$

is an isomorphism of groups, see [**Ros11**]. More generally, let w be a weight structure on a triangulated category $(\mathscr{A}, \boldsymbol{\Sigma}, \mathscr{T})$, as in Theorem 1.4.23. Then, the inclusion $\heartsuit\mathsf{w} \hookrightarrow \mathscr{A}$ induces an isomorphism

$$\mathtt{K}_{\oplus}(\heartsuit\mathsf{w}) \cong \mathtt{K}_{\triangle}(\mathscr{A}),$$

see [**Bon10**, Th.5.3.1].

CHAPTER 2

Motivic Spaces and Complexes

Several theories in algebraic geometry are $\mathbb{A}^1$-invariant, which made it desirable to have a homotopy theory for schemes in which the affine line is contractable. Quillen's model structures provide a well-established machinery for homotopy theories; however, they are restricted to (finitely) bicomplete categories. The $\mathbb{A}^1$-homotopy theory of schemes is obtained by first taking the free cocompletion of the considered category through the Yoneda embedding. This comes at the cost of loosing colimits that already exist at the level of schemes, for instance a Zariski open covering does not give a covering of presheaves, due to the Yoneda embedding not preserving colimits. Hence, the category of presheaves is localised with respect to hypercovers for a topology τ that recovers enough colimits needed to obtain a well-behaved theory.

For a base scheme S, the category $\mathsf{Shv}_\tau(\mathsf{Sm}/S)$ admits a model structure in which every projection $\mathscr{X} \times \mathbb{A}^1_S \to \mathscr{X}$ is a weak equivalence for every τ-sheaf $\mathscr{X}$, *i.e.* the affine line is contracted. The homotopy category $\mathcal{H}_{\tau,\mathbb{A}^1_S}(S)$ of $\mathsf{Shv}_\tau(\mathsf{Sm}/S)$ with respect to this model structure is Quillen equivalent to the homotopy category of simplicial τ-sheaves $\mathsf{sShv}_\tau(\mathsf{Sm}/S)$ with respect to some model structure in which the projection $\mathscr{X} \times \mathbb{A}^1_S \to \mathscr{X}$ is a weak equivalence for every simplicial τ-sheaf $\mathscr{X}$. One may then consider the homotopy category of simplicial τ-sheaves which is technically more feasible, compared to the homotopy category of sheaves.

The machinery mentioned above can be run for different categories of schemes and different topologies on them. However, some of the important results, like the Gluing Theorem 2.3.1 and the Purity Theorem 2.3.3, are obtained only for topologies that are as fine as the Nisnevich topology on smooth schemes over a Noetherian base of finite Krull dimension.

2.1. Homotopy Theories of Simplicial (Pre)sheaves

Throughout this section, let $\mathscr{C}_\tau$ be an essentially small Grothendieck site, see §.A.4. Let $\mathsf{sPSh}(\mathscr{C})$ denote the category of *simplicial presheaves* on $\mathscr{C}$, *i.e.* the functor category $\mathsf{Fun}\big(\Delta^{op}, \mathsf{PSh}(\mathscr{C})\big)$, where Δ is the simplex category, see §.1.2.4.1. Since $\mathscr{C}$ is essentially small, there exist canonical isomorphisms

$$\mathsf{Fun}(\Delta^{op}, \mathsf{PSh}(\mathscr{C})) \xrightarrow{\cong} \mathsf{Fun}(\mathscr{C}^{op} \times \Delta^{op}, \mathsf{Set}) \xrightarrow{\cong} \mathsf{Fun}(\mathscr{C}^{op}, \mathsf{sSet}), \tag{10}$$

which allows considering simplicial presheaves as simplicial objects in presheaves on $\mathscr{C}$, presheaves of sets on $\mathscr{C} \times \Delta$, or presheaves of simplicial sets on $\mathscr{C}$.

The Yoneda embedding induces full embeddings

$$\mathsf{sSet} \hookrightarrow \mathsf{sPSh}(\mathscr{C}) \quad , \quad \mathscr{C} \hookrightarrow \mathsf{PSh}(\mathscr{C}) \hookrightarrow \mathsf{sPSh}(\mathscr{C}) \quad \text{and} \quad \Delta^{\bullet}_{-} : \Delta \times \mathscr{C} \hookrightarrow \mathsf{sPSh}(\mathscr{C}).$$

Denote the simplicial presheaf represented by $([n], U) \in \Delta \times \mathscr{C}$ by Δ^n_U. Then, the Yoneda lemma implies that for a simplicial presheaf $\mathscr{X}$ on $\mathscr{C}$, one has a isomorphism

$$\mathscr{X}_m(V) \cong \mathsf{sPSh}(\mathscr{C})\big(\Delta^m_V, \mathscr{X}\big) \qquad \text{for} \qquad [m] \in \Delta, V \in \mathscr{C}.$$

In particular, one has an isomorphism

$$(\Delta^n_U)_m(V) \cong \Delta^n_m \times \mathscr{C}(V, U) \qquad \text{and} \qquad (\Delta^n_U)_{\nu,\psi} = \nu^* \times \psi^*$$

for every $[n], [m] \in \Delta$, $U, V \in \mathscr{C}$, and morphisms $\nu : [l] \to [m]$ in Δ and $\psi : W \to V$ in $\mathscr{C}$; thus, Δ^0_{-} coincides with the embedding $\mathscr{C} \to \mathsf{sPSh}(\mathscr{C})$.

Let $\mathscr{X}$ be a simplicial presheaf on $\mathscr{C}$, and let $U_{\mathscr{X}} : (\Delta^{\bullet}_{-} \downarrow \mathscr{X}) \to \mathsf{sPSh}(\mathscr{C})$ be the canonical projection, then there exist an isomorphism

$$\mathscr{X} \cong \operatorname*{\mathtt{colim}}_{(\Delta^{\bullet}_{-} \downarrow \mathscr{X})} U_{\mathscr{X}}.$$

Boundedness is essential for the existence of model structures on simplicial presheaves.

Definition 2.1.1. Let κ be an infinite cardinal. A simplicial presheaf $\mathscr{X}$ on $\mathscr{C}$ is said to be *κ-bounded* if and only if $|\mathscr{X}_n(U)| < \kappa$, for every $n \geq 0, U \in \mathscr{C}$.

Example 2.1.2. Let κ be an infinite cardinal, such that $\kappa > 2^{|Mor(\mathscr{C})|}$. Then, Δ^n_U is κ-bounded, for every $n \geq 0, U \in \mathscr{C}$.

The category $\mathsf{sShv}_\tau(\mathscr{C})$ of *simplicial τ-sheaves* over $\mathscr{C}_\tau$ is the category of simplicial objects in the category of τ-sheaves $\mathsf{Shv}_\tau(\mathscr{C})$.

2.1.1. Symmetric Monoidal Structure. The category of simplicial τ-sheaves on $\mathscr{C}$ admits a closed symmetric monoidal structure, whose monoidal product (resp. unit object) is the Cartesian product (resp. terminal object), and whose internal $\underline{\mathtt{Hom}}$ is given for simplicial τ-sheaves $\mathscr{Y}$ and $\mathscr{Z}$ by the τ-sheafification of the simplicial presheaf $\underline{\mathtt{Hom}}_{\mathrm{pre}}(\mathscr{Y}, \mathscr{Z})$, given by

$$\underline{\mathtt{Hom}}_{\mathrm{pre}}(\mathscr{Y}, \mathscr{Z})_n(U) \coloneqq \mathsf{sShv}_\tau(\mathscr{C})(\mathscr{Y} \times \Delta^n_U, \mathscr{Z}) \qquad \text{for} \qquad [n] \in \Delta, U \in \mathscr{C}.$$

Denote the coproduct in the pointed category $\mathsf{sShv}_{\tau,\bullet}(\mathscr{C})$ by $\vee$, and define the smash product of pointed simplicial τ-sheaves $(\mathscr{X}, x)$ and $(\mathscr{Y}, y)$ to be the cofibre of the canonical morphism $(\mathscr{X}, x) \vee (\mathscr{Y}, y) \to (\mathscr{X}, x) \times (\mathscr{Y}, y)$ in the category of pointed simplicial τ-sheaves. The category $\mathsf{sShv}_{\tau,\bullet}(\mathscr{C})$ admits a closed symmetric monoidal structure, with a monoidal product (resp. a unit object) given by the smash product (resp. the pointed simplicial τ-sheaf $\mathbb{1}_\wedge = *_+$), and whose internal $\underline{\mathtt{Hom}}_\bullet$ is induced from $\underline{\mathtt{Hom}}$ as in Proposition 1.2.56.

2.1.2. The Model Structures on Simplicial (Pre)sheaves. It is well-known that the category of simplicial presheaves on an essentially small site admits several model structures. Some of these structures depend on the topology, defined stalk-wise or using alternative local conditions, and they are said to be *local structures*; whereas the others, defined object-wise, and they are said to be *global structures*. These structures are usually defined based on Kan-Quillen's model structure on simplicial sets. Structures with cofibrations induced directly from cofibrations of simplicial sets are said to be *injective structures*; whereas structures with fibrations induced directly from fibrations of simplicial sets are said to be *projective structures*.

The global structures considered here are Quillen equivalent to each other. Also, the local structures are Quillen equivalent, assuming the site has enough points, and hence they give rise to the same homotopy theory. Moreover, local structures are left Bousfield localisations of their corresponding global ones.

Definition 2.1.3. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of simplicial presheaves on $\mathscr{C}$ is said to be an *object-wise* (or *section-wise*) weak equivalence, cofibration, or fibration if $f_U : \mathscr{X}(U) \to \mathscr{Y}(U)$ is a weak equivalence, cofibration, or fibration of simplicial sets, respectively, for every $U \in \mathscr{C}$.

The essential difference between local and global structures lies in weak equivalences, local weak equivalences are recalled in §.2.1.3. Assuming the site has enough points, local weak equivalences are morphisms that induce weak equivalences of simplicial sets stalk-wise.

The category $\mathsf{sPSh}(\mathscr{C})$ admits a proper cofibrantly generated simplicial model structure with weak equivalences, cofibrations and fibrations given by

- object-wise weak equivalences, object-wise cofibrations, RLP with respect to object-wise weak cofibrations, respectively; called the *global injective structure*, see [**Hel88**];
- object-wise weak equivalences, LLP with respect to object-wise weak fibrations, object-wise fibrations, respectively; called the *global projective structure* or *Bousfield-Kan model structure*, see [**BK72**, §.XI.8] and [**Dug01**];
- local weak equivalences, object-wise cofibrations, RLP with respect to local weak equivalences that are object-wise cofibrations, respectively; called the *local injective structure* or the *Joyal-Jardine model structure*, see [**Jar87**]; and
- local weak equivalences, LLP with respect to local weak equivalences that are object-wise fibrations, object-wise fibrations, respectively; called the *local projective structure*, see [**Bla01**].

LEMMA 2.1.4. The identity functor is a left Quillen equivalence from the global projective to the global injective model structures on $\mathsf{sPSh}(\mathscr{C})$.

PROOF. See [**DHI04**]. $\square$

The category of simplicial τ-sheaves admits model structures corresponding to the local model structures on simplicial presheaves, with weak equivalences, cofibrations, fibrations being weak equivalences, cofibrations, fibrations, respectively, in the corresponding local model structure of simplicial presheaves. These structures give equivalent homotopy theories.

The local injective model structure on simplicial presheaves is cofibrantly generated with sets
$\mathbf{I} \coloneqq \{\partial\Delta^n_U \hookrightarrow \Delta^n_U \mid n \geq 0, U \in \mathscr{C}\}$ and
$\mathbf{J} \coloneqq \{j : \mathscr{U} \hookrightarrow \mathscr{V} \mid j$ is a local injective weak cofibration, and $\mathscr{V}$ is κ-bounded$\}$
of generating cofibrations and generating weak cofibrations, respectively, for a cardinal $\kappa > 2^{|Mor(\mathscr{C})|}$. The set of cofibrations in the local injective model structure consists of inclusions of simplicial presheaves. This structure is a left Bousfield localisation of the global injective model stricture with respect to τ-hypercovers, see [**DHI04**].

2.1.3. Local Weak Equivalences. Local weak equivalences should be defined in a way that accounts for the site's topology. That can be achieved in different ways, using different topological and simplicial homotopy sheaves, stalks, or local lifting conditions, see [**DI04**].

The definitions due to Jardine in [**Jar87**] and Morel and Voevodsky in [**MV99**] depend on different functors of homotopy sheaves, and they are recalled below.

2.1.3.1. *Joyal's Homotopy Sheaves.* The *presheaves of path connected components* functor

$$\pi_0^{pre}(-) : \mathsf{sPSh}(\mathscr{C}) \to \mathsf{PSh}(\mathscr{C})$$

is given on an object $\mathscr{X} \in \mathsf{sPSh}(\mathscr{C})$ by

$$\pi_0^{pre}(\mathscr{X})(U) \coloneqq \pi_0^{\mathsf{Top}}(|\mathscr{X}(U)|),$$

for $U \in \mathscr{C}$; and the *τ-sheaves of path connected components* functor $\pi_0^{top}(-)$ is given by the the composition of the τ-sheafification with $\pi_0^{pre}(-)$.

Similarly, for an integer $n \geq 1$ and an object $U \in \mathscr{C}$, the *presheaves of the n^{th}-homotopy groups* functor

$$\pi_n^{pre}(-|_U, *) : \mathsf{sPSh}(\mathscr{C})_U \to \mathsf{PSh}^{\mathsf{Grp}}(\mathscr{C}\downarrow U)$$

is given on a pointed simplicial presheaf $(\mathscr{X}, x)$ over U by

$$\pi_n^{pre}(\mathscr{X}|_U, x)(V) \coloneqq \pi_n^{\mathsf{Top}}(|\mathscr{X}(V)|, x),$$

for every $V \in \mathscr{C}\downarrow U$; and the *$\tau$-sheaves of the n^{th}-homotopy groups* functor $\pi_n^{top}(-|_U, *)$ is given by the composition of the τ-sheafification with $\pi_n^{pre}(-|_U, *)$.

DEFINITION 2.1.5. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of simplicial presheaves on $\mathscr{C}$ is called a *topological local weak equivalence* if

- $\pi_0^{top}(f)$ is an isomorphism of τ-sheaves; and
- for every integer $n \geq 1$ and an object $U \in \mathscr{C}$, the morphism $\pi_n^{top}(f|_U, x)$ is an isomorphism of τ-sheaves, for all $x \in \mathscr{X}_0(U)$.

Topological local weak equivalences satisfy the two-out-of-three property, and contain object-wise weak equivalences.

2.1.3.2. *Morel and Voevodsky's Homotopy Sheaves.* The main difference between Joyal's homotopy sheaves and Morel and Voevodsky's homotopy sheaves is that the latter considers all base points at once. That is, it is defined on a bigger site that encodes both the category $\mathscr{C}$ and all the vertices of $\mathscr{X}$, namely $\mathsf{h}_- \downarrow \mathscr{X}_0$, where $\mathsf{h}_- : \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ is the Yoneda embedding, instead of the sites $(\mathscr{C}_\tau \downarrow U)$ for every object $U \in \mathscr{C}$.

There exists a bijection between the objects in $\mathsf{h}_- \downarrow \mathscr{X}_0$ and all the vertices of $\mathscr{X}$, which sends a morphism $f : \mathsf{h}_U \to \mathscr{X}_0$ of presheaves to the vertex $x_f \coloneqq f_U(id_U) \in \mathscr{X}_0(U)$.

DEFINITION 2.1.6. Let $\mathscr{X}$ be a simplicial presheaf on $\mathscr{C}$. For an integer $n \geq 0$, the *n-homotopy presheaf of $\mathscr{X}$* is defined to be the functor

$$\Pi_n^{pre}(\mathscr{X}) : (\mathsf{h}_- \downarrow \mathscr{X}_0)^{op} \to \mathsf{Set}_*, \tag{11}$$

given on a object $f : \mathsf{h}_U \to \mathscr{X}_0$ by

$$\Pi_n^{pre}(\mathscr{X})(f) \coloneqq \pi_n^{\mathsf{Top}}(|\mathscr{X}(U)|, x_f).$$

The *n-homotopy τ-sheaf $\Pi_n^{top}(\mathscr{X})$ of $\mathscr{X}$* is defined to be the τ-sheafification of $\Pi_n^{pre}(\mathscr{X})$, with respect to the induced topology on $\mathsf{h}_- \downarrow \mathscr{X}_0$, that is the coarser topology with respect to which the projection $p^{-1} : \mathsf{h}_- \downarrow \mathscr{C} \to \mathscr{C}$ is a continuous functor.

Let $p : \mathscr{C}_\tau \to (\mathsf{h}_- \downarrow \mathscr{X}_0)_{\mathsf{h}_- \downarrow \tau}$ be the continuous map of sites given by p^{-1}. For a simplicial presheaf $\mathscr{X}$ on $\mathscr{C}$, the direct image $p_*(\mathscr{X})$ is the presheaf on $\mathsf{h}_- \downarrow \mathscr{X}_0$, given on an object $f : \mathsf{h}_U \to \mathscr{X}_0$ by

$$p_*(\mathscr{X})(f) = \mathscr{X}_0(U),$$

which can be pointed canonically by x_f. Hence, it defines a pointed presheaf $p_*(\mathscr{X}_0)_*$ on $\mathsf{h}_- \downarrow \mathscr{X}_0$, endued with a natural transformation

$$p_{\mathscr{X}} : \Pi_n^{top}(\mathscr{X}) \to p_*(\mathscr{X}_0)_*^{\mathsf{a}_\tau}.$$

For a morphism of simplicial presheaves $f : \mathscr{X} \to \mathscr{Y}$, the induced functor

$$f^{-1} \coloneqq h_- \downarrow f : (h_- \downarrow \mathscr{X}_0) \to (h_- \downarrow \mathscr{Y}_0)$$

is continuous with respect to the induced topologies, and hence it defines a continuous map of sites $f : (\mathsf{h}_- \downarrow \mathscr{Y}_0)_{\mathsf{h}_-\downarrow\tau} \to (\mathsf{h}_- \downarrow \mathscr{X}_0)_{\mathsf{h}_-\downarrow\tau}$. In particular, the direct image

$$f_* : \mathsf{PSh}(h_- \downarrow \mathscr{Y}_0) \to \mathsf{PSh}(h_- \downarrow \mathscr{X}_0)$$

preserves τ-sheaves, giving rise to a natural transformation

$$\Pi_n^{top}(f) : \Pi_n^{top}(\mathscr{X}) \to f_*(\Pi_n^{top}(Y)).$$

For every integer $n \geq 0$, the square

$$\begin{tikzcd} \Pi_n^{top}(\mathscr{X}) \arrow[r, "\Pi_n^{top}(f)"] \arrow[d, "p_{\mathscr{X}}"'] \arrow[dr, phantom, "\circlearrowleft"] & f_*(\Pi_n^{top}(\mathscr{Y})) \arrow[d, "p_{\mathscr{Y}}"] \\ p_*(\mathscr{X}_0)_*^{\mathsf{a}_\tau} \arrow[r, "p_*(f_0)_*^{\mathsf{a}_\tau}"'] & p_*(\mathscr{Y}_0)_*^{\mathsf{a}_\tau} \end{tikzcd} \tag{12}$$

of pointed τ-sheaves on $(\mathsf{h}_- \downarrow \mathscr{X}_0)_{\mathsf{h}_-\downarrow\tau}$ commutes. In fact, the square above is Cartesian in $\mathsf{Shv}_{\mathsf{h}_-\downarrow\tau}(\mathsf{h}_- \downarrow \mathscr{X}_0)$ if and only if Joyal's morphism of sheaves $\pi_n^{top}(f|U, x_{0,U})$ is an isomorphism for every $U \in \mathscr{C}$ and $x_{0,U} \in \mathscr{X}_0(U)$. Therefore, f is a topological local weak equivalence if and only if for every integer $n \geq 0$ the square (12) is Cartesian in $\mathsf{Shv}_{\mathsf{h}_-\downarrow\tau}(\mathsf{h}_- \downarrow \mathscr{X}_0)$, see [**MV99**, §.2.Rem.1.3].

2.1.3.3. *Stalks of Simplicial Presheaves.* Let p be a point of the site $\mathscr{C}_\tau$, and let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of simplicial presheaves on $\mathscr{C}$. Then, the stalks functor at p induces a morphism of simplicial sets $p^* f : p^* \mathscr{X} \to p^* \mathscr{Y}$. The morphism $f : \mathscr{X} \to \mathscr{Y}$ is called a *stalk-wise weak equivalence* if $p^* f$ is a weak equivalence of simplicial sets for every point p of $\mathscr{C}_\tau$.

Topological local weak equivalences are point-wise weak equivalences, but the inverse is not true in general.

Recall that, for every point p of $\mathscr{C}_\tau$, the stalks functor p^* is given by filtered colimits, see §.A.4.2.1. The geometric realisation commutes with colimits as it is a left adjoint, and the connected component functor $\pi_0(-)$ and homotopy group functors $\pi_n(-, *)$ commute with filtered colimits. Then, for every morphism $f : \mathscr{X} \to \mathscr{Y}$ of simplicial presheaves on $\mathscr{C}$, there exist canonical bijections

$$p^*\big(\pi_0^{top}(f)\big) \cong \pi_0(|p^*(f)|) \qquad \text{and} \qquad p^*_{|U}\big(\pi_0^{top}(f_{|U}, x_0)\big) \cong \pi_n\big(\big|p^*_{|U}(f_{|U})\big|, x_0\big),$$

for every $U \in \mathscr{C}$ and for every integer $n \geq 1$. Therefore, when the site $\mathscr{C}_\tau$ has enough points, the notions of topological local weak equivalences and stalk-wise weak equivalences coincide.

2.1.4. The Homotopy Theory of Simplicial Sheaves for Sites with Intervals. The fibrant replacement for Bousfield localised model categories is rather complicated and difficult to work with. Morel and Voevodsky provided in [**MV99**] a relatively simpler fibrant repentance of Bousfield localised model with respect to an interval.

The main idea here is to use the contractability of the interval $\left|\Delta^1\right|$ in the classical homotopy theory of topological spaces, or equivalently the contractability of the simplicial set Δ^n, for every integer $n \geq 0$, and 'alter' (simplicial) τ-sheaves 'replacing' the τ-sheaf represented by the affine space $\mathbb{A}^n$ by the simplicial τ-sheaf of the contractable simplicial set Δ^n, for every integer $n \geq 0$.

Definition 2.1.7. An *interval* in the essentially small $\mathscr{C}_\tau$ is a quadruple (I, μ, i_0, i_1), where I is an τ-sheaf on $\mathscr{C}$, $\mu : I \times I \to I$ is a morphism in $\mathsf{Shv}_\tau(\mathscr{C})$, and i_0 and i_1 are two distinct morphisms $i_0, i_1 : * \to I$ in $\mathsf{Shv}_\tau(\mathscr{C})$, such that

- for the terminal morphism $p : I \to *$ in $\mathsf{Shv}_\tau(\mathscr{C})$, one has

$$\mu \circ (i_0 \times \mathrm{id}_I) \cong \mu \circ (\mathrm{id}_I \times i_0) \cong i_0 \circ p \qquad \text{and} \qquad \mu \circ (i_1 \times \mathrm{id}_I) \cong \mu \circ (\mathrm{id}_I \times i_1) \cong \mathrm{id}_I; \text{ and}$$

- the induced morphism $i_0 \coprod i_1 : * \coprod * \to I$ is a monomorphism.

Then, the morphism μ is called the *multiplication of I*.

An interval I in $\mathscr{C}_\tau$ defines a cosimplicial τ-sheaf $\Delta_I^\bullet : \mathit{\Delta} \to \mathsf{Shv}_\tau(C)$, given for an object $[n] \in \mathit{\Delta}$ by the n-fold Cartesian product $I^{\times n}$ in $\mathsf{Shv}_\tau(C)$, with codegeneracies

$$s_n^i : \Delta_I^{n+1} \to \Delta_I^n \qquad \text{for} \qquad 0 \leq i \leq n,$$

given by projecting out the $(i+1)^{\text{th}}$-term, *i.e.* $s_n^i = \mathrm{id}_I^{\times i} \times p \times \mathrm{id}_I^{\times(n-i)}$, and cofaces

$$d_n^i : \Delta_I^{n-1} \to \Delta_I^n \qquad \text{for} \qquad 0 \leq i \leq n,$$

given by

$$d_n^i = \begin{cases} i_1 \times \mathrm{id}_I^{\times(n-1)} & i = 0; \\ \mathrm{id}_I^{\times(i-1)} \times \delta_I \times \mathrm{id}_I^{\times(n-1-i)} & 0 < i < n; \\ \mathrm{id}_I^{\times(n-1)} \times i_0 & i = n; \end{cases}$$

where δ_I is the diagonal morphism $\delta_I : I \to I \times I$. The cosimplicial τ-sheaf $\Delta_I^\bullet$ is called the *cubical cosimplicial τ-sheaf associated to I*. A description of the morphism Δ_I^μ for any morphism $\mu : [m] \to [n]$ in $\mathit{\Delta}$ is given in [**Voe96**, p.88].

2.1.4.1. *Simplified Fibrant Replacement.* Assume that τ is subcanonical on $\mathscr{C}$, let I be an interval in $\mathscr{C}_\tau$, and let $\Delta_I^\bullet : \mathit{\Delta} \to \mathsf{Shv}_\tau(\mathscr{C})$ be the cubical cosimplicial τ-sheaf associated to the interval I. Since $\mathscr{C}_\tau$ is subcanonical, the Yoneda embedding $\mathsf{h}_- : \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ factorises though the category of τ-sheaves on $\mathscr{C}$, let $\underline{\Delta}_I^\bullet : \mathit{\Delta} \to \mathsf{sShv}_\tau(\mathscr{C})$ be the diagonal of the bicosimplicial simplicial τ-sheaf $\Delta_I^\bullet \times \Delta^\bullet : \mathit{\Delta} \times \mathit{\Delta} \to \mathsf{sShv}_\tau(\mathscr{C})$, and let $\underline{\Delta}_{I,-}^\bullet$ denote the functor $\underline{\Delta}_I^\bullet \times \mathsf{h}_- : \mathit{\Delta} \times \mathscr{C} \to \mathsf{sShv}_\tau(\mathscr{C})$. Since the category $\mathsf{sShv}_\tau(\mathscr{C})$

is cocomplete, there exists a $\underline{\Delta}^{\bullet}_{I,-}$-tensor and $\underline{\mathtt{Hom}}$ adjunction, given by the left Kan extensions for the span

$$\begin{tikzcd} \mathit{\Delta} \times \mathscr{C} \arrow[r, "\underline{\Delta}^{\bullet}_{I,-}"] \arrow[d, "\Delta^{\bullet}_{-}"'] & \mathsf{sShv}_{\tau}(\mathscr{C}) \\ \mathsf{sShv}_{\tau}(\mathscr{C}), & \end{tikzcd}$$

similar to Example A.3.8.

Denote the $\underline{\Delta}^{\bullet}_{I,-}$-tensor and $\underline{\mathtt{Hom}}$ functors by $\underline{|-|}_I$ and $\underline{\mathtt{Sing}}_I$, respectively. The functors $\underline{|-|}_I$ and $\underline{\mathtt{Sing}}_I$ are isomorphic to the functors $|-|_{\Delta^{\bullet}_I \times \Delta^{\bullet}}$ and $\mathtt{Sing}^I_*$, given in [**MV99**, p.90 and p.88]. That is, for a simplicial τ-sheaf $\mathscr{X} \in \mathsf{sShv}_{\tau}(\mathscr{C})$, one has

$$\begin{aligned} \underline{\mathtt{Hom}}^{\mathit{\Delta}\times\mathscr{C}}(\underline{\Delta}^{\bullet}_{I,-}, \mathscr{X})_n\big(U\big) &= \mathsf{sShv}_{\tau}(\mathscr{C})\big(\Delta^n_I \times \Delta^n \times \mathtt{h}_U, \mathscr{X}\big) \\ &\cong \mathsf{sShv}_{\tau}(\mathscr{C})\big(\Delta^n_I \times \mathtt{h}_U, \underline{\mathtt{Hom}}_{\mathsf{sShv}_{\tau}(\mathscr{C})}(\Delta^n, \mathscr{X})\big) \\ &\cong \mathsf{Shv}_{\tau}(\mathscr{C})\big(\Delta^n_I \times \mathtt{h}_U, \underline{\mathtt{Hom}}_{\mathsf{sShv}_{\tau}(\mathscr{C})}(\Delta^n, \mathscr{X})_0\big) \\ &\cong \mathsf{Shv}_{\tau}(\mathscr{C})\big(\Delta^n_I \times \mathtt{h}_U, \mathscr{X}_n\big) = \underline{\mathtt{Hom}}_{\mathsf{Shv}_{\tau}(\mathscr{C})}\big(\Delta^n_I, \mathscr{X}_n\big)(U) \\ &= \mathtt{Sing}^I_*(\mathscr{X})_n(U), \end{aligned}$$

for every $[n] \in \mathit{\Delta}$ and $U \in \mathscr{C}$. On the other hand,

$$\begin{aligned} \mathscr{X} \otimes_{\mathit{\Delta}\times\mathscr{C}} \underline{\Delta}^{\bullet}_{I,-} &= \int^{([n],U)\in\mathit{\Delta}\times\mathscr{C}} \coprod_{\mathsf{sShv}_{\tau}(\mathscr{C})(\Delta^n_U, \mathscr{X})} \Delta^n_I \times \Delta^n \times \mathtt{h}_U \\ &\cong \int^{[n]\in\mathit{\Delta}} \int^{U\in\mathscr{C}} \coprod_{\mathsf{Shv}_{\tau}(\mathscr{C})(\mathtt{h}_U, \mathscr{X}_n)} \Delta^n_I \times \Delta^n \times \mathtt{h}_U \\ &\cong \int^{[n]\in\mathit{\Delta}} \Delta^n_I \times \Delta^n \times \int^{U\in\mathscr{C}} \coprod_{\mathsf{Shv}_{\tau}(\mathscr{C})(\mathtt{h}_U, \mathscr{X}_n)} \mathtt{h}_U \\ &\cong \int^{[n]\in\mathit{\Delta}} \Delta^n_I \times \Delta^n \times \mathscr{X}_n = |\mathscr{X}|_{\Delta^{\bullet}_I \times \Delta^{\bullet}}. \end{aligned}$$

Similarly, considering the left Kan extensions for the span

$$\begin{tikzcd} \mathit{\Delta} \times \mathscr{C} \arrow[r, "\Delta^{\bullet}_{I,-}"] \arrow[d, "\Delta^{\bullet}_{-}"'] & \mathsf{Shv}_{\tau}(\mathscr{C}) \\ \mathsf{sShv}_{\tau}(\mathscr{C}) & \end{tikzcd}$$

yields a $\Delta^{\bullet}_{I,-}$-tensor and $\underline{\mathtt{Hom}}$ adjunction. Denote the $\Delta^{\bullet}_{I,-}$-tensor and $\underline{\mathtt{Hom}}$ functors by $|-|_I$ and $\mathtt{Sing}_I$, respectively. The functors $|-|_I$ and $\mathtt{Sing}_I$ are isomorphic to the functor $|-|_{\Delta^{\bullet}_I}$ and the restriction of the functor $\mathtt{Sing}^I_*$, given in [**MV99**, p.90 and p.88], to the category $\mathsf{Shv}_{\tau}(\mathscr{C})$.

The functors $\underline{\mathtt{Sing}}_I$ and $\mathtt{Sing}_I$ preserve filtered colimits, as representable simplicial τ-sheaves (resp. representable τ-sheaves) are compact objects in $\mathsf{sShv}_{\tau}(\mathscr{C})$ (resp.

$\mathsf{Shv}_\tau(\mathscr{C})$). Moreover, the functor $\underline{\mathtt{Sing}}_I$ and $\mathtt{Sing}_I$ commute with limits for being right adjoints, and hence they are symmetric monoidal. However, neither $\underline{|-|}_I$ nor $|-|_I$ preserves finite product or is symmetric monoidal. That is, for a pair of integers $n, m > 0$,

$$|\Delta^n \times \Delta^m|_I \cong \int^{[p]\in\Delta} (\Delta^n \times \Delta^m)_p \times \Delta^p_I \cong \int^{[p]\in\Delta} \Delta^{nm+n+m}_p \times \Delta^p_I \cong \Delta^{nm+n+m}_I \ncong \Delta^{n+m}_I \cong \Delta^n_I \times \Delta^m_I.$$

Similar to Kan's $\mathtt{Ex}$ functor in §.1.2.4.1, the split epimorphism $\underline{\Delta}^\bullet_{I,-} \to \Delta^\bullet_-$ induces a monomorphism

$$X \hookrightarrow \underline{\mathtt{Sing}}_I(\mathscr{X})$$

of simplicial presheaves that is also an I-weak equivalence, for every $\mathscr{X} \in \mathsf{sShv}_\tau(\mathscr{C})$, see [**MV99**, §.2.Cor.3.8]. The functor $\underline{\mathtt{Sing}}_I$ takes the projection $\mathscr{X} \times I \to \mathscr{X}$ to a local weak equivalence, for every $\mathscr{X} \in \mathsf{sShv}_\tau(\mathscr{C})$, see [**MV99**, §.2.Cor.3.5]. Moreover, for a fibrant replacement R_τ for the model category $\mathsf{sShv}_\tau(\mathscr{C})_{\mathtt{inj}}$ and for a large enough ordinal λ, the transfinite composition

$$(R_\tau \circ \underline{\mathtt{Sing}}_I)^\lambda \circ R_\tau$$

is a fibrant replacement for the left Bousfield localisation of $\mathsf{sShv}_\tau(\mathscr{C})_{\mathtt{inj}}$ with respect to the set of projections $\{\mathscr{X} \times I \to \mathscr{X} \mid \mathscr{X} \in \mathsf{sShv}_\tau(\mathscr{C})\}$, see [**MV99**, §.2.Lem.3.21].

2.2. τ-Local Homotopy of Schemes

Let S be a Noetherian scheme of finite Krull dimension. Recall the conventions and notations in §.0.2 and suppose that τ is a subcanonical topology on the category Sm/S of smooth S-scheme. Let $\mathsf{sShv}_\tau(\mathsf{Sm}/S)^{\mathrm{loc}}_{\mathtt{inj}}$ and $\mathsf{sShv}_{\tau,\bullet}(\mathsf{Sm}/S)^{\mathrm{loc}}_{\mathtt{inj}}$ be the local injective model categories of simplicial τ-sheaves and pointed simplicial τ-sheaves, respectively, and denote their homotopy categories by $\mathcal{H}^s_\tau(S)$ and $\mathcal{H}^s_{\tau,\bullet}(S)$, respectively.

The Cartesian and smash products preserve τ-local weak equivalences, and they induce derived closed symmetric monoidal structures on the homotopy categories $\mathcal{H}^s_\tau(S)$ and $\mathcal{H}^s_{\tau,\bullet}(S)$, respectively.

2.2.1. The B.G.-Property in the Nisnevich Topology. A simplicial presheaf $\mathscr{X} \in \mathsf{sPSh}(\mathsf{Sm}/S)$ is said to have the **B.G.**-property if it sends Nisnevich distinguished squares to homotopy Cartesian squares of simplicial set, see Definition A.4.28. This is an analogue of Brown and Gersten construction in Zariski topology, as in [**BG73**].

Example 2.2.1. Fibrant simplicial Nisnevich sheaves have the **B.G.**-property, see [**MV99**, §.3.Rem.1.15].

Lemma 2.2.2. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of simplicial presheaves on Sm/S that has the **B.G.**-property whose Nisnevich sheafification $\mathsf{a}_{\mathrm{Nis}}(f) : \mathsf{a}_{\mathrm{Nis}}(\mathscr{X}) \to \mathsf{a}_{\mathrm{Nis}}(\mathscr{Y})$ is a Nisnevich-local weak equivalence. Then, f is an object-wise weak equivalence.

Proof. See [**MV99**, §.3.Lem.1.18]. □

2.2.2. Functoriality. For a morphism $f : S \to T$ of Noetherian schemes of finite Krull dimension, the functor $f^{-1} : \mathsf{Sm}/T \to \mathsf{Sm}/S$ given by base change along f defines a continuous map of sites $f^{-1} : \mathsf{Sm}/S_{\mathrm{Nis}} \to \mathsf{Sm}/T_{\mathrm{Nis}}$. The induced adjunction

$$f^* : \mathsf{sShv}_{\mathrm{Nis}}(\mathsf{Sm}/T) \rightleftarrows \mathsf{sShv}_{\mathrm{Nis}}(\mathsf{Sm}/S) : f_*$$

is a Quillen pair, by [**MV99**, §.3.Prop.1.20], and hence it induces total derived functors

$$\mathbf{L}\, f^* : \mathcal{H}^s_{\mathrm{Nis}}(S) \to \mathcal{H}^s_{\mathrm{Nis}}(T) : \mathbf{R}\, f_*.$$

Moreover, for a smooth morphism $f : S \to T$, the functor f^* admits a left adjoint $f_\#$ given by composing with f, and it induces an adjunction $\mathbf{L}\, f_\# \dashv \mathbf{L}\, f^*$.

2.3. The Unstable $\mathbb{A}^1$-Homotopy of Schemes

The $\mathbb{A}^1_S$-*model categories* $\mathsf{sShv}_\tau(\mathsf{Sm}/S)_{\mathbb{A}^1_S}$ and $\mathsf{sShv}_{\tau,\bullet}(\mathsf{Sm}/S)_{\mathbb{A}^1_S}$ are defined to be the left Bousfield localisation of $\mathsf{sShv}_\tau(\mathsf{Sm}/S)^{\mathrm{loc}}_{\mathtt{inj}}$ and $\mathsf{sShv}_{\tau,\bullet}(\mathsf{Sm}/S)^{\mathrm{loc}}_{\mathtt{inj}}$, respectively, with respect to the set of projections $\left\{\mathscr{X} \times \mathbb{A}^1_S \to \mathscr{X} \mid \mathscr{X} \in \mathsf{sShv}_\tau(\mathsf{Sm}/S)\right\}$. The resulting local weak equivalences are called $\mathbb{A}^1_S$-*weak equivalences*. Then, the *unpointed* and (resp. *pointed*) *homotopy category* of schemes over S is defined to be the homotopy category of $\mathsf{sShv}_\tau(\mathsf{Sm}/S)_{\mathbb{A}^1_S}$ (resp. $\mathsf{sShv}_{\tau,\bullet}(\mathsf{Sm}/S)_{\mathbb{A}^1_S}$), and it is denoted by $\mathcal{H}(S)$ (resp. $\mathcal{H}_\bullet(S)$). Objects of $\mathcal{H}(S)$ and $\mathcal{H}_\bullet(S)$ are called *unpointed* and *pointed motivic spaces*, respectively.

These homotopy categories are reflective localisations of the Nisnevich local homotopy categories. In particular, for an $\mathbb{A}^1_S$-fibrant simplicial Nisnevich sheaves $\mathscr{X}$ and $\mathscr{Y}$, there exists a bijection

$$\mathcal{H}(S)\,(\mathscr{X}, \mathscr{Y}) \cong \mathcal{H}^s_{\mathrm{Nis}}(S)\,(\mathscr{X}, \mathscr{Y})\,.$$

In fact, the category $\mathcal{H}(S)$ is equivalent to the full subcategory $\mathcal{H}^s_{\mathrm{Nis},\mathbb{A}^1_S}(S)$ of $\mathbb{A}^1_S$-local simplicial sheaves in $\mathcal{H}^s_{\mathrm{Nis}}(S)$. That is, the inclusion $\mathcal{H}^s_{\mathrm{Nis},\mathbb{A}^1_S}(S) \hookrightarrow \mathcal{H}^s_{\mathrm{Nis}}(S)$ admits a left adjoint

$$\mathsf{L}_{\mathbb{A}^1_S} : \mathcal{H}^s_{\mathrm{Nis}}(S) \to \mathcal{H}^s_{\mathrm{Nis},\mathbb{A}^1_S}(S),$$

called the $\mathbb{A}^1_S$-*localisation functor*, which sends $\mathbb{A}^1_S$-weak equivalences to topological Nisnevich local equivalences.

The Cartesian and smash products also preserve $\mathbb{A}^1_S$-weak equivalences, and hence the induce closed symmetric monoidal structures on the categories $\mathcal{H}^s_\tau(S)$ and $\mathcal{H}^s_{\tau,\bullet}(S)$, respectively. In particular, for the *simplicial sphere*

$$S^1_s \coloneqq (S^1, [\delta^1_1]) = (\Delta^1, \delta^1_1)/(\Delta^0, \mathrm{id}_{[0]}) \in \mathsf{sShv}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/S),$$

the Quillen pair

$$- \wedge S^1_s : \mathsf{sShv}_{\tau,\bullet}(\mathsf{Sm}/S) \rightleftarrows \mathsf{sShv}_{\tau,\bullet}(\mathsf{Sm}/S) : \underline{\mathtt{Hom}}_\bullet(S^1_s, -)$$

induces derived functors

$$\mathbf{\Sigma} : \mathcal{H}_\bullet(S) \to \mathcal{H}_\bullet(S) : \mathbf{\Omega},$$

called the *simplicial suspension* and the *simplicial loop spaces* functors, respectively.

2.3.1. Functoriality. A morphism $f : S \to T$ of Noetherian schemes of finite Krull dimension induces an adjunction of total derived functors

$$\mathbf{L}\, f^* : \mathcal{H}^s_{\mathrm{Nis}}(T) \rightleftarrows \mathcal{H}^s_{\mathrm{Nis}}(S) : \mathbf{R}^{\mathbb{A}^1_S} f_*,$$

as the functor f^* preserves $\mathbb{A}^1_S$-weak equivalences. Moreover, when f is smooth, the functor $\mathbf{L}\, f_\#$ preserves $\mathbb{A}^1_S$-weak equivalences, and it induces an adjunction

$$\mathbf{L}\, f_\# : \mathcal{H}^s_{\mathrm{Nis}}(S) \rightleftarrows \mathcal{H}^s_{\mathrm{Nis}}(T) : \mathbf{L}\, f^*,$$

see [**MV99**, §.3.Prop.2.8-9].

Theorem 2.3.1 (Gluing Theorem). Let S be a Noetherian scheme of finite Krull dimension, let $i : Z \hookrightarrow S$ be a closed immersion with an open complement $j : U \hookrightarrow S$, and let $\mathscr{X}$ be a Nisnevich simplicial sheaf on Sm/S. Then, the square

$$\begin{tikzcd}
\mathbf{L}\, j_\# j^* \mathscr{X} \arrow[rrr] \arrow[d] & & & \mathscr{X} \arrow[d] \\
U \cong \mathbf{L}\, j_\# j^* S \arrow[r] & S \arrow[r] & i_* \mathbf{L}\, i^* S \arrow[r] & i_* \mathbf{L}\, i^* \mathscr{X},
\end{tikzcd}$$

in $\mathcal{H}(S)$, induced by the unit and counit of the adjunctions, is homotopy cocartesian.

Proof. See [**MV99**, §.3.Th.2.21]. ☐

Lemma 2.3.2. For a proper *cdh*-square

$$\begin{tikzcd}
z \times_x y \arrow[r, hook, "i'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\ulcorner"] & y \arrow[d, "p"] \\
z \arrow[r, hook, "i"'] & x
\end{tikzcd}$$

in Sm/S, its simplicial suspension is a homotopy cocartesian square in $\mathcal{H}(S)$.

Proof. See [**MV99**, §.3.Rem.2.30]. ☐

However, it does not seem to be known whether the square, without suspension, is a homotopy cocartesian square in $\mathcal{H}(S)$, see [**Voe10b**, p.1406].

Theorem 2.3.3 (Purity Theorem). Let S be a Noetherian scheme of finite Krull dimension, let $i : Z \hookrightarrow S$ be a closed immersion, and let $\mathscr{N}_{S,Z} \to Z$ be the normal vector bundle associated to i, with zero sections i_0. Then, the quotients ${}^{X}\!/_{(X - i(Z))}$ and ${}^{\mathscr{N}_{S,Z}}\!/_{(\mathscr{N}_{S,Z} - i_0(Z))}$ in $\mathcal{H}(S)$ are isomorphic.

Proof. See [**MV99**, §.3.Th.2.23]. ☐

2.3.2. Motivic Spheres. The cocartesian square

$$\begin{tikzcd} \mathbb{G}\mathrm{m}_S \arrow[r] \arrow[d] & \mathbb{A}^1_S \arrow[d] \\ \mathbb{A}^1_S \arrow[r] & \mathbb{P}^1_S \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in Sm/S induces an isomorphism of pointed Nisnevich sheaves $T_S \coloneqq \mathbb{A}^1_S/\mathbb{G}\mathrm{m}_S \cong \mathbb{P}^1_S/\mathbb{A}^1_S$. After contracting $\mathbb{A}^1_S$ in $\mathcal{H}_\bullet(S)$, the square above induce isomorphisms $T_S \cong (\mathbb{P}^1_S, \infty) \cong \mathbf{\Sigma}_s(\mathbb{G}\mathrm{m}_S, 1) = S^1_s \wedge (\mathbb{G}\mathrm{m}_S, 1)$, see [**MV99**, §.3.Cor.2.18] and [**Hov99**, §.8.1].

The pointed Nisnevich sheaf $(\mathbb{G}\mathrm{m}_S, 1)$ is called the *Tate sphere*, and denoted by S^1_t. For a pair of integers $p, q \in \mathbb{Z}_{\geq 0}$, the *mixed sphere* $S^{p,q}$ is defined as

$$S^{p,q} \coloneqq (S^1_s)^{\wedge p-q} \wedge (S^1_t)^{\wedge p}.$$

In particular, one has $(\mathbb{P}^1_S, \infty) \cong S^{2,1}$.

2.4. Stable Motivic Homotopy Theories

2.4.1. Stable $\mathbb{A}^1$-Homotopy Theory of S^1-Spectra. Let k be a field, and let $\mathsf{Spt}^\Sigma_s(k)$ denote the category of S^1_s-symmetric spectra of pointed simplicial Nisnevich sheaves on Sm/k, *i.e.*

$$\mathsf{Spt}^\Sigma_s(k) = \mathsf{Spt}^\Sigma_T(\mathsf{sShv}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/k), S^1_s),$$

and let $\mathbf{\Sigma}^\infty_s : \mathsf{sShv}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/k) \hookrightarrow \mathsf{Spt}^\Sigma_s(k)$ be the associated S^1_s-*symmetric suspension spectrum*. Model structures on $\mathsf{sShv}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/k)$ induce corresponding level and S^1_s-stable model structures on $\mathsf{Spt}^\Sigma_s(k)$, as seen in §.1.3.1.

Let $\mathsf{Spt}^\Sigma_s(k)_{\mathrm{stab}}$ be the S^1_s-stable model category induced by the local injective model structure on $\mathsf{sShv}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/k)$. Weak equivalences in $\mathsf{Spt}^\Sigma_s(k)_{\mathrm{stab}}$ are called S^1_s-*stable weak equivalences*. Let $\mathcal{SH}_{S^1_s}(k)$ be the homotopy category of $\mathsf{Spt}^\Sigma_s(k)_{\mathrm{stab}}$, called the S^1_s-*stable homotopy category*. Then, the left derived simplicial suspension $\mathbf{L}\,\mathbf{\Sigma}_s : \mathcal{SH}_{S^1_s}(k) \to \mathcal{SH}_{S^1_s}(k)$ is an equivalence of categories, see Theorem 1.3.5.

Since $\mathsf{Spt}^\Sigma_s(k)_{\mathrm{stab}}$ is a left proper model category, it admits Bousfield localisations with respect to small sets of its morphisms. Let $\mathsf{Spt}^\Sigma_s(k)^{\mathbb{A}^1_k}_{\mathrm{stab}}$ be the left Bousfield localisation of $\mathsf{Spt}^\Sigma_s(k)_{\mathrm{stab}}$ with respect to the set of projections $\{\mathbf{\Sigma}^\infty_s U_+ \wedge \mathbf{\Sigma}^\infty_s(\mathbb{A}^1_k, 0) \to \mathbf{\Sigma}^\infty_s U_+ \mid U \in \mathsf{Sm}/k\}$, and let $\mathcal{SH}_s(k)$ be its homotopy category, called the *motivic S^1_s-stable homotopy category*, see [**VRØ07**, §.2.2]. Weak equivalences in $\mathsf{Spt}^\Sigma_s(k)^{\mathbb{A}^1_k}_{\mathrm{stab}}$ are called $\mathbb{A}^1_k$-*stable weak equivalences of S^1_s-symmetric spectra*. Also, the simplicial suspension $\mathbf{\Sigma}_s$ induces an equivalence of categories $\mathbf{L}\,\mathbf{\Sigma}_s : \mathcal{SH}_s(k) \to \mathcal{SH}_s(k)$.

2.4.2. Stable $\mathbb{A}^1$-Homotopy Theory of $\mathbb{P}^1$-Spectra. Let $\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)$ denote the category of $(\mathbb{P}^1_k,\infty)$-symmetric spectra of pointed simplicial Nisnevich sheaves on Sm/k, and let $\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)_{\mathbb{A}^1,\mathrm{stab}}$ be the $(\mathbb{P}^1_k,\infty)$-stable model category induced by the $\mathbb{A}^1_k$-model structure on $\mathsf{sShv}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/k)$, recalled in §.2.3. Its weak equivalences are called *motivic stable weak equivalence*. Let $\mathcal{SH}(k)$ be the homotopy category of $\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)_{\mathbb{A}^1,\mathrm{stab}}$, called the *motivic stable homotopy category*.

Alternatively, $\mathcal{SH}(k)$ may be defined from $\mathcal{SH}_s(k)$, using bigraded homotopy sheaves. For a $(\mathbb{P}^1_k,\infty)$-symmetric spectrum $E\in\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)$, an S^1_s-spectrum $F\in\mathsf{Spt}^{\Sigma}_s(k)$, and an integer $m\geq 0$, there exists a canonical composite morphism

$$\begin{aligned}\pi_m(F,E) \coloneqq{}& \mathcal{SH}_s(k)\big(F \overset{\mathsf{L}}{\wedge}\ \mathbf{\Sigma}^{\infty}_s(\mathbb{P}^1_k,\infty)^m, \mathbf{\Sigma}^{\infty}_s E_m\big)\to\\ & \mathcal{SH}_s(k)\big(F \overset{\mathsf{L}}{\wedge}\ \mathbf{\Sigma}^{\infty}_s(\mathbb{P}^1_k,\infty)^{m+1}, \mathbf{\Sigma}^{\infty}_s \left(E_m\ \wedge\ (\mathbb{P}^1_k,\infty)\right)\big)\to\\ & \mathcal{SH}_s(k)\big(F \overset{\mathsf{L}}{\wedge}\ \mathbf{\Sigma}^{\infty}_s(\mathbb{P}^1_k,\infty)^{m+1}, \mathbf{\Sigma}^{\infty}_s E_{m+1}\big) = \pi_{m+1}(F,E),\end{aligned}$$

and hence a sequence $\pi_\bullet(F,E):\mathbb{Z}_{\geq 0}\to\mathsf{Set}$. Then, for integers $p,q\geq 0$, define the bigraded *homotopy presheaves functor* $\pi^{\mathrm{pre}}_{p,q}$ to be the functor $\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)\to\mathsf{PSh}(\mathsf{Sm}/k)$ given on an object $E\in\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)$ by

$$\pi^{\mathrm{pre}}_{p,q}(E)(-) \coloneqq \mathtt{colim}\,\pi_\bullet(\mathbf{\Sigma}^{\infty}_s(-)_+ \overset{\mathsf{L}}{\wedge}\ \mathbf{\Sigma}^{\infty}_s S^{p,q}, E),$$

and let the bigraded *homotopy Nisnevich sheaves functor* $\pi_{p,q}$ be the composition of the Nisnevich sheafification with $\pi^{\mathrm{pre}}_{p,q}$.

LEMMA 2.4.1. A morphism $f:E\to F$ in $\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)$ is a motivic stable weak equivalence if and only if $\pi_{p,q}(f)$ is an isomorphism of Nisnevich sheaves for every pair of integers $p,q\geq 0$.

PROOF. See [**VRØ07**, §.5]. ☐

$\mathtt{hom}$-sets in $\mathcal{SH}(k)$ can be expressed in terms of colimits of $\mathtt{hom}$-sets in $\mathcal{SH}_s(k)$. That is, for a k-scheme $X\in\mathsf{Sm}/k$ and for a symmetric spectrum $E\in\mathsf{Spt}^{\Sigma}_{\mathbb{P}^1_k}(k)$, there exists a sequence $(X,E)_\bullet:\mathbb{Z}_{\geq 0}\to\mathsf{Set}$, given for every integer $n\geq 0$ by the canonical composite morphism

$$\begin{aligned}(X,E)_m \coloneqq{}& \mathcal{SH}_s(k)\big(\mathbf{\Sigma}^{\infty}_s(X_+\wedge S_t^{\wedge m}), \mathbf{\Sigma}^{\infty}_s E_m\big)\to\\ & \mathcal{SH}_s(k)\big(\mathbf{\Sigma}^{\infty}_s\left(X_+\wedge S_t^{\wedge m+1}\right), \mathbf{\Sigma}^{\infty}_s\left(E_m\wedge S^1_t\right)\big)\to\\ & \mathcal{SH}_s(k)\big(\mathbf{\Sigma}^{\infty}_s\left(X_+\wedge S_t^{\wedge m+1}\right), \mathbf{\Sigma}^{\infty}_s E_{m+1}\big) = (X,E)_{m+1},\end{aligned}$$

which induces an isomorphism

$$\mathcal{SH}(k)\big(\mathbf{\Sigma}^{\infty}_{\mathbb{P}^1_k} X_+, E\big)\cong \mathtt{colim}\ (X,E)_\bullet,$$

see [**VRØ07**, Prop.2.13].

Triangulated Structure. The motivic stable homotopy categories are triangulated, whose suspensions are always taken to be the simplicial suspensions, and whose distinguished triangles are symmetric suspension spectra of cofibre sequences of simplicial sheaves.

2.5. Motivic Complexes

Throughout this section, let k be a field and let R be a commutative unitary ring. Recall the conventions and notations in §.0.2. In particular, the category of schemes of finite type over k is denoted by $\mathsf{Sch}^{\mathrm{ft}}/k$. Also, the subcategory in $\mathsf{Sch}^{\mathrm{ft}}/k$ of smooth (resp. smooth projective) schemes over k is denoted by Sm/k (resp. SmProj/k). An k-scheme (resp. a smooth k-scheme) refers to an object in $\mathsf{Sch}^{\mathrm{ft}}/k$ (resp. Sm/k).

2.5.1. Finite Correspondences. In the construction of pure motives over k, one considers algebraic correspondences modulo rational equivalence for Chow motives, or an adequate equivalence relation for other pure motives, in order to obtain a well-defined composition homomorphism. This approaches depends on the Moving Lemma, and it is restricted to smooth proper k-schemes. Instead of considering such quotients, one may restrict correspondences to subgroups that provide a well-defined composition, *i.e.* they guarantee proper intersections in the corresponding product schemes. That has been realised using finite correspondences, as in [**VSF00**].

Definition 2.5.1. Let S be a smooth k-scheme, and let $X \to S$ a morphism of schemes of finite type. A prime cycle $Z \in \mathrm{C}^*(X)$ is said to be *elementary over S* if the composition $f_Z : \mathsf{Supp}(Z) \hookrightarrow X \to S$ is a finite morphism that is surjective on a connected component of S. Then, let $\mathrm{c}(X/S)$ denote the abelian group generated by elementary cycles on X over S, called the *group of finite cycles* on X over S.

Definition 2.5.2. Let X and Y be smooth k-schemes. An algebraic correspondence $\Gamma : X \vdash Y$ is said to be *finite* if it is a finite cycle on $X \times Y$ over X, along the canonical projection $X \times Y \overset{pr_X}{\to} X$. Denote the group of *finite correspondences* from X to Y by

$$\mathsf{FCor}(X, Y) \coloneqq \mathrm{c}(X \times Y/X).$$

When X is irreducible, one has $\mathtt{dim}\,\mathsf{Supp}\,\Gamma = \mathtt{dim}\,X$. Since connected components of smooth k-schemes coincide with their irreducible components, on has a decomposition

$$\mathsf{FCor}(X, Y) = \bigoplus_{i \in I} \mathsf{FCor}(X_i, Y) \subset \bigoplus_{i \in I} \mathrm{C}_{\mathtt{dim}\,X_i}(X_i \times Y),$$

where $\{X_i \mid i \in I\}$ is the set of connected components of X.

Example 2.5.3. Let $f : X \to Y$ be a morphism of smooth k-schemes. The morphism $\Gamma_f \hookrightarrow X \times Y \twoheadrightarrow X$ is an isomorphism, and the algebraic correspondence $[\Gamma_f] : X \vdash Y$ is a finite correspondence from X to Y.

LEMMA 2.5.4. Let X, Y, and Z be smooth k-schemes, and let

$$\Gamma \in \mathsf{FCor}(X, Y) \qquad \text{and} \qquad \Theta \in \mathsf{FCor}(Y, Z)$$

be elementary correspondences. Then, the cycles $\Gamma \times Z$ and $X \times \Theta$ intersect properly in $X \times Y \times Z$. Moreover, the pushforward

$$\Theta \circ \Gamma = (pr_{XZ})_* \big((\Gamma \times Z) \cdot (X \times \Theta) \big)$$

is a finite correspondence from X to Z.

PROOF. See [**MVW06**, Lem.1.7]. $\square$

Therefore, consecutive elementary correspondences are composable. Extending bilinearly, one has a well-defined group homomorphism

$$\circ : \mathsf{FCor}(Y, Z) \times \mathsf{FCor}(X, Y) \to \mathsf{FCor}(X, Z). \tag{13}$$

DEFINITION 2.5.5. The *category of finite correspondences over k with coefficients in R* is defined to be the R-linear category $\mathsf{SmCor}(k, R)$, given by

- a set of objects $\mathtt{Ob}(\mathsf{SmCor}(k, R)) = \mathsf{Sm}/k$;
- for a pair (X, Y) of objects in $\mathtt{Ob}(\mathsf{SmCor}(k, R))$, the R-module of finite correspondences with coefficient in R

$$\mathsf{SmCor}(k, R)(X, Y) \coloneqq \mathsf{FCor}(X, Y) \bigotimes_{\mathbb{Z}} R;$$

- for a triple (X, Y, Z) of objects in $\mathtt{Ob}(\mathsf{SmCor}(k, R))$, the composition R-bilinear homomorphism induced from (13); and
- for an object $X \in \mathsf{SmCor}(k, R)$, the R-linear homomorphism of modules

$$1_X : R \to \mathsf{SmCor}(k, R)(X, X)$$

sending 1 to $[\Gamma_{id_X}] = [\Delta_X] : X \vdash X$.

For $R = \mathbb{Z}$, write $\mathsf{SmCor}(k) \coloneqq \mathsf{SmCor}(k, R)$.

There exists a well-defined covariant faithful (but not full) functor

$$\begin{array}{cccc} [-]_{k,R} : & \mathsf{Sm}/k & \to & \mathsf{SmCor}(k, R) \\ & X & \mapsto & X \\ & f & \mapsto & [\Gamma_f]. \end{array}$$

The category $\mathsf{SmCor}(k, R)$ is additive, with a direct sum given by disjoint union of schemes. Also, it is a symmetric monoidal category, with a monoidal product given by the Cartesian product of smooth k-schemes.

2.5.2. Geometric Motives. Let $\mathcal{K}^b(\mathsf{SmCor}_k)$ be the bounded homotopy category of complexes in $\mathsf{SmCor}(k)$, and let

$$\begin{array}{cccccc} [-] : & \mathsf{Sm}/k & \longrightarrow & \mathsf{SmCor}_k & \longrightarrow & \mathcal{K}^b(\mathsf{SmCor}_k) \\ & X & \mapsto & X & \mapsto & \cdots \to \mathbf{0} \to X \to \mathbf{0} \to \cdots \\ & f & \mapsto & [\Gamma_f] & \mapsto & \cdots \to \mathbf{0} \to [\Gamma_f] \to \mathbf{0} \to \cdots \end{array}$$

be the evident functor sending each smooth k-schemes X to the complex concentrated on X in degree zero, and let T be the set of complexes in $\mathcal{K}^b(\mathsf{SmCor}_k)$ of the form

- $\cdots \to \mathbf{0} \to [X \times \mathbb{A}^1_k] \overset{[\Gamma_{pr_X}]}{\to} [X] \to \mathbf{0} \to \cdots$, for $X \in \mathsf{Sm}/k$; and
- $\cdots \to \mathbf{0} \to [U \cap V] \overset{[\Gamma_{j_U}] \oplus [\Gamma_{j_V}]}{\longrightarrow} [U] \oplus [V] \overset{[\Gamma_{i_U}] \oplus (-[\Gamma_{i_V}])}{\longrightarrow} [X] \to \mathbf{0} \to \cdots$, for $X \in \mathsf{Sm}/k$ and an open covering $X = U \cup V$.

Then, let $\mathscr{T}$ be the thick closure of T in $\mathcal{K}^b(\mathsf{SmCor}_k)$.

Definition 2.5.6. The category $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{R})$ of *effective geometric motives* over k with R-coefficients is defined to be the Karoubian envelope of the the Verdier quotient $\mathcal{K}^b(\mathsf{SmCor}_k)/\mathscr{T}$ of $\mathcal{K}^b(\mathsf{SmCor}_k)$ with respect to the tick subcategory $\mathscr{T}$. The composition functor

$$\mathsf{Sm}/k \xrightarrow{[-]} \mathcal{K}^b(\mathsf{SmCor}_k) \xrightarrow{-/\mathscr{T}} \mathcal{K}^b(\mathsf{SmCor}_k)/\mathscr{T} \xrightarrow{-^\natural} \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \coloneqq (\mathcal{K}^b(\mathsf{SmCor}_k)/\mathscr{T})^\natural$$

is denoted by $M_{\mathrm{gm},R}$, and called the *geometric motive* functor. For $R = \mathbb{Z}$, write $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \coloneqq \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R)$ and $M_{\mathrm{gm}} \coloneqq M_{\mathrm{gm},R}$.

The *reduced motive* $\widetilde{M}_{\mathrm{gm}}(X)$ of $X \in \mathsf{Sm}/k$ is defined to be the cocone

$$\widetilde{M}_{\mathrm{gm},R}(X) \coloneqq \mathsf{Cocone}\left(\cdots \to M_{\mathrm{gm},R}(X) \to M_{\mathrm{gm},R}(\mathsf{Spec}\, k) \to \cdots\right)$$

of the complex centred in degrees 0 and 1, see [**Voe00**, p.192]. Let,

$$R(0) \coloneqq M_{\mathrm{gm},R}(\mathsf{Spec}\, k) \qquad \text{and} \qquad R(1) \coloneqq \widetilde{M}_{\mathrm{gm},R}(\mathbb{P}^1_k)[-2].$$

The geometric motive $R(1)$ is called the *Tate motive* over k.

The category of effective geometric motives $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R)$ is R-linear additive with a symmetric monoidal structure. Moreover, the functor

$$- \otimes R(1) : \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R) \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R)$$

is triangulated, and the category $\mathsf{DM}_{\mathrm{gm}}(k, R)$ of *geometric motives* over k with R-coefficients is defined to be the Spanier-Whitehead stabilisation of $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R)$ with respect to the functor $- \otimes R(1)$, *i.e.*

$$\mathsf{DM}_{\mathrm{gm}}(k, R) \coloneqq \mathsf{SW}^{R(1)} \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R).$$

Proposition 2.5.7 (Gysin Triangles). Let X be a smooth k-scheme, let $i : Z \hookrightarrow X$ be a smooth closed immersion that is everywhere of codimension c, with complementary open immersion $j : U \hookrightarrow X$. Then, there is a canonical distinguished triangle

$$M_{\mathrm{gm}}(U) \xrightarrow{M_{\mathrm{gm}}(j)} M_{\mathrm{gm}}(X) \xrightarrow{g_Z} M_{\mathrm{gm}}(Z)(c)[2c] \longrightarrow M_{\mathrm{gm}}(U)[1]$$

in $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Proof. See [**Voe00**, Prop.3.5.4]. $\square$

Corollary 2.5.8. Let k be a perfect field of exponential characteristic p. Then, the triangulated category $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\big[\mathbb{Z}[\frac{1}{p}]\big](k)$ is generated by direct summands of geometric motives for smooth projective k-schemes.

Proof. See [**Kel12**, Prop.5.5.3]. $\square$

Motives of Singular Varieties. Let k be a perfect field. The functors

$$M_{\mathrm{gm},p} : \mathsf{Sm}/k \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big) \quad \text{and} \quad M^{\mathrm{prop}}_{\mathrm{gm},p} : \mathsf{SmProp}/k \hookrightarrow \mathsf{Sm}/k \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big)$$

extend to functors

$$M_{\mathrm{gm},p} : \mathsf{Sch}^{\mathrm{ft}}/k \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big) \qquad \text{and} \qquad M^{\mathrm{c}}_{\mathrm{gm},p} : \mathsf{Sch}^{\mathrm{ft\,prop}}/k \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big),$$

called the *geometric motive* and *geometric motive with proper support* functors, respectively, see [**Kel12**, Lem.5.5.2 and Lem.5.5.6]. These functors satisfy the following homological properties, among others,

- (Homotopy invariance) the morphism $M_{\mathrm{gm},p}(pr_X) : M_{\mathrm{gm},p}(X \times \mathbb{A}^1_k) \to M_{\mathrm{gm},p}(X)$ is an isomorphism for every $X \in \mathsf{Sch}^{\mathrm{ft}}/k$, see [**Kel12**, Cor.5.5.9];
- (Blow-up) there exists a canonical distinguished triangle

$$M_{\mathrm{gm},p}(p_Z^{-1}(Z)) \to M_{\mathrm{gm},p}(Z) \bigoplus M_{\mathrm{gm},p}(X_Z) \to M_{\mathrm{gm},p}(X) \to M_{\mathrm{gm},p}(p_Z^{-1}(Z))[1]$$

  in $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big)$, for every closed subscheme $Z \hookrightarrow X$ in $\mathsf{Sch}^{\mathrm{ft}}/k$, where $p_Z : X_Z \to X$ is the blow-up of X centred at Z, see [**Kel12**, Cor.5.5.4];
- there exists a canonical isomorphism

$$M^{\mathrm{c}}_{\mathrm{gm},p}(X) \cong M_{\mathrm{gm},p}(X),$$

  for every proper k-scheme $X \in \mathsf{Sch}^{\mathrm{ft}}/k$, see [**Kel12**, Prop.5.5.5];
- there exists a canonical isomorphism

$$M^{\mathrm{c}}_{\mathrm{gm},p}(Y)(n)[2n] \to M^{\mathrm{c}}_{\mathrm{gm},p}(X),$$

  for every flat equidimensional morphism $f : X \to Y$ in $\mathsf{Sch}^{\mathrm{ft}}/k$, where $n = \mathtt{dim}_Y X$, see [**Kel12**, Prop.5.5.11]; in particular, there exists a canonical isomorphism

$$M^{\mathrm{c}}_{\mathrm{gm},p}(X \times \mathbb{A}^1_k) \cong M^{\mathrm{c}}_{\mathrm{gm},p}(X)(1)[2],$$

  induced by the projection $pr_X : X \times \mathbb{A}^1_k \to X$;
- there exist canonical isomorphisms

$$M_{\mathrm{gm},p}(X) \cong M_{\mathrm{gm},p}(X_{\mathrm{red}}) \qquad \text{and} \qquad M^{\mathrm{c}}_{\mathrm{gm},p}(X) \cong M^{\mathrm{c}}_{\mathrm{gm},p}(X_{\mathrm{red}}),$$

  for every $X \in \mathsf{Sch}^{\mathrm{ft}}/k$;
- there exists a canonical distinguished triangle

$$M^{\mathrm{c}}_{\mathrm{gm},p}(Z) \to M^{\mathrm{c}}_{\mathrm{gm},p}(X) \to M^{\mathrm{c}}_{\mathrm{gm},p}(U) \to M^{\mathrm{c}}_{\mathrm{gm},p}(Z)[1],$$

  for every closed immersion $Z \hookrightarrow X$ in $\mathsf{Sch}^{\mathrm{ft}}/k$ with complementary open immersion $j : U \hookrightarrow X$, see [**Kel12**, Prop.5.5.5]; and

- (Künneth Formula) there exist canonical isomorphisms

$$M_{\mathrm{gm},p}(X \times Y) \cong M_{\mathrm{gm},p}(X) \otimes M_{\mathrm{gm},p}(Y)$$

$$M^{\mathrm{c}}_{\mathrm{gm},p}(X \times Y) \cong M^{\mathrm{c}}_{\mathrm{gm},p}(X) \otimes M^{\mathrm{c}}_{\mathrm{gm},p}(Y),$$

for every $X, Y \in \mathsf{Sch}^{\mathrm{ft}}/k$, see [**Kel12**, Prop.5.5.8].

These results were obtained by Voevodsky, in [**Voe00**, §.4], with integral coefficients for fields admitting resolution of singularities, and by S. Kelly, in [**Kel12**], with $\mathbb{Z}[\frac{1}{p}]$ coefficients for any perfect field of exponential characteristic p.

In particular, the last three properties imply that the functor $M^{\mathrm{c}}_{\mathrm{gm},p}$ induces a motivic measure

$$\begin{array}{rccc} \mu_{\mathsf{DM}}: & \mathbb{K}_0(\mathsf{Var}/k) & \to & \mathbb{K}_\triangle(\mathsf{DM}^{\mathrm{eff}}(k, \mathbb{Z}[\tfrac{1}{p}])) \\ & [X] & \mapsto & [M^{\mathrm{c}}_{\mathrm{gm},p}(X)]. \end{array} \tag{14}$$

2.5.2.1. *Chow Motives.* Recall the construction of the category of pure Chow motives with R-coefficients over k, starting with the category of covariant (resp. contravariant) Chow correspondences of degree zero on smooth projective k-schemes, applying the Karoubian envelope yields the category of covariant (resp. contravariant) effective Chow motives, then the Spanier-Whitehead stabilisation of effective Chow motives with respect to the Lefschetz motive $\mathbb{L}$ results in the category of covariant (resp. contravariant) pure Chow motives. This was first developed by Grothendieck in 1964, within the general framework of motives.

The R-linear categories $\mathsf{Cor}_R(k)$ and $\mathsf{Cor}^R(k)$ of *covariant and contravariant Chow correspondences*, respectively, of degree zero over SmProj/k have the same objects of SmProj/k, and their morphisms are given by the R-modules of *Chow correspondences*[1]

$$\mathsf{Cor}_R(k)(X, Y) \coloneqq \bigoplus_{Y_j \in \mathrm{irr}(Y)} \mathrm{CH}^{\mathtt{dim}\, Y_j}(X \times Y_j) \bigotimes_{\mathbb{Z}} R$$

and

$$\mathsf{Cor}^R(k)(X, Y) \coloneqq \bigoplus_{X_i \in \mathrm{irr}(X)} \mathrm{CH}^{\mathtt{dim}\, X_i}(X_i \times Y) \bigotimes_{\mathbb{Z}} R,$$

for every pairs (X, Y) of smooth projective k-schemes, with compositions given by the intersection product on the Chow rings.

There are well-defined functors

$$\mathsf{SmProj}/k \to \mathsf{Cor}_R(k) \qquad \text{and} \qquad \mathsf{SmProj}/k^{\mathrm{op}} \to \mathsf{Cor}^R(k),$$

[1]Algebraic correspondences modulo rational equivalence, see [**Ful98**].

sending a morphism f of smooth projective k-schemes to the Chow correspondence $[\Gamma_f]$ of its graph and to its transpose $[\Gamma_f]^\top$, respectively. Moreover, since connected components of smooth k-schemes coincide with their irreducible components, the transpose defines an isomorphism of categories

$$\mathsf{Cor}_R(k)^{\mathrm{op}} \to \mathsf{Cor}^R(k),$$

that is the identity on objects and given by the transpose on correspondences.

The categories $\mathsf{Cor}_R(k)$ and $\mathsf{Cor}^R(k)$ are additive, whose direct sum (resp. zero object) is given by the disjoint union of schemes (resp. the scheme $\mathsf{Spec}\, k$). Moreover, these categories are symmetric monoidal, whose monoidal product is induced by the Cartesian product of smooth projective k-schemes. However, neither of the categories $\mathsf{Cor}_R(k)$ and $\mathsf{Cor}^R(k)$ is Karoubian. The categories of *covariant and contravariant effective Chow motives* $\mathsf{CHM}_R^{\mathrm{eff}}(k)$ and $\mathsf{CHM}_{\mathrm{eff}}^R(k)$, respectively, are defined to be the Karoubian completion of $\mathsf{Cor}_R(k)$ and $\mathsf{Cor}^R(k)$, respectively. In particular, objects of $\mathsf{CHM}_R^{\mathrm{eff}}(k)$ are pairs of the form (X, Ξ), for a smooth projective k-scheme X and an idempotent Chow correspondence $\Xi : X \vdash X$ in $\mathsf{Cor}_R(k)$; and

$$\mathsf{CHM}_R^{\mathrm{eff}}(k)\big((X,\Xi),(Y,\Psi)\big) \cong \Psi \circ \mathsf{Cor}_R(k)(X,Y) \circ \Xi.$$

The categories $\mathsf{CHM}_R^{\mathrm{eff}}(k)$ and $\mathsf{CHM}_{\mathrm{eff}}^R(k)$ inherit the R-linear additive and symmetric monoidal structures of $\mathsf{Cor}_R(k)$ and $\mathsf{Cor}^R(k)$, respectively.

Lefschetz motive. Let $p : \mathbb{P}^1_k \twoheadrightarrow \mathsf{Spec}\, k$ be the structure morphism of $\mathbb{P}^1_k$, let $x : \mathsf{Spec}\, k \hookrightarrow \mathbb{P}^1_k$ be a rational point of $\mathbb{P}^1_k$, let $e \coloneqq x \circ p : \mathbb{P}^1_k \to \mathbb{P}^1_k$, and let $\Gamma \coloneqq [\Gamma_e] : \mathbb{P}^1_k \vdash \mathbb{P}^1_k$ be the idempotent endomorphism in $\mathsf{Cor}_R(k)$ corresponding to e. The endomorphism Γ induces an idempotent endomorphism $\Gamma : (\mathbb{P}^1_k, [\Delta_{\mathbb{P}^1_k}]) \vdash (\mathbb{P}^1_k, [\Delta_{\mathbb{P}^1_k}])$ in $\mathsf{CHM}_R^{\mathrm{eff}}(k)$. Hence, $(\mathbb{P}^1_k, [\Delta_{\mathbb{P}^1_k}])$ decomposes in $\mathsf{CHM}_R^{\mathrm{eff}}(k)$ as

$$(\mathbb{P}^1_k, [\Delta_{\mathbb{P}^1_k}]) \cong \mathbf{ker}\,\Gamma \bigoplus \mathbf{im}\,\Gamma.$$

Let $\Gamma_1 = [\mathsf{Spec}\, k \times \mathbb{P}^1_k]$ and $\Gamma_2 = [\mathbb{P}^1_k \times \mathsf{Spec}\, k]$ be the Chow correspondences spanning

$$\mathsf{CHM}_R^{\mathrm{eff}}(k)(\mathbb{P}^1_k, \mathbb{P}^1_k) = \mathsf{Cor}_R(k)(\mathbb{P}^1_k, \mathbb{P}^1_k) = \mathrm{CH}^1_R(\mathbb{P}^1_k \times \mathbb{P}^1_k) \cong R \bigoplus R,$$

then $\Gamma_1 + \Gamma_2 = [\Delta_{\mathbb{P}^1_k}] \in \mathsf{CHM}_R^{\mathrm{eff}}(k)(\mathbb{P}^1_k, \mathbb{P}^1_k)$, and $\Gamma = \Gamma_1 = [\mathsf{Spec}\, k \times \mathbb{P}^1_k]$. Therefore,

$$\mathbf{ker}\,\Gamma \cong (\mathbb{P}^1_k, [\Delta_{\mathbb{P}^1_k}] - \Gamma_1) = (\mathbb{P}^1_k, \Gamma_2) \qquad \text{and} \qquad \mathbf{im}\,\Gamma \cong (\mathbb{P}^1_k, \Gamma_1).$$

The motive $(\mathbb{P}^1_k, \Gamma_2) = (\mathbb{P}^1_k, [\mathbb{P}^1_k \times \mathsf{Spec}\, k])$ is called the *covariant Lefschetz Motive* and it is denoted by $\mathbb{L}$. Then, one has

$$(\mathbb{P}^1_k, [\Delta_{\mathbb{P}^1_k}]) \cong (\mathsf{Spec}\, k, [\Delta_{\mathsf{Spec}\, k}]) \bigoplus \mathbb{L} = \mathbb{1} \bigoplus \mathbb{L}, \tag{15}$$

in $\mathsf{CHM}_R^{\mathrm{eff}}(k)$. The *contravariant Lefschetz* Motive is defined dually in $\mathsf{CHM}_{\mathrm{eff}}^R(k)$. The categories of *covariant and contravariant Chow motives* $\mathsf{CHM}_R(k)$ and $\mathsf{CHM}^R(k)$ are defined, respectively, to be the Spanier-Whitehead categories

$$\mathsf{CHM}_R(k) \coloneqq \mathsf{SW}^{\mathbb{L}}\mathsf{CHM}_R^{\mathrm{eff}}(k) \qquad \text{and} \qquad \mathsf{CHM}^R(k) \coloneqq \mathsf{SW}^{\mathbb{L}}\mathsf{CHM}_{\mathrm{eff}}^R(k).$$

Embedding of Chow Motives in Geometric Motives. Denote the functor $\mathsf{SmProj}/k \to \mathsf{CHM}^{\mathrm{eff}}(k)$ that sends a smooth projective k-scheme to its covariant effective Chow motive, with integral coefficients, by Chow.

THEOREM 2.5.9. There exists a functor $M_{\mathrm{gm}}^{\mathsf{CHM}} : \mathsf{CHM}^{\mathrm{eff}}(k) \to \mathsf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ for which the square

$$\begin{tikzcd} \mathsf{SmProj}/k \arrow[r, hook] \arrow[d, "\mathrm{Chow}"'] & \mathsf{Sm}/k \arrow[d, "M_{\mathrm{gm}}"] \\ \mathsf{CHM}^{\mathrm{eff}}(k) \arrow[r, "M_{\mathrm{gm}}^{\mathsf{CHM}}"'] & \mathsf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \end{tikzcd}$$

commutes.

PROOF. See [**Voe00**, Prop.2.1.4]. □

This theorem is the main reason beyond using the Karoubian envelope in the construction of the category of geometric motives.

Voevodsky deduced in [**Voe00**], when k is a field that admits resolutions of singularities, that the functor $M_{\mathrm{gm}}^{\mathsf{CHM}} : \mathsf{CHM}^{\mathrm{eff}}(k) \to \mathsf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is a full embedding. Then, Bondarko used Gabber's refined uniformisation to prove the existence of such full embedding for a perfect field, but with $\mathbb{Z}[\frac{1}{p}]$-coefficients. In fact, in [**Bon11**, Th.2.2.1.(2)], Bondarko showed the existence of a bounded weight structure on $\mathsf{DM}_{\mathrm{gm}}^{\mathrm{eff}}\big(k, \mathbb{Z}[\frac{1}{p}]\big)$, whose heart is isomorphic to $\mathsf{CHM}_{\mathbb{Z}[\frac{1}{p}]}^{\mathrm{eff}}(k)$, which implies the full faithfulness of the induced functor $M_{\mathrm{gm}}^{\mathsf{CHM}} : \mathsf{CHM}^{\mathrm{eff}}(k) \to \mathsf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Also, it induces an exact conservative functor

$$\mathsf{DM}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big) \to \mathcal{K}^b\left(\mathsf{CHM}_{\mathbb{Z}[\frac{1}{p}]}(k)\right),$$

see [**Bon11**, Prop.2.3.2.(1)]. Hence, it establishes an isomorphism

$$\mathrm{K}_{\triangle}\left(\mathsf{DM}_{\mathrm{gm}}\big(k, \mathbb{Z}[\frac{1}{p}]\big)\right) \to \mathrm{K}_{\oplus}\left(\mathsf{CHM}_{\mathbb{Z}[\frac{1}{p}]}(k)\right). \tag{16}$$

Also, Bondarko showed, in [**Bon11**, Th.2.2.1.(1)], that $\mathsf{DM}_{\mathrm{gm}}(k, \mathbb{Z}[\frac{1}{p}]$ is generated, as a triangulated category, by summands of objects in $\mathsf{CHM}_{\mathbb{Z}[\frac{1}{p}]}^{\mathrm{eff}}(k)$.

2.5.3. Motivic Complexes.

2.5.3.1. *(Pre)sheaves with Transfers.* A *presheaf with transfers* with coefficients in R over k is an additive functor $F : \mathsf{SmCor}_k^{op} \to R\text{-}\mathsf{Mod}$, and the category of presheaves with transfers with coefficients in R over k is the full subcategory in the functor category $\mathsf{Fun}(\mathsf{SmCor}_k^{op}, R\text{-}\mathsf{Mod})$, whose objects are presheaves with transfers, it is denoted by $\mathsf{PST}(k, R)$. The category $\mathsf{PST}(k, R)$ is abelian and has enough injectives and projectives, see [**MVW06**, Th.2.3].

A presheaf with transfers $F : \mathsf{SmCor}_k^{op} \to R\text{-}\mathsf{Mod}$ is said to be a *Nisnevich sheaf with transfers*, if its restriction to the category Sm/k of smooth k-schemes is a Nisnevich sheaf. Let $\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k, R))$ be the category of Nisnevich sheaves with transfers with coefficients in R over k. Then, the triangulated category of *effective motivic complexes* is defined to be the full triangulated subcategory

$$\mathsf{DM}^{\mathrm{eff}}(k,R) \subset \mathcal{D}^{-}\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right),$$

whose cohomology sheaves are *homotopy invariant*, *i.e.* $F^{\bullet} \in \mathcal{D}^{-}\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right)$ if

$$\mathrm{H}^{n}(F^{\bullet})(X) \cong \mathrm{H}^{n}(F^{\bullet})(X \times \mathbb{A}^{1}_{k}),$$

for every $X \in \mathsf{Sm}/k$ and $n \in \mathbb{Z}$. The triangulated category $\mathsf{DM}(k,R)$ inherits the standard t-structure on $\mathcal{D}^{-}\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right)$, given by canonical truncations, whose heart is the abelian category of homotopy invariant Nisnevich sheaves with transfers with coefficients in R over k, see [**Voe00**, p.205].

Let L be the Yoneda embedding functor

$$\begin{array}{rccc} L: & \mathsf{SmCor}(k,R) & \to & \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R)) \subset \mathsf{PST}(k,R) \\ & X & \mapsto & L(X) = \mathrm{c}(-,X)_{R}. \end{array}$$

The category of presheaves with transfers $\mathsf{PST}(k,R)$ is symmetric monoidal. However, its monoidal product differs from the restriction of the Cartesian product of the category of presheaves of R-modules over k. To recall the symmetric monoidal structure on $\mathsf{PST}(k,R)$, one needs Suslin simplicial complexes. For a presheaf $P : \mathsf{Sm}/k^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$, the *Suslin simplicial complex* of P is given by

$$\underline{\mathrm{C}}^{-\bullet}(P)(-) = \underline{\mathrm{C}}_{\bullet}(P)(-) \coloneqq \mathsf{PST}(k,R)(L(\Delta^{\bullet}_{k} \times -), P),$$

where $\Delta^{\bullet}_{k}$ is the *standard algebraic cosimplicial object* in Sm/k, *i.e.*

$$\Delta^{n}_{k} = \mathsf{Spec}\left(k[x_0, x_1, \ldots, x_n] / \sum_{i=0}^{n} x_i = 1\right),$$

for $[n] \in \Delta$, with evident face and degeneracy morphisms, see [**Voe98**, §.3]. Then, for $X \in \mathsf{Sm}/k$, one has

$$\underline{\mathrm{C}}^{-\bullet}(P)(X) = \mathsf{PST}(k,R)(L(\Delta^{\bullet}_{k} \times X), P) \cong P(\Delta^{\bullet}_{k} \times X).$$

Since one has an isomorphism $\Delta^{n}_{k} \cong \mathbb{A}^{n}_{k}$ for every integer $n \geq 0$, the complex $\underline{\mathrm{C}}^{-\bullet}(P)$ has homotopy invariant cohomology sheaves $\underline{h}_{\bullet}(F) \coloneqq \mathrm{H}\left(\underline{\mathrm{C}}^{-\bullet}(P)\right)$, see [**Voe00**, 3.2.1]. Moreover, if P is a presheaf (resp. Nisnevich sheaf) with transfers, then $\underline{\mathrm{C}}^{-\bullet}(P)$ is a complex of presheaves (resp. Nisnevich sheaves) with transfers, whose cohomology sheaves are homotopy invariant, see [**Voe00**, Th.3.1.12].

Let $P \in \mathsf{PST}(k,R)$, then one has a canonical isomorphism

$$P \cong \operatorname*{\mathtt{colim}}_{\mathsf{El}(P)} L \circ \pi,$$

where $\mathsf{El}(P)$ is the category of elements of P, and $\pi : \mathsf{El}(P) \to \mathsf{Sm}/k$ is the evident projection. Hence, the canonical morphism

$$\bigoplus_{(X,\phi)\in \mathsf{Ob}(\mathsf{El}(P))} L(X) \to P$$

is a surjection of presheaves with transfers. There exists a canonical left resolution $\mathcal{L}(P)$ of P in $\mathsf{PST}(k,R)$, whose morphisms are the canonical surjections with

$$\mathcal{L}_1(P) \coloneqq \bigoplus_{(X,\phi)\in \mathsf{Ob}(\mathsf{El}(P))} L(X) \qquad \text{and} \qquad \mathcal{L}_{n+1}(P) \coloneqq \mathcal{L}_1(\mathcal{L}_n(P)) \qquad \text{for} \qquad n \geq 1.$$

The monoidal structure on $\mathsf{PST}(k,R)$ is defined by $L(X) \otimes L(Y) \coloneqq L(X \times Y)$ for $X, Y \in \mathsf{Sm}/k$, and

$$P \otimes G \coloneqq \underline{h}_0\big(\mathcal{L}(F) \otimes \mathcal{L}(G)\big) \qquad \text{for} \qquad P, Q \in \mathsf{PST}(k,R).$$

In fact, $\mathsf{PST}(k,R)$ is monoidal closed with internal $\underline{\mathtt{Hom}}$ given by

$$\underline{\mathtt{Hom}}(P,Q)(-) \coloneqq \mathsf{PST}(k,R)(P \otimes L(-), G) \qquad \text{for} \qquad P, G \in \mathsf{PST}(k,R).$$

In particular, $\underline{C}^{-\bullet}(P) \cong \underline{\mathtt{Hom}}(L(\Delta^\bullet_k), P)$, for every $P \in \mathsf{PST}(k,R)$. Moreover, if Q is a Nisnevich sheaf with transfers, so is $\underline{\mathtt{Hom}}(P,Q)$, for any presheaf with transfers P.

On the other hand, the Suslin simplicial complex extends to a functor

$$\underline{C}^{-\bullet} : \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R)) \to \mathsf{DM}^{\mathrm{eff}}(k,R),$$

with a right derived functor

$$\mathbf{R}\underline{C}^{-\bullet} : \mathcal{D}^-\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right) \to \mathsf{DM}^{\mathrm{eff}}(k,R),$$

that is a left adjoint of the inclusion $\mathsf{DM}^{\mathrm{eff}}(k,R) \subset \mathcal{D}^-\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right)$, see [**Voe00**, Prop.3.2.3]. This functor is particularly useful in lifting the monoidal structure of $\mathsf{PST}(k,R)$ to $\mathsf{DM}^{\mathrm{eff}}(k,R)$, whose monoidal product is given by

$$P^\bullet \otimes Q^\bullet \coloneqq \mathbf{R}\underline{C}^{-\bullet}(P^\bullet \otimes^{\mathbf{R}} Q^\bullet) \qquad \text{for} \qquad P^\bullet, Q^\bullet \in \mathsf{DM}^{\mathrm{eff}}(k,R).$$

Also, it induces an embedding of geometric motives into effective motivic complexes, for a perfect field k.

Theorem 2.5.10. Let k be a perfect field. The functor

$$L : \mathcal{K}^b(\mathsf{SmCor}(k,R)) \to \mathcal{D}^-\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right),$$

induced by the Yoneda embedding, has a fully faithful symmetric monoidal triangulated right derived functor $\mathbf{R}L$, with a dense image, that makes the square

$$\begin{tikzcd}
\mathcal{K}^b(\mathsf{SmCor}(k,R)) \arrow[r, "L"] \arrow[d, "Q^\natural_{\mathbb{A}^1_k, MV}"'] & \mathcal{D}^-\left(\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{SmCor}(k,R))\right) \arrow[d, "\mathbf{R}\underline{C}^{-\bullet}"] \\
\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,R) \arrow[r, "\mathbf{R}L"] & \mathsf{DM}^{\mathrm{eff}}(k,R),
\end{tikzcd}$$

of symmetric monoidal triangulated functors, commute.

Proof. See [**Voe00**, Th.3.2.6]. □

The Hurewicz functor. Let $\mathscr{A}$ be an abelian category, the Dold-Kan Theorem implies that the Moore's normalisation functor

$$\mathsf{C}_{\mathtt{N}} : \Delta^{\mathrm{op}}\mathscr{A} \to \mathsf{Ch}_{\geq 0}(\mathscr{A})$$

is an equivalence of categories, with a quasi-inverse $\mathtt{K}$, called the *Dold-Kan correspondence*, see [**Wei94**, Th.8.4.1]. In fact, the adjunctions $\mathtt{K} \dashv \mathsf{C}_{\mathtt{N}}$ and $\mathsf{C}_{\mathtt{N}} \dashv \mathtt{K}$ are Quillen equivalences with respect to the standard model structures on both categories.

The Dold-Kan correspondence gives rise to a monoidal triangulated adjunction

$$\mathtt{Hu} : \mathcal{SH}_s(k) \rightleftarrows \mathsf{DM}^{\mathrm{eff}}(k) : \mathtt{H}. \tag{17}$$

The functor $\mathtt{Hu}$ is called the *Hurewicz functor*, it sends the S^1-symmetric spectrum $\boldsymbol{\Sigma}^{\infty}_{S^1} X_+$ to the effective geometric motive $M_{\mathrm{gm}}(X)$ for every $X \in \mathsf{Sm}/k$. The right adjoint $\mathtt{H}$ to $\mathtt{Hu}$ is called the *Eilenberg-Mac Lane spectrum functor*, see the discussion in [**AH11**, §.2.1]. Moreover, for a perfect field k, the adjunction (17) induces an equivalence of the $\mathbb{Q}$-localised triangulated categories

$$\mathtt{Hu}_{\mathbb{Q}} : \mathcal{SH}(k, \mathbb{Q}) \rightleftarrows \mathsf{DM}(k, \mathbb{Q}) : \mathtt{H}_{\mathbb{Q}}, \tag{18}$$

see [**Mor06**, Th.4.1 and Rem.1.5].

CHAPTER 3

Motivic Measures

Motivic measures are connected to fundamental questions in algebraic geometry. For instance, the motivic measure of counting points over a finite field gives rise to the Hasse-Weil zeta function through applying it to symmetric powers, as it was first shown by Kapranov in [**Kap00**]. Also, Larsen-Lunts motivic measure, introduced in [**LL03**], has important applications in birational algebraic geometry, see [**GS14**]. Other important questions are tackled through the universal motivic measures, called the Grothendieck ring of varieties, see [**NS11**] and [**DL04**].

In this chapter we recall the basics of motivic measures, then we restrict our attention to the classical motivic measure of counting points over a finite field.

Throughout this chapters, let S be a Noetherian scheme. Recall the conventions and notations in §.0.2. In particular, the category of schemes of finite type over S is denoted by $\mathsf{Sch}^{\mathrm{ft}}/S$, and its subcategory of reduced such schemes is denoted by Var/S. An S-scheme (resp. S-variety) refers to an object in $\mathsf{Sch}^{\mathrm{ft}}/S$ (resp. Var/S).

Definition 3.0.1 (Euler-Poincaré characteristic). Let $(G, +)$ be a group. A *generalised Euler-Poincaré characteristic* over S with values in $(G, +)$ is a map

$$\chi : \mathtt{Ob}(\mathsf{Sch}^{\mathrm{ft}}/S) \to G$$

that is invariant under isomorphisms and respects the *scissors relations*, *i.e.*

$$\chi(x) = \chi(z) + \chi(u), \tag{19}$$

if there exists a closed immersion $z \hookrightarrow\!\!\!\!\to x$ in $\mathsf{Sch}^{\mathrm{ft}}/S$ with complementary open immersion $u \hookrightarrow\!\!\!\circ\!\!\to x$, see [**Mus13**, p.73] and [**DL01**, p.5].

In particular, the scissors relations imply that $\chi(\varnothing_S) = 0_G$, and

$$\chi(z) = \chi(x) \tag{20}$$

if there exits a surjective closed immersion $z \hookrightarrow\!\!\!\!\to x$ in $\mathsf{Sch}^{\mathrm{ft}}/S$.

The relation (20) shows one may equivalently define generalised Euler-Poincaré characteristics over S to be maps from $\mathtt{Ob}(\mathsf{Var}/S)$ that are invariant under isomorphisms and respect the scissors relations.

Recall that the category $\mathsf{Sch}^{\mathrm{ft}}/S$ admits a Cartesian product given by the fibre product over S, *i.e.* for S-schemes $x : X \to S$ and $y : Y \to S$ in $\mathsf{Sch}^{\mathrm{ft}}/S$, the Cartesian product $x \times y$ is the S-scheme $X \times_S Y$, with the canonical structure morphism over S.

Definition 3.0.2 (Motivic Measure). Let $(R, +, 0)$ be a commutative ring. A *motivic measure* (or *multiplicative Euler-Poincaré characteristic*) over S with values in $(R, +, 0)$ is a generalised Euler-Poincaré characteristic

$$\mu : \mathtt{Ob}(\mathsf{Sch}^{\mathrm{ft}}/S) \to R$$

with value in $(R, +)$ that respects the Cartesian product of $\mathsf{Sch}^{\mathrm{ft}}/S$, *i.e.* $\mu(x \times y) = \mu(x) \cdot \mu(y)$, for every $x, y \in \mathsf{Sch}^{\mathrm{ft}}/S$. In particular, when μ is surjective, one has $\mu(\mathrm{id}_S) = 1_R$.

Similarly, since the category Var/S admits a Cartesian product given by the reduced induced structure on the fibre product over S, one may equivalently define motivic measures as generalised Euler-Poincaré characteristics from $\mathtt{Ob}(\mathsf{Var}/S)$ that respect the Cartesian product of Var/S, due to the relation (20).

Example 3.0.3 (Counting points). Let $\mathbb{F}_q$ be a finite field with q elements, for every finite field extension $\mathbb{F}_{q^s}$ of $\mathbb{F}_q$, there exists a motivic measure $\mu^s_\# : \mathtt{Ob}(\mathsf{Sch}^{\mathrm{ft}}/k) \to \mathbb{Z}$, given by $\mu^s_\#(X) \coloneqq \#X(\mathbb{F}_{q^s})$, it is called the *counting $\mathbb{F}_{q^s}$-point motivic measure*, see §.3.2. In particular, for $s = 1$, the motivic measure $\mu^s_\#$ is the *counting rational points motivic measure*, and it is denoted by $\mu_\#$.

Definition 3.0.4 (Kapranov Motivic Zeta Functions). Let k be a field, and let

$$\mu : K_0(\mathsf{Sch}^{\mathrm{ft}}/k) \to R$$

be a motivic measure, and let X be a quasi-projective k-variety. The *motivic zeta-function* of X with respect to μ is the formal power series

$$\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu\left(\mathtt{Sym}^n X\right) t^n \in R[\![t]\!],$$

where $\mathtt{Sym}^n x$ is the n^{th}-symmetric power of X.

Example 3.0.5. The *Hasse-Weil zeta function* is the motivic zeta-function with respect to the counting rational points motivic measure over a finite field, see [**Kap00**].

3.1. Grothendieck Ring of Varieties

Let $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ (resp. $\mathtt{K}_0(\mathsf{Var}/S)$) be the abelian group generated by isomorphism classes of S-schemes (resp. S-varieties) module the scissors relations

$$\left\{ [\mathrm{x}]{=}[\mathrm{z}]{+}[\mathrm{u}] \;\middle|\; \begin{array}{l}\text{there exists a closed immersion } z \hookrightarrow x \text{ in } \mathsf{Sch}^{\mathrm{ft}}/S \text{ (resp.}\\ \mathsf{Var}/S\text{) with complementary open immersion } u \hookrightarrow x\end{array} \right\},$$

where $[y]$ denotes the isomorphism class of an S-scheme (resp. an S-variety) y. The group $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ (resp. $\mathtt{K}_0(\mathsf{Var}/S)$) is called the *Grothendieck group of S-schemes* (resp.

S*-varieties*). We abuse notations, and we also use $[y]$ to refer to the element in the Grothendieck group that is represented by the isomorphism class of the S-scheme y.

The Cartesian product of the category $\mathsf{Sch}^{\mathrm{ft}}/S$ (resp. Var/S) defines a commutative ring structure on the group $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ (resp. $\mathtt{K}_0(\mathsf{Var}/S)$), whose multiplication $\cdot$ is given by

$$[x] \cdot [y] \coloneqq [x \times y],$$

for every $x, y \in \mathsf{Sch}^{\mathrm{ft}}/S$ (resp. $x, y \in \mathsf{Var}/S$). The resulting ring is called *Grothendieck ring of* S*-schemes* (resp. S*-varieties*).

The rings $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ and $\mathtt{K}_0(\mathsf{Var}/S)$ are isomorphic, in which one has $0 = [\varnothing_S]$, $1 = [\mathrm{id}_S]$, and $[z] = [x]$ if there exits a surjective closed immersion $z \hookrightarrow\!\!\!\!\rightarrow x$ of S-schemes.

For a field k, one may show using Noetherian induction that the ring $\mathtt{K}_0(\mathsf{Var}/k)$ is isomorphic to the subring generated by quasi-projective k-varieties.

The canonical map

$$\begin{array}{rccc} [-]: & \mathtt{Ob}(\mathsf{Sch}^{\mathrm{ft}}/S) & \to & \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \\ & x & \mapsto & [x] \end{array}$$

is an initial universal generalised Euler-Poincaré characteristic over S. In fact, it is an initial universal motivic measure over S. Hence, one might abuse notation and call any ring homomorphism from $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ a motivic measure over S.

The Grothendieck ring of varieties was first introduced by Grothendieck in a letter to Serre in 1964. Yet, it was not until 2002, when it was shown to contain zero divisors over a field of characteristic zero, see [**Poo02**]. Also, the class of the affine line was not proven to be a zero divisor until recently. In 2014, Borisov constructed two smooth Calabi-Yau varieties over the complex numbers, and showed that a multiple of the difference between their classes annihilate the class of the affine line, see [**Bor15**, Th.2.12]. That, in particular, answers negatively the cut-and-past question of Larsen and Lunts, proposed in [**LL03**, Question 1.2].

3.1.1. Grothendieck Ring of Varieties in Characteristic Zero. For a field k of characteristic zero, the ring $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/k)$ admits alternative presentations as quotients with better-behaved generators.

LEMMA 3.1.1. Let k be a field of characteristic zero. Then, the group $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/k)$ is isomorphic to the abelian group generated by isomorphism classes of smooth connected projective k-varieties modulo the scissors relations.

PROOF. See [**Mus13**, Lem.7.9]. □

THEOREM 3.1.2. Let k be a field of characteristic zero. Then, the group $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/k)$ is isomorphic to the abelian group generated by isomorphism classes of smooth connected projective (resp. proper) k-varieties modulo the relations

- $[\varnothing] = 0$; and
- $[\mathtt{Bl}_Y X] - [E] = [X] - [Y]$ for every smooth connected projective (resp. proper) k-variety X and a closed smooth subvariety $Y \hookrightarrow X$, where $\mathtt{Bl}_Y X$ is the blow-up of X along Y with an exceptional divisor E.

PROOF. See [**Bit04**, Th.3.1]. $\square$

3.1.2. The Modified Grothendieck Ring of Varieties. In several situations, it is rather difficult to utilise the Grothendieck Ring of varieties. In these cases, it is usually more convenient to consider a modified version, see [**NS11**] and [**Har16**].

DEFINITION 3.1.3. Let I_S^{uh} be the ideal in $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ generated by the set

$$\left\{ \text{[x]-[y]} \;\middle|\; \text{there exists a universal homeomorphism of } S\text{-schemes between } x \text{ and } y \right\}.$$

The modified Grothendieck ring of S-schemes, denoted by $\mathtt{K}_0^{\mathrm{uh}}(\mathsf{Sch}^{\mathrm{ft}}/S)$ is the quotient ring

$$\mathtt{K}_0^{\mathrm{uh}}(\mathsf{Sch}^{\mathrm{ft}}/S) \coloneqq \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S)/I_S^{\mathrm{uh}}. \tag{21}$$

PROPOSITION 3.1.4. Let $f : x \to y$ be a universal homeomorphism of $\mathbb{Q}$-schemes. Then, f is a piecewise isomorphism. Thus, for a Q-scheme S, the quotient projection

$$\mu_{\mathrm{uh}} : \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0^{\mathrm{uh}}(\mathsf{Sch}^{\mathrm{ft}}/S)$$

is an isomorphism.

PROOF. See [**NS11**, Prop.3.10 and Cor.3.11]. $\square$

3.2. Counting Points over a Finite Field

Let K/k be an algebraic field extension. The cardinality of the set of K-points in a k-scheme X is given by

$$\#X(K) = \#\mathsf{Sch}^{\mathrm{ft}}/k(\mathsf{Spec}\, K, X) = \# \bigsqcup_{x \in X} k\text{-}\mathsf{Alg}(\kappa(x), K),$$

which is finite for a finite extension K/k of a finite field k, see [**Mus13**, Prop.2.1, Rem.2.2 and Rem.2.3].

Fix a finite field $\mathbb{F}_q$, of characteristic p with $q = p^r$ elements, and fix an algebraic closure $\mathbb{F}_q \subset \overline{\mathbb{F}_q}$. Then, for an integer $s \geq 1$, fix a field $\mathbb{F}_q \subset \mathbb{F}_{q^s} \subset \overline{\mathbb{F}_q}$ of degree s over

$\mathbb{F}_q$. The corepresentable functor

$$\begin{array}{rccccc} \mathsf{h}^{\mathsf{Spec}\,\mathbb{F}_{q^s}}: & \overset{\mathrm{ft}}{\mathsf{Sch}}/\mathbb{F}_q & \longrightarrow & \mathsf{FSet} & & \\ & X & \longmapsto & X(\mathbb{F}_{q^s}) & & \\ & f: X \to Y & \longmapsto & f(\mathbb{F}_{q^s}): \quad X(\mathbb{F}_{q^s}) & \to & Y(\mathbb{F}_{q^s}) \\ & & & x & \mapsto & f \circ x \end{array} \tag{22}$$

defines a map $\mu^s_{\#} : \mathtt{Ob}(\overset{\mathrm{ft}}{\mathsf{Sch}}/\mathbb{F}_q) \to \mathbb{Z}$, given for an $\mathbb{F}_q$-scheme X by

$$\mu^s_{\#}(X) := \#X(\mathbb{F}_{q^s}) = \# \bigsqcup_{\substack{x \in X \\ \mathtt{deg}(x) | s}} \mathbb{F}_q\text{-}\mathsf{Alg}(\kappa(x), \mathbb{F}_{q^s}),$$

where $\mathtt{deg}(x) = [\kappa(x) : \mathbb{F}_q]$, because the residue field of an $\mathbb{F}_{q^s}$-point is an intermediate field extension between $\mathbb{F}_q$ and $\mathbb{F}_{q^s}$.

Remark 3.2.1. Notice that, for an $\mathbb{F}_q$-scheme X, every closed point $x \in X$ with $\kappa(x) \cong \mathbb{F}_{q^t}$ corresponds to t different $\mathbb{F}_{q^t}$-points in X over $\mathbb{F}_q$, given by the t different $\mathbb{F}_q$-automorphism of $\mathbb{F}_{q^t}$. Moreover, the group $\mathtt{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ is cyclic, generated by the restriction of the arithmetic Frobenius automorphism to $\mathbb{F}_{q^t}$, whose t^{th}-power fixes $\mathbb{F}_{q^t}$.

Since functors preserve isomorphisms, the map $\mu^s_{\#}$ is invariant under isomorphisms. The scheme-theoretic image of an $\mathbb{F}_{q^s}$-point is a closed point. Then, for a closed immersion $i : Z \hookrightarrow X$ in $\overset{\mathrm{ft}}{\mathsf{Sch}}/\mathbb{F}_q$ with complementary open immersion $j : U \hookrightarrow X$, each $\mathbb{F}_{q^s}$-point in X factorises either through the closed immersion i or through the open immersion j, and hence $\mu^s_{\#}$ respects the scissors relations. For a locally small category with Cartesian product, the definition of the Cartesian product implies that corepresentable functors are strong symmetric monoidal, with respect to the Cartesian monoidal structure. In particular, for $\mathbb{F}_q$-schemes X_0 and X_1, we have a bijection

$$(X_0 \times X_1)(\mathbb{F}_{q^s}) = \overset{\mathrm{ft}}{\mathsf{Sch}}/\mathbb{F}_q(\mathsf{Spec}\,\mathbb{F}_{q^s}, X_0 \times X_1) \cong X_0(\mathbb{F}_{q^s}) \times X_1(\mathbb{F}_{q^s}),$$

and hence

$$\mu^s_{\#}(X_0 \times X_1) = \#(X_0 \times X_1)(\mathbb{F}_{q^s}) = \#\big(X_0(\mathbb{F}_{q^s}) \times X_1(\mathbb{F}_{q^s})\big) = \mu^s_{\#}(X_0) \cdot \mu^s_{\#}(X_1)$$

Thus, the map $\mu^s_{\#}$ is a motivic measure. The notation $\mu^s_{\#}$ is also used to denote the induced ring homomorphism

$$\mu^s_{\#} : (K_0(\overset{\mathrm{ft}}{\mathsf{Sch}}/\mathbb{F}_q), +, \cdot) \to (\mathbb{Z}, +, \cdot). \tag{23}$$

For $s = 1$, we write $\mu_{\#} := \mu^1_{\#}$.

Counting points over a finite field can be realised using the Frobenius endomorphism, by means of the trace formula. That is of a particular interest in §3.2.1, to extend counting points to effective Chow motives.

Counting points and the Frobenius Endomorphism. Hereby, we briefly recall the Frobenius endomorphism, the arithmetic Frobenius, and recall how it gives the number of points of a given degree.

Recall that for an $\mathbb{F}_q$-scheme X, the Frobenius endomorphism $\mathrm{Fr}_X : X \to X$ of X is such an endomorphism of schemes whose underlying continuous map $|\mathrm{Fr}_X| : |X| \to |X|$ is the identity map, with comorphisms

$$\begin{array}{rccc} \mathrm{Fr}^{\#}_{X,U} : & \mathcal{O}_X(U) & \to & \mathcal{O}_X(U), \\ & a & \mapsto & a^q \end{array}$$

for every open affine subset $U \subset X$. Iterating the Fermat-Euler Theorem shows that $a^q = a$ for every $a \in \mathbb{F}_q$, and hence Fr_X is a morphism of $\mathbb{F}_q$-schemes. Also, for every point $x \in X$, one has induced morphisms

$$\mathrm{Fr}^{\#}_{X,x} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \qquad \text{and} \qquad \mathrm{Fr}^{\#}_{\kappa(x)} : \kappa(x) \to \kappa(x)$$

over $\mathbb{F}_q$, sending elements to their q^{th}-power.

Recall that the absolute Galois group $\mathtt{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is generated as a topological group by an $\overline{\mathbb{F}_q}$-automorphism over $\mathbb{F}_q$ sending each element to its q^{th}-power, called the *arithmetic Frobenius automorphism.* For a closed point $x \in X$, the extension $\kappa(x)/\mathbb{F}_q$ is algebraic because $\mathtt{tr.deg}(\kappa(x)/k) = \mathtt{dim}\, x = 0$. Then, the group $\mathtt{Gal}(\kappa(x)/\mathbb{F}_q)$ is cyclic of order $[\kappa(x) : \mathbb{F}_q]$, and it is generated by the restriction of the arithmetic Frobenius automorphism to $\kappa(x)$, that is $\mathrm{Fr}^{\#}_{\kappa(x)}$.

For a integer $s \geq 1$, the s^{th}-power of the arithmetic Frobenius automorphism generates a subgroup in $\mathtt{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, and hence determines a unique subextension of $\mathbb{F}_q$ in $\overline{\mathbb{F}_q}$ of degree s, namely $\mathbb{F}_{q^s} = \{a \in \overline{\mathbb{F}_q} \mid a^{q^s} = a\}$. Hence, for a closed point $x \in X$, the field $\kappa(x)$ is fixed by the $[\kappa(x) : \mathbb{F}_q]^{\mathrm{th}}$-power of $\mathrm{Fr}^{\#}_{\kappa(x)}$, but not by any smaller power.

In general, an $\mathbb{F}_q$-scheme X may have infinity many geometric points. However, the lemma below shows that only finitely many of them are fixed by a given power of the Frobenius endomorphism.

Lemma 3.2.2. Let X be an $\mathbb{F}_q$-scheme and let $s \geq 1$ be an integer. Then, the set $X(\mathbb{F}_{q^s})$ of $\mathbb{F}_{q^s}$-points in X over $\mathbb{F}_q$ is in bijection with the set of all $\overline{\mathbb{F}_q}$-points in X over $\mathbb{F}_q$ that are fixed by the s^{th}-power of the Frobenius endomorphism, *i.e.* $X(\mathbb{F}_{q^s})$ is in bijection with the set $\left(\mathrm{equi}_{\mathsf{Sch}^{\mathrm{ft}}/\mathbb{F}_q}(\mathrm{Fr}^s_X, \mathrm{id}_X)\right)(\overline{\mathbb{F}_q})$, and hence

$$\mu^s_{\#}(X) = \#\{x \in X(\overline{\mathbb{F}_q}) \mid \mathrm{Fr}^s_X \circ x = x = \mathrm{id}_X \circ x\},$$

where $\mathrm{equi}_{\mathsf{Sch}^{\mathrm{ft}}/\mathbb{F}_q}(\mathrm{Fr}^s_X, \mathrm{id}_X)$ is the equaliser of Fr^s_X and id_X in $\mathsf{Sch}^{\mathrm{ft}}/\mathbb{F}_q$.

Proof. Consider the map

$$\Phi : X(\mathbb{F}_{q^s}) \to \{x \in X(\overline{\mathbb{F}_q}) \mid \mathrm{Fr}^s_X \circ x = x = \mathrm{id}_X \circ x\}$$

given by precomposition with the morphism $\mathsf{Spec}\,\overline{\mathbb{F}_q} \to \mathsf{Spec}\,\mathbb{F}_{q^s}$ over $\mathbb{F}_q$, induced by the fixed embeddings $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^s} \hookrightarrow \overline{\mathbb{F}_q}$.

The map Φ is well-defined. To see that, let $x : \mathsf{Spec}\,\mathbb{F}_{q^s} \to X$ be an $\mathbb{F}_{q^s}$-point in X over $\mathbb{F}_q$, and let $\overline{x}$ be its scheme-theoretic image. Since the s^{th}-power of the arithmetic Frobenius automorphism fixes $\mathbb{F}_{q^s}$, it fixes $\kappa(\overline{x})$ for having field extensions $\mathbb{F}_{q^s}/\kappa(\overline{x})/\mathbb{F}_q$. The endomorphism $\mathrm{Fr}^{\#}_{\kappa(\overline{x})}$ is the restriction of the arithmetic Frobenius automorphism to $\kappa(\overline{x})$. Hence, one has

$$(\mathrm{Fr}^s_{\kappa(\overline{x})})^{\#} = (\mathrm{Fr}^{\#}_{\kappa(\overline{x})})^s = \mathrm{id}_{\kappa(\overline{x})} \qquad \text{, i.e.} \qquad x^{\#}_{\kappa(\overline{x})} \circ (\mathrm{Fr}^s_{\kappa(\overline{x})})^{\#} = x^{\#}_{\kappa(\overline{x})}.$$

Since $\overline{x}$ is the scheme-theoretic image of x, one has $\mathrm{Fr}^s_X \circ x = x$. Let $\underline{x}$ be the precomposition of x with the canonical morphism $\mathsf{Spec}\,\overline{\mathbb{F}_q} \to \mathsf{Spec}\,\mathbb{F}_{q^s}$, then $\mathrm{Fr}^s_X \circ \underline{x} = \underline{x}$.

Since $\mathsf{Spec}\,\overline{\mathbb{F}_q} \to \mathsf{Spec}\,\mathbb{F}_{q^s}$ is an epimorphism, the map Φ is injective. To show the surjectivity of Φ, let $\underline{x} : \mathsf{Spec}\,\overline{\mathbb{F}_q} \to X$ be an $\overline{\mathbb{F}_q}$-point in X over $\mathbb{F}_q$ that is fixed by Fr^s_X, and let $\overline{x}$ be its scheme-theoretic image, then one has field extensions $\overline{\mathbb{F}_q}/\kappa(\overline{x})/\mathbb{F}_q$. Since Fr^s_X fixes x, one has

$$x^{\#}_{\kappa(\overline{x})} \circ (\mathrm{Fr}^s_{\kappa(\overline{x})})^{\#} = x^{\#}_{\kappa(\overline{x})}.$$

The homomorphism $x^{\#}_{\kappa(\overline{x})}$ is an immersion of fields, and hence a monomorphism. Thus, one has $(\mathrm{Fr}^{\#}_{\kappa(\overline{x})})^s = \mathrm{id}_{\kappa(\overline{x})}$, and the restriction of the s^{th}-power of the arithmetic Frobenius automorphism fixes $\kappa(\overline{x})$. Since the s^{th}-power of the arithmetic Frobenius automorphism only fixes subfields of $\mathbb{F}_{q^s}$, one has field extensions $\overline{\mathbb{F}_q}/\mathbb{F}_{q^s}/\kappa(\overline{x})/\mathbb{F}_q$, and $\underline{x}$ factors through the canonical morphism $\mathsf{Spec}\,\overline{\mathbb{F}_q} \to \mathsf{Spec}\,\mathbb{F}_{q^s}$. □

For an $\mathbb{F}_q$-scheme X, let $\overline{X} \coloneqq X \times_{\mathsf{Spec}\,\mathbb{F}_q} \mathsf{Spec}\,\overline{\mathbb{F}_q}$. There is a bijection $\Phi : \overline{X}(\overline{\mathbb{F}_q}) \to X(\overline{\mathbb{F}_q})$, between the sets of $\overline{\mathbb{F}_q}$-points in $\overline{X}$ over $\overline{\mathbb{F}_q}$ and $\overline{\mathbb{F}_q}$-points in X over $\mathbb{F}_q$, given by the composition with the projection $\overline{X} \to X$. Define the *relative Frobenius endomorphism* $\mathrm{Fr}_{\overline{X},q}$ of $\overline{X}$ over $\mathsf{Spec}\,\overline{\mathbb{F}_q}$ to be $\mathrm{Fr}_X \times_{\mathsf{Spec}\,\mathbb{F}_q} \mathrm{id}_{\mathsf{Spec}\,\overline{\mathbb{F}_q}}$. Then, Lemma 3.2.2 can be restated as follows.

Corollary 3.2.3. Let X be an $\mathbb{F}_q$-scheme and let $s \geq 1$ be an integer. Then, the set $X(\mathbb{F}_{q^s})$ of $\mathbb{F}_{q^s}$-points in X over $\mathbb{F}_q$ is in bijection with the set of all $\overline{\mathbb{F}_q}$-points in $\overline{X}$ over $\overline{\mathbb{F}_q}$ that are fixed by the s^{th}-power of the relative Frobenius endomorphism $\mathrm{Fr}_{\overline{X},q}$, *i.e.* the set $X(\mathbb{F}_{q^s})$ is in bijection with the set $\left(\mathrm{equi}_{\mathsf{Sch}^{\mathrm{ft}}/\overline{\mathbb{F}_q}}(\mathrm{Fr}^s_{\overline{X},q}, \mathrm{id}_{\overline{X}})\right)(\overline{\mathbb{F}_q})$, and hence

$$\mu^s_{\#}(X) = \#\{\underline{x} \in \overline{X}(\overline{\mathbb{F}_q}) \mid \mathrm{Fr}^s_{\overline{X},q} \circ \underline{x} = \underline{x} = \mathrm{id}_{\overline{X}} \circ \underline{x}\}.$$

The main advantage of the corollary above is that geometric points in $\overline{X}$ coincide with its closed points, and hence one counts closed points in $\overline{X}$ fixed by powers of the relative Frobenius endomorphism.

One may would like to consider the *absolute Frobenius endomorphism* of $\overline{X}$ over $\mathbb{F}_q$, given by

$$\mathrm{Fr}_{\overline{X}} \coloneqq \mathrm{Fr}_X \times_{\mathsf{Spec}\,\mathbb{F}_q} \mathrm{Fr}_{\mathsf{Spec}\,\overline{\mathbb{F}_q}},$$

instead of $\mathrm{Fr}_{\overline{X},q}$. However, the corollary above does not hold for $\mathrm{Fr}_{\overline{X}}$.

3.2.1. Counting Points on Effective Chow Motives. Kleiman extended counting point to the category of contravariant effective Chow motives with coefficients in a ring R, in his survey on the theory of motives [**Kle72**]. Combined with the work of Gillet and Soulé, in [**GS09**], this extends the motive measure of counting points over a finite field to the Grothendieck ring of effective Chow motives with rational coefficients.

Let R be a ring of characteristic zero. Recall that $\mathsf{SmProj}/\mathbb{F}_q$ denotes the category of smooth projective $\mathbb{F}_q$-varieties and that $\mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q)$ denotes the category of contravariant effective Chow motives with R-coefficients over $\mathbb{F}_q$, as in §.2.5.2.1. There exists a functor

$$\begin{array}{rccc} \underline{-}: & (\mathsf{SmProj}/\mathbb{F}_q)^{\mathrm{op}} & \to & \mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q) \\ & X & \mapsto & (X, \underline{\mathrm{id}_X}) \\ & f : X \to Y & \mapsto & \underline{f} \coloneqq [\Gamma_f]^{\top}, \end{array}$$

where Γ_f is the scheme-theoretic image of the graph morphism associated to f in $X \times Y$.

PROPOSITION 3.2.4. Let X be a smooth projective $\mathbb{F}_q$-variety. Then, the scheme-theoretic intersection $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ in $X \times X$ is reduced, and the set $X(\mathbb{F}_{q^s})$ is in bijection with the set of all geometric $\overline{\mathbb{F}_q}$-points in $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ over $\mathbb{F}_q$. Moreover, the algebraic cycles $\underline{\mathrm{Fr}^s_X}$ and $\underline{\mathrm{id}_X} = [\Delta_X]^{\top}$ intersect properly in the group of algebraic cycles $\mathrm{C}^*(X \times X)$; and hence

$$\mu^s_{\#}(X) = \langle (\underline{\mathrm{Fr}^s_X} \cdot \underline{\mathrm{id}_X}) \rangle, \tag{24}$$

where the intersection product is taken in the Chow ring $\mathrm{CH}^*(X \times X)$, and $\langle - \rangle$ is the degree morphism over $\mathbb{F}_q$.

PROOF. Consider the solid commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dl, "\mathrm{id}_X"'] \arrow[d, dashed, "\cong"] \arrow[dr, "\mathrm{Fr}^s_X"] & \\
 & & \Gamma_{\mathrm{Fr}^s_X} \arrow[dl, "\cong"'] \arrow[d] \arrow[dr] & \\
X'_s \arrow[uurr, dotted, bend left, "p_{\mathrm{Fr}}"] \arrow[urr, bend left, "\mathrm{pr}_{\mathrm{Fr}}"] \arrow[drr, bend right, "\mathrm{pr}_{\Delta}"'] \arrow[ddrr, dotted, bend right, "p_{\Delta}"'] & X & X \times_{\mathbb{F}_q} X \arrow[l] \arrow[r] & X. \\
 & & \Delta_X \arrow[ul, "\cong"] \arrow[u] \arrow[ur, "\cong"'] & \\
 & & X \arrow[ul, "\mathrm{id}_X"] \arrow[u, dashed, "\cong"'] \arrow[ur, "\mathrm{id}_X"'] &
\end{tikzcd} \tag{25}$$

of $\mathbb{F}_q$-schemes defining the intersection $X'_s \coloneqq \Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$. Since X is smooth over $\mathbb{F}_q$, it is a reduced Noetherian scheme whose irreducible components coincide with its connected components. Hence, there exists a finite covering family $\{U_i \mid i \in I\}$ of connected open affine subschemes of X. In particular, the ring $\mathcal{O}_X(U_i)$ is an integral domain for every $i \in I$. The diagonal morphism Δ_X and the graph morphism $\daleth_{\mathrm{Fr}^s_X}$ are induced respectively by the ring homomorphisms

$$\begin{array}{rccc} \Delta_X^{\#}(U_i): & \mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i) & \to & \mathcal{O}_X(U_i) \\ & a \otimes b & \mapsto & a \cdot b \end{array}$$

and

$$\begin{array}{rccc} \daleth_{\mathrm{Fr}^s_X}^{\#}(U_i): & \mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i) & \to & \mathcal{O}_X(U_i) \\ & a \otimes b & \mapsto & a \cdot b^{q^s} \end{array}$$

for every $i \in I$. Since the ring $\mathcal{O}_X(U_i)$ is integral, one has

$$\ker \Delta_X^{\#}(U_i) \cong (a \otimes 1 - 1 \otimes a)\big(\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)\big) \text{ and}$$
$$\ker \daleth_{\mathrm{Fr}^s_X}^{\#}(U_i) \cong (a^{q^s} \otimes 1 - 1 \otimes a)\big(\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)\big).$$

Thus, the closed subschemes Δ_X and $\Gamma_{\mathrm{Fr}^s_X}$ in $X \times X$ are given respectively by the unions

$$\Delta_X \cong \bigcup_{i \in I} \Delta_X(U_i) \qquad \text{and} \qquad \Gamma_{\mathrm{Fr}^s_X} \cong \bigcup_{i \in I} \Gamma_{\mathrm{Fr}^s_X}(U_i)$$

in $X \times X$, where $\Delta_X(U_i)$ and $\Gamma_{\mathrm{Fr}^s_X}(U_i)$ are the closed subschemes in $X \times X$ given by

$$\Delta_X(U_i) = \mathsf{Spec}\, {}^{\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)}\!/\!_{(a \otimes 1 - 1 \otimes a)} \text{ and}$$
$$\Gamma_{\mathrm{Fr}^s_X}(U_i) = \mathsf{Spec}\, {}^{\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)}\!/\!_{(a^{q^s} \otimes 1 - 1 \otimes a)},$$

for every $i \in I$. Hence, the scheme-theoretic intersection $X'_s = \Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ is given by the union $\bigcup_{i \in I} X'_{s,i}$ in $X \times X$, where $X'_{s,i}$ is the closed subscheme

$$\mathsf{Spec}\left({}^{\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)}\!/\!_{(a \otimes 1 - 1 \otimes a)} \otimes_{\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)} {}^{\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)}\!/\!_{(a^{q^s} \otimes 1 - 1 \otimes a)} \right)$$
$$\cong \mathsf{Spec}\, {}^{\mathcal{O}_X(U_i) \otimes_{\mathbb{F}_q} \mathcal{O}_X(U_i)}\!/\!_{(a^{q^s} \otimes 1 - 1 \otimes a,\, a \otimes 1 - 1 \otimes a)} \cong \mathsf{Spec}\, {}^{\mathcal{O}_X(U_i)}\!/\!_{(a^{q^s} - a)}.$$

in $X \times X$, for every $i \in I$. Since $\mathcal{O}_X(U_i)$ is an $\mathbb{F}_q$-algebra and $a^{q^s} - a$ has only simple roots over $\mathbb{F}_q$, the scheme $X'_{s,i}$ is reduced for every $i \in I$, and so is the union $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$.

The morphisms labelled with $\cong$ in diagram (25) are isomorphisms of schemes. Therefore, there exist the dotted morphisms $p_{\mathrm{Fr}}, p_\Delta : X'_s \to X$ over $\mathbb{F}_q$ that make the whole diagram commute, which in turn implies that $p_{\mathrm{Fr}} = p_\Delta$. Chasing diagram (25), one sees that the scheme-theoretic intersection of $\Gamma_{\mathrm{Fr}^s_X}$ and Δ_X in $X \times X$ coincides with the equaliser of $\mathrm{Fr}^s_X, \mathrm{id}_X : X \to X$ in the category of $\mathbb{F}_q$-schemes. Then, one has the desired bijection, by Lemma 3.2.2.

The bijection between the set $X(\mathbb{F}_{q^s})$ and the set of all $\overline{\mathbb{F}_q}$-points in $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ over $\mathbb{F}_q$ shows, in particular, that $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ has finitely many $\overline{\mathbb{F}_q}$-points over $\mathbb{F}_q$, as $\mu^s_{\#}(X)$ is finite. Thus, the scheme $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ is zero-dimensional, has finitely many closed points, and the residue field of each of these points is a finite field extension of $\mathbb{F}_q$. Therefore, the algebraic cycles $\underline{\mathrm{Fr}^s_X}$ and $\underline{\mathrm{id}_X}$ intersect properly in $X \times X$, and hence their intersection product is given by

$$\underline{\mathrm{Fr}^s_X} \cdot \underline{\mathrm{id}_X} = [\Gamma_{\mathrm{Fr}^s_X} \bigcap \Delta_X]^\top.$$

Since X is projective over $\mathbb{F}_q$, the structure morphism $p : X \times X \to \mathsf{Spec}\,\mathbb{F}_q$ induces a well-defined degree morphism $\langle - \rangle = p_* : \mathrm{CH}_0(X \times X) \otimes R \to \mathrm{CH}_0(\mathsf{Spec}\,\mathbb{F}_q) \otimes R \cong R$. The morphism p_* sends each integral component x of $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ to $[\kappa(x) : \mathbb{F}_q] 1_{\mathrm{CH}_0(\mathsf{Spec}\,\mathbb{F}_q)}$. Since R is of characteristic zero and $[\kappa(x) : \mathbb{F}_q]$ is the number of $\overline{\mathbb{F}_q}$-points in $\Gamma_{\mathrm{Fr}^s_X} \cap \Delta_X$ over $\mathbb{F}_q$ with the scheme-theoretic image x, the equality (24) holds. □

Extending the base field to its algebraic closure allows a reformulation of the proposition above using closed points and powers of the relative Frobenius endomorphism.

Corollary 3.2.5. Let X be a smooth projective $\mathbb{F}_q$-variety. Then, the scheme-theoretic intersection $\Gamma_{\mathrm{Fr}^s_{\overline{X},q}} \cap \Delta_{\overline{X}}$ in $\overline{X} \times \overline{X}$ is reduced, and the set $X(\mathbb{F}_{q^s})$ is in bijection with the set of all $\overline{\mathbb{F}_q}$-points (equivalently closed points) in $\Gamma_{\mathrm{Fr}^s_{\overline{X},q}} \cap \Delta_{\overline{X}}$ over $\overline{\mathbb{F}_q}$. Moreover, the algebraic cycles $\underline{\mathrm{Fr}^s_{\overline{X},q}}$ and $\underline{\mathrm{id}_{\overline{X}}} = [\Delta_{\overline{X}}]^\top$ intersect properly in the group of algebraic cycles $\mathrm{C}^*(\overline{X} \times \overline{X})$; and hence

$$\mu^s_\#(X) = \langle \underline{\mathrm{Fr}^s_{\overline{X},q}} \cdot \underline{\mathrm{id}_{\overline{X}}} \rangle,$$

where the intersection product is taken in the Chow ring $\mathrm{CH}^*(\overline{X} \times \overline{X})$, and $\langle - \rangle$ is the degree morphism over $\overline{\mathbb{F}_q}$.

For an effective Chow motive $\mathscr{M} = (X, \Xi)$ over $\mathbb{F}_q$, both $\underline{\mathrm{Fr}^s_X}$ and $\Xi^\top$ belong to the group of Chow correspondences

$$\mathsf{Cor}^{\mathrm{CH}}_R(X, X) = \bigoplus_{X_i \in \mathrm{irr}(X)} \mathrm{CH}^{\dim X_i}_R(X_i \times X) \overset{\psi}{\cong} \bigoplus_{X_i \in \mathrm{irr}(X)} \bigoplus_{X_j \in \mathrm{irr}(X)} \mathrm{CH}^{\dim X_i}_R(X_i \times X_j).$$

Let $\underline{\mathrm{Fr}^s_X}_{i,j}$ and $\Xi^\top_{i,j}$ be the (i,j)-components of the images of $\underline{\mathrm{Fr}^s_X}$ and $\Xi^\top$, respectively, along ψ. Since the underlying continuous map of Fr^s_X is the identity map, $\underline{\mathrm{Fr}^s_X}_{i,j}$ vanishes for $i \neq j$ and $\underline{\mathrm{Fr}^s_X}_{i,i} = \underline{\mathrm{Fr}^s_{X_i}}$, for every $X_i, X_j \in \mathrm{irr}(X)$. Therefore, the intersection product $\underline{\mathrm{Fr}^s_X} \cdot \Xi^\top$ in $\mathrm{CH}^*_R(X \times X)$ is given by

$$\underline{\mathrm{Fr}^s_X} \cdot \Xi^\top = \psi^{-1}\Big(\sum_{X_i \in \mathrm{irr}(X)} \underline{\mathrm{Fr}^s_{X_i}} \cdot \Xi^\top_{i,i} \Big),$$

as the intersection product vanishes cross different connected components. Then, in particular, $\underline{\mathrm{Fr}^s_X} \cdot \Xi^\top$ is the Chow class of a zero-cycle in $X \times X$, *i.e.* it is contained in $\mathrm{CH}_{0,R}(X \times X)$. There exists a well-defined degree ring homomorphism

$$\langle - \rangle = p_* : \mathrm{CH}_{0,R}(X \times X) \to \mathrm{CH}_{0,R}(\mathsf{Spec}\,\mathbb{F}_q) \cong R,$$

induced by the proper structure morphism $p : X \times X \to \mathtt{Spec}\,\mathbb{F}_q$. Define the function

$$\begin{array}{rccc} \mu^s_{\#,M} : & \mathsf{Ob}(\mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q)) & \to & R \\ & \mathscr{M} = (X, \Xi) & \mapsto & \mu^s_{\#,M}(\mathscr{M}) \coloneqq \langle \underline{\mathrm{Fr}^s_X} \cdot \Xi^\top \rangle = \sum_{X_j \in \mathrm{irr}(X)} \langle \underline{\mathrm{Fr}^s_{X_i}} \cdot \Xi^\top_{i,i} \rangle. \end{array} \tag{26}$$

Following Kleiman's [**Kle72**], we recall that $\mu^s_{\#,M}$ defines a ring homomorphism from the additive Grothendieck ring $\mathbb{K}_\oplus \left(\mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q)\right)$ to R, as in Corollary 3.2.10.

Lemma 3.2.6. Let X be an irreducible smooth projective $\mathbb{F}_q$-variety of dimension n, and let Z be an integral closed subscheme in X of dimension m. Then,

$$(\mathrm{Fr}_X)_*(\mathrm{Fr}_X)^*(\Gamma) = q^n\Gamma \qquad \text{and} \qquad (\mathrm{Fr}_X)_*(\Gamma) = q^m\Gamma,$$

where Γ is the Chow class in $\mathrm{CH}^*(X)$ of the fundamental cycle of Z.

Proof. Let η be the unique generic point of the integral scheme X. Then, the local ring homomorphism

$$\begin{array}{rccc} \mathrm{Fr}^\#_{X,\eta} : & \mathcal{O}_{X,\eta} & \to & \mathcal{O}_{X,\eta} \\ & a & \mapsto & a^q \end{array}$$

is a monomorphism of local rings. Moreover, the local ring $\mathcal{O}_{X,\eta}$ is a field isomorphic to the function field of X, and hence it is of transcendental degree n over $\mathbb{F}_q$. Thus, the image of the morphism $\mathrm{Fr}^\#_{X,\eta}$, denoted $\mathcal{O}^q_{X,\eta}$, is a subfield of $\mathcal{O}_{X,\eta}$. Since $\mathbb{F}_q$ is perfect,

$$\mathsf{deg}(\mathrm{Fr}_X) = [\mathcal{O}_{X,\eta} : \mathcal{O}^q_{X,\eta}] = q^n,$$

by [**Kle68**, Lem.4.3]. Then, one has $(\mathrm{Fr}_X)_*([X]) = q^n[X]$. Applying the projection formula [**Ful98**, Ex.8.1.7], one has

$$(\mathrm{Fr}_X)_*(\mathrm{Fr}_X)^*(\Gamma) = (\mathrm{Fr}_X)_*\left((\mathrm{Fr}_X)^*(\Gamma) \cdot [X]\right) = ((\mathrm{Fr}_X)_*([X])) \cdot \Gamma = q^n[X] \cdot \Gamma = q^n\Gamma.$$

Also, since $[R(Z) : R(Z)^q] = q^m$, one has $(\mathrm{Fr}_X)_*(\Gamma) = q^m\Gamma$. $\square$

Lemma 3.2.7. Let X and Y be smooth projective $\mathbb{F}_q$-varieties, and let $\Gamma : X \vdash Y$ be a Chow correspondence with R-coefficients of degree zero. Then, one has

$$\Gamma \circ \underline{\mathrm{Fr}_X} = \underline{\mathrm{Fr}_Y} \circ \Gamma$$

in $\mathsf{Cor}^{\mathrm{CH}}_R(\mathbb{F}_q)$.

Proof. One may first prove the statement for irreducible $\mathbb{F}_q$-varieties, and use it to deduce the general statement.

Assume that X and Y are irreducible smooth projective $\mathbb{F}_q$-varieties of dimensions n and m, respectively. Let $\Gamma \subset X \times Y$ be a generator for the group of Chow correspondences from X to Y with R-coefficients of degree zero, *i.e.* an integral closed subscheme of $X \times Y$ of dimension m. Then, by Lemma 3.2.6, one has

$$(\mathrm{Fr}_{X \times Y})_*(\Gamma) = q^m\Gamma \qquad \text{and} \qquad (\mathrm{Fr}_{X \times Y})^*(\Gamma) = q^n\Gamma.$$

Recall that $\mathrm{Fr}_{X\times Y} = \mathrm{Fr}_X \times \mathrm{Fr}_Y$. Indeed, for open affine subsets $U \subset X$ and $V \subset Y$, one has $\mathcal{O}_{X\times Y}(U\times V) = \mathcal{O}_X(U)\otimes_{\mathbb{F}_q}\mathcal{O}_Y(V)$, and hence

$$\begin{array}{rccc} \mathrm{Fr}^{\#}_{X\times Y, U\times V}: & \mathcal{O}_X(U)\otimes_{\mathbb{F}_q}\mathcal{O}_Y(V) & \to & \mathcal{O}_X(U)\otimes_{\mathbb{F}_q}\mathcal{O}_Y(V), \\ & a\otimes b & \mapsto & (a\otimes b)^q = (a^q\otimes b^q). \end{array}$$

Therefore,

$$(\mathrm{Fr}_X \times \mathrm{Fr}_Y)_*(\Gamma) = q^m\Gamma \qquad \text{and} \qquad (\mathrm{Fr}_X\times\mathrm{Fr}_Y)^*(\Gamma) = q^n\Gamma. \tag{27}$$

Since the pushforward along a morphism of schemes coincide with the pushforward along the Chow correspondence of its graph, applying Lieberman's Lemma [**MNP13**, Lem.2.1.3] and [**Ful98**, Prop.16.1.1] to (27) one has

$$\begin{aligned} q^m\Gamma &= (\mathrm{Fr}_X\times\mathrm{Fr}_Y)_*(\Gamma) = \left([\Gamma_{(\mathrm{Fr}_X\times\mathrm{Fr}_Y)}]\right)(\Gamma) = ([\Gamma_{\mathrm{Fr}_X}]\boxtimes[\Gamma_{\mathrm{Fr}_Y}])(\Gamma) = \left(\underline{\mathrm{Fr}_X}^{\top}\boxtimes\underline{\mathrm{Fr}_Y}^{\top}\right)(\Gamma) \\ &= (pr^{XYXY}_{XY})_*\left((pr^{XYXY}_{YY})^*(\underline{\mathrm{Fr}_Y}^{\top})\cdot(pr^{XYXY}_{XX})^*(\underline{\mathrm{Fr}_X}^{\top})\cdot(pr^{XYXY}_{XY})^*(\Gamma)\right) \\ &= \underline{\mathrm{Fr}_Y}^{\top}\circ\Gamma\circ\underline{\mathrm{Fr}_X}. \end{aligned}$$

Also, the pullback along a morphism of schemes is given by the pushforward along the transpose of the correspondence of its graph, and hence

$$\begin{aligned} q^n\Gamma &= (\mathrm{Fr}_X\times\mathrm{Fr}_Y)^*(\Gamma) = \left([\Gamma_{(\mathrm{Fr}_X\times\mathrm{Fr}_Y)}]^{\top}\right)(\Gamma) = \left([\Gamma_{\mathrm{Fr}_X}]^{\top}\boxtimes[\Gamma_{\mathrm{Fr}_Y}]^{\top}\right)(\Gamma) = \left(\underline{\mathrm{Fr}_X}\boxtimes\underline{\mathrm{Fr}_Y}\right)(\Gamma) \\ &= \underline{\mathrm{Fr}_Y}\circ\Gamma\circ\underline{\mathrm{Fr}_X}^{\top}. \end{aligned}$$

Since the group $\mathsf{Cor}^{\mathrm{CH}}_R(X,Y)$ is R-linearly generated by fundamental classes of integral closed subschemes of $X\times Y$, the relations above extend R-linearly, and one has

$$q^m\Gamma = \underline{\mathrm{Fr}_Y}^{\top}\circ\Gamma\circ\underline{\mathrm{Fr}_X} \qquad \text{and} \qquad q^n\Gamma = \underline{\mathrm{Fr}_Y}\circ\Gamma\circ\underline{\mathrm{Fr}_X}^{\top}, \tag{28}$$

for every Chow correspondence $\Gamma : X \vdash Y$ with R-coefficients of degree zero.

In particular, for $\Gamma = [\Delta_X] = [\Delta_X]^{\top} = \underline{\mathrm{id}_X}$, one has

$$q^n\underline{\mathrm{id}_X} = \underline{\mathrm{Fr}_X}^{\top}\circ\underline{\mathrm{id}_X}\circ\underline{\mathrm{Fr}_X} = \underline{\mathrm{Fr}_X}^{\top}\circ\underline{\mathrm{Fr}_X} \qquad \text{and} \qquad q^n\underline{\mathrm{id}_X} = \underline{\mathrm{Fr}_X}\circ\underline{\mathrm{id}_X}\circ\underline{\mathrm{Fr}_X}^{\top} = \underline{\mathrm{Fr}_X}\circ\underline{\mathrm{Fr}_X}^{\top},$$

i.e. $\underline{\mathrm{Fr}_X}$ and $\underline{\mathrm{Fr}_X}^{\top}/q^n$ are mutually inverses in the ring $\mathsf{Cor}^{\mathrm{CH}}_R(\mathbb{F}_q)(X,X)$. Then, composing the first equality in (28) with $\underline{\mathrm{Fr}_Y}/q^m$ or precomposing the second equality in (28) with $\underline{\mathrm{Fr}_X}/q^n$ yields

$$\Gamma\circ\underline{\mathrm{Fr}_X} = \underline{\mathrm{Fr}_Y}\circ\Gamma, \tag{29}$$

for every Chow correspondence $\Gamma : X\vdash Y$ with R-coefficients of degree zero.

More generally, for any smooth projective $\mathbb{F}_q$-varieties X and Y, not necessarily irreducible, one has

$$\underline{\mathrm{Fr}_X} = \sum_{X_i\in\mathrm{irr}(X)}\underline{\mathrm{Fr}_{X_i}} \qquad \text{and} \qquad \underline{\mathrm{Fr}_Y} = \sum_{Y_j\in\mathrm{irr}(Y)}\underline{\mathrm{Fr}_{Y_j}}.$$

A Chow correspondence $\Gamma : X \vdash Y$ with R-coefficients of degree zero decomposes as

$$\Gamma = \sum_{\substack{X_i \in \mathrm{irr}(X) \\ Y_j \in \mathrm{irr}(Y)}} \Gamma_{i,j} \qquad \text{for} \qquad \Gamma_{i,j} \in \mathsf{Cor}_R^{\mathrm{CH}}(\mathbb{F}_q)(X_i, Y_j).$$

Since the composition in $\mathsf{Cor}_R^{\mathrm{CH}}(\mathbb{F}_q)$ is R-bilinear, applying (29) we have

$$\Gamma \circ \underline{\mathrm{Fr}_X} = \sum_{\substack{X_i \in \mathrm{irr}(X) \\ Y_j \in \mathrm{irr}(Y)}} \Gamma_{i,j} \circ \underline{\mathrm{Fr}_{X_i}} = \sum_{\substack{X_i \in \mathrm{irr}(X) \\ Y_j \in \mathrm{irr}(Y)}} \underline{\mathrm{Fr}_{Y_j}} \circ \Gamma_{i,j} = \underline{\mathrm{Fr}_Y} \circ \Gamma. \tag{30}$$

□

Lemma 3.2.8. Let $\mathscr{M} = (X, \Xi)$ and $\mathscr{N} = (Y, \Upsilon)$ be isomorphic effective Chow motives over $\mathbb{F}_q$, with R-coefficients. Then,

$$\mu^s_{\#,M}(\mathscr{M}) = \mu^s_{\#,M}(\mathscr{N}).$$

Proof. Let $\Gamma : (X, \Xi) \vdash (Y, \Upsilon)$ be an isomorphism of effective Chow motives, with an inverse $\Theta = \Gamma^{-1}$. Since $\Theta \circ \Gamma = \Xi$, composing (30) with Θ yields $\Xi \circ \underline{\mathrm{Fr}_X} = \Theta \circ \underline{\mathrm{Fr}_Y} \circ \Gamma$, and hence

$$\Xi \circ \underline{\mathrm{Fr}^s_X} = \Xi \circ \underline{\mathrm{Fr}_X}^s = \Theta \circ \underline{\mathrm{Fr}_Y}^s \circ \Gamma = \Theta \circ \underline{\mathrm{Fr}^s_Y} \circ \Gamma.$$

Thus, [**Kle72**, p.80.Lem] implies

$$\begin{aligned}\mu^s_{\#,M}(\mathscr{M}) &= \langle \underline{\mathrm{Fr}^s_X} \cdot \Xi^\top \rangle = \langle (\Xi \circ \underline{\mathrm{Fr}^s_X}) \cdot \Xi^\top \rangle = \langle (\Theta \circ \underline{\mathrm{Fr}^s_Y} \circ \Gamma) \cdot \Xi^\top \rangle = \langle \underline{\mathrm{Fr}^s_Y} \cdot (\Theta^\top \circ \Xi^\top \circ \Gamma^\top) \rangle \\ &= \langle \underline{\mathrm{Fr}^s_Y} \cdot (\Gamma \circ \Xi \circ \Theta)^\top \rangle = \langle \underline{\mathrm{Fr}^s_Y} \cdot (\Gamma \circ \Theta)^\top \rangle = \langle \underline{\mathrm{Fr}^s_Y} \cdot \Upsilon^\top \rangle = \mu^s_{\#,M}(\mathscr{N}).\end{aligned}$$

□

Lemma 3.2.8 does not hold if one tries to define $\mu^s_{\#,M}(\mathscr{M})$ to be $\langle \underline{\mathrm{Fr}^s_X} \cdot \Xi \rangle$, that comes down to having to show that $\Gamma^\top$ is a morphism of effective Chow motives when Γ is, which is not the case in general. That, in particular, explains why (26) uses the transpose $\Xi^\top$ instead of Ξ.

Lemma 3.2.9. Let $\mathscr{M}_0 = (X_0, \Xi_0)$ and $\mathscr{M}_1 = (X_1, \Xi_1)$ be effective Chow motives over $\mathbb{F}_q$, with R-coefficients. Then,

$$\begin{aligned}\mu^s_{\#,M}(\mathscr{M}_0 \oplus \mathscr{M}_1) &= \mu^s_{\#,M}(\mathscr{M}_0) + \mu^s_{\#,M}(\mathscr{M}_1) \text{ and} \\ \mu^s_{\#,M}(\mathscr{M}_0 \otimes \mathscr{M}_1) &= \mu^s_{\#,M}(\mathscr{M}_0) \cdot \mu^s_{\#,M}(\mathscr{M}_1).\end{aligned}$$

Proof. Recall that the category $\mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q)$ of effective Chow motives over $\mathbb{F}_q$, with R-coefficients, is an additive R-linear category with

$$\mathscr{M}_0 \oplus \mathscr{M}_1 = (X_0 \oplus X_1, \Xi_0 \oplus \Xi_1) = (X_0 \bigsqcup X_1, i_0 \circ \Xi_0 \circ p_0 + i_1 \circ \Xi_1 \circ p_1),$$

where $X_0 \oplus X_1 = X_0 \bigsqcup X_1$ is the biproduct of X_0 and X_1 in $\mathsf{Cor}_R^{\mathrm{CH}}(\mathbb{F}_q)$, and i_0, i_1 and p_0, p_1 are its injections and projections, respectively. Then,

$$\mu^s_{\#,M}(\mathscr{M}_0 \oplus \mathscr{M}_1) = \langle \underline{\mathrm{Fr}^s_{X_0 \bigsqcup X_1}} \cdot (\Xi_0 \oplus \Xi_1)^\top \rangle,$$

where the intersection product is taken in the LHS ring of the isomorphism

$$\mathrm{CH}^*\left((X_0 \bigsqcup X_1) \times_{\mathbb{F}_q} (X_0 \bigsqcup X_1)\right) \otimes R \cong \bigoplus_{0 \le i,j \le 1} \mathrm{CH}^*(X_i \times_{\mathbb{F}_q} X_j) \otimes R.$$

Since intersection product vanishes cross different connected components, calculating the intersection product in the RHS ring yields

$$\begin{aligned}\mu^s_{\#,M}(\mathscr{M}_0 \bigoplus \mathscr{M}_1) &= \langle (\underline{\mathrm{Fr}^s_{X_0}} + 0 + 0 + \underline{\mathrm{Fr}^s_{X_1}}) \cdot (\Xi_0^\top + 0 + 0 + \Xi_1^\top) \rangle \\ &= \langle \underline{\mathrm{Fr}^s_{X_0}} \cdot \Xi_0^\top + 0 + 0 + \underline{\mathrm{Fr}^s_{X_1}} \cdot \Xi_1^\top \rangle = \langle \underline{\mathrm{Fr}^s_{X_0}} \cdot \Xi_0^\top \rangle + \langle \underline{\mathrm{Fr}^s_{X_1}} \cdot \Xi_1^\top \rangle \\ &= \mu^s_{\#,M}(\mathscr{M}_0) + \mu^s_{\#,M}(\mathscr{M}_1).\end{aligned}$$

Also, the category $\mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q)$ is symmetric monoidal whose monoidal product is given by

$$\mathscr{M}_0 \otimes \mathscr{M}_1 = (X_0 \times X_1, \Xi_0 \boxtimes \Xi_1),$$

where $\boxtimes$ is the symmetric monoidal product of morphisms in $\mathsf{Cor}^{\mathrm{CH}}_R(\mathbb{F}_q)$. Then,

$$\begin{aligned}\mu^s_{\#,M}(\mathscr{M}_0 \otimes \mathscr{M}_1) &= \langle \underline{\mathrm{Fr}^s_{X_0 \times X_1}} \cdot (\Xi_0 \boxtimes \Xi_1)^\top \rangle = \langle (\underline{\mathrm{Fr}^s_{X_0}} \boxtimes \underline{\mathrm{Fr}^s_{X_1}}) \cdot (\Xi_0^\top \times' \Xi_1^\top) \rangle \\ &= \langle (\underline{\mathrm{Fr}^s_{X_0}} \cdot \Xi_0^\top) \boxtimes (\underline{\mathrm{Fr}^s_{X_1}} \cdot \Xi_1^\top) \rangle,\end{aligned}$$

where the intersection product is taken in the LHS ring of the isomorphism

$$\mathrm{CH}^*\left((X_0 \times X_1) \times (X_0 \times X_1)\right) \otimes R \cong \mathrm{CH}^*\left((X_0 \times X_0) \times (X_1 \times X_1)\right) \otimes R.$$

Then, calculating the intersection product in the RHS ring yields

$$\begin{aligned}\mu^s_{\#,M}(\mathscr{M}_0 \otimes \mathscr{M}_1) &= \langle (\underline{\mathrm{Fr}^s_{X_0}} \cdot \Xi_0^\top) \times (\underline{\mathrm{Fr}^s_{X_1}} \cdot \Xi_1^\top) \rangle = \langle \underline{\mathrm{Fr}^s_{X_0}} \cdot \Xi_0^\top \rangle \cdot \langle \underline{\mathrm{Fr}^s_{X_1}} \cdot \Xi_1^\top \rangle \\ &= \mu^s_{\#,M}(\mathscr{M}_0) \cdot \mu^s_{\#,M}(\mathscr{M}_1).\end{aligned}$$

□

Corollary 3.2.10. The map (26) induces a ring homomorphism

$$\begin{array}{rccc} \mu^s_{\#,M}: & \mathbb{K}_\oplus\left(\mathsf{CHM}^R_{\mathrm{eff}}(\mathbb{F}_q)\right) & \to & R \\ & [\mathscr{M}] = [(X, \Xi)] & \mapsto & \mu^s_{\#,M}([\mathscr{M}]) \coloneqq \langle \underline{\mathrm{Fr}^s_X} \cdot \Xi^\top \rangle. \end{array} \tag{31}$$

3.2.2. Gillet-Soulé Motivic Measure. H. Gillet and C. Soulé used Hironaka's resolution of singularities, for a field k of characteristic zero, to define a functor

$$W : (\mathsf{Var}^{\mathrm{prop}}/k)^{\mathrm{op}} \to \mathcal{K}^b(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Z}}(k)),$$

called the *contravariant weight complex*, which sends a smooth projective k-variety to the complex concentrated at its effective Chow motive. Using Gersten complexes, they show in [**GS96**, Th.2] that the contravariant weight complex functor induces a motivic measure

$$\mu_{GS} : \mathsf{Ob}(\mathsf{Var}/k) \to \mathbb{K}_\triangle\left(\mathcal{K}^b(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Z}}(k))\right) \cong \mathbb{K}_\oplus\left(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Z}}(k)\right).$$

In the sequel [**GS09**], they use De Jong's alterations of singularities [**dJ97**] to define a *covariant weight complex* functor

$$W : (\mathsf{Var}^{\mathrm{prop}}/k)^{\mathrm{op}} \to \mathcal{K}^b(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Q}}(k)),$$

for an arbitrary field k, in fact that was achieved even in greater generality. Then, using the $\mathtt{K}$-theory of coherent sheaves, they show in [**GS09**, Th.5.9 and Cor.5.13] that the covariant weight complex functor induces a motivic measure

$$\mu_{GS,\mathbb{Q}} : \mathtt{Ob}(\mathsf{Var}/k) \to \mathtt{K}_{\triangle}\left(\mathcal{K}^b(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Q}}(k))\right) \cong \mathtt{K}_{\oplus}\left(\mathsf{CHM}^{\mathrm{eff}}_{\mathbb{Q}}(k)\right),$$

for arbitrary field k, we call it *Gillet-Soulé motivic measure*. Bondarko's isomorphism (16) is an isomorphism between the motivic measures $\mu_{GS,\mathbb{Q}}$ and $\mu_{DM,\mathbb{Q}} \coloneqq \mu_{\mathsf{DM}} \otimes \mathbb{Q}$.

CHAPTER 4

Motivic Measures through Waldhausen $\mathtt{K}$-Theories

Several motivic measures arise from (co)homology theories with proper support. For instance, the Hodge measure and the Hodge characteristic arise from the polarised mixed Hodge structure on singular cohomology with rational coefficients and proper support over the complex numbers, see [**Sri14**]; whereas the ℓ–adic motivic measure arises from the ℓ-adic cohomology with proper support over a perfect field. The latter, also gives rise to the classical measure of counting points through the trace formula, see [**Mus13**]. Also, the motivic measure (14) is induced from Voevodsky's geometric motives with proper support.

Each of these motivic measures can be realised as a decategorification of a cohomology theory functor

$$G_\mu : (\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S, \times, \mathrm{id}_S) \to (\mathscr{C}, \wedge, \mathbb{1}),$$

to a symmetric monoidal Waldhausen category, where $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S$ is the category of schemes of finite type over a scheme S whose morphisms are finite compositions of proper morphisms and formal inverses of open immersions[1], such that G_μ is weak monoidal and satisfies the excision property, *i.e.*

(**WM**) G_μ is lax monoidal, such that the coherence morphism

$$G_\mu(x) \wedge G_\mu(y) \to G_\mu(x \times y) \tag{32}$$

is a weak equivalence for every $x, y \in \mathsf{Sch}^{\mathrm{ft}}/S$, and so is the coherence morphism $\mathbb{1} \to G(\mathrm{id}_S)$; and

(**E**) for every closed immersion $i : v \hookrightarrow x$ in $\mathsf{Sch}^{\mathrm{ft}}/S$ with complementary open immersion $j : u \hookrightarrow x$, the sequence

$$G_\mu(v) \overset{i_!}{\rightarrowtail} G_\mu(x) \overset{j^!}{\twoheadrightarrow} G_\mu(u) \tag{33}$$

is a cofibre sequence in $\mathscr{C}$, where $i_! \coloneqq G(i)$ and $j^! \coloneqq G(j^{\mathrm{op}})$.

In fact, every weak monoidal functor

$$G : (\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S, \times, \mathrm{id}_S) \to (\mathscr{C}, \wedge, \mathbb{1}),$$

[1]The category $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S$ is not a subcategory of the localisation of $\mathsf{Sch}^{\mathrm{ft}}/S$ with respect to open immersions, as for a closed open immersion $j : u \hookrightarrow x$ we do not ask for j^{op} to be an inverse of j in $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S$. However, Example 4.1.19 shows that we may impose the relation $j^{\mathrm{op}} \circ j = \mathrm{id}_u$, without affecting the argument, but we may not impose $j \circ j^{\mathrm{op}} = \mathrm{id}_x$, see Example 4.1.28.

that satisfies the excision property, induces a motivic measure $\mu_G : \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0(\mathscr{C})$ that sends the class of an S-scheme x to the class of $G(x)$. For the motivic measure μ_G to exist, it suffices that the weak equivalences in (**WM**) exist, not necessarily for the coherence morphisms, and the cofibre sequence in (**E**) exists, not necessarily for $i_!$ and $j^!$. Of course, for G to induce a meaningful motivic measure, $\mathtt{K}(\mathscr{C})$ should not be connected.

The aforementioned cohomology theories have *plain* versions (do not satisfy the excision property). Plane and properly supported versions of a cohomology theory coincide for proper schemes over the base. For a Noetherian scheme S of a finite Krull dimension, there exists a plain motivic spaces functor $M : \mathsf{Sch}^{\mathrm{ft}}/S \to \mathsf{sShv}_{\bullet,\mathrm{Nis}}(S)_{\mathbb{A}^1}$, given by the left Kan extension of the functor $\Delta^0_{-,+} : \mathsf{Sm}/S \to \mathsf{sShv}_{\bullet,\mathrm{Nis}}(S)_{\mathbb{A}^1}$ along the inclusion $\mathsf{Sm}/S \hookrightarrow \mathsf{Sch}^{\mathrm{ft}}/S$. Then, one may ask if there exists a properly supported motivic spaces functor $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S \to \mathsf{sShv}_{\bullet,\mathrm{Nis}}(S)_{\mathbb{A}^1}$, which coincides with M for proper schemes and gives rise to a motivic measure

$$\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0\left(\mathsf{sShv}^c_{\bullet,\mathrm{Nis}}(S)_{\mathbb{A}^1}\right),$$

where $\mathsf{sShv}^c_{\bullet,\mathrm{Nis}}(S)_{\mathbb{A}^1}$ is a Waldhausen subcategory in $\mathsf{sShv}_{\bullet,\mathrm{Nis}}(S)_{\mathbb{A}^1}$ with a non-connected $\mathtt{K}$-theory.

When $S = \mathtt{Spec}\, k$ for a field k of characteristic zero, Theorem 3.1.2 shows that it is sufficient for $\Delta^0_{-,+}$ to map blow up squares of smooth projective schemes in $\mathsf{Sch}^{\mathrm{ft}}/k$ to homotopy pushout squares and to map the empty scheme to the zero object, for it to induces the desired Euler-Poincaré characteristic. In fact the scissors relations (19) show that these conditions are also necessary. Although $\Delta^0_{-,+}$ does not seem to satisfy these conditions, its S^1-symmetric suspension $\Sigma^\infty_{S^1}\Delta^0_{-,+} : \mathsf{Sm}/k \to \mathsf{Spt}^{\Sigma}_{S^1}\left(\mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/k)\right)^{\mathbb{A}^1}_{\mathrm{stab}}$ does, see [**Voe10b**] and [**MV99**, §.3.Rem.2.30]. Hence, it gives rise to an Euler-Poincaré characteristic

$$\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0\left(\mathsf{Spt}^{\Sigma,c}_{S^1}\left(\mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/k)\right)^{\mathbb{A}^1}_{\mathrm{stab}}\right),$$

which is surjective as shown in [**Rön16**, Th.5.2]. The superscript ^{c} refers to a suitable Waldhausen subcategory, with a non-connected Waldhausen $\mathtt{K}$-theory, see [**Rön16**, Def.2.9].

For a more general Noetherian base scheme S, the question persists, due to the absence of an analogue of Theorem 3.1.2 over S. We find it more convenient to consider a more general question. That is,

> when does a weak monoidal functor $F : \mathsf{Prop}/S \to \mathscr{C}$, to a symmetric monoidal Waldhausen category, give rise to a weak monoidal functor $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S \to \mathscr{C}$ that satisfies the excision property, with the same restriction to Prop/S, and

hence defines a motivic measure

$$\mu_F : \mathrm{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathrm{K}_0(\mathscr{C}),$$

that sends the class of a proper S-scheme x to the class of $F(x)$?

If such a functor F^{c} exits, the excision property implies that, for a scheme $x \in \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S$ and a closed immersion $j : z \hookrightarrow p$ in Prop/S with complementary open immersion $i : x \hookrightarrow p$, the morphism $F(i) : F(z) \to F(p)$ is a cofibration in $\mathscr{C}$ and the cofibre of $F(i)$ is independent of the choice of such a closed immersion i. We refer to this property by saying that F is independent of compactifications[2]. Since the restriction of F^{c} to Prop/S coincides with F, it also implies $F(\varnothing) \cong \mathbf{0}$. This, in addition to Theorem 3.1.2 and the constructions in [**GS09**], led us to distinguish the properties:

(**PS1**) F maps closed immersions in Prop/S to cofibrations in $\mathscr{C}$;

(**PS2**) F maps the empty scheme to a zero object in $\mathscr{C}$; and

(**PS3**) F maps *cdp*-squares in Prop/S to pushout squares[3] in $\mathscr{C}$, where a *cdp*-square is is a Cartesian square

$$\begin{tikzcd} w \arrow[r, hook, "\underline{i}"] \arrow[d, "\underline{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & q \arrow[d, "f"] \\ z \arrow[r, hook, "i"'] & p \end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, where f is a proper morphism, i is a closed immersion and the induced morphism $\left(q \smallsetminus \underline{i}\right) \to \left(p \smallsetminus i\right)$ is an isomorphism, see Definition A.4.28.

Proposition 4.1.5 shows that (**PS3**) implies that F is independent of compactifications. Moreover, using Nagata's compactifications, we show that a (weak monoidal) functor $F : \mathsf{Prop}/S \to \mathscr{C}$ that satisfies the properties (**PS1**)-(**PS3**) induces a (weak monoidal) functor $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S \to \mathscr{C}$ that satisfies the proper support property, see Theorem 4.1.32.

Remark 4.0.1. One may only ask for F to send the empty scheme to an object weakly equivalent to the zero object in $\mathscr{C}$. Also, when $\mathscr{C}$ is induced from a model category, one may ask for F to send a *cdp*-square to a homotopy pushout square and drop the property (**PS1**). However, in that case, for the statements proven in this section to hold, one needs to assume that cofiltered limits preserve weak equivalences in $\mathscr{C}$.

Starting with a functor $F : \mathsf{Prop}/S \to \mathscr{C}$ that does not satisfy the properties (**PS1**)-(**PS3**), one may look for a localising exact functor $\mathscr{C} \to \mathscr{C}'$ of Waldhausen categories, for which the composition $F' : \mathsf{Prop}/S \to \mathscr{C} \to \mathscr{C}'$ satisfies the properties (**PS1**)-(**PS3**).

[2]For a justification of the terminology, see Definition 4.1.1.

[3]Since the *cdp*-topology is generated by *cdp*-squares, as in §.A.4.3, the properties (**PS2**) and (**PS3**) imply that F maps *cdp*-coving sieves to colimit cocones, *i.e.* F is a *cdp*-cosheaf.

We apply this construction to the Yoneda embedding in §4.2, and we recover a spectrum that we expect its path components to be isomorphic to the modified Grothendieck ring of S-schemes. Then, localising the affine line, we recover a motivic measure to a variant of the simplicially stable motivic homotopy category, with the *cdh*-topology. Over a field of characteristic zero, this measure coincides with the motivic measure defined in [**Rön16**].

We interpret such a weak monoidal functor $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}/S \to \mathscr{C}$, that satisfies the excision property, as a mean to provide a minimal compactification of S-schemes in $\mathscr{C}$. One does not seem to have a good notion of a minimal compactification in the category of schemes over a field k, for instance each of the Hirzebruch surfaces, particularly $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$, are good candidates to be minimal compactifications of the affine plane $\mathbb{A}^2_k$, yet none of them is minimal, even in the weakest sense[4].

EXAMPLE 4.0.2. Recall the motivic measure (14), for a perfect field k of exponential characteristic p. The geometric motive with proper support $M^{\mathrm{c}}_{\mathrm{gm},p}$ induces a motivic measure

$$\mathtt{K}_0(\mathsf{Var}/k) \to \mathtt{K}_{\triangle}\left(\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Z}[\frac{1}{p}])\right).$$

Then, one may think of $M^{\mathrm{c}}_{\mathrm{gm}}(\mathbb{A}^2)$ as a minimal compactification of $\mathbb{A}^2$ in $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Z}[\frac{1}{p}])$, as one realises cohomology theories with proper support through $M^{\mathrm{c}}_{\mathrm{gm}}$.

Through this chapter, assume that S is a Noetherian scheme of finite Krull dimension, and recall the conventions and notations in §0.2. In particular, the category of schemes of finite type over S is denoted by $\mathsf{Sch}^{\mathrm{ft}}/S$, and an S-scheme refers to an object in $\mathsf{Sch}^{\mathrm{ft}}/S$. Also, the full subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ of proper S-schemes is denoted by Prop/S. We use small Latin letters to denote S-schemes, and capital letters to denote their underlying schemes.

4.1. Properly Supported Extensions

The aim of this section is to show that a weak monoidal functor $F : \mathsf{Prop}/S \to \mathscr{C}$, to a symmetric monoidal Waldhausen category, admits a properly supported extension $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S \to \mathscr{C}$, when it satisfies the properties (**PS1**)-(**PS3**). In which case, it defines a motivic measure to $\mathtt{K}_0(\mathscr{C})$, given by sending the class of a proper S-scheme to the class of its image along F, see Theorem 4.1.32.

In this section, we begin by defining compactifications of S-schemes, and we show the category of compactifications to be cofiltered[5], as in Corollary 4.1.7, which is the main ingredient used to define properly supported functors on morphisms. Then, in §4.1.2, we define the extension F^{c}, and study its properties leading to the construction

[4] I have learnt about this example from [**vDdB16**].

[5] Our notion of a morphism of compactifications differs from that usually used in the literature.

of the desired motivic measure in §.4.1.2.7. Finally, we describe how one may proceed when F is not weak monoidal or does not satisfy the properties (**PS2**)-(**PS3**).

4.1.1. Compactifications. A compactification of an S-scheme $x : X \to S$ is usually defined as factorisation (j, p) of x in $\mathsf{Sch}^{\mathrm{ft}}/S$ as a (dominant) open immersion j followed by a proper morphism p, as in [**CD13**, §.2.2.8]; whereas a morphism of compactifications $(j, p) \to (l, q)$ is usually defined to be a commutative square

$$\begin{tikzcd} x \arrow[r, hook, "j"] \arrow[d, "f"'] & p \arrow[d, "g"] \\ Y \arrow[r, hook, "l"'] & q \end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$. In an early version of this thesis, morphisms of compactifications were restricted to Cartesian such squares, which was motivated by the argument in [**GS96**], Corollary 4.1.7, and Definition 4.1.15. However, after becoming aware of [**Cam17**], we adopt a different notion inspired by *subtraction sequences*, *loc.cit.*, which both strengthens our result and simplifies the proofs.

Definition 4.1.1. Let x be an S-scheme, a *compactification* of x is a closed immersion $i : z \hookrightarrow p$ of proper S-schemes, with complementary open immersion $j_i : x \hookrightarrow p$. Let $i : z \hookrightarrow p$ and $l : w \hookrightarrow q$ be a pair of compactifications of S-schemes x and y, respectively. A *morphism of compactifications* $(f, g) : i \to l$ is a solid commutative square

$$\begin{tikzcd} z \arrow[dr, hook, dotted] \arrow[drr, hook, bend left, "i"] \arrow[ddr, bend right, "f"'] & & \\ & w \times_q p \arrow[r, hook, dashed] \arrow[d, dashed] & p \arrow[d, "g"] \\ & w \arrow[r, hook, "l"'] & q \end{tikzcd} \tag{34}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, for which the unique morphism $z \hookrightarrow w \times_q p$ of S-schemes, induced by the universal property of pullbacks, is surjective.

In particular, the morphisms f and g are proper, by [**Gro61**, Cor.5.4.3(i)]. Compactifications of S-schemes and their morphisms form a category, with the evident composition and identity maps, and we denote it by Comp_S. For an S-scheme x, let $\mathsf{Comp}_S(x)$ denote the subcategory in Comp_S whose objects are compactifications of x and whose morphisms are morphisms of compactifications that restrict to isomorphisms on x. That is, a morphism $(f, g) : i' \to i$ of compactifications of x belongs to $\mathsf{Comp}_S(x)$ if and only if id_x is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_i. The restriction imposed on the morphisms of $\mathsf{Comp}_S(x)$ is needed for Corollary 4.1.7, and for the cofibres in Remark 4.1.11 to be independent of the choice of compactifications.

For a morphism of compactifications $(f, g) : i \to l$, the morphism f is uniquely determined by g, when it exists, due to l being a monomorphism in $\mathsf{Sch}^{\mathrm{ft}}/S$. Therefore, when no confusion arise, we may denote this morphism of compactifications by $g : i \to l$.

A compactification $i : z \hookrightarrow p$ of an S-scheme x induces a complementary open immersion $x \hookrightarrow p$, which is unique up to isomorphisms, and we denote by j_i. Since open complements are closed under pullbacks and both z and $w \times_q p$ in (34) have the same open complement in p, the morphism of compactifications $(f, g) : i \to l$ induces a Cartesian square

$$\begin{tikzcd} x \arrow[r, hook, "j_i"] \arrow[d, "g_{|x}"'] \arrow[dr, phantom, very near start, "\ulcorner"] & p \arrow[d, "g"] \\ y \arrow[r, hook, "j_l"'] & q \end{tikzcd} \tag{35}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$. One may alternatively define the morphism of compactifications $(f, g) : i \to l$ to be the solid outer square in (34) that induces the Cartesian square (35).

Remark 4.1.2. Although, the existence of the Cartesian square (35) does not imply the existence of a morphism of compactifications $(f, g) : i \to l$, it defines a morphism of compactifications $i_{\mathrm{red}} \to l$, where i_{red} is the composition of i with the surjective closed immersion $z_{\mathrm{red}} \hookrightarrow z$. One may be tempted to define a morphism of compactifications as a Cartesian square, without invoking the additional surjective closed immersion. However, our need to induce a morphism of compactifications from the Cartesian square (35), to prove Proposition 4.1.5 and Proposition 4.1.6, is the reason for the adopted notion of a morphism of compactifications.

Before we proceed, we need to recall the following technical result that we need to utilise on multiple occasions.

Lemma 4.1.3. Let $i : v \hookrightarrow x$ be a closed immersion and $j : x \hookrightarrow q$ be an open immersion of S-schemes, and let $i' : p \hookrightarrow q$ be the scheme-theoretic image of the immersion $j \circ i$. Then, the unique morphism $j' : v \to p$ of S-schemes for which $j \circ i = i' \circ j'$ is an open immersion. Moreover, the square

$$\begin{tikzcd} v \arrow[r, hook, "j'"] \arrow[d, hook, "i"'] & p \arrow[d, hook, "i'"] \\ x \arrow[r, hook, "j"'] & q \end{tikzcd} \tag{36}$$

is Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
v \arrow[drr, bend left, "j'"] \arrow[dr, dotted, "l"] \arrow[ddr, bend right, "i"'] & & \\
& x \times_q p \arrow[r, hook, "\underline{j}"'] \arrow[d, "\underline{i}'"] \arrow[dr, phantom, "\lrcorner", very near start] & p \arrow[d, "i'"] \\
& x \arrow[r, hook, "j"'] & q
\end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, where l is the unique morphism $v \to x \times_q p$ of S-schemes that makes the diagram commute. Since $j \circ i$ is an immersion, so is j'. All the underlying schemes of the S-schemes in the diagram above are Noetherian; hence the immersion $j' = \underline{j} \circ l$ is quasi-compact, see [**Sta17**, Tags 01OX and 01T6], and it factorises in $\mathsf{Sch}^{\mathrm{ft}}/S$ as an open immersion followed by a closed immersion, see [**Sta17**, Tag 01QV]. Then, j' is an open immersion, as i' is the scheme-theoretic image of $j \circ i = i' \circ j'$. Hence, l is also an open immersion. On the other hand, since i and $\underline{i}'$ are closed immersions, so is l, which is also surjective because i' the scheme-theoretic image of $j \circ i$. Therefore, l is a surjective open immersion, and hence an isomorphism. $\square$

The Category of Compactification. Since the notions of compactifications and their morphisms used here differ from those in the literature, we need to prove that the category $\mathsf{Comp}_S(x)$, and certain subcategories of which, are cofiltered, for every S-scheme x. This is the main tool used to extend a functor $F : \mathsf{Prop}/S \to \mathscr{C}$, that satisfies the properties (**PS1**)-(**PS3**), to a functor $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft\,prop}}_{\mathrm{open}}/S \to \mathscr{C}$ that satisfies the excision property.

Recall that a category $\mathscr{J}$ is *cofiltered* if it is nonempty and

- for every $X_0, X_1 \in \mathscr{J}$ there exists a *span* $X_0 \leftarrow X \to X_1$ in $\mathscr{J}$; and
- for every parallel morphisms $f_0, f_1 : X_0 \rightrightarrows X_1$ in $\mathscr{J}$, there exists a *refining morphism* $f : X \to X_0$ in $\mathscr{J}$ for which $f_0 \circ f = f_1 \circ f$.

For every S-scheme x, we start by showing the category $\mathsf{Comp}_S(x)$ to be nonempty, then Proposition 4.1.5 provides the existence of the desired spans, and Proposition 4.1.6 gives the refining morphisms.

Remark 4.1.4. Due to Nagata's Compactification Theorem, as in [**Nag62**] and [**Nag63**], every S-scheme x admits an open immersion $j : x \hookrightarrow p$ into a proper S-scheme p. Let $i_j : z \hookrightarrow p$ be the complementary closed immersion of j, endued with the reduced induced structure. Then, i_j is a compactification of x, and hence $\mathsf{Comp}_S(x) \neq \varnothing$. In particular, when p is a proper S-scheme, the category $\mathsf{Comp}_S(p)$ has an initial object, namely $\varnothing_p : \varnothing_S \hookrightarrow p$.

Proposition 4.1.5. Assume that $f : x \to y$ is a proper morphism in $\mathsf{Sch}^{\mathrm{ft}}/S$, and let $i : z \hookrightarrow p$ and $l : w \hookrightarrow q$ be compactifications of x and y, respectively. Then, there exists a compactification $i' : z' \hookrightarrow p'$ of x and morphisms of compactifications $h' : i' \to i$ and $g' : i' \to l$, such that id_x (resp. f) is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of h' (resp. g') along j_i (resp. j_l), where $j_i : x \hookrightarrow p$ and $j_l : y \hookrightarrow q$ are complementary open immersions of i and l, respectively.

Proof. In line with the argument of [**GS96**, §.2.3,p.141] and [**Sta17**, Tags 0ATU and 0A9Z], consider the solid commutative diagram

$$\begin{tikzcd}
x \arrow[rrr, hook, "j_i"] \arrow[ddd, "f"'] & & & p \\
& \Gamma_f \arrow[ul, "h"'] \arrow[ddl, "g"'] \arrow[r, dashed, hook, "j"] \arrow[d, hook, "c"] & \overline{\Gamma_f} \arrow[ur, dotted, "h'"] \arrow[d, dashed, hook, "c'"'] \arrow[ddr, dotted, "g'"] & \\
& x \times y \arrow[uul, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[r, hook, "j_i \times j_l"'] & p \times q \arrow[uur, two heads] \arrow[dr, two heads] & \\
y \arrow[rrr, hook, "j_l"'] & & & q,
\end{tikzcd} \tag{37}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$ that is induced by the existence of Cartesian products in $\mathsf{Sch}^{\mathrm{ft}}/S$ and the definition of the graph Γ_f of f. Since open immersions are closed under pullbacks and compositions, the morphism $j_i \times j_l$ is an open immersion. Let h (resp. g) be the composition of the Cartesian product projection $x \times y \twoheadrightarrow x$ (resp. $x \times y \twoheadrightarrow y$) with the closed immersion $c : \Gamma_f \hookrightarrow x \times y$, let $c' : \overline{\Gamma_f} \hookrightarrow p \times q$ be the scheme-theoretic image of $(j_i \times j_l) \circ c$, and let h' (resp. g') be the composition of the Cartesian product projection $p \times q \twoheadrightarrow p$ (resp. $p \times q \twoheadrightarrow q$) with the closed immersion $c' : \overline{\Gamma_f} \hookrightarrow p \times q$.

There exists an open immersion j for which $(j_i \times j_l) \circ c = c' \circ j$, by Lemma 4.1.3. The composition $h : \Gamma_f \hookrightarrow x \times y \twoheadrightarrow x$ is an isomorphism, see [**Gro60**, p.134]. Thus, there exists a compactification $i' : z' \hookrightarrow \overline{\Gamma_f}$ of x, where i' is a complementary closed immersion of $j \circ h^{-1} : x \hookrightarrow \overline{\Gamma_f}$, endued with the reduced induced structure.

Consider the commutative diagram (38), on the next page, induced by the universal property of pullbacks in $\mathsf{Sch}^{\mathrm{ft}}/S$. Since $j \circ h^{-1}$ is an open immersion, so is j'. The morphism g' is proper, and so is $\underline{g}'$, by [**Gro61**, Prop.5.4.2]. Since f is also proper, [**Gro61**, Cor.5.4.3(i)] implies that the immersion j' is proper, and hence a closed immersion, by [**Gro67**, Cor.18.12.6]. Since c' is the scheme-theoretic image of $(j_i \times j_l) \circ c \circ h^{-1}$ and j' is a closed immersion, j' is also surjective. Thus, j' is an isomorphism for being a surjective open immersion. Therefore, g' defines a morphism of compactifications $g' : i' \to l$, as in Remark 4.1.2, because the underlying scheme of z' is reduced. Moreover, f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g' along j_l.

$$
\begin{tikzcd}
x \arrow[rrd, bend left, hook, "j \circ h^{-1}"] \arrow[rd, dashed, hook, "j'"] \arrow[rdd, bend right, "f"'] & & \\
& \overline{\Gamma_f} \times_q y \arrow[r, hook, "\underline{j_l}"'] \arrow[d, "\underline{g'}"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\Gamma_f} \arrow[d, "g'"] \\
& y \arrow[r, hook, "j_l"'] & q
\end{tikzcd}
\tag{38}
$$

Similarly, one sees that there exists a morphism of compactifications $h' : i' \to i$ such that id_x is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of h' along j_i. $\square$

Proposition 4.1.6. Let x and y be S-schemes, let $i : z \hookrightarrow p$ and $l : w \hookrightarrow q$ be compactifications of x and y, respectively, and suppose that $(f_0, g_0), (f_1, g_1) : i \to l$ are parallel morphisms of compactifications. Then, there exist an S-scheme x', a compactification $i' : z' \hookrightarrow p'$ of x', and a morphism of compactifications $(f, g) : i' \to i$ for which

$$(f_0, g_0) \circ (f, g) = (f_1, g_1) \circ (f, g).$$

Moreover, when $g_{0|_x} = g_{1|_x}$, the S-scheme x' can be chosen to be x, and the morphism g can be chosen such that id_x is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_i.

Proof. Let $j_i : x \overset{\circ}{\hookrightarrow} p$ and $j_l : y \overset{\circ}{\hookrightarrow} q$ be the complementary open immersions of i and l, respectively, and let $g_{k|_x} : x \to y$ be a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g_k along j_l, for $k = 0, 1$. Consider the solid diagram (39) of S-schemes, on the next page, which is induced by the definition of the graphs $\Gamma_{g_{k|_x}}$ and Γ_{g_k} of $g_{k|_x}$ and g_k, respectively, for $k = 0, 1$. In the solid diagram (39), the side subdiagrams are commutative, but the front and back faces are not necessarily commutative. The morphisms h_k and $\overline{h}_k$ are the unique morphisms that factorise $(\mathrm{id}_x, g_{k|_x})$ and (id_p, g_k) in $\mathsf{Sch}^{\mathrm{ft}}/S$ as $i_k \circ h_k$ and $\overline{i}_k \circ \overline{h}_k$, respectively, for $k = 0, 1$. Whereas, the morphisms π_y and π_q are the Cartesian products projections.

The proof is based on basic constructions on this solid diagram, and follows through commutative subdiagrams chase; yet we spell it out for the reader's convenience.

The morphisms h_k and $\overline{h}_k$ are isomorphisms with inverses $\pi_x \circ i_k$ and $\pi_p \circ \overline{i}_k$, respectively, for $k = 0, 1$. Hence, in particular, we have an open immersion $j_k : \Gamma_{g_{k|_x}} \overset{\circ}{\hookrightarrow} \Gamma_{g_k}$, given by $j_k = \overline{h}_k \circ j_i \circ h_k^{-1}$, for $k = 0, 1$. Then, the horizontal square containing j_0 and the vertical square containing j_1 are commutative, *i.e.* $\overline{i}_k \circ j_k = (j_i \times j_l) \circ i_k$, for $k = 0, 1$.

Let x' be the fibre product $\Gamma_{g_{0|x}} \times_{x\times y} \Gamma_{g_{1|x}}$, with the fibre product projections i'_0 and i'_1, and let p' be the fibre product $\Gamma_{g_0} \times_{p\times q} \Gamma_{g_1}$, with the fibre product projections $\bar{i}'_0$ and $\bar{i}'_1$. Then, there exists a unique morphism $x' \to p'$ of S-schemes, induced by the universal property of fibre products, making the squares containing it commute, which we denote by j'.

$$\begin{tikzcd}
 & & & p \arrow[drrr, bend left, "\bar{h}_1"] \arrow[ddddr, bend right, "\bar{h}_0"'] & & & & \\
 & & & & p' \arrow[rr, hook, dotted, "\bar{i}'_0"] \arrow[ddd, hook, dotted, "\bar{i}'_1"] \arrow[dr, phantom, "\ulcorner", very near start] & & \Gamma_{g_1} \arrow[ddd, hook, "\bar{i}_1"] \arrow[ddddr, bend left] & \\
x \arrow[uurrr, hook, "j_i"] \arrow[drrr, bend left, "h_1"] \arrow[ddddr, bend right, "h_0"'] & & & & & & & \\
 & x' \arrow[uurrr, dotted, "j'"] \arrow[rr, hook, dotted, "i'_0"'] \arrow[ddd, hook, dotted, "i'_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & & \Gamma_{g_{1|x}} \arrow[uurrr, hook, dashed, "j_1"] \arrow[ddd, hook, "i_1"'] \arrow[ddddr, bend left] & & & & \\
 & & & & \Gamma_{g_0} \arrow[rr, hook, "\bar{i}_0"] \arrow[drrr, bend right] & & p\times q \arrow[dr, "\pi_q"] & \\
 & & & & & & & q \\
 & \Gamma_{g_{0|x}} \arrow[uurrr, hook, dashed, "j_0"'] \arrow[rr, hook, "i_0"'] \arrow[drrr, bend right] & & x\times y \arrow[uurrr, hook, "j_i \times j_l"'] \arrow[dr, "\pi_y"'] & & & & \\
 & & & & y \arrow[uurrr, hook, "j_l"'] & & &
\end{tikzcd}$$

(39)

In fact the two squares that contain the morphism j' are Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$. To see that, fix $k \in \{0,1\}$, let u be an S-scheme and suppose that $\alpha_k : u \to \Gamma_{g_{k|x}}$ and $\beta : u \to p'$ are morphisms in $\mathsf{Sch}^{\mathrm{ft}}/S$ for which $j_k \circ \alpha_k = \bar{i}'_{1-k} \circ \beta$. To establish the desired unique morphism $\gamma_k : u \to x'$ of S-schemes for which $\alpha_k = i'_{1-k} \circ \gamma_k$ and $\beta = j' \circ \gamma_k$, we first deduce the existence of a morphism $\delta_k : u \to \Gamma_{g_{1-k|x}}$ satisfying some uniqueness property, and we use it to establish the desired morphism $\gamma_k : u \to x'$.

Composing the given relation with $\pi_q \circ \bar{i}_k$, one has

$$j_l \circ (\pi_y \circ i_k \circ \alpha_k) = \pi_q \circ \bar{i}_k \circ j_k \circ \alpha_k = \pi_q \circ \bar{i}_k \circ \bar{i}'_{1-k} \circ \beta = (\pi_q \circ \bar{i}_{1-k}) \circ (\bar{i}'_k \circ \beta).$$

Since $(f_{1-k}, g_{1-k}) : i \to l$ is a morphism of compactifications and h_{1-k} and $\bar{h}_{1-k}$ are isomorphisms, the square containing the morphisms $\pi_y \circ i_{1-k}$ and $\pi_q \circ \bar{i}_{1-k}$ is Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$. Thus, there exists a unique morphism $\delta_k : u \to \Gamma_{g_{1-k|x}}$ of S-schemes for which

$$\pi_y \circ i_k \circ \alpha_k = (\pi_y \circ i_{1-k}) \circ \delta_k \qquad \text{and} \qquad \bar{i}'_k \circ \beta = j_{1-k} \circ \delta_k.$$

Notice that

$$(j_i \times j_l) \circ i_{1-k} \circ \delta_k = \bar{i}_{1-k} \circ j_{1-k} \circ \delta_k = \bar{i}_{1-k} \circ \bar{i}'_k \circ \beta = \bar{i}_k \circ \bar{i}'_{1-k} \circ \beta = \bar{i}_k \circ j_k \circ \alpha_k = (j_i \times j_l) \circ i_k \circ \alpha_k.$$

Since $j_i \times j_l$ is a monomorphism in $\mathsf{Sch}^{\mathrm{ft}}/S$, one has $i_{1-k} \circ \delta_k = i_k \circ \alpha_k$. Then, by the universal property of pullbacks, there exists a unique morphism $\gamma_k : u \to x'$ of S-schemes for which $\alpha_k = i'_{1-k} \circ \gamma_k$ and $\delta_k = i'_k \circ \gamma_k$. Thus,

$$\overline{i}'_{1-k} \circ j' \circ \gamma_k = j_k \circ i'_{1-k} \circ \gamma_k = j_k \circ \alpha_k = \overline{i}'_{1-k} \circ \beta.$$

Since $\overline{i}'_{1-k}$ is a monomorphism, $\beta = j' \circ \gamma_k$. To prove the uniqueness, let $\gamma'_k : u \to x'$ be a morphism of S-schemes for which $\alpha_k = i'_{1-k} \circ \gamma'_k$ and $\beta = j' \circ \gamma'_k$. Since i'_{1-k} is a monomorphism in $\mathsf{Sch}^{\mathrm{ft}}/S$, one has $\gamma'_k = \gamma_k$. Therefore, the squares that contain j' are Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$.

Then, in particular, j' is an open immersion, and there exists a compactification $i' : z' \hookrightarrow p'$ of x', where i' is a complementary closed immersion of j', endued with the reduced induced structure.

To establish the desired morphism of compactifications, notice that

$$h_1^{-1} \circ i'_0 = (\pi_x \circ i_1 \circ h_1) \circ (h_1^{-1} \circ i'_0) = \pi_x \circ i_1 \circ i'_0 = \pi_x \circ i_0 \circ i'_1 = (\pi_x \circ i_0 \circ h_0) \circ (h_0^{-1} \circ i'_1) = h_0^{-1} \circ i'_1,$$

and similarly, $\overline{h}_1^{-1} \circ \overline{i}'_0 = \overline{h}_1^{-1} \circ \overline{i}'_1$. Let $g \coloneqq \overline{h}_1^{-1} \circ \overline{i}'_0 = \overline{h}_0^{-1} \circ \overline{i}'_1$, then the morphism $h_1^{-1} i'_0 = h_0^{-1} i'_1$ is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_i, which we denote by $g_{|_{x'}}$. Since the square

$$\begin{tikzcd}
x' \arrow[r, hook, "j'"] \arrow[d, "g_{|_{x'}}"'] \arrow[dr, phantom, very near start, "\ulcorner"] & p' \arrow[d, "g"] \\
x \arrow[r, hook, "j_i"'] & p
\end{tikzcd}$$

is Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$ and the underlying scheme of z' is reduced, there exists a morphism of compactifications $(f, g) : i' \to i$, for the unique morphism $f : z' \to z$ that factorises $g \circ i'$ in $\mathsf{Sch}^{\mathrm{ft}}/S$ as $i \circ f$, see Remark 4.1.2.

Then, one has

$$g_0 \circ g = (\pi_q \circ \overline{i}_0 \circ \overline{h}_0) \circ (\overline{h}_0^{-1} \circ \overline{i}'_1) = \pi_q \circ \overline{i}_0 \circ \overline{i}'_1 = \pi_q \circ \overline{i}_1 \circ \overline{i}'_0 = (\pi_q \circ \overline{i}_1 \circ \overline{h}_1) \circ (\overline{h}_1^{-1} \circ \overline{i}'_0) = g_1 \circ g,$$

and

$$l \circ f_0 \circ f = g_0 \circ g \circ i' = g_1 \circ g \circ i' = l \circ f_1 \circ f.$$

Since l is a monomorphism in $\mathsf{Sch}^{\mathrm{ft}}/S$, one has $(f_0, g_0) \circ (f, g) = (f_1, g_1) \circ (f, g)$.

Moreover, when $g_{0|_x} = g_{1|_x}$, the universal property of pullbacks implies the existence of a morphism $x \to x'$ in $\mathsf{Sch}^{\mathrm{ft}}/S$ that factorises the isomorphism $h_0 = h_1$. Since i'_0 is a closed immersion, such a morphism $x \to x'$ is an isomorphism. Pullbacks are determined up to isomorphisms; thus, we may choose $x' = x$, in which case id_x is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_i. □

Corollary 4.1.7. Let x be an S-scheme. Then, the category $\mathsf{Comp}_S(x)$ is cofiltered.

Proof. Since S is a Noetherian scheme[6], Nagata's Compactification Theorem implies that $\mathsf{Comp}_S(x)$ is nonempty, as seen in Remark 4.1.4. Then, the statement of the corollary is a direct result of Proposition 4.1.5, for $f = \mathrm{id}_x$, and Proposition 4.1.6. $\square$

Let $f : x \to y$ be a morphism of S-schemes, and let $l : w \hookrightarrow\!\!\!\!\!\!\not\;\; q$ be a compactification of y. Denote by $\mathsf{Comp}_S(f, l)$ the full subcategory in $\mathsf{Comp}_S(x)$ that satisfies the property

> a compactification i of x belongs to $\mathsf{Comp}_S(f, l)$ if and only if it admits a morphism of compactifications $g : i \to l$ such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_l.

Also, let $\mathsf{Comp}_S(f)$ denote the full subcategory in $\mathsf{Comp}_S(x)$ of compactifications of x that belong to $\mathsf{Comp}_S(f, l)$ for some compactification l of y.

Corollary 4.1.8. Assume that $f : x \to y$ is a proper morphism of S-schemes, and let $l : w \hookrightarrow\!\!\!\!\!\!\not\;\; q$ be a compactification of y. Then, the category $\mathsf{Comp}_S(f, l)$ is co-cofinal in $\mathsf{Comp}_S(x)$, and so is $\mathsf{Comp}_S(f)$. Moreover, the categories $\mathsf{Comp}_S(f, l)$ and $\mathsf{Comp}_S(f)$ are cofiltered.

Proof. A direct consequence of Proposition 4.1.5, Proposition 4.1.6, and [**Tam94**, Ch.0.§.3.2-3]. $\square$

4.1.2. Extensions of Compactifiable Functors. For the rest of this subsection, let $F : \mathsf{Prop}/S \to \mathscr{C}$ be a functor to a Waldhausen category that satisfies the properties (**PS1**)-(**PS3**). We will show that F extends to a functor $F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S \to \mathscr{C}$ that satisfies the excision property (**E**), as in Proposition 4.1.29. Moreover, for a symmetric monoidal Waldhausen category $\mathscr{C}$, if F is weak monoidal, then so is F^{c}, as in Proposition 4.1.31. The main statement in this subsection is Theorem 4.1.32.

Definition 4.1.9. A functor $\mathsf{Prop}/S \to \mathscr{C}$ to a Waldhausen category that satisfies the properties (**PS1**)-(**PS3**) is called a *cdp-functor*.

This terminology is motivated by Definition A.4.28 and Proposition A.4.30.

Remark 4.1.10. The properties (**PS1**)-(**PS3**) imply that

(**PS4**) F maps every surjective closed immersion in Prop/S to an isomorphism.

That is, for a surjective closed immersion $i : z \hookrightarrow\!\!\!\!\!\!\not\;\; p$ in Prop/S, the square

$$\begin{tikzcd} \varnothing_S \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \varnothing_S \arrow[d, hook] \\ z \arrow[r, hook, "i"'] & p \end{tikzcd} \tag{40}$$

[6] In the light of [**Con07**], one may generalise most statements in this section for a quasi-compact quasi-separated base scheme S.

is a *cdp*-square in Prop/S, which is mapped by F to a square of cofibrations in $\mathscr{C}$, by (**PS1**). Then, (**PS3**) and (**PS2**) imply that i_*[7] is the composite isomorphism $F(z) \cong F(z)/F(\varnothing_S) \cong F(p)/F(\varnothing_S) \cong F(p)$.

Assume that x and y are S-schemes, let $i : z \hookrightarrow\!\!\!\!\to p$ and $l : w \hookrightarrow\!\!\!\!\to q$ be compactifications of x and y, respectively, and let $(f, g) : i \to l$ be a morphism of compactifications, as in (34). The morphism (f, g) is mapped to the solid commutative square

$$\begin{tikzcd}
F(z) \arrow[r, tail, "i_*"] \arrow[d, "f_*"'] & F(p) \arrow[r, dashed, two heads, "\epsilon_i"] \arrow[d, "g_*"] & C_F(i) \arrow[d, dotted, "(f_*, g_*)"] \\
F(w) \arrow[r, tail, "l_*"'] & F(q) \arrow[r, dashed, two heads, "\epsilon_l"'] & C_F(l)
\end{tikzcd} \tag{41}$$

in $\mathscr{C}$. Sine F satisfies (**PS1**), both i_* and l_* are cofibrations in $\mathscr{C}$. Let $C_F(i)$ and $C_F(l)$ be the cofibres of i_* and l_*, respectively. Since the left solid square commutes, there exists a unique morphism $C_F(i) \to C_F(l)$ in $\mathscr{C}$ that makes the whole diagram commute, which we denote by (f_*, g_*).

That defines a functor

$$C_F : \mathsf{Comp}_S \to \mathscr{C}, \tag{42}$$

given on objects and morphisms in (41).

Remark 4.1.11. For every S-scheme x, let $C_{F,x}$ be the composition of the functor C with the inclusion of the subcategory $\mathsf{Comp}_S(x) \hookrightarrow \mathsf{Comp}_S$. For a morphism of compactifications $(f, g) : i' \to i$ in $\mathsf{Comp}_S(x)$, *i.e.* a commutative diagram

$$\begin{tikzcd}
z' \arrow[dr, dashed, hook, "c"] \arrow[drr, bend left, hook, "i'"] \arrow[ddr, bend right, "f"'] & & \\
& z \times_p p' \arrow[r, dashed, hook, "\underline{i}"'] \arrow[d, dashed, "\underline{g}"] & p' \arrow[d, "g"] \\
& z \arrow[r, hook, "i"'] & p,
\end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, in which c is a surjective closed immersion. The pullback square, in the diagram above, is a *cdp*-square in $\mathsf{Sch}^{\mathrm{ft}}/S$, as g is a proper morphism, i is a closed immersion, and id_x is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_i. Since F satisfies (**PS1**) and (**PS3**), the morphism $(\underline{g}_*, g_*)$ is an isomorphism in $\mathscr{C}$. Also, c_* is an isomorphism in $\mathscr{C}$, as F satisfies (**PS4**). Hence, (f_*, g_*) is an isomorphism. Therefore, $C_{F,x}$ is a diagram of isomorphisms, and hence $\mathtt{lim}\, C_{F,x}$ exists in $\mathscr{C}$. For a compactification $i : z \hookrightarrow\!\!\!\!\to p$ of x, we denote the limit projection $\mathtt{lim}\, C_{F,x} \to C_F(i)$ by ι_i.

[7] For a morphism f in Prop/S, we denote $F(i)$ by f_*. Also, we adopt the same notation for other (contravariant) functors, when no confusion arises.

Since $\mathtt{lim}\, C_{F,x}$ is independent of the choice of compactifications of x and satisfies the excision property for proper S-schemes, we will define F^{c} on objects by $F^{\mathrm{c}}(x) \coloneqq \mathtt{lim}\, C_{F,x}$. Then, in §.4.1.2.1 and §.4.1.2.3, we define F^{c} on proper morphisms and formal inverses of open immersions, respectively.

4.1.2.1. *Proper Pushforwards.* In order to define the desired functor F^{c} on proper morphisms, one needs to assign for every proper morphism $f : x \to y$ of S-schemes a unique morphism $\mathtt{lim}\, C_{F,x} \to \mathtt{lim}\, C_{F,y}$, that is independent of the choice of compactifications and morphisms between them. We show below that the canonical choices of such morphisms coincide, see Corollary 4.1.14.

Lemma 4.1.12. Assume that $f : x \to y$ is a proper morphism of S-schemes, and let $l : w \hookrightarrow\!\!\!\to q$ be a compactification of y. Then, the morphism

$$g_* \circ \iota_i : \mathtt{lim}\, C_{F,x} \to C_F(l) \tag{43}$$

is independent of the choice of the compactification $i : z \hookrightarrow\!\!\!\to p$ in $\mathsf{Comp}_S(f, l)$ and of the morphism of compactifications $g : i \to l$ such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_l. We denote this morphism by ϱ_l^f.

Proof. Since f is proper, the category $\mathsf{Comp}_S(f, l)$ is nonempty, by Proposition 4.1.5. Suppose that $i_k : z_k \hookrightarrow\!\!\!\to p_k$ is a compactification in $\mathsf{Comp}_S(f, l)$, and let $g_k : i_k \to l$ be a morphism of compactifications such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g_k along j_l, for $k = 0, 1$. Since the category $\mathsf{Comp}_S(f, l)$ is cofiltered, there exists a compactification $i : z \hookrightarrow\!\!\!\to p$ in $\mathsf{Comp}_S(f, l)$ and a morphism of compactifications $g'_k : i \to i_k$ such that id_x is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g'_k along j_{i_k}, for $k = 0, 1$. Proposition 4.1.6 implies that i, g'_0, and g'_1 can be chosen such that $g_0 \circ g'_0 = g_1 \circ g'_1$. Thus,

$$g_{0*} \circ \iota_{i_0} = g_{0*} \circ g'_{0*} \circ \iota_i = g_{1*} \circ g'_{1*} \circ \iota_i = g_{1*} \circ \iota_{i_1}.$$

On the other hand, suppose that $i : z \hookrightarrow\!\!\!\to p$ is a compactification in $\mathsf{Comp}_S(f, l)$ and let $g_0, g_1 : i \to l$ be parallel morphisms of compactifications such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g_k along j_l, for $k = 0, 1$. By Proposition 4.1.6, there exists a refining compactification $i' : z' \to p'$ of x and a morphism of compactifications $g : i' \to i$ in $\mathsf{Comp}_S(f, l)$ such that $g_0 \circ g = g_1 \circ g$, by Corollary 4.1.8. Thus,

$$g_{0*} \circ \iota_i = g_{0*} \circ g_* \circ \iota_{i'} = g_{1*} \circ g_* \circ \iota_{i'} = g_{1*} \circ \iota_i.$$

□

Lemma 4.1.13. Assume that $f : x \to y$ is a proper morphism of S-schemes, let $l_k : w_k \hookrightarrow\!\!\!\to q_k$ be a compactification of y, for $k = 0, 1$, and let $g : l_0 \to l_1$ be a morphism of compactifications in $\mathsf{Comp}_S(y)$. Then,

$$\varrho_{l_1}^f = g_* \circ \varrho_{l_0}^f.$$

PROOF. Since f is proper, the category $\mathsf{Comp}_S(f, l_k)$ is nonempty, for $k = 0, 1$. Let $i_k : z_k \hookrightarrow\!\!\!\rightarrow p_k$ be a compactification in $\mathsf{Comp}_S(f, l_k)$, and let $g_k : i_k \to l_k$ be a morphism of compactifications such that f is a base change in $\overset{\mathrm{ft}}{\mathsf{Sch}}/S$ of g_k along j_{l_k}, for $k = 0, 1$. Since $\mathsf{Comp}_S(x)$ is cofiltered, there exists a compactification $i : z \hookrightarrow\!\!\!\rightarrow p$ of x and a morphism of compactifications $g'_k : i \to i_k$ in $\mathsf{Comp}_S(x)$, for $k = 0, 1$, by Corollary 4.1.7.

Since $g \circ g_0 \circ g'_0$ and $g_1 \circ g'_1$ are parallel morphisms of compactifications and f is a base change in $\overset{\mathrm{ft}}{\mathsf{Sch}}/S$ of both $g \circ g_0 \circ g'_0$ and $g_1 \circ g'_1$ along j_l, Lemma 4.1.12 implies that

$$\varrho^f_{l_1} = g_{1*} \circ \iota_{i_1} = g_{1*} \circ g'_{1*} \circ \iota_i = g_* \circ g_{0*} \circ g'_{0*} \circ \iota_i = g_* \circ g_{0*} \circ \iota_{i_0} = g_* \circ \varrho^f_{l_0}.$$

□

COROLLARY 4.1.14. Assume that $f : x \to y$ is a proper morphism of S-schemes. Then, there exists a unique morphism $f_! : \lim C_{F,x} \to \lim C_{F,y}$ in $\mathscr{C}$ for which

$$\iota_l \circ f_! = \varrho^f_l, \tag{44}$$

for every compactification $l : w \hookrightarrow\!\!\!\rightarrow q$ of y.

The uniqueness of the morphism $f_!$ implies the functoriality of pushforward along proper morphisms. That is, for proper morphisms $f : x \to y$ and $g : y \to z$ of S-schemes, one has

$$(g \circ f)_! = g_! \circ f_! \qquad \text{and} \qquad (\mathrm{id}_x)_! = \mathrm{id}_{\lim C_{F,x}}. \tag{45}$$

DEFINITION 4.1.15. The *properly supported counterpart* of F to be a functor

$$F^{\mathrm{c}} : \overset{\mathrm{ft\,prop}}{\mathsf{Sch}}/S \to \mathscr{C} \tag{46}$$

that sends an S-schemes x to $\lim C_{F,x}$, as in Remark 4.1.11, and sends a proper morphism $f : x \to y$ of S-schemes to the unique morphism $f_! : \lim C_{F,x} \to \lim C_{F,y}$ satisfying the statement of Corollary 4.1.14.

REMARK 4.1.16. When $\mathscr{C}$ has cofiltered limits, the functor F^{c} may be defined on proper morphisms similarly, even when F is not a *cdp*-functor. However, such a functor does not necessarily satisfy the excision property.

EXAMPLE 4.1.17. Assume that p is a proper S-scheme. Since the category $\mathsf{Comp}_S(p)$ admits an initial object, namely $\varnothing_p : \varnothing_S \hookrightarrow\!\!\!\rightarrow p$, and F satisfies (**PS2**), one has

$$F^{\mathrm{c}}(p) \cong C_F(\varnothing_p) \cong F(p).$$

The functor F^{c} satisfies generalisations of the properties (**PS1**)-(**PS4**) to the category $\overset{\mathrm{ft\,prop}}{\mathsf{Sch}}/S$, as seen in the following proposition.

PROPOSITION 4.1.18. The functor $F^{\mathrm{c}} : \overset{\mathrm{ft\,prop}}{\mathsf{Sch}}/S \to \mathscr{C}$

(**PS1′**) maps closed immersions in $\overset{\mathrm{ft\,prop}}{\mathsf{Sch}}/S$ to cofibrations in $\mathscr{C}$;
(**PS2′**) maps *cdp*-squares in $\overset{\mathrm{ft\,prop}}{\mathsf{Sch}}/S$ to pushout squares in $\mathscr{C}$;

(PS3′) maps the empty S-scheme to the zero object in $\mathscr{C}$; and

(PS4′) maps surjective closed immersions in $\mathsf{Sch}^{\mathrm{ft\,prop}}/S$ to isomorphisms.

Proof. The statement **(PS3′)** is evident; whereas **(PS4′)** follows from the other statements, as seen in (40).

(PS1′) Assume that $i : v \hookrightarrow x$ is a closed immersion of S-schemes. Let $l : w \hookrightarrow q$ be a compactification of x with complementary open immersion $j_l : x \hookrightarrow q$, and let $i' : p \hookrightarrow q$ be the scheme-theoretic image of the immersion $j_l \circ i$. Then, Lemma 4.1.3 implies the existence of an open immersion $j_l' : v \hookrightarrow p$ for which the solid square

$$
\begin{tikzcd}
v \arrow[r, hook, "j_l'"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & p \arrow[d, hook, "i'"] & z \arrow[l, hook', dashed, "\underline{l}"'] \arrow[d, hook, dashed, "\underline{i}"] \arrow[dl, phantom, "\urcorner", very near start] \\
x \arrow[r, hook, "j_l"'] & q & w \arrow[l, hook', "l"]
\end{tikzcd}
\tag{47}
$$

is Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$. Let $\underline{l}$ be a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of l along i'. Since open complements are closed under pullbacks and j_l is a complementary open immersion to l, one finds that j_l' is a complementary open immersion to $\underline{l}$. Hence, $\underline{l} : z \hookrightarrow p$ is a compactification in $\mathsf{Comp}_S(i, l)$ and $(\underline{i}, i') : \underline{l} \to l$ is a morphism of compactifications such that i is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of i' along j_l.

There exists a pushout of the closed immersions $\underline{l}$ and $\underline{i}$ in Prop/S, which we denote by $\underline{q}$. In fact, since the right square in (47) is Cartesian, there exists a bicartesian square

$$
\begin{tikzcd}
z \arrow[r, hook, "\underline{l}"] \arrow[d, hook, "\underline{i}"'] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near end] & p \arrow[d, hook, "\underline{i}'"] \\
w \arrow[r, hook, "\underline{l}'"'] & \underline{q},
\end{tikzcd}
$$

of closed immersions in Prop/S, see [**Sch05**, Th.3.11] and [**Sta17**, Tag 0B7M]. In particular, it is a *cdp*-square in Prop/S, and hence it is mapped by F to a pushout square of cofibrations in $\mathscr{C}$. Moreover, the unique morphism $k : \underline{q} \to q$, for which $k \circ \underline{i}' = i'$ and $k \circ \underline{l}' = l$, is a closed immersion. Consider the solid diagram (48) of cofibrations in $\mathscr{C}$, on the next page. Since $\epsilon_{\underline{l}} \circ \underline{l}_* = 0$, there exists a unique morphism $\gamma : F(\underline{q}) \to C_F(\underline{l})$ in $\mathscr{C}$, for which $\gamma \circ \underline{l}'_* = 0$ and $\gamma \circ \underline{i}'_* = \epsilon_{\underline{l}}$. Since ϵ_l is an epimorphism in $\mathscr{C}$, a diagram chase shows that $(\underline{i}_*, i'_*)$ is a cobase change in $\mathscr{C}$ of k_* along γ. Recall that $\iota_l \circ i_! = (\underline{i}_*, i'_*) \circ \iota_{\underline{l}}$ and that $\iota_{\underline{l}}$ is an isomorphisms in $\mathscr{C}$. Thus, $i_!$ is a cobase change in $\mathscr{C}$ of k_* along $\iota_{\underline{l}}^{-1} \circ \gamma$.

Since k_* is a cofibration in $\mathscr{C}$, as k is a closed immersion, the morphism $i_!$ is a cofibration in $\mathscr{C}$.

$$\begin{tikzcd}
F(z) \arrow[rr, tail, "\underline{l}_*"] \arrow[dd, tail, "\underline{i}_*"'] & & F(p) \arrow[r, dashed, two heads, "\epsilon_{\underline{l}}"] \arrow[dl, tail, "\underline{i'}_*"'] \arrow[dd, tail, "i'_*"] & C_F(\underline{l}) \arrow[dd, dotted, "(\underline{i}_*, i'_*)"] \\
& F(\underline{q}) \arrow[dr, tail, "k_*"] & & \\
F(w) \arrow[rr, tail, "l_*"'] \arrow[ur, tail, "\underline{l'}_*"] & & F(q) \arrow[r, dashed, two heads, "\epsilon_l"] & C_F(l)
\end{tikzcd} \tag{48}$$

(PS2′) A *cdp*-square in $\mathsf{Sch}^{\mathrm{ft}}/S$ defines a Cartesian cube, in which the ambient proper S-schemes fit into a *cdp*-square. Then, a diagram chase on the Cartesian cube imply the statement.

□

EXAMPLE 4.1.19. Assume that $j_x : x \hookrightarrow x \bigsqcup y$ is a closed open immersion of S-schemes with complementary closed open immersion $j_y : y \hookrightarrow x \bigsqcup y$. Then, the square

$$\begin{tikzcd}
\varnothing_S \arrow[r, "\varnothing_y"] \arrow[d, "\varnothing_x"'] & y \arrow[d, hook, "j_y"] \\
x \arrow[r, hook, "j_x"'] & x \bigsqcup y
\end{tikzcd}$$

is a *cdp*-square in $\mathsf{Sch}^{\mathrm{ft}}/S$. Hence, by Proposition 4.1.18, one finds that

$$F^{\mathrm{c}}(x \bigsqcup y) \cong F^{\mathrm{c}}(x) \coprod F^{\mathrm{c}}(y).$$

4.1.2.2. *A Comparison Morphism.* Let $G : \mathsf{Sch}^{\mathrm{ft}}/S \to \mathscr{C}$ be a functor to a Waldhausen category, whose restriction to Prop/S is a *cdp*-functor. We abuse notation, and use G to also denote its restrictions to $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}/S$ and Prop/S. We see below that there exists a canonical natural transformation from G to its properly supported counterpart. This natural transformation is particularly useful to define the monoidal coherence morphisms in §.4.1.2.6.

LEMMA 4.1.20. Assume that $G : \mathsf{Sch}^{\mathrm{ft}}/S \to \mathscr{C}$ is a functor to a Waldhausen category, whose restriction to Prop/S is a *cdp*-functor. Then, there exists a unique natural transformation $\varphi : G \Rightarrow G^{\mathrm{c}}$ for which

$$\iota_i \circ \varphi_x = \epsilon_i \circ j_{i*},$$

for every S-scheme x and for every compactification $i : z \hookrightarrow\!\!\!\!\rightarrow p$ of x.

PROOF. Suppose that x is an S-scheme x, let $i_k : z_k \to p_k$ be a compactification of x with complementary open immersion $j_{i_k} : x \hookrightarrow p_k$, for $k = 0, 1$, and let $g : i_0 \to i_1$ be a morphism of compactifications in $\mathsf{Comp}_S(x)$. Then,

$$g_* \circ \epsilon_{i_0} \circ j_{i_0 *} = \epsilon_{i_1} \circ g_* \circ j_{i_0 *} = \epsilon_{i_1} \circ (g \circ j_{i_0})_* = \epsilon_{i_1} \circ j_{i_1 *}.$$

Therefore, by the universal property of limits, there exists a unique morphism $\varphi_x : G(x) \to G^{\mathrm{c}}(x)$ in $\mathscr{C}$ for which

$$\iota_i \circ \varphi_x = \epsilon_i \circ j_{i *}, \tag{49}$$

for every compactification $i : z \hookrightarrow p$ of x.

Suppose that $f : x \to y$ is a proper morphism of S-schemes and let $l : w \hookrightarrow q$ be a compactification of y, then the category $\mathsf{Comp}_S(f, l)$ is nonempty, see Proposition 4.1.5. Assume that $i : z \hookrightarrow p$ is a compactification of x in $\mathsf{Comp}_S(f, l)$, let $g : i \to l$ be a morphism of compactifications such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_l, and consider the diagram

$$\begin{tikzcd}
G(x) \arrow[r, "\varphi_x"] \arrow[d, "f_*"'] & G^{\mathrm{c}}(x) \arrow[r, "\iota_i"] \arrow[d, "f_!"] & C_G(i) \arrow[d, "g_*"] \\
G(y) \arrow[r, "\varphi_y"'] & G^{\mathrm{c}}(y) \arrow[r, "\iota_l"'] & C_G(l)
\end{tikzcd}$$

in $\mathscr{C}$. The right square, in the diagram above, is commutative due to Corollary 4.1.14. Then, one has

$$\begin{aligned}
\iota_l \circ \varphi_y \circ f_* &= \epsilon_l \circ j_{l *} \circ f_* = \epsilon_l \circ (j_l \circ f)_* = \epsilon_l \circ (g \circ j_i)_* = \epsilon_l \circ g_* \circ j_{i *} \\
&= g_* \circ \epsilon_i \circ j_{i *} = g_* \circ \iota_i \circ \varphi_x = \iota_l \circ f_! \circ \varphi_x.
\end{aligned}$$

By the universal property of limits, one has $\varphi_y \circ f_* = f_! \circ \varphi_x$. Therefore, there exists a natural transformation $\varphi : G \Rightarrow G^{\mathrm{c}}$, whose component at an S-scheme x is given by the unique morphism φ_x in $\mathscr{C}$ that satisfies (49).

Assume that $\varphi' : G \Rightarrow G^{\mathrm{c}}$ is a natural transformation for which

$$\iota_i \circ \varphi'_x = \epsilon_i \circ j_{i *} = \iota_i \circ \varphi_x$$

for every S-scheme x and for every compactification $i : z \hookrightarrow p$ of x. Then, the universal property of limits implies that $\varphi'_x = \varphi_x$, and hence $\varphi' = \varphi$. $\square$

COROLLARY 4.1.21. There exists a unique natural isomorphism

$$\varphi : F \overset{\sim}{\Rightarrow} F^{\mathrm{c}}_{|\mathsf{Prop}/S} : \mathsf{Prop}/S \to \mathscr{C},$$

such that, for every proper S-scheme p and for every compactification i of p, one has

$$\iota_i \circ \varphi_p = \epsilon_i \circ j_{i *}.$$

4.1.2.3. *Open Pullbacks.* The functor F^{c} in Definition 4.1.15 can be extended to the category $\mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S$ whose objects are S-schemes and whose morphisms are finite compositions of proper morphisms and formal inverses of open immersions. We define below the pullbacks along open immersions, and we show in §.4.1.2.4 the comparability between proper pushforwards and open pullbacks.

Suppose that $f : x \overset{\circ}{\hookrightarrow} y$ is an open immersion of S-schemes, let $l : w \overset{/}{\hookrightarrow} q$ be a compactification of y with complementary open immersion $j_l : y \overset{\circ}{\hookrightarrow} q$, and let l_{red} be the composition of l with the surjective closed immersion $w_{\mathrm{red}} \overset{/}{\hookrightarrow} w$.

The open immersion $j_l \circ f : x \overset{\circ}{\hookrightarrow} q$ defines a compactification $i_l^f : z \overset{/}{\hookrightarrow} q$, that is a complementary closed immersion of $j_l \circ f$, endued with the reduced induced structure. Then, l_{red} factorises uniquely in $\mathsf{Sch}^{\mathrm{ft}}/S$ as $i_l^f \circ c$ for a closed immersion $c : w_{\mathrm{red}} \overset{/}{\hookrightarrow} z$. The resulting commutative square

$$\begin{tikzcd}
z \arrow[r, hook, "i_l^f"] & q \\
w_{\mathrm{red}} \arrow[u, hook, "c"] \arrow[r, hook, "l_{\mathrm{red}}"'] & q \arrow[u, "\mathrm{id}_q"']
\end{tikzcd} \tag{50}$$

of closed immersions in $\mathsf{Sch}^{\mathrm{ft}}/S$ induces the solid commutative square

$$\begin{tikzcd}
F(z) \arrow[r, tail, "i^f_{l\,*}"] & F(q) \arrow[r, dashed, two heads, "\epsilon_{i_l^f}"] & C_F(i_l^f) \\
F(w_{\mathrm{red}}) \arrow[u, tail, "c_*"] \arrow[r, tail, "l_{\mathrm{red}\,*}"'] & F(q) \arrow[u, "\mathrm{id}_{F(q)}"'] \arrow[r, dashed, two heads, "\epsilon_{l_{\mathrm{red}}}"'] & C_F(l_{\mathrm{red}}). \arrow[u, dotted, "f_l^*"']
\end{tikzcd}$$

of cofibrations in $\mathscr{C}$. Let f_l^* denote the unique morphism $C_F(l_{\mathrm{red}}) \to C_F(i_l^f)$, induced the universal property of cokernels, which makes the diagram commute; and denote the morphism

$$f_l^* \circ \iota_{l_{\mathrm{red}}} : F^{\mathrm{c}}(y) \to C_F(i_l^f) \tag{51}$$

by ρ_l^f. Notice that the square (50) is not a morphism of compactifications unless f is an isomorphism.

LEMMA 4.1.22. Assume that $f : x \overset{\circ}{\hookrightarrow} y$ is an open immersion of S-schemes, let $l_k : w_k \overset{\circ}{\hookrightarrow} q_k$ be a compactification of y, for $k = 0, 1$, and let $g : l_0 \to l_1$ be a morphism of compactifications in $\mathsf{Comp}_S(y)$. Then, g induces a morphism of compactifications $g : i^f_{l_0} \to i^f_{l_1}$ in $\mathsf{Comp}_S(x)$ for which

$$\rho^f_{l_1} = g_* \circ \rho^f_{l_0}. \tag{52}$$

Proof. The morphism of compactifications $g : l_0 \to l_1$ defines a morphism compactifications $g : l_{0_{\mathrm{red}}} \to l_{1_{\mathrm{red}}}$ in $\mathsf{Comp}_S(y)$. Let $i^f_{l_k} : z_k \mathrel{\hookrightarrow\!\!\!\!\!\!/\;} q_k$ be a complementary closed immersion of $j_{l_k} \circ f$, endued with the reduced induced structure, and hence a compactification of x, and let $c_k : w_{k_{\mathrm{red}}} \mathrel{\hookrightarrow\!\!\!\!\!\!/\;} z_k$ be the unique closed immersion for which $l_{k_{\mathrm{red}}} = i^f_{l_k} \circ c_k$, for $k = 0, 1$. Since $g : l_{0_{\mathrm{red}}} \to l_{1_{\mathrm{red}}}$ is a morphism of compactifications in $\mathsf{Comp}_S(y)$, the morphism id_y is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_{l_1}, and hence id_x is a base change of g along $j_{i^f_{l_1}} = j_{l_1} \circ f$. Therefore, g induces a morphism of compactifications $g : i^f_{l_0} \to i^f_{l_1}$ in $\mathsf{Comp}_S(x)$, see Remark 4.1.2. Consider the commutative diagram

$$\begin{tikzcd}
& z_1 \arrow[rr, hook, "i^f_{l_1}"] & & q_1 \\
z_0 \arrow[ur, hook] \arrow[rr, hook, "i^f_{l_0}"'] & & q_0 \arrow[ur, "g"] & \\
& w_{1_{\mathrm{red}}} \arrow[uu, hook, "c_1"' near start] \arrow[rr, hook, "l_{1_{\mathrm{red}}}"] & & q_1 \arrow[uu, "\mathrm{id}"'] \\
w_{0_{\mathrm{red}}} \arrow[uu, hook, "c_0"] \arrow[ur] \arrow[rr, hook, "l_{0_{\mathrm{red}}}"'] & & q_0 \arrow[uu, crossing over, "\mathrm{id}" near end] \arrow[ur, "g"] &
\end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, where the commutativity of the left face is a result of the commutativity of the other faces and having $i^f_{l_1}$ a monomorphism in $\mathsf{Sch}^{\mathrm{ft}}/S$. The diagram above induces the solid commutative diagram

$$\begin{tikzcd}[column sep=small]
& F(z_1) \arrow[rr, tail, "{i^f_{l_1}}_*"] & & F(q_1) \arrow[rr, dashed, two heads, "\epsilon_{i^f_{l_1}}"] & & C_F(i^f_{l_1}) \\
F(z_0) \arrow[ur] \arrow[rr, tail, "{i^f_{l_0}}_*"] & & F(q_0) \arrow[ur, "g_*"] \arrow[rr, dashed, two heads, "\epsilon_{i^f_{l_0}}"] & & C_F(i^f_{l_0}) \arrow[ur, dotted, "g_*"] & \\
& F(w_{1_{\mathrm{red}}}) \arrow[uu, "c_{1*}"' near start] \arrow[rr, tail, "l_{1_{\mathrm{red}}*}"] & & F(q_1) \arrow[uu, "\mathrm{id}"' near end] \arrow[rr, dashed, two heads, "\epsilon_{l_{1_{\mathrm{red}}}}"] & & C_F(l_{1_{\mathrm{red}}}). \arrow[uu, dotted, "f^*_{l_1}"'] \\
F(w_{0_{\mathrm{red}}}) \arrow[uu, "c_{0*}"] \arrow[ur] \arrow[rr, tail, "l_{0_{\mathrm{red}}*}"] & & F(q_0) \arrow[uu, crossing over, "\mathrm{id}" near end] \arrow[ur, "g_*"] \arrow[rr, dashed, two heads, "\epsilon_{l_{0_{\mathrm{red}}}}"] & & C_F(l_{0_{\mathrm{red}}}) \arrow[uu, dotted, crossing over, "f^*_{l_0}" near end] \arrow[ur, dotted, "g_*"] &
\end{tikzcd}$$

in $\mathscr{C}$. Since $\epsilon_{l_{0_{\mathrm{red}}}}$ is an epimorphism in $\mathscr{C}$, the universal propriety of cokernels implies

$$f^*_{l_1} \circ g_* = g_* \circ f^*_{l_0},$$

i.e. the whole diagram commutes. Then,

$$\rho^f_{l_1} = f^*_{l_1} \circ \iota_{l_{1_{\mathrm{red}}}} = f^*_{l_1} \circ g_* \circ \iota_{l_{0_{\mathrm{red}}}} = g_* \circ f^*_{l_0} \circ \iota_{l_{0_{\mathrm{red}}}} = g_* \circ \rho^f_{l_0}.$$

$\square$

The proof above, in particular, shows that there exists a faithful (not necessarily full) functor $\theta^f : \mathsf{Comp}_S(y) \to \mathsf{Comp}_S(x)$ that sends a compactification l of y to the compactification i_l^f of x, and sends a morphism of compactifications $g : l_0 \to l_1$ in $\mathsf{Comp}_S(y)$ to the morphism of compactifications $g : i_{l_0}^f \to i_{l_1}^f$ in $\mathsf{Comp}_S(x)$. Let $C_{F,x}^f$ be the composition of the functor $C_{F,x}$, given in Remark 4.1.11, with the functor θ^f. The universal properties of limits induces a canonical morphism $\vartheta^f : \mathtt{lim}\, C_{F,x} \to \mathtt{lim}\, C_{F,x}^f$ in $\mathscr{C}$. Since F is a *cdp*-functor, the morphism ϑ^f is an isomorphism, which allows us to deduce the following corollary.

Corollary 4.1.23. Assume that $f : x \overset{\circ}{\hookrightarrow} y$ is an open immersion of S-schemes. Then, there exists a unique morphism $f^! : F^{\mathrm{c}}(y) \to F^{\mathrm{c}}(x)$ in $\mathscr{C}$ for which

$$\iota_{i_l^f} \circ f^! = f_l^* \circ \iota_{l_{\mathrm{red}}}, \tag{53}$$

for every compactification $l : w \hookrightarrow q$ of y.

The uniqueness of the morphism $f^!$ implies the functoriality of pullbacks along open immersions. That is, for open immersions $f : x \overset{\circ}{\hookrightarrow} y$ and $g : y \overset{\circ}{\hookrightarrow} z$ of S-schemes, one has

$$(g \circ f)^! = f^! \circ g^! \qquad \text{and} \qquad (\mathrm{id}_x)^! = \mathrm{id}_{\mathtt{lim}\, C_{F,x}}.$$

Corollary 4.1.24. The functor F^{c}, in (46), extends to a functor

$$F^{\mathrm{c}} : \mathsf{Sch}^{\mathrm{ft\,prop}}_{\mathrm{open}}/S \to \mathscr{C}$$

that sends f^{op}, for an open immersion $f : x \overset{\circ}{\hookrightarrow} y$ of S-schemes, to the unique morphism $f^! : \mathtt{lim}\, C_{F,y} \to \mathtt{lim}\, C_{F,x}$ satisfying the statement of Corollary 4.1.23.

Remark 4.1.25. In contrast to the pushforwards along proper morphisms, pullbacks along open immersions do not necessarily exist when F is not a *cdp*-functor, even if $\mathscr{C}$ is has cofiltered limits. That is because the morphism ϑ^f is not necessarily an isomorphism, as the functor θ^f does not have to be co-cofinal. For instance, when p is a proper S-scheme and $f : p \overset{\circ}{\hookrightarrow} q$ is a non-isomorphic open immersion of S-schemes, the initial compactification $\varnothing_p : \varnothing_S \hookrightarrow p$ does not coincide with i_l^f for any compaction l of q.

4.1.2.4. *Base Change.* The open pullback and proper pushforward satisfy the *proper-base change compatibility formula*, as in the following lemma.

Lemma 4.1.26. Let

$$\begin{tikzcd} x' \arrow[r, hook, "j'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & x \arrow[d, "f"] \\ y' \arrow[r, hook, "j"'] & y \end{tikzcd} \tag{54}$$

be a Cartesian square in $\mathsf{Sch}^{\mathrm{ft}}/S$ in which j is an open immersion and f is a proper morphism, and hence j' is an open immersion and f' is proper. Then,

$$j^! \circ f_! = f'_! \circ j'^!. \tag{55}$$

Proof. Let $l : w \hookrightarrow\!\!\!\!\to q$ be a compactification of y, since f is proper there exists a compactification $i : z \hookrightarrow\!\!\!\!\to p$ of x in $\mathsf{Comp}_S(f, l)$ and a morphism of compactifications $g : i \to l$ such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_l. Given that j and j' are open immersions, let $i^j_l : w' \hookrightarrow\!\!\!\!\to q$ (resp. $i^{j'}_i : z' \hookrightarrow\!\!\!\!\to p$) be the compactification of y' and (resp. x') induced from l (resp. i), as in §.4.1.2.3. The morphism of compactification $g : i \to l$ defines a morphism compactifications $g : i_{\mathrm{red}} \to l_{\mathrm{red}}$. Since the square (54) is Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$, the morphism f' is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along $j_{i^j_l} = j_l \circ j$. Therefore, The morphism of compactification $g : i_{\mathrm{red}} \hookrightarrow\!\!\!\!\to l_{\mathrm{red}}$ defines a morphism compactifications $g : i^{j'}_i \to i^j_l$, see Remark 4.1.2. Then, one has

$$\iota_{i^j_l} \circ j^! \circ f_! = j^*_l \circ \iota_{l_{\mathrm{red}}} \circ f_! = j^*_l \circ g_* \circ \iota_{i_{\mathrm{red}}} \qquad \text{and} \qquad \iota_{i^j_l} \circ f'_! \circ j'^! = g_* \circ \iota_{i^{j'}_i} \circ j'^! = g_* \circ j'^*_i \circ \iota_{i_{\mathrm{red}}}.$$

Consider the commutative diagram

$$\begin{tikzcd}
z_{\mathrm{red}} \arrow[r, hook] \arrow[d] \arrow[rr, bend left, hook, "i_{\mathrm{red}}"] & z' \arrow[r, hook, "i^{j'}_i"'] \arrow[d] & p \arrow[d, "g"] \\
w_{\mathrm{red}} \arrow[r, hook] \arrow[rr, bend right, hook, "l_{\mathrm{red}}"'] & w' \arrow[r, hook, "i^j_l"] & q
\end{tikzcd} \tag{56}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, where the commutativity of the left square is a result of the commutativity of the other squares and having i^j_l a monomorphism in $\mathsf{Sch}^{\mathrm{ft}}/S$. The diagram above induces the solid commutative diagram

$$\begin{tikzcd}
& F(w') \arrow[rr, tail, "i^j_{l*}"] & & F(q) \arrow[rr, dashed, two heads, "\epsilon_{i^j_l}"] & & C_F(i^j_l) \\
F(z') \arrow[ur] \arrow[rr, tail, "i^{j'}_{i*}"] & & F(p) \arrow[ur, "g_*"] \arrow[rr, dashed, two heads, "\epsilon_{i^{j'}_i}"] & & C_F(i^{j'}_i) \arrow[ur, dotted, "g_*"] & \\
& F(w_{\mathrm{red}}) \arrow[uu, tail] \arrow[rr, tail, "l_{\mathrm{red}*}"] & & F(q) \arrow[uu, "\mathrm{id}"] \arrow[rr, dashed, two heads, "\epsilon_{l_{\mathrm{red}}}"] & & C_F(l_{\mathrm{red}}). \arrow[uu, dotted, "j^*_l"] \\
F(z_{\mathrm{red}}) \arrow[uu, tail] \arrow[ur] \arrow[rr, tail, "i_{\mathrm{red}*}"] & & F(p) \arrow[uu, "\mathrm{id}"] \arrow[ur, "g_*"] \arrow[rr, dashed, two heads, "\epsilon_{i_{\mathrm{red}}}"] & & C_F(i_{\mathrm{red}}) \arrow[uu, dotted, "j'^*_i"] \arrow[ur, dotted, "g_*"] &
\end{tikzcd}$$

in $\mathscr{C}$. Since $\epsilon_{i_{\mathrm{red}}}$ is an epimorphism in $\mathscr{C}$, the universal propriety of cokernels implies

$$j^*_l \circ g_* = g_* \circ j'^*_i.$$

Since $\iota_{i_l^j}$ is an isomorphism, as F is a *cdp*-functor, one has $j^! \circ f_! = f'_! \circ j'^!$. $\square$

EXAMPLE 4.1.27. Assume that $i : v \hookrightarrow\!\!\!\!\!\!\!\not\;\;\; x$ is a closed immersion of S-schemes with complementary open immersion $j : u \hookrightarrow x$. Then, one has a Cartesian square

$$\begin{tikzcd} \varnothing_S \arrow[r, hook] \arrow[d, hook] & v \arrow[d, hook, "i"] \\ u \arrow[r, hook, "j"'] & x \end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$. Since $F^{\mathrm{c}}(\varnothing_S) \cong F(\varnothing_S) \cong \mathbf{0}$, one has

$$j^! \circ i_! = 0.$$

In fact, we see in §.4.1.2.5 that the sequence

$$F^{\mathrm{c}}(v) \overset{i_!}{\rightarrowtail} F^{\mathrm{c}}(x) \overset{j^!}{\twoheadrightarrow} F^{\mathrm{c}}(u)$$

is a cofibre sequence in $\mathscr{C}$.

EXAMPLE 4.1.28. Recall Example 4.1.19, and assume that $j_x : x \hookrightarrow x \bigsqcup y$ is a closed open immersion of S-schemes with complementary closed open immersion $j_y : y \hookrightarrow x \bigsqcup y$. Then, there exists a Cartesian square

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, hook, "j_x"] \\ x \arrow[r, hook, "j_x"'] & x \bigsqcup y \end{tikzcd}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, and hence

$$j_x{}^! \circ j_{x_!} = (\mathrm{id}_x)_! \circ (\mathrm{id}_x)^! = \mathrm{id}_{F^{\mathrm{c}}(x)}.$$

Similarly $j_y{}^! \circ j_{y_!} = \mathrm{id}_{F^{\mathrm{c}}(y)}$. By Example 4.2.22, one also has $j_x{}^! \circ j_{y_!} = 0$ and $j_y{}^! \circ j_{x_!} = 0$. Therefore, when $\mathscr{C}$ is additive, $F^{\mathrm{c}}(x \bigsqcup y)$ is a direct sum in $\mathscr{C}$ of $F^{\mathrm{c}}(x)$ and $F^{\mathrm{c}}(y)$.

4.1.2.5. *Excision.* A functor $G : \mathsf{Sch}^{\mathrm{ft\,prop}}_{\mathrm{open}}/S \to \mathscr{C}$, to a Waldhausen category, induces a group homomorphism $\mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0(\mathscr{C})$ only when the evident composition map $\mathtt{Ob}(\mathsf{Sch}^{\mathrm{ft\,prop}}_{\mathrm{open}}/S) \to \mathtt{Ob}(\mathscr{C}) \to \mathtt{K}_0(\mathscr{C})$ respects the scissors relations (19). That holds when G satisfies the excision property (**E**). We will see below that the functor F^{c}, given in Corollary 4.1.24, satisfies the excision property, and hence it induces an Euler-Poincaré characteristic $\mu_F : \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0(\mathscr{C})$.

PROPOSITION 4.1.29. The functor F^{c}, given in Corollary 4.1.24, satisfies the excision property (**E**), *i.e.* for every closed immersion $i : v \hookrightarrow\!\!\!\!\!\!\!\not\;\;\; x$ of S-schemes with complementary open immersion $j : u \hookrightarrow x$, the sequence

$$F^{\mathrm{c}}(v) \overset{i_!}{\rightarrowtail} F^{\mathrm{c}}(x) \overset{j^!}{\twoheadrightarrow} F^{\mathrm{c}}(u) \tag{57}$$

is a cofibre sequence in $\mathscr{C}$. In particular, $j^!$ is an epimorphism in $\mathscr{C}$.

Proof. Assume that $i : v \hookrightarrow x$ is a closed immersion of S-schemes with complementary open immersion $j : u \hookrightarrow x$, and let $l : w \hookrightarrow q$ be a compactification of x in which the underlying scheme of w is reduced. The open immersion $j_l \circ j : u \hookrightarrow q$ induces a compactification $l' : w' \hookrightarrow q$ of u in which the underlying scheme of w' is reduced, *i.e.* $l' = i_l^j$ using the notation of §.4.1.2.3. Hence, there exists a unique closed immersion $c : w \hookrightarrow w'$ for which $l = l' \circ c$.

On the other hand, let $i' : p \hookrightarrow q$ be the closed immersion of the scheme-thoracic image of $j_l \circ i$. Then, Lemma 4.1.3 implies the existence of an open immersion $j_l' : v \hookrightarrow p$ for which the solid square containing it in the commutative diagram

$$\begin{tikzcd}
u \arrow[r, hook, "j"] & x \arrow[r, hook, "j_l"] & q & w' \arrow[l, hook', "l'"'] & w \arrow[l, hook', "c"'] \\
\varnothing_S \arrow[r, hook] \arrow[u, hook, "\varnothing_u"] & v \arrow[r, hook, "j_l'"'] \arrow[u, hook, "i"] & p \arrow[u, hook, "i'"] & p' \arrow[l, hook', dashed, "\pi_p'"] \arrow[u, hook, dashed, "\pi_{w'}"] & z \arrow[l, hook', dashed, "e"] \arrow[u, hook, dashed, "\pi_w"]
\end{tikzcd}$$

is Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$. The leftmost square, in the above diagram, is also Cartesian in $\mathsf{Sch}^{\mathrm{ft}}/S$, by the definition of u. Let $(\pi_{w'}, p', \pi_p')$ (resp. (π_w, z, π_p)) be the pullback of i' along l' (resp. $l = l' \circ c$). Then, there exists a unique morphism $e : z \to p'$ that makes the diagram commute, in particular, $\pi_p = \pi_p' \circ e$. Moreover, e is a closed immersion.

Since open complements are closed under pullbacks and j_l is a complementary open immersion to l, one finds that j_l' is a complementary open immersion to π_p. Hence, $\pi_p : z \hookrightarrow p$ is a compactification in $\mathsf{Comp}_S(i, l)$ and $(\pi_w, i') : \pi_p \to l$ is a morphism of compactifications such that i is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of i' along j_l. Also, since $j_l \circ j$ is a complementary open immersion to l', the projection π_p' is a surjective closed immersion. Hence, $\pi_p' : p' \hookrightarrow p$ is a compactification in $\mathsf{Comp}_S(\varnothing_u, l')$ and $(\pi_{w'}, i') : \pi_p' \to l'$ is a morphism of compactifications such that $\varnothing_u$ is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of i' along $j_{l'}$.

The Pullback Lemma implies that the square

$$\begin{tikzcd}
z \arrow[r, hook, "\pi_w"] \arrow[d, hook, "e"'] & w \arrow[d, hook, "c"] \\
p' \arrow[r, hook, "\pi_{w'}"'] & w'
\end{tikzcd} \tag{58}$$

is Cartesian in Prop/S. Since π'_p is a surjective closed immersion, the morphism c induces an isomorphism

$$w' \smallsetminus \pi_{w'} \cong (q \smallsetminus (j_l \circ j))_{\mathrm{red}} \smallsetminus \pi_{w'} = \big((q \smallsetminus (j_l \circ j)) \smallsetminus i'\big)_{\mathrm{red}} = \big((q \smallsetminus j_l) \smallsetminus i'\big)_{\mathrm{red}} = (q \smallsetminus j_l)_{\mathrm{red}} \smallsetminus \pi_w \cong w \smallsetminus \pi_w.$$

Therefore, the square (58) is a *cdp*-square in Prop/S, and hence the functor F sends it to a pushout square in $\mathscr{C}$.

Since $(\pi_w, i') : \pi_p \to l$ and $(\pi_{w'}, i') : \pi'_p \to l'$ are morphism of compactifications, and i (resp. $\varnothing_u$) is a base change of i' along j_l (resp. $j_{l'} = j_l \circ j$), there exists a commutative diagram

$$\begin{tikzcd}
F(z) \arrow[r, tail, "\pi_{w*}"] \arrow[drr, tail, "\pi_{p*}"'] \arrow[dd, tail, "e_*"'] & F(w) \arrow[drr, tail, "l_*"] \arrow[dd, tail, "c_*"] & & & & \\
& & F(p) \arrow[r, tail, "i'_*"] \arrow[dd, equal] \arrow[drr, two heads, "\epsilon'_{\pi_p}"'] & F(q) \arrow[dd, equal] \arrow[drr, two heads, "\epsilon'_l"] & & \\
F(p') \arrow[r, tail, "\pi_{w'*}"] \arrow[drr, tail, "\pi'_{p*}"'] & F(w') \arrow[ul, phantom, "\ulcorner", very near start] \arrow[drr, tail, "l'_*"] & & & F^{\mathrm{c}}(v) \arrow[r, tail, "i_!"] \arrow[dd, "0"] & F^{\mathrm{c}}(x) \arrow[dd, "j^!"] \\
& & F(p) \arrow[r, tail, "i'_*"] \arrow[drr, two heads, "\epsilon'_{\pi'_p}"'] & F(q) \arrow[drr, two heads, "\epsilon'_{l'}"] & & \\
& & & & F^{\mathrm{c}}(\varnothing_S) \arrow[r, tail, "0"] & F^{\mathrm{c}}(u)
\end{tikzcd}$$

in $\mathscr{C}$, where $\epsilon'_l = \iota_l^{-1} \circ \epsilon_l$, $\epsilon'_{l'} = \iota_{l'}^{-1} \circ \epsilon_{l'}$, $\epsilon'_{\pi_p} = \iota_{\pi_p}^{-1} \circ \epsilon_{\pi_p}$, and $\epsilon'_{\pi'_p} = \iota_{\pi'_p}^{-1} \circ \epsilon_{\pi'_p}$. Since π'_p is a surjective closed immersion of proper S-schemes, the morphism π'_{p*} is an isomorphism. Also, $F^{\mathrm{c}}(\varnothing_S) \cong \mathbf{0}$.

Let $\alpha : F^{\mathrm{c}}(x) \to A$ be a morphism in $\mathscr{C}$ for which $\alpha \circ i_! = 0 = \beta \circ 0$, for the unique morphism $\beta : F^{\mathrm{c}}(\varnothing) \to A$ in $\mathscr{C}$. Since the functor F send the square (58) to a pushout square in $\mathscr{C}$ and $\alpha \circ \epsilon'_l \circ l_* \circ \pi_{w*} = \beta \circ \epsilon'_{\pi'_p} \circ \pi'_{p*} \circ e_* = 0$, one has $\alpha \circ \epsilon'_l \circ l'_* = 0$. Then, there exists a unique morphism $\gamma : F^{\mathrm{c}}(u) \to A$ such that $\alpha \circ \epsilon'_l = \gamma \circ \epsilon'_{l'} = \gamma \circ j^! \circ \epsilon'_l$. The morphism ϵ'_l is an epimorphism in $\mathscr{C}$, and hence $\alpha = \gamma \circ j^!$. Also, $\epsilon'_{l'}$ is an epimorphism, and so is $j^!$. Thus, for any morphism $\gamma' : F^{\mathrm{c}}(u) \to A$ in $\mathscr{C}$ for which $\gamma' \circ j^! = \alpha = \gamma \circ j^!$, one has $\gamma' = \gamma$. Therefore, the sequence

$$F^{\mathrm{c}}(v) \overset{i_!}{\rightarrowtail} F^{\mathrm{c}}(x) \overset{j^!}{\twoheadrightarrow} F^{\mathrm{c}}(u)$$

is a cokernel sequence in $\mathscr{C}$, and hence a cofibre sequence because $i_!$ is a cofibration, by Proposition 4.1.18.(**PS1′**). $\square$

4.1.2.6. *Weak Monoidal.* When the Waldhausen category $\mathscr{C}$ is symmetric monoidal and the *cdp*-functor $F : \mathsf{Prop}/S \to \mathscr{C}$ is weak monoidal with respect to the Cartesian product in Prop/S, we show that F^{c} is also weak monoidal. The weak monoidality of the functor F^{c} is a formal consequence of Proposition 4.1.29 and the weak monoidality of F, which is based on the following lemma.

Lemma 4.1.30. Let the diagram (59) be a commutative diagram in a Waldhausen category $\mathscr{C}$, in which the vertical and horizontal sequences are cokernel sequences, (ς, F, σ) is a pushout of the span $F_{2,1} \xleftarrow{\sigma_1} F_{1,1} \xrightarrow{\varsigma_1} F_{1,2}$, and the morphism λ is the unique morphism $F \to F_{2,2}$ induced by the universal property of pushouts making the diagram commute. Then, the sequence $F \xrightarrow{\lambda} F_{2,2} \xrightarrow{\pi} F_{3,3}$ is a cokernel sequence in $\mathscr{C}$.

$$\begin{tikzcd}
F_{1,1} \arrow[rr, tail, "\varsigma_1"] \arrow[dd, "\sigma_1"'] & & F_{1,2} \arrow[rr, two heads, "\pi_1"] \arrow[dl, "\sigma"'] \arrow[dd, "\sigma_2"] & & F_{1,3} \arrow[dd, tail, "\sigma_3"] \\
& F \arrow[dr, dashed, "\lambda"] & & & \\
F_{2,1} \arrow[ur, tail, "\varsigma"] \arrow[rr, tail, "\varsigma_2"'] & & F_{2,2} \arrow[rr, two heads, "\pi_2"] \arrow[ddrr, "\pi"] & & F_{2,3} \arrow[dd, two heads, "\pi_3"] \\
& & & & \\
& & & & F_{3,3}
\end{tikzcd} \tag{59}$$

Proof. The proof is an elementary diagram chase; yet we spell it out for the reader's convenience.

On the one hand, the universal property of pushouts implies $\pi \circ \lambda = 0$, due to having

$$\pi \circ \lambda \circ \varsigma = \pi \circ \varsigma_2 = \pi_3 \circ \pi_2 \circ \varsigma_2 = 0 \qquad \text{and} \qquad \pi \circ \lambda \circ \sigma = \pi \circ \sigma_2 = \pi_3 \circ \sigma_3 \circ \pi_1 = 0.$$

On the other hand, let $\alpha : F_{2,2} \to Z$ be a morphism in $\mathscr{C}$ such that $\alpha \circ \lambda = 0$. Then, in particular, $\alpha \circ \varsigma_2 = \alpha \circ \lambda \circ \varsigma = 0$, and hence there exists a unique morphism $\beta : F_{2,3} \to Z$ in $\mathscr{C}$ for which $\alpha = \beta \circ \pi_2$. Thus,

$$\beta \circ \sigma_3 \circ \pi_1 = \beta \circ \pi_2 \circ \sigma_2 = \alpha \circ \lambda \circ \sigma = 0.$$

Since π_1 is an epimorphism in $\mathscr{C}$, one has $\beta \circ \sigma_3 = 0$, and hence there exists a unique morphism $\gamma : F_{3,3} \to Z$ in $\mathscr{C}$ for which $\beta = \gamma \circ \pi_3$. Thus, $\alpha = \gamma \circ \pi_3 \circ \pi_2 = \gamma \circ \pi$.

Let $\gamma' : F_{3,3} \to Z$ be a morphism in $\mathscr{C}$ for which $\alpha = \gamma' \circ \pi$. Since π is an epimorphism in $\mathscr{C}$, one has $\gamma' = \gamma$. Therefore, π is a cokernel injection of λ in $\mathscr{C}$. $\square$

Proposition 4.1.31. Let $(\mathscr{C}, \wedge, \mathbb{1})$ be a symmetric monoidal Waldhausen category, and suppose that the *cdp*-functor $F : (\mathsf{Prop}/S, \times, \mathrm{id}_S) \to (\mathscr{C}, \wedge, \mathbb{1})$ is weak monoidal. Then, the functor $F^{\mathrm{c}} : \mathsf{Sch}^{\text{ft prop}}_{\text{open}} \to \mathscr{C}$ is weak monoidal. Moreover, F^{c} is strong monoidal when F is.

Proof. Since F is weak monoidal, it coherence morphisms

$$\phi_{p,q} : F(p) \wedge F(q) \to F(p \times q) \qquad \text{and} \qquad \phi_S : \mathbb{1} \to F(\mathrm{id}_S)$$

are weak equivalences in $\mathscr{C}$, for every pair of proper S-schemes p and q. Let $\phi^{\mathrm{c}}_S : \mathbb{1} \to F^{\mathrm{c}}(\mathrm{id}_S)$ be the composite weak equivalence $\phi^{\mathrm{c}}_S \coloneqq \varphi_S \circ \phi_S$ in $\mathscr{C}$, where φ is the natural isomorphism asserted by Corollary 4.1.21.

For $k = 0, 1$, assume that x_k is an S-scheme, let $i_k : z_k \hookrightarrow p_k$ be a compactification of x_k with complementary open immersion $j_{i_k} : x_k \hookrightarrow p_k$. The functors $- \wedge C$ and $C \wedge -$ preserve cofibre sequences for every object $C \in \mathscr{C}$, see Definition 1.5.20. Thus, Proposition 4.1.29 induces a cofibre sequence

$$F^{\mathrm{c}}(z_k) \wedge F^{\mathrm{c}}(y) \xrightarrow{i_{k!}} F^{\mathrm{c}}(p_k) \wedge F^{\mathrm{c}}(y) \xrightarrow{j^!_{i_k}} F^{\mathrm{c}}(x_k) \wedge F^{\mathrm{c}}(y)$$

in $\mathscr{C}$, for every S-scheme y. Since both p_k and z_k are proper S-schemes, and φ is a natural isomorphism, the sequence

$$F(z_k) \wedge F(y) \xrightarrow{i_{k*}} F(p_k) \wedge F(y) \xrightarrow{j^!_{i_k}} F^{\mathrm{c}}(x_k) \wedge F(y),$$

is a cofibre sequence in $\mathscr{C}$ for a proper S-scheme y.

On the one hand, the monoidal product bifunctor $\wedge : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ induces the solid commutative diagram

$$\begin{tikzcd}
F(z_0) \wedge F(z_1) \arrow[r, tail, "i_{1*}"] \arrow[dd, tail, "i_{0*}"'] & F(z_0) \wedge F(p_1) \arrow[r, two heads, "j^!_{i_1}"] \arrow[dd, tail, "i_{0*}"] \arrow[d, dashed, "\lambda_0"'] & F(z_0) \wedge F^{\mathrm{c}}(x_1) \arrow[dd, tail, "i_{0*}"] \\
 & \underline{F} \arrow[d, dotted, "\lambda"] & \\
F(p_0) \wedge F(z_1) \arrow[r, tail, "i_{1*}"'] \arrow[dd, two heads, "j^!_{i_0}"'] \arrow[ur, dashed, "\lambda_1"] & F(p_0) \wedge F(p_1) \arrow[r, two heads, "j^!_{i_1}"] \arrow[dd, two heads, "j^!_{i_0}"'] \arrow[ddr, "\pi"] & F(p_0) \wedge F^{\mathrm{c}}(x_1) \arrow[dd, two heads, "j^!_{i_0}"] \\
 & & \\
F^{\mathrm{c}}(x_0) \wedge F(z_1) \arrow[r, tail, "i_{1*}"'] & F^{\mathrm{c}}(x_0) \wedge F(p_1) \arrow[r, two heads, "j^!_{i_1}"'] & F^{\mathrm{c}}(x_0) \wedge F^{\mathrm{c}}(x_1)
\end{tikzcd}$$

in $\mathscr{C}$, in which horizontal and vertical sequences consist of cofibre sequences. Let $\lambda : \underline{F} \to F(p_0) \wedge F(p_1)$ be the pushout-product of i_{0*} and i_{1*}, and let $\pi \coloneqq j^!_{i_0} \circ j^!_{i_1}$. Since $\mathscr{C}$ is a symmetric monoidal Waldhausen category and i_{0*} and i_{1*} are cofibrations in $\mathscr{C}$, the morphisms λ is also a cofibration in $\mathscr{C}$. Thus, Lemma 4.1.30 implies that the sequence

$$\underline{F} \xrightarrow{\lambda} F(p_0) \wedge F(p_1) \xrightarrow{\pi} F^{\mathrm{c}}(x_0) \wedge F^{\mathrm{c}}(x_1)$$

is a cofibre sequence in $\mathscr{C}$.

On the other hand, there exits a pushout in Prop/S of the closed immersions $\mathrm{id}_{z_0} \times i_1$ and $i_0 \times \mathrm{id}_{z_1}$. In fact, there exists a bicartesian square

$$\begin{tikzcd}
z_0 \times z_1 \arrow[r, hook, "\mathrm{id}_{z_0} \times i_1"] \arrow[d, hook, "i_0 \times \mathrm{id}_{z_1}"'] & z_0 \times p_1 \arrow[d, hook, "i'_0"] \\
p_0 \times z_1 \arrow[r, hook, "i'_1"'] & z,
\end{tikzcd}$$

of closed immersions in Prop/S, see [**Sch05**, Th.3.11] and [**Cam17**, §.2]. In particular, it is a *cdp*-square in Prop/S, and hence it is mapped by F to a pushout square of cofibrations in $\mathscr{C}$. Since $\mathscr{C}$ is a Waldhausen category and F is weak monoidal, the unique morphism $\lambda' : \underline{F} \to F(z)$, that makes the diagram

$$\begin{tikzcd}
F(z_0) \wedge F(z_1) \arrow[rr, tail, "i_{1*}"] \arrow[dd, tail, "i_{0*}"'] \arrow[dr, "\phi_{z_0,z_1}"] & & F(z_0) \wedge F(p_1) \arrow[dd, tail, "i_{0*}" near start] \arrow[dr, "\phi_{z_0,p_1}"] & \\
& F(z_0 \times z_1) \arrow[rr, tail, crossing over] & & F(z_0 \times p_1) \arrow[dd, tail, "i'_{0*}"] \\
F(p_0) \wedge F(z_1) \arrow[rr, tail, "i_{1*}" near start] \arrow[dr, "\phi_{p_0,z_1}"'] & & \underline{F} \arrow[dr, dotted, "\lambda'"] & \\
& F(p_0 \times z_1) \arrow[rr, tail, "i'_{1*}"'] \arrow[from=uu, tail, crossing over] & & F(z)
\end{tikzcd}$$

commute, is a weak equivalence in $\mathscr{C}$. Let p denote the proper S-scheme $p_0 \times p_1$, denote the closed immersion $z \hookrightarrow p$ by i, and denote its complementary open immersion $x_0 \times x_1 \hookrightarrow p$ by j. The morphism i_* is a cofibration in $\mathscr{C}$, and $\phi_{p_0,p_1} \circ \lambda = i_* \circ \lambda'$. Thus, there exists the solid commutative diagram (60) in $\mathscr{C}$, on the next page, in which the horizontal sequences are cofibre sequences, and vertical morphisms are weak equivalences. Thus, the universal property of cokernels implies the existence of the unique morphism

$$\phi^{\mathrm{c}}_{x_0,x_1} : F^{\mathrm{c}}(x_0) \wedge F^{\mathrm{c}}(x_1) \to F^{\mathrm{c}}(x_0 \times x_1),$$

that makes the diagram commute; which is a weak equivalence in $\mathscr{C}$, see Definition 1.5.8. The uniqueness of the morphism $\phi^{\mathrm{c}}_{x_0,x_1}$ implies the existence of a natural transformation $\phi^{\mathrm{c}} : F^{\mathrm{c}} \wedge F^{\mathrm{c}} \Rightarrow F^{\mathrm{c}}(\times)$, with components $\phi^{\mathrm{c}}_{x_0,x_1}$ for every pair of S-schemes x_0 and x_1. Also, a diagram chase of the associativity hexagons and unitality squares shows that F^{c} is weak monoidal with the coherence natural morphism ϕ^{c}. Moreover, when ϕ is a natural isomorphism, so is ϕ^{c}.

$$\begin{tikzcd}
\underline{F} \arrow[r, tail, "\lambda"] \arrow[d, "\lambda'"'] & F(p_0) \wedge F(p_1) \arrow[r, two heads, "\pi"] \arrow[d, "\phi_{p_0,p_1}"] & F^{\mathrm{c}}(x_0) \wedge F^{\mathrm{c}}(x_1) \arrow[dd, dashed, "\phi^{\mathrm{c}}_{x_0,x_1}"] \\
F(z) \arrow[r, tail, "i_*"'] \arrow[d, "\varphi_z"'] & F(p) \arrow[d, "\varphi_p"] & \\
F^{\mathrm{c}}(z) \arrow[r, tail, "i_!"'] & F^{\mathrm{c}}(p) \arrow[r, two heads, "j^!"'] & F^{\mathrm{c}}(x_0 \times x_1)
\end{tikzcd} \tag{60}$$

□

4.1.2.7. *Motivic Measures.* Theorem 4.1.32 collects the main statements of the argument in §.4.1.2, which allows one to associate motivic measures to *cdp*-functors from proper S-schemes to Waldhausen categories.

THEOREM 4.1.32. Assume that $F : (\mathsf{Prop}/S, \times, \mathrm{id}_S) \to (\mathscr{C}, \wedge, \mathbb{1})$ is a weak monoidal *cdp*-functor to a symmetric monoidal Waldhausen category. Then, there exists a functor

$$F^{\mathrm{c}} : (\mathsf{Sch}^{\mathrm{ft\,prop}}_{\mathrm{open}}/S, \times, \mathrm{id}_S) \to (\mathscr{C}, \wedge, \mathbb{1}),$$

where $\mathsf{Sch}^{\mathrm{ft\,prop}}_{\mathrm{open}}/S$ is the category whose objects are S-schemes and whose morphisms are finite compositions of proper morphisms and formal inverses of open immersions, defined on proper morphisms in (44) and on open immersions in (53), such that

- there exists a natural isomorphism $\varphi : F \overset{\sim}{\Rightarrow} F^{\mathrm{c}}_{|\mathsf{Prop}/S}$;
- F^{c} satisfies the excision property, *i.e.* for every closed immersion $i : v \hookrightarrow x$ of S-schemes with complementary open immersion $j : u \hookrightarrow x$, the sequence

$$F^{\mathrm{c}}(v) \xrightarrow{i_!} F^{\mathrm{c}}(x) \xrightarrow{j^!} F^{\mathrm{c}}(u)$$

  is a cofibre sequence in $\mathscr{C}$; and
- F^{c} is weak monoidal, *i.e.* there exist natural transformations

$$\phi^{\mathrm{c}} : F^{\mathrm{c}} \wedge F^{\mathrm{c}} \Rightarrow F^{\mathrm{c}}(\times) \qquad \text{and} \qquad \phi^{\mathrm{c}}_S : \mathbb{1} \Rightarrow F(\mathrm{id}_S),$$

  which satisfy the associativity and unitality axioms, whose components are weak equivalences in $\mathscr{C}$.

Therefore, there exists a motivic measure

$$\mu_F : \mathbb{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathbb{K}_0(\mathscr{C}),$$

that sends the class of a proper S-scheme p to the class of $F(p)$.

EXAMPLE 4.1.33. Suppose that k is a field. Then, a closed immersion $i : \mathsf{Spec}\, k \hookrightarrow \mathbb{P}^1_k$ is an initial object in the category of compactifications $\mathsf{Comp}_k(\mathbb{A}^1_k)$, and hence

$$F^{\mathrm{c}}(\mathbb{A}^1_k) = \mathtt{coker}\, i_*.$$

Remark 4.1.34. For a field k, Zakharevich introduced the notion an assembler in [**Zak17**], and associated to the category of k-varieties a spectrum whose group of path components is isomorphic to the Grothendieck group $\mathtt{K}_0(\mathsf{Var}/k)$. Then, Campbell provided an E_∞-ring spectrum $\mathtt{K}(\mathsf{Var}/k)$ whose ring of path components is isomorphic to the Grothendieck ring $\mathtt{K}_0(\mathsf{Var}/k)$, which is conjectured to be equivalent to Zakharevich's spectrum, see [**Cam17**]. Lemma 4.1.26 and Proposition 4.1.29 imply that a *cdp*-functor $F : \mathsf{Prop}/S \to \mathscr{C}$, to a Waldhausen category, defines a map of spectra

$$\mathtt{K}(F) : \mathtt{K}(\mathsf{Var}/k) \to \mathtt{K}(\mathscr{C})$$

that sends a point in the class $[P] \in \mathtt{K}_0(\mathsf{Var}/k)$ to a point in the class $[F(P)] \in \mathtt{K}_0(\mathscr{C})$, for every proper k-scheme P, see [**Cam17**, Def.5.2 and Prop.5.3]. This will be explored further in §.4.2.1.

4.1.3. Functors Compactification. Given a functor $F : \mathsf{Prop}/S \to \mathscr{C}$ to a Waldhausen category, that is not a *cdp*-functor, one may would like to 'universally' associate to F a *cdp*-functor, and hence define an associated motivic measure that is closely related to F. Recall that the properties (**PS2**) and (**PS3**) imply that *cdp*-functors are *cdp*-cosheaves on Prop/S, as the *cdp*-topology is generated by *cdp*-squares, see §.A.4.3. Hence, a natural choice of such an association is the *cdp*-cosheafification, when it exists, which is the dual of the *cdp*-sheafification, see [**Pra16**]. We will restrain ourself from discussing the general process here, and only focus on the aspects relevant to §.4.2.

Assume that F satisfies (**PS1**), *i.e.* it sends closed immersions of proper S-schemes to cofibrations in $\mathscr{C}$. Then, for a *cdp*-square

$$\begin{tikzcd} z \arrow[r, hook, "\underline{i}"] \arrow[d, "\underline{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & p \arrow[d, "f"] \\ w \arrow[r, hook, "i"'] & q \end{tikzcd}$$

in Prop/S, the morphism $\underline{i}_*$ is a cofibration, and the pushout of $\underline{i}_*$ along $\underline{f}_*$ exists in $\mathscr{C}$. Denote the canonical morphism $F(w)\coprod_{F(z)} F(p) \to F(q)$ in $\mathscr{C}$ induced by the universal property of pushouts by $\alpha_{i,f}$, and consider the set of morphisms

$$\overline{\Lambda} \coloneqq \left\{ \alpha_{i,f} : F(w) \coprod_{F(z)} F(p) \to F(q) \mid (i,f) \in \Lambda \right\} \bigcup \left\{ \mathbf{0} \to F(\varnothing) \right\}$$

in $\mathscr{C}$, where Λ is the set of all *cdp*-squares in Prop/S. If there exists an exact functor of Waldhausen categories $\mathscr{C} \to \mathscr{C}'$ that sends all morphisms in $\overline{\Lambda}$ to isomorphisms in $\mathscr{C}'$, the composition $F' : \mathsf{Prop}/S \to \mathscr{C} \to \mathscr{C}'$ satisfies the properties (**PS1**)-(**PS3**), *i.e.* it is a *cdp*-functor. When the functor $\mathscr{C} \to \mathscr{C}'$ is a localisation with respect to $\overline{\Lambda}$, we say that the induced functor $F' : \mathsf{Prop}/S \to \mathscr{C}'$ is a *cdp-compactification* of F.

When $\mathscr{C}$ is a symmetric monoidal Waldhausen category and F is only lax monoidal, one seeks a localisation for which the composite functor is also weak monoidal.

In the next section, we apply this argument to the most natural functor there is, that is the Yoneda embedding[8].

4.2. Applications

For an essentially small category $\mathscr{C}$, the Yoneda embedding into the category of presheaves $\mathsf{PSh}(\mathscr{C})$ gives a free cocompletion of $\mathscr{C}$. Whereas, for a Grothendieck topology τ on $\mathscr{C}$, the τ-cosheafification of the Yoneda embedding gives a cocompletion of $\mathscr{C}$ in the category of τ-sheaves $\mathsf{Shv}_\tau(\mathscr{C})$, with the relations imposed by declaring τ-coving sieves to be colimit cocones, see Remark A.4.7. The category of pointed τ-sheaves admits a symmetric monoidal Waldhausen structure, whose cofibrations are monomorphisms, weak equivalences are isomorphisms, and monoidal product is given by the smash product, recall Example 1.5.22. In particular, when $\mathscr{C}$ is the category of proper S-schemes and τ is a topology on Prop/S, it is interesting to consider when the τ-cosheafification of the pointed Yoneda embedding is a *cdp*-functor, and to use such a *cdp*-functor, if it exists, to better understand the Grothendieck ring $\mathtt{K}_0(\overset{\mathrm{ft}}{\mathsf{Sch}}/S)$, and probably its higher $\mathtt{K}$-theory.

In this direction, we utilise the *cdp*-topology to construct a monoidal proper-fibred Waldhausen category $\mathscr{C}^\omega_\tau$ over Noetherian schemes of finite Krull dimensions, in §.4.2.1.2, for a topology τ that is finer than the *cdp*-topology. For every Noetherian scheme T of finite Krull dimension, there exists a *cdp*-functor $\underline{\mathsf{h}}^\tau : \mathsf{Prop}/T \to \mathscr{C}^\omega_\tau(T)$, given by the τ-cosheafification of the pointed Yoneda embedding, as in (65). This functor induces, in (66), a surjective motivic measure

$$\mu_\tau : \mathtt{K}_0(\overset{\mathrm{ft}}{\mathsf{Sch}}/T) \to \mathtt{K}_0\left(\mathscr{C}^\omega_\tau(T)\right).$$

On the other hand, giving the role the class of the affine line plays in the study of the Grothendieck ring $\mathtt{K}_0(\overset{\mathrm{ft}}{\mathsf{Sch}}/S)$ and that the category of (simplicial) sheaves is the home for motivic homotopy theory, it is desired to have motivic measures obtained from Waldhausen $\mathtt{K}$-theories of models for the (un)stable motivic homotopy categories.

4.2.1. Waldhausen $\mathtt{K}$-Theories of Noetherian Schemes. For the rest of this subsection, let τ be an additively-saturated pretopology on the category Noetherian schemes of finite Krull dimensions that is finer than the *cdp*-pretopology and coarser than the proper pretopology, *cf.* Remark A.4.39.

Recall that the category of proper S-schemes is essentially small, and the forgetful functor $\mathsf{PSh}_\bullet(\mathsf{Prop}/S) \to \mathsf{PSh}(\mathsf{Prop}/S)$, that forgets the base point, admits a faithful (but not full) left adjoint $\mathsf{PSh}(\mathsf{Prop}/S) \to \mathsf{PSh}_\bullet(\mathsf{Prop}/S)$, given by adjoining a disjoint

[8] In fact, we need to consider a pointed version of the Yoneda embedding.

base point, *i.e.* $\mathscr{X}_+ = (\mathscr{X} \coprod *, *)$, see [**Hov99**, p.4]. Let $\mathsf{h}_{-,+}$ denote the composite functor

$$-_+ \circ \mathsf{h} : \mathsf{Prop}/S \to \mathsf{PSh}_\bullet(\mathsf{Prop}/S).$$

The gluing of a pair of closed subschemes of a proper S-scheme, along their scheme-theoretic intersection, defines a pushout square in Prop/S, which is a *cdp*-square, see the proof of Proposition 4.1.18. Then, the functor $\mathsf{h}_{-,+}$ is not a *cdp*-functor, as it forgets all colimits. However, for every closed immersion $i : z \hookrightarrow p$ in Prop/S, the morphism $\mathsf{h}_{i,+}$ is a monomorphism. Following the argument in §.4.1.3, we may consider a localisation of the category $\mathsf{PSh}_\bullet(\mathsf{Prop}/S)$ with respect to the set of morphisms

$$\overline{\Lambda} \coloneqq \left\{ \alpha_{i,f} : \mathsf{h}_{w,+} \coprod_{\mathsf{h}_{z,+}} \mathsf{h}_{p,+} \to \mathsf{h}_{p,+} \mid (i,f) \in \Lambda \right\} \bigcup \left\{ \mathbf{0} \to \mathsf{h}_{\varnothing,+} \right\},$$

where Λ is the set of all *cdp*-squares in Prop/S.

The *cdp*-sheafification functor

$$-^{\mathsf{a}_{\mathrm{cdp}}} : \mathsf{PSh}_\bullet(\mathsf{Prop}/S) \to \mathsf{Shv}_{\bullet,\mathrm{cdp}}(\mathsf{Prop}/S)$$

provides such a localisation, see Definition A.4.28. That is,

(**PS1**) the *cdp*-sheafification functor preserve monomorphisms;

(**PS2**) the *cdp*-sheafification of $\mathsf{h}_{\varnothing,+}$ is isomorphic to $\mathbf{0}$, as $\mathsf{h}^{\mathrm{cdp}}_{\varnothing,+}(\varnothing) = *$; and

(**PS3**) the functor $\mathsf{h}^{\mathsf{a}_{\mathrm{cdp}}}$, *i.e.* the composition of the *cdp*-sheafification functor with the the Yoneda embedding, sends *cdp*-squares to pushout squares, see [**Voe10a**, Lem.2.11 and Cor.2.16] and [**Voe10b**, Th.2.2]; also, the left adjoint functor $-_+^{\mathsf{a}_{\mathrm{cdp}}}$ preserves colimits.

Therefore, the functor

$$\underline{\mathsf{h}} \coloneqq -^{\mathsf{a}_{\mathrm{cdp}}} \circ \mathsf{h}_{-,+} : \mathsf{Prop}/S \to \mathsf{Shv}_{\bullet,\mathrm{cdp}}(\mathsf{Prop}/S) \tag{61}$$

is a *cdp*-functor. Moreover, Remark A.4.7 shows that the *cdp*-topology is the coarsest topology τ on Prop/S for which the composite functor $-^{\mathsf{a}_\tau} \circ \mathsf{h}_{-,+}$ is a *cdp*-functor. The τ-sheafification functor preserves monomorphisms and colimits, and it factorises through the *cdp*-sheafification functor, as τ is finer than the *cdp*-pretopology. Hence, the functor

$$\underline{\mathsf{h}}^\tau \coloneqq -^{\mathsf{a}_\tau} \circ \mathsf{h}_{-,+} : \mathsf{Prop}/S \to \mathsf{Shv}_{\bullet,\tau}(\mathsf{Prop}/S) \tag{62}$$

is a *cdp*-functor. To avoid bulky notations, we let

$$\mathscr{C}_\tau(S) \coloneqq \mathsf{Shv}_{\bullet,\tau}(\mathsf{Prop}/S).$$

In particular, when τ is the *cdp*-pretopology, denote $\mathscr{C}_\tau(S)$, $\underline{\mathsf{h}}^\tau$, and $\mathsf{h}^{\mathsf{c}_\tau}$, by $\mathscr{C}(S)$, $\underline{\mathsf{h}}$, and h^{c}, respectively.

The functor $\mathsf{h}_{-,+}$ is strong monoidal, with respect to the Cartesian product of proper S-schemes and the smash product of pointed presheaves. Since the τ-sheafification functor is a left exact reflector, it preserves the smash product, as the smash product of pointed (pre)sheaves only involves finite limits and colimits, recall Example 1.5.22. Thus, the functor $\underline{\mathbf{h}}^\tau$ is also strong monoidal.

Since the Waldhausen category $\mathscr{C}_\tau(S)$ is cocomplete, its $\mathtt{K}$-theory is connected, *i.e.* it has a trivial group of path components, recall Lemma 1.5.15. To establish a non-trivial motivic measure, we need to consider a Waldhausen subcategory in $\mathscr{C}_\tau(S)$, with a non-connected $\mathtt{K}$-theory, which contains the essential image of $\underline{\mathbf{h}}^\tau$. We construct this subcategory by mathematical induction

- Let $\mathscr{C}_\tau^0(S)$ be the full subcategory in $\mathscr{C}_\tau(S)$ in which $\mathscr{X} \in \mathscr{C}_\tau^0(S)$ if and only if $\mathscr{X} \cong \underline{\mathbf{h}}_p^\tau$ for a proper S-scheme p.
- For an integer $n \geq 1$, let $\mathscr{C}_\tau^n(S)$ be the full subcategory in $\mathscr{C}_\tau(S)$ in which $\mathscr{X} \in \mathscr{C}_\tau^n(S)$ if and only if there exists a pushout square

$$\begin{tikzcd} \mathscr{Y}' \arrow[r, tail, "\iota"] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{X}' \arrow[r, tail] & \mathscr{X} \arrow[ul, phantom, "\lrcorner"] \end{tikzcd} \tag{63}$$

  in $\mathscr{C}_\tau(S)$, in which $\mathscr{X}', \mathscr{Y}$, and $\mathscr{Y}'$ belong to $\mathscr{C}_\tau^{n-1}(S)$, and ι is a monomorphism of pointed τ-sheaves.

One has $\mathscr{C}_\tau^n(S) \subset \mathscr{C}_\tau^{n+1}(S)$ for every $n \in \mathbb{N}$. Then, the full subcategory

$$\mathscr{C}_\tau^\omega(S) \coloneqq \bigcup_{n\in\mathbb{N}} \mathscr{C}_\tau^n(S) \tag{64}$$

in $\mathscr{C}_\tau(S)$ admits a Waldhausen structure, whose cofibrations (resp. weak equivalences) are morphisms in $\mathscr{C}_\tau^\omega(S)$ that are cofibrations (resp. weak equivalences) in $\mathscr{C}_\tau(S)$, *i.e.* monomorphisms (resp. isomorphisms). Indeed,

- the zero object in $\mathscr{C}_\tau(S)$ is given by $\mathbf{0} \cong \underline{\mathbf{h}}_\varnothing^\tau$ and $\mathscr{C}_\tau^\omega(S)$ is a full subcategory in $\mathscr{C}_\tau(S)$; thus, the category $\mathscr{C}_\tau^\omega(S)$ contains a zero object, namely $\underline{\mathbf{h}}_\varnothing^\tau$;
- for every $\mathscr{X} \in \mathscr{C}_\tau^\omega(S)$, the zero morphism $\underline{\mathbf{h}}_\varnothing^\tau \to \mathscr{X}$ is a monomorphism in $\mathscr{C}_\tau(S)$, and hence a monomorphism in the subcategory $\mathscr{C}_\tau^\omega(S)$;
- for a pair of a monomorphism $\iota : \mathscr{U} \rightarrowtail \mathscr{V}$ and a morphism $\varphi : \mathscr{U} \to \mathscr{X}$ in $\mathscr{C}_\tau^\omega(S)$, there exists an integer n such that ι and φ belong to $\mathscr{C}_\tau^n(S)$, and hence the cobase change of ι along φ exists in $\mathscr{C}_\tau^{n+1}(S) \subset \mathscr{C}_\tau^\omega(S)$, and it is a monomorphism; thus, cofibrations in $\mathscr{C}_\tau^\omega(S)$ are closed under pushouts; and
- the gluing axiom holds, as pushouts are determined up to isomorphisms.

The inclusion functor $\mathscr{C}_\tau^\omega(S) \hookrightarrow \mathscr{C}_\tau(S)$ is an exact functor of Waldhausen categories, *i.e.* $\mathscr{C}_\tau^\omega(S)$ is a Waldhausen subcategory in $\mathscr{C}_\tau(S)$. In fact, $\mathscr{C}_\tau^\omega(S)$ is a the smallest full Waldhausen subcategory in $\mathscr{C}_\tau(S)$ that contains the essential image of $\underline{\mathbf{h}}^\tau$. We abuse notation and denote by $\underline{\mathbf{h}}^\tau$ the unique *cdp*-functor

$$\mathsf{Prop}/S \to \mathscr{C}_\tau^\omega(S) \tag{65}$$

that factorises $\underline{\mathsf{h}}^\tau$. Since the category Prop/S is essentially small, one can use mathematical induction to show that $\mathscr{C}_\tau^\omega(S)$ is also essentially small.

LEMMA 4.2.1. The symmetric monoidal structure on $\mathscr{C}_\tau(S)$, given by the smash product, restricts to $\mathscr{C}_\tau^\omega(S)$, making $\mathscr{C}_\tau^\omega(S)$ into a symmetric monoidal Waldhausen category.

PROOF. Since $\mathscr{C}_\tau^\omega(S)$ is a full Waldhausen subcategory in $\mathscr{C}_\tau(S)$ and the unit $\mathbb{1}_\wedge = \underline{\mathsf{h}}^\tau_{\mathrm{id}_S}$ belongs to $\mathscr{C}_\tau^\omega(S)$, it is sufficient to show that the smash product restricts to $\mathscr{C}_\tau^\omega(S)$.

For every $n \in \mathbb{N}$, we use mathematical induction to show that the smash product restricts to a functor

$$\wedge : \mathscr{C}_\tau^n(S) \times \mathscr{C}_\tau^n(S) \to \mathscr{C}_\tau^{2n}(S),$$

- Let $\mathscr{X}_0$ and $\mathscr{X}_1$ belong to $\mathscr{C}_\tau^0(S)$, *i.e.* there exists a proper S-scheme p_k for which $\mathscr{X}_k \cong \underline{\mathsf{h}}^\tau_{p_k}$, for $k = 0, 1$. Since $\underline{\mathsf{h}}^\tau$ is a strong monoidal functor, one has
$$\mathscr{X}_0 \wedge \mathscr{X}_1 \;\cong\; \underline{\mathsf{h}}^\tau_{p_0} \wedge \underline{\mathsf{h}}^\tau_{p_1} \cong \underline{\mathsf{h}}^\tau_{p_0 \times p_1} \in \mathscr{C}_\tau^0(S).$$
- For an integer $n \geq 1$, assume that the smash product restricts to a functor
$$\wedge : \mathscr{C}_\tau^{n-1}(S) \times \mathscr{C}_\tau^{n-1}(S) \to \mathscr{C}_\tau^{2n-2}(S),$$
and let $\mathscr{X}_0$ and $\mathscr{X}_1$ belong to $\mathscr{C}_\tau^n(S)$, *i.e.* there exists a pushout square
$$\begin{tikzcd} \mathscr{Y}'_k \arrow[r, tail, "\iota_k"] \arrow[d] & \mathscr{Y}_k \arrow[d] \\ \mathscr{X}'_k \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & \mathscr{X}_k \end{tikzcd}$$
in $\mathscr{C}_\tau(S)$, in which $\mathscr{X}'_k, \mathscr{Y}_k$, and $\mathscr{Y}'_k$ belong to $\mathscr{C}_\tau^{n-1}(S)$, and ι_k is a monomorphism, for $k = 0, 1$. Since $\mathscr{C}_\tau(S)$ is a symmetric monoidal Waldhausen category, the smash product with any object in $\mathscr{C}_\tau(S)$ preserves such a pushout square, and hence there exists a pushout square
$$\begin{tikzcd} \mathscr{U} \wedge \mathscr{Y}'_k \arrow[r, tail, "\mathrm{id}_{\mathscr{X}} \wedge \iota_k"] \arrow[d] & \mathscr{U} \wedge \mathscr{Y}_k \arrow[d] \\ \mathscr{U} \wedge \mathscr{X}'_k \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & \mathscr{U} \wedge \mathscr{X}_k \end{tikzcd}$$
in $\mathscr{C}_\tau(S)$, for every $\mathscr{U} \in \mathscr{C}_\tau(S)$. In particular, for $k = 0$ and $\mathscr{U} \in \{\mathscr{X}'_1, \mathscr{Y}'_1, \mathscr{Y}_1\}$, one finds that $\mathscr{X}_0 \wedge \mathscr{U} \cong \mathscr{U} \wedge \mathscr{X}_0$ belongs to $\mathscr{C}_\tau^{2n-1}(S)$. Similarly, for $k = 1$ and $\mathscr{U} = \mathscr{X}_0$, one finds that $\mathscr{X}_0 \wedge \mathscr{X}_1$ belongs to $\mathscr{C}_\tau^{2n}(S)$.

Therefore, the smash product restricts to $\mathscr{C}_\tau^\omega(S)$. $\square$

The functor $\underline{\mathrm{h}}^{\tau} : \mathsf{Prop}/S \to \mathscr{C}^{\omega}_{\tau}(S)$ is a strong monoidal *cdp*-functor. Then, by Theorem 4.1.32, there exists a strong monoidal functor

$$\mathrm{h}^{c_\tau} : \mathsf{Sch}^{\mathrm{ft}\,\mathrm{prop}}_{\mathrm{open}}/S \to \mathscr{C}^{\omega}_{\tau}(S)$$

that satisfies the excision property, and coincides with $\underline{\mathrm{h}}^{\tau}$ for proper S-schemes. Hence, it induces a motivic measure

$$\mu_{\tau} : \mathrm{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathrm{K}_0\left(\mathscr{C}^{\omega}_{\tau}(S)\right), \tag{66}$$

which sends the class of a proper S-scheme p to the class of $\underline{\mathrm{h}}^{\tau}_{p}$. Moreover, for a field k, the functor h^{c_τ} induces a map of spectra

$$\mathrm{K}(\mathrm{h}^{c_\tau}) : \mathrm{K}(\mathsf{Sch}^{\mathrm{ft}}/k) \to \mathrm{K}\left(\mathscr{C}^{\omega}_{\tau}(k)\right),$$

from the spectrum $\mathrm{K}(\mathsf{Sch}^{\mathrm{ft}}/k)$ defined in [**Cam17**].

Lemma 4.2.2. The motivic measure μ_{τ} is surjective.

Proof. To show that μ_{τ} is surjective, it suffices to show that for every pointed τ-sheaf $\mathscr{X} \in \mathscr{C}^{\omega}_{\tau}(S)$ the class $[\mathscr{X}] \in \mathrm{K}_0\left(\mathscr{C}^{\omega}_{\tau}(S)\right)$ belongs to the image of μ_{τ}, which follows by mathematical induction.

- For every $\mathscr{X} \in \mathscr{C}^{0}_{\tau}(S)$, there exists a proper S-scheme p for which $\mathscr{X} \cong \underline{\mathrm{h}}^{\tau}_{p}$. Hence, one has
$$[\mathscr{X}] = [\underline{\mathrm{h}}^{\tau}_{p}] = \mu_{\tau}([p]) \in \mathtt{im}\,\mu_{\tau}.$$
- For an integer $n \geq 1$, assume that for every $\mathscr{U} \in \mathscr{C}^{n-1}_{\tau}(S)$, one has $[\mathscr{U}] \in \mathtt{im}\,\mu_{\tau}$, and let $\mathscr{X}$ belongs to $\mathscr{C}^{n}_{\tau}(S)$, *i.e.* there exists a pushout square (63) in $\mathscr{C}_{\tau}(S)$, in which $\mathscr{X}', \mathscr{Y}$, and $\mathscr{Y}'$ belong to $\mathscr{C}^{n-1}_{\tau}(S)$, and ι is a monomorphism. Then, the cofibres $\mathscr{X}/\mathscr{X}'$ and $\mathscr{Y}/\mathscr{Y}'$ are canonically isomorphic, and hence
$$[\mathscr{X}] - [\mathscr{X}'] = [\mathscr{X}/\mathscr{X}'] = [\mathscr{Y}/\mathscr{Y}'] = [\mathscr{Y}] - [\mathscr{Y}'],$$
i.e.
$$[\mathscr{X}] = [\mathscr{X}'] + [\mathscr{Y}] - [\mathscr{Y}'] \in \mathtt{im}\,\mu_{\tau}.$$

□

Proposition 4.2.3. The modified Grothendieck ring (21) factorises the motivic measure μ_{τ}.

Proof. Let $f : x \to y$ be a universal homeomorphism of S-schemes, and let $l : w \hookrightarrow\!\!\!\rightarrow q$ be a compactification of y. Recall our conventions, in §.0.2, which require S-schemes to be of finite type. Hence, the morphism f is finite, universally injective, and surjective, by [**Gro65**, Prop.2.4.5]. Since f is proper, the category $\mathsf{Comp}_S(f, l)$ is nonempty, by Proposition 4.1.5. Suppose that $i : z \hookrightarrow\!\!\!\rightarrow p$ is a compactification in $\mathsf{Comp}_S(f, l)$, and let $g : i \to l$ be a morphism of compactifications such that f is a base change in $\mathsf{Sch}^{\mathrm{ft}}/S$ of g along j_l. For simplicity and without loss of generality, we may choose i that fits into the left Cartesian square

$$\begin{tikzcd}
z \arrow[r, hook, "i"] \arrow[d, "\underline{g}"'] \arrow[dr, phantom, "\ulcorner", very near start] & p \arrow[d, "g"] & x \arrow[l, hook', "j_i"'] \arrow[d, "f"] \arrow[dl, phantom, "\urcorner", very near start] \\
w \arrow[r, hook, "l"'] & q & y \arrow[l, hook', "j_l"]
\end{tikzcd} \tag{67}$$

in Prop/S. Then, the set $\{l : w \hookrightarrow q, g : p \to q\}$ is a *cdp*-covering family of q in Prop/S. Indeed, let k be a field, and let $b : \mathsf{Spec}\, k \to Q$ be a morphism of schemes in $\mathsf{Noe}^{\mathrm{fd}}$, where Q is the underlying scheme of q. Either b lifts along l or j_l. When b lifts along j_l, there exist a morphism $a : \mathsf{Spec}\, k \to Y$, where Y is the underlying scheme of y, such that $b = j_l \circ a$. Consider the Cartesian square

$$\begin{tikzcd}
T \arrow[r, "\underline{a}"] \arrow[d, "\underline{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\
\mathsf{Spec}\, k \arrow[r, "a"'] & Y
\end{tikzcd}$$

in the category $\mathsf{Noe}^{\mathrm{fd}}$. The morphism f is a finite universal homeomorphism, and hence T is a one-point scheme $\mathsf{Spec}\, R$ and $\underline{f}$ is induced by a finite ring homomorphism $\psi : k \hookrightarrow R$, to a local ring R of Krull dimension zero. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa \coloneqq {}^{R}/_{\mathfrak{m}}$. Then, the induced homomorphism $k \hookrightarrow \kappa$ is a finite field extension. Assuming that $[\kappa : k] \neq 1$, there exist distinct ring homeomorphisms $\kappa \to \kappa$ over k, which contradicts with $\underline{f}$ being universally injective. Thus, one has $[\kappa : k] = 1$, *i.e.* the residue field of T at its unique point is isomorphic to k. Hence, a lifts along f, which lifts b along g, and $\{l, q\}$ is a *cdp*-covering family.

In order to show that μ_τ factorises through the modified Grothendieck ring, it suffices to show that $f_! : \mathsf{h}_x^{\mathrm{c}\tau} \to \mathsf{h}_y^{\mathrm{c}\tau}$ is an isomorphism in $\mathscr{C}^{\omega}_{\tau}(S)$, which isomorphic to the square

$$\begin{tikzcd}
\underline{\mathsf{h}}_z^{\tau} \arrow[r, tail, "i_*"] \arrow[d, "\underline{g}_*"'] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathsf{h}}_p^{\tau} \arrow[d, "g_*"] \\
\underline{\mathsf{h}}_w^{\tau} \arrow[r, tail, "l_*"'] & \underline{\mathsf{h}}_q^{\tau}
\end{tikzcd} \tag{68}$$

being a pushout square in $\mathscr{C}^{\omega}_{\tau}(S)$. Since the τ-sheafification functor preserves colimits, it suffices to show that the canonical morphism $\Theta : \mathsf{h}_{w,+} \coprod_{\mathsf{h}_{z,+}} \mathsf{h}_{p,+} \to \mathsf{h}_{q,+}$ of pointed presheaves is a τ-local isomorphism.

Let $t : T \to S$ be a proper S-scheme, and let $b \in \mathsf{h}_{q,+}(t)$. Either $b = *$, in which case b belongs to the image of Θ_t, or b is a morphism $t \to q$ of proper S-schemes. Assuming the latter case, consider the Cartesian squares

$$
\begin{tikzcd}
t_l \arrow[r, hook, "l_b"] \arrow[d, "b_l"'] \arrow[dr, phantom, "\ulcorner", very near start] & t \arrow[d, "b"] \\
w \arrow[r, hook, "l"'] & q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
t_g \arrow[r, "g_b"] \arrow[d, "b_g"'] \arrow[dr, phantom, "\ulcorner", very near start] & t \arrow[d, "b"] \\
p \arrow[r, "g"'] & q
\end{tikzcd}
$$

in Prop/S. The set $\{l_b, g_b\}$ is a *cdp*-covering family of t in Prop/S. Let $[b_l]$ and $[b_g]$ denote the classes of $b_l \in \mathsf{h}_{w,+}(t_l)$ and $b_g \in \mathsf{h}_{p,+}(t_g)$ in $\left(\mathsf{h}_{w,+} \coprod_{\mathsf{h}_{z,+}} \mathsf{h}_{p,+}\right)(t_l)$ and $\left(\mathsf{h}_{w,+} \coprod_{\mathsf{h}_{z,+}} \mathsf{h}_{p,+}\right)(t_g)$, respectively, then, one has

$$\Theta_{t_l}([b_l]) = l_*(b_l) = l_b^*(b) \qquad \text{and} \qquad \Theta_{t_g}([b_g]) = g_*(b_g) = g_b^*(b).$$

Since the pretopology τ is finer than the *cdp*-pretopology, the morphism Θ is a τ-local epimorphism, by Corollary A.4.11.

On the other hand, suppose that $t : T \to S$ is a proper S-scheme, and let $a_0, a_1 \in \left(\mathsf{h}_{w,+} \coprod_{\mathsf{h}_{z,+}} \mathsf{h}_{p,+}\right)(t)$ such that $\Theta_t(a_0) = \Theta_t(a_1)$. When either a_0 or a_1 coincides with the base point $*$ in $\left(\mathsf{h}_{w,+} \coprod_{\mathsf{h}_{z,+}} \mathsf{h}_{p,+}\right)(t)$, so does the other. Assume that $a_0 \neq *$ and $a_1 \neq *$, and distinguish the following cases.

(1) There are a'_0 and a'_1 in $\mathsf{h}_{w,+}(t)$ such that $a_0 = [a'_0]$ and $a_1 = [a'_1]$. Then, $l_*(a'_0) = \Theta_t(a_0) = \Theta_t(a_1) = l_*(a'_1)$. Since $a_0 \neq *$ and $a_1 \neq *$, the sections a'_0 and a'_1 are morphisms $t \to w$ of proper S-schemes such that $l \circ a'_0 = l \circ a'_1$, which implies that $a'_0 = a'_1$, as l is a monomorphism of proper S-schemes. Hence, $a_0 = a_1$.

(2) Either there exists a'_0 or a'_1 in $\mathsf{h}_{w,+}(t)$ such that $a_0 = [a'_0]$ or $a_1 = [a'_1]$, but not both. Without loss of generality, assume that the section a'_0 (resp. a'_1) is a morphism $t \to w$ (resp. $t \to p$) of proper S-schemes, as $a_0 \neq *$ and $a_1 \neq *$. Then, $l \circ a'_0 = \Theta_t(a_0) = \Theta_t(a_1) = g \circ a'_1$, and hence there exists a morphism $a' : t \to z$ of proper S-schemes such that $a'_0 = \underline{g}_*(a')$ and $a'_1 = i_*(a')$, see (67). Thus, $a_0 = [a'_0] = [a'_1] = a_1$.

(3) There does not exist a'_0 or a'_1 in $\mathsf{h}_{w,+}(t)$ such that $a_0 = [a'_0]$ or $a_1 = [a'_1]$. As $a_0 \neq *$ and $a_1 \neq *$, let a'_0 and a'_1 be morphisms $t \to p$ of proper S-schemes such that $a_0 = [a'_0]$ or $a_1 = [a'_1]$. Then, $g \circ a'_0 = \Theta_t(a_0) = \Theta_t(a_1) = g \circ a'_1$. Consider the Cartesian squares (70) in $\mathsf{Sch}^{\mathrm{ft}}/S$, on the next page, for $k = 1, 0$. Since $g \circ a'_0 = g \circ a'_1$, there exist such Cartesian squares with

$$t_l \coloneqq t_{0,l} = t_{1,l} \quad , \quad \underline{l} \coloneqq l'_0 = l'_1 \quad , \quad t_{j_l} \coloneqq t_{0,j_l} = t_{1,j_l} \quad , \text{and} \quad j_{\underline{l}} \coloneqq j'_{0,l} = j'_{1,l}.$$

Then, one has $f \circ a'_{0,j_l} = f \circ a'_{1,j_l}$, which implies the existence of a *cdp*-cover $\sigma_{j_l} : t'_{j_l} \to t_{j_l}$ such that

$$a'_{0,j_l} \circ \sigma_{j_l} = a'_{1,j_l} \circ \sigma_{j_l}, \tag{69}$$

as $f_* : \mathsf{h}_x \to \mathsf{h}_y$ is a *cdp*-local monomorphism, by Proposition A.4.37.

$$
\begin{tikzcd}
t_{k,l} \arrow[r, hook, "l_k"] \arrow[d, "a'_{k,l}"'] \arrow[dr, phantom, "\ulcorner", very near start] & t \arrow[d, "a'_k"'] & t_{k,j_l} \arrow[l, hook', "j_{k,l}"'] \arrow[d, "a'_{k,j_l}"] \arrow[dl, phantom, "\urcorner", very near start] \\
z \arrow[r, hook, "i"] \arrow[d, "\underline{g}"'] \arrow[dr, phantom, "\ulcorner", very near start] & p \arrow[d, "g"'] & x \arrow[l, hook', "j_i"'] \arrow[d, "f"] \arrow[dl, phantom, "\urcorner", very near start] \\
w \arrow[r, hook, "l"'] & q & y \arrow[l, hook', "j_l"]
\end{tikzcd}
\tag{70}
$$

Since σ_{j_l} is proper, the category $\mathsf{Comp}_S(\sigma_{j_l}, \underline{l})$ is nonempty, and there exists a compactification $l' : t'_l \hookrightarrow t'$ of t'_{j_l} that fits into Cartesian squares

$$
\begin{tikzcd}
t'_l \arrow[r, hook, "l'"] \arrow[d, "\sigma_l"'] \arrow[dr, phantom, "\ulcorner", very near start] & t' \arrow[d, "\sigma"'] & t'_{j_l} \arrow[l, hook', "j_{l'}"'] \arrow[d, "\sigma_{j_l}"] \arrow[dl, phantom, "\urcorner", very near start] \\
t_l \arrow[r, hook, "\underline{l}"'] & t & t_{j_l} \arrow[l, hook', "j_{\underline{l}}"]
\end{tikzcd}
$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$. Since σ_{j_l} is a *cdp*-cover, the set $\{\underline{l} : t_l \hookrightarrow t, \sigma : t' \to t\}$ is a *cdp*-covering family of t in Prop/S. Indeed, let k be a field, and let $b : \mathsf{Spec}\, k \to T$ be a morphism of schemes in $\mathsf{Noe}^{\mathrm{fd}}$, then either b lifts along $\underline{l}$ or $j_{\underline{l}}$. In the latter case, the lift $\mathsf{Spec}\, k \to T_{j_{\underline{l}}}$, where $T_{j_{\underline{l}}}$ is the underlying scheme of $t_{j_{\underline{l}}}$, lifts along the *cdp*-cover σ_{j_l}, which lifts b along σ. Since $\underline{l}$ is a monomorphism of schemes and $j_{\underline{l}}$ is a complementary open immersion to $\underline{l}$, one sees that the morphism σ_l is a *cdp*-cover.

Let $\{i_\alpha : t_\alpha \to t' \mid \alpha \in A\}$ be the *cdp*-covering family of t' in Prop/S by its integral components, and consider the Cartesian squares

$$
\begin{tikzcd}
t_{\alpha,l} \arrow[r, hook, "l_\alpha"] \arrow[d, "i_{\alpha,l}"'] \arrow[dr, phantom, "\ulcorner", very near start] & t_\alpha \arrow[d, "i_\alpha"'] & t_{\alpha,j_l} \arrow[l, hook', "j_{l_\alpha}"'] \arrow[d, "i_{\alpha,j_l}"] \arrow[dl, phantom, "\urcorner", very near start] \\
t'_l \arrow[r, hook, "l'"'] & t' & t'_{j_l} \arrow[l, hook', "j_{l'}"]
\end{tikzcd}
$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$, for every $\alpha \in A$. Then, $\{i_{\alpha,l} : t_{\alpha,l} \to t'_l \mid \alpha \in A\}$ is a *cdp*-covering family of t'_l in Prop/S, and hence the set

$$
\mathscr{U} := \left\{\underline{l} \circ \sigma_l \circ i_{\alpha,l} : t_{\alpha,l} \to t \mid \alpha \in A\right\} \bigcup \left\{\sigma \circ i_\alpha : t_\alpha \to t \mid \alpha \in A\right\}
$$

is a *cdp*-covering family in Prop/S. For every $\alpha \in A$, we distinguish two cases.

(a) When t_{α,j_l} is nonempty, the open immersion j_{l_α} is dominant, and hence $a_0' \circ \sigma \circ i_\alpha = a_1' \circ \sigma \circ i_\alpha$, by [**Vak15**, Reduced-to-Separated Th.10.2.2] and (69). Thus, one has $(\sigma \circ i_\alpha)^*(a_0) = [a_0' \circ \sigma \circ i_\alpha] = [a_1' \circ \sigma \circ i_\alpha] = (\sigma \circ i_\alpha)^*(a_1)$.

(b) When $t_{\alpha,j_l} \cong \varnothing_S$, the morphism i_α factorises through l', and hence there exists $a'_{\alpha,k}$ in $\mathtt{h}_{w,+}(t_\alpha)$ with $[a'_{\alpha,k}] = (\sigma \circ i_\alpha)^*(a_k) \in \left(\mathtt{h}_{w,+} \coprod_{\mathtt{h}_{z,+}} \mathtt{h}_{p,+}\right)(t_\alpha)$, for $k = 0,1$. Since $\Theta_t(a_0) = \Theta_t(a_1)$, one has $l_*(a'_{\alpha,0}) = l_*(a'_{\alpha,1})$, and hence $a'_{\alpha,0} = a'_{\alpha,1}$ as l is a monomorphism of proper S-schemes. Thus,

$$(\sigma \circ i_\alpha)^*(a_0) = (\sigma \circ i_\alpha)^*(a_1).$$

Thus, for every $\alpha \in A$, one has

$$(\sigma \circ i_\alpha)^*(a_0) = (\sigma \circ i_\alpha)^*(a_1) \qquad \text{and} \qquad (\underline{l} \circ \sigma_l \circ i_{\alpha,l})^*(a_0) = (\underline{l} \circ \sigma_l \circ i_{\alpha,l})^*(a_1).$$

There always exists a *cdp*-covering family $\mathscr{V}$ of t in Prop/S such that $\delta^*(a_0) = \delta^*(a_1)$, for every $\delta \in \mathscr{V}$. Thus, the morphism Θ is a τ-local monomorphism, by Corollary A.4.14.

Therefore, the square (68) is a pushout square in $\mathscr{C}_\tau^\omega(S)$, *i.e.* the morphism $f_!$ is an isomorphism in $\mathscr{C}_\tau^\omega(S)$, and μ_τ factorises through the modified Grothendieck ring. ☐

Conjecture 4.2.4. The motivic measure μ_{cdp} is isomorphic to the quotient map

$$\mu_{\mathrm{uh}} : \mathtt{K}_0(\mathsf{Sch}^{\mathrm{ft}}/S) \to \mathtt{K}_0^{\mathrm{uh}}(\mathsf{Sch}^{\mathrm{ft}}/S).$$

Let k be a field, recall Example 4.1.33, and consider a closed immersion $i : \mathtt{Spec}\, k \hookrightarrow \mathbb{P}^1_k$ with complementary open immersion $j : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. Then,

$$\mathtt{h}^{\mathtt{c}_\tau}_{\mathbb{A}^1_k} = \mathtt{coker}\left(\underline{\mathtt{h}}^\tau_{\mathtt{Spec}\, k} \longrightarrow \underline{\mathtt{h}}^\tau_{\mathbb{P}^1_k}\right) = \left(\mathtt{h}_{\mathbb{P}^1_k}, \infty\right)^{\mathtt{a}_\tau},$$

where ∞ denotes the unique k-rational point in $\left(\mathbb{P}^1_k \smallsetminus j(\mathbb{A}^1_k)\right)_{\mathrm{red}}$.

4.2.1.1. *The Commutative Ring Spectrum Structure.* The spectrum $\mathtt{K}\left(\mathscr{C}_\tau^\omega(S)\right)$ admits a canonical commutative ring spectrum structure, *i.e.* a homotopy commutative monoid structure in the category of S^1-spectra of pointed topological spaces.

The functor $* \to \mathsf{Prop}/S$ that sends the unique object of $*$ to id_S is continuous, with respect to the indiscrete topology on $*$ and any topology on Prop/S. Hence, it induces an exact functor of Waldhausen categories

$$u_S^\tau : \mathsf{PSh}_\bullet(*) \cong \mathsf{Set}_\bullet \to \mathscr{C}_\tau(S),$$

which is a left adjoint to the the global section functor $\mathscr{C}_\tau(S) \to \mathsf{Set}_\bullet$, *i.e.* u_S^τ is given by sending a pointed sets to its constant pointed τ-sheaf.

The category $\mathsf{FSet}_\bullet$ of pointed finite sets admits a symmetric monoidal Waldhausen structure, as in Example 1.5.21. The category $\mathscr{C}_\tau^\omega(S)$ contains the unit $1_\wedge = \underline{\mathtt{h}}^\tau_{\mathrm{id}_S}$, admits all finite colimits, and the τ-sheafification functor commutes with colimits. Thus, the functor u_S^τ restricts to an exact functor of Waldhausen categories

$$\upsilon_S^\tau : \mathsf{FSet}_\bullet \to \mathscr{C}_\tau^\omega(S). \tag{71}$$

The exact functor υ_S^τ induces a map of spectra

$$\mathtt{K}(\upsilon_S^\tau) : \mathbb{S} \cong \mathtt{K}(\mathsf{FSet}_\bullet) \to \mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big),$$

which induces the ring characteristic $\mathtt{K}_0(\upsilon_S^\tau) : \mathbb{Z} \to \mathtt{K}_0\big(\mathscr{C}_\tau^\omega(S)\big)$.

On the other hand, $\mathscr{C}_\tau^\omega(S)$ is a symmetric monoidal Waldhausen category, by Lemma 4.2.1. Hence, there exists a paring

$$\otimes : \mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big) \wedge \mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big) \to \mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big),$$

see [**Wal85**, p.342]. That makes $\mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big)$ into a commutative ring spectrum, see [**BM11**, Cor.2.8].

4.2.1.2. *Monoidal Proper-Fibred Waldhausen Category.* The $\mathtt{K}$-theory commutative ring spectrum $\mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big)$, for a Noetherian scheme S of finite Krull dimension, arises from a fibre of a *monoidal proper-fibred Waldhausen category* over Noetherian schemes of finite Krull dimensions[9]. That is, there exists a strong monoidal pseudofunctor

$$\mathscr{C}_\tau^\omega : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{op}} \to \mathsf{Wald}_2^\wedge,$$

where $\mathsf{Wald}_2^\wedge$ is the 2-category of essentially small symmetric monoidal Waldhausen categories, weak monoidal exact functors between them, and monoidal natural transformations between the latter, such that

- for every scheme $S \in \mathsf{Noe}^{\mathrm{fd}}$ the fibre $\mathscr{C}_\tau^\omega(S)$ is the symmetric monoidal Waldhausen category constructed in (64), as in Lemma 4.2.7;
- for every proper morphism $f : S \to T$ in $\mathsf{Noe}^{\mathrm{fd}}$, the pullback $f^* : \mathscr{C}_\tau^\omega(T) \to \mathscr{C}_\tau^\omega(S)$ admits a left adjoint $f_\# : \mathscr{C}_\tau^\omega(S) \to \mathscr{C}_\tau^\omega(T)$, as in Lemma 4.2.12;
- $\mathscr{C}_\tau^\omega$ satisfies the proper-base change property, as in Lemma 4.2.20; and
- $\mathscr{C}_\tau^\omega$ satisfies the proper-projection formula, as in Lemma 4.2.23.

Then, applying the Waldhausen's $\mathtt{K}$-theory 2-functor[10] induces a monoidal proper-fibred commutative ring spectrum

$$\mathtt{K}(\mathscr{C}_\tau^\omega) : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{op}} \to \mathsf{CRingSpt}_2.$$

In fact, the strong monoidal *cdp*-functor $\underline{\mathsf{h}}^\tau : \mathsf{Prop}/S \to \mathscr{C}_\tau^\omega(S)$, given in (65), arises from the *geometric section* of $\mathscr{C}_\tau^\omega$, see [**CD13**, §.1.1.c].

Remark 4.2.5. The statements in the rest of this subsection were motivated by Dan Petersen's answer in [**Pet14**], which recalls Ekedahl's approach to higher Grothendieck

[9]See [**CD13**, §.1] for a treatment of $\mathscr{P}$-fibred categories, for a set $\mathscr{P}$ of morphisms of schemes.
[10]See [**FP17**, §.1] for the treatment of the 2-categorical Waldhausen's $\mathtt{K}$-theory.

groups of varieties. The statements of Lemma 4.2.20 and Lemma 4.2.23 are essentially consequences of [**CD13**, Ex.1.1.11 and Ex.1.1.28].

Inverse Image. Recall the canonical proper-fibred category $\mathsf{Prop}/- : \mathsf{No\ddot{e}}^{\mathrm{fd}} \to \mathsf{CAT}_2$, as in [**CD13**, Ex.1.1.4 and Ex.1.1.11]. A morphism $f : S \to T$ in $\mathsf{No\ddot{e}}^{\mathrm{fd}}$ induces a functor

$$f^{-1} : \mathsf{Prop}/T \to \mathsf{Prop}/S \tag{72}$$

that sends a proper T-scheme to its base change along f. That in turn induces a direct image functor

$$f_* : \mathsf{PSh}_\bullet(\mathsf{Prop}/S) \to \mathsf{PSh}_\bullet(\mathsf{Prop}/T),$$

given by precomposition with $(f^{-1})^{\mathrm{op}}$, *i.e.* for a presheaf $\mathscr{X} \in \mathsf{PSh}_\bullet(\mathsf{Prop}/S)$ and for a proper T-scheme q, one has $f_*(\mathscr{X})(q) = \mathscr{X}\big(f^{-1}(q)\big)$. The functor f_* admits a left adjoint

$$f^*_{\mathrm{pre}} : \mathsf{PSh}_\bullet(\mathsf{Prop}/T) \to \mathsf{PSh}_\bullet(\mathsf{Prop}/S),$$

called the *inverse image functor* along f, and it is given by a left Kan extension along $(f^{-1})^{\mathrm{op}}$, see §.A.4.2. The functor $-_+$ commutes with colimits, for being a left adjoint. Then, using the coend formula (87), one sees that

$$\begin{aligned}
f^*_{\mathrm{pre}}(\mathtt{h}_{q,+})(p) &= \int^{x^{\mathrm{op}} \in (\mathsf{Prop}/T)^{\mathrm{op}}} \coprod_{(\mathsf{Prop}/S)^{\mathrm{op}}\left((f^{-1})^{\mathrm{op}}(x^{\mathrm{op}}),p^{\mathrm{op}}\right)} \mathtt{h}_{q,+}(x) \\
&\cong \left(\int^{x \in \mathsf{Prop}/T} \coprod_{\mathsf{Prop}/S\left(p,f^{-1}(x)\right)} \mathtt{h}_q(x) \right)_+ \\
&\cong \left(\int^{x \in \mathsf{Prop}/T} \mathsf{Prop}/S\big(p, f^{-1}(x)\big) \times \mathsf{Prop}/T(x,q) \right)_+ \\
&\cong \left(\int^{x \in \mathsf{Prop}/T} \mathsf{Prop}/S\big(p, f^{-1}(x)\big) \times \mathsf{Prop}/S\big(f^{-1}(x), f^{-1}(q)\big) \right)_+ \\
&\cong \Big(\mathsf{Prop}/S\big(p, f^{-1}(q)\big)\Big)_+ \cong \mathtt{h}_{f^{-1}(q),+}(p),
\end{aligned}$$

for every proper T-scheme q and for every proper S-scheme p, *i.e.*

$$f^*_{\mathrm{pre}}(\mathtt{h}_{q,+}) \cong \mathtt{h}_{f^{-1}(q),+}\,.$$

In fact, this is a defining property for f^*_{pre}, as every object in $\mathsf{PSh}_\bullet(\mathsf{Prop}/T)$ is a colimit of a diagram in the essential image of $\mathtt{h}_{-,+}$.

Since the base change functor f^{-1} commutes with fibre products, it is continuous with respect to the τ-pretopology, see [**SGA73**, §.III.Prop.1.6]. Thus, the direct image functor f_* preserves τ-sheaves, and it restricts to a functor

$$f_* : \mathscr{C}_\tau(S) \to \mathscr{C}_\tau(T),$$

which admits a left adjoint f^*, given by the composition of f^*_{pre} with the associated τ-sheaf functor $-^{\mathrm{a}_\tau}$. Since the τ-sheafification functor commutes with colimits, one has

$$f^*(\underline{\mathtt{h}}^\tau_q) \cong \underline{\mathtt{h}}^\tau_{f^{-1}(q)}, \tag{73}$$

for every proper T-scheme p.

Lemma 4.2.6. Assume that $f : S \to T$ is a morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor $f^* : \mathscr{C}_\tau(T) \to \mathscr{C}_\tau(S)$ is a strong monoidal exact functor of symmetric monoidal Waldhausen categories.

Proof. The functor f^{-1} is Cartesian, as limits commute with each other, and hence the functor f^*_{pre} is left exact, see [**Joh02**, A.Ex.4.1.10]. Also both the τ-sheafification functor and the inclusion functor, of τ-sheaves into presheaves, are left exact. Thus, the left adjoint functor f^* is left exact. In particular, the functor f^* preserves monomorphisms, finite colimits, the unit of the monoidal structure, and smash products of pointed τ-sheaves, as the latter only involves finite limits and colimits of pointed τ-sheaves. □

Although the functor $f_* : \mathsf{PSh}_\bullet(\mathsf{Prop}/S) \to \mathsf{PSh}_\bullet(\mathsf{Prop}/T)$ admits a right adjoint given by the right Kan extension, it is resection $f_* : \mathscr{C}_\tau(S) \to \mathscr{C}_\tau(T)$ does not necessarily admit a right adjoint. In particular, it is not necessarily exact.

Lemma 4.2.7. Assume that $f : S \to T$ is a morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor $f^* : \mathscr{C}_\tau(T) \to \mathscr{C}_\tau(S)$ restricts to a strong monoidal exact functor of Waldhausen categories

$$f^{*,\omega} : \mathscr{C}^\omega_\tau(T) \to \mathscr{C}^\omega_\tau(S). \tag{74}$$

Proof. The statement follows from Lemma 4.2.6, provided the restriction $f^{*,\omega}$ exists. Since f^* commutes with pushout squares, it suffices to show that f^* restricts to a functor $f^{*,0} : \mathscr{C}^0_\tau(T) \to \mathscr{C}^\omega_\tau(S)$, in order to induce a strong monoidal exact functor $f^{*,\omega} : \mathscr{C}^\omega_\tau(T) \to \mathscr{C}^\omega_\tau(S)$, which holds by (73). □

When no confusion arises, we abuse notation, and refer to $f^{*,\omega}$ by f^*.

Corollary 4.2.8. Assume that $f : S \to T$ is a morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the there exists a morphism of commutative ring spectra

$$f^* : \mathtt{K}\left(\mathscr{C}^\omega_\tau(T)\right) \to \mathtt{K}\left(\mathscr{C}^\omega_\tau(S)\right),$$

that sends a point in the component $[\underline{\mathtt{h}}^\tau_q]$ to a point in the component $[\underline{\mathtt{h}}^\tau_{f^{-1}(q)}]$, for every proper T-scheme q.

Example 4.2.9. Assume that $f : S \to T$ is a morphism in $\mathsf{Noe}^{\mathrm{fd}}$, let y be a T-scheme, and let $l : w \hookrightarrow\!\!\!\!\to q$ be a compactification of y. Since f^* is exact and complementary open immersions are closed under pullbacks, one has

$$f^*(\mathtt{h}^{\mathrm{c}_\tau}_y) \cong \mathtt{coker}\left(f^*(\underline{\mathtt{h}}^\tau_w) \rightarrowtail f^*(\underline{\mathtt{h}}^\tau_q)\right) \cong \mathtt{coker}\left(\underline{\mathtt{h}}^\tau_{f^{-1}(w)} \rightarrowtail \underline{\mathtt{h}}^\tau_{f^{-1}(q)}\right) \cong \mathtt{h}^{\mathrm{c}_\tau}_{f^{-1}(y)},$$

where the S-scheme $f^{-1}(y)$ is a base change in $\mathsf{Sch}^{\mathrm{ft}}/T$ of y along f. Thus, in particular,

$$\pi_0(f^*)\big([\mathtt{h}_y^{c_\tau}]\big) = [f^*(\mathtt{h}_y^{c_\tau})] = [\mathtt{h}_{f^{-1}(y)}^{c_\tau}]. \tag{75}$$

Since the motivic measure $\mu_{\tau,T}$ is surjective, by Lemma 4.2.2, one sees that the ring homomorphisms

$$\pi_0(f^*) : \mathtt{K}_0\left(\mathscr{C}_\tau^\omega(T)\right) \to \mathtt{K}_0\left(\mathscr{C}_\tau^\omega(S)\right)$$

is determined by (75). We may abuse notation and refer to $\pi_0(f^*)$ by f^*.

Suppose that $f : S \to T$ and $g : T \to U$ are morphisms in $\mathsf{Noe}^{\mathrm{fd}}$. Since Kan extensions are determined up to canonical natural isomorphisms, one has canonical monoidal natural isomorphisms

$$(g \circ f)^* \overset{\sim}{\Rightarrow} f^* \circ g^* \qquad \text{and} \qquad (\mathrm{id}_S)^* \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathscr{C}_\tau^\omega(S)},$$

which satisfy the cocycle condition.

Corollary 4.2.10. There exists a pseudofunctor

$$\mathscr{C}_\tau^\omega : {\mathsf{Noe}^{\mathrm{fd}}}^{\mathrm{op}} \to \mathsf{Wald}_2^\wedge,$$

which sends a Noetherian scheme S of finite Krull dimension to the Waldhausen category $\mathscr{C}_\tau^\omega(S)$, given in (64), and sends f^{op}, for a morphism $f : S \to T$ in $\mathsf{Noe}^{\mathrm{fd}}$, to the strong monoidal exact functor $f^{*,\omega}$, as in Lemma 4.2.7. Then, the Waldhausen's $\mathtt{K}$-theory 2-functor induces a pseudofunctor

$$\mathtt{K}(\mathscr{C}_\tau^\omega) : {\mathsf{Noe}^{\mathrm{fd}}}^{\mathrm{op}} \to \mathsf{CRingSpt}_2.$$

Proper Direct Image. Suppose that $f : S \to T$ is a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor f^{-1}, given in (72), admits a left adjoint

$$f_\circ : \mathsf{Prop}/S \to \mathsf{Prop}/T, \tag{76}$$

given by composition with f. Thus, the functor

$$f^! : \mathsf{PSh}_\bullet(\mathsf{Prop}/T) \to \mathsf{PSh}_\bullet(\mathsf{Prop}/S),$$

given by precomposition with $f_\circ$, is a left adjoint to $f_* : \mathsf{PSh}_\bullet(\mathsf{Prop}/S) \to \mathsf{PSh}_\bullet(\mathsf{Prop}/T)$, and hence $f^!$ is canonically isomorphic to f^*_{pre}. The functor $f^!$ admits a left adjoint $f_{\#,\mathrm{pre}}$, given by a left Kan extension along $f_\circ^{\mathrm{op}}$. Since the functor $f_\circ$ preserves τ-coving families, $f^!$ restricts to a functor $f^! : \mathscr{C}_\tau(T) \to \mathscr{C}_\tau(S)$, which is canonically isomorphic to f^*. The functor $f^!$ admits a left adjoint

$$f_\# : \mathscr{C}_\tau(S) \to \mathscr{C}_\tau(T), \tag{77}$$

called the *proper direct image functor* along f, and it is given by the τ-sheafification of $f_{\#,\mathrm{pre}}$. Similar to the inverse image functor, for a proper S-scheme p, one has

$$f_\#(\underline{\mathtt{h}}_p^\tau) \cong \underline{\mathtt{h}}_{f_\circ(p)}^\tau, \tag{78}$$

which is a defining property of $f_\#$.

Lemma 4.2.11. Assume that $f : S \to T$ is a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor $f_\# : \mathscr{C}_\tau(S) \to \mathscr{C}_\tau(T)$ is an exact functor of Waldhausen categories.

Proof. It is sufficient to show the functor $f_\#$ commutes with monomorphisms, as it commutes with colimits for being a left adjoint.

Assume that $\iota : \mathscr{X} \to \mathscr{Y}$ is a monomorphism in $\mathscr{C}_\tau(S)$, let q be a proper T-scheme, and let $t_0, t_1 \in f_\#(\mathscr{X})(q)$ such that $f_\#(\iota)_q(t_0) = f_\#(\iota)_q(t_1)$.

By the definition of the τ-sheafification functor, as in [**Vis08**, Def.2.63], there exists a τ-covering family $\mathscr{U} = \{\sigma_i : q_i \to q \mid i \in I\}$, and there exists a section $t_{k,i} \in f_{\#,\mathrm{pre}}(\mathscr{X})(q_i)$ such that $\sigma_i^*(t_k) = t_{k,i}^{\mathsf{a}}$ for every $i \in I$, for $k = 0, 1$. For $i \in I$, pulling back along σ_i yields

$$\big(f_\#(\iota)_{q_i}(t_{0,i})\big)^{\mathsf{a}} = f_\#(\iota)_{q_i}(t_{0,i}^{\mathsf{a}}) = f_\#(\iota)_{q_i}(t_{1,i}^{\mathsf{a}}) = \big(f_{\#,\mathrm{pre}}(\iota)_{q_i}(t_{1,i})\big)^{\mathsf{a}}.$$

Thus, there exits a τ-covering family $\mathscr{U}_i = \{\sigma_{i,j} : q_{i,j} \to q_i \mid j \in J_i\}$ for which

$$\begin{aligned} f_{\#,\mathrm{pre}}(\iota)_{q_{i,j}}\big(\sigma_{i,j}^*(t_{0,i})\big) &= \sigma_{i,j}^*\big(f_{\#,\mathrm{pre}}(\iota)_{q_i}(t_{0,i})\big) = \sigma_{i,j}^*\big(f_{\#,\mathrm{pre}}(\iota)_{q_i}(t_{1,i})\big) \\ &= f_{\#,\mathrm{pre}}(\iota)_{q_{i,j}}\big(\sigma_{i,j}^*(t_{1,i})\big). \end{aligned}$$

The functor $f_{\#,\mathrm{pre}}$ is a left Kan extension along $f_\circ^{\mathrm{op}}$. Hence, for a proper T-scheme q', the coend formula (87) implies that the underlying set of $f_{\#,\mathrm{pre}}(\mathscr{X})(q')$ can be given by

$$\Big(\coprod_{p \in \mathsf{Prop}/S} \mathsf{Prop}/T(q', f \circ p) \times \mathscr{X}(p)\Big) \Big/ \sim,$$

where $\sim$ is the smallest equivalence relation that identifies $(g, s) \in \mathsf{Prop}/T(q', f \circ p) \times \mathscr{X}(p)$ and $(g', s') \in \mathsf{Prop}/T(q', f \circ p') \times \mathscr{X}(p')$ whenever there exists a morphism $h : p \to p'$ of proper S-schemes for which

$$g' = f_\circ(h) \circ g \qquad \text{and} \qquad s = h^*(s');$$

whereas the point of $f_{\#,\mathrm{pre}}(\mathscr{X})(q')$ is given by the unique class $[(g, *)]$, which is independent of the choice of the proper S-scheme p and the morphism $g : q' \to f \circ p$ of proper T-schemes. Also, one has

$$f_{\#,\mathrm{pre}}(\iota)_{q'}\left([(g, s)]\right) = \big[\big(g, \iota_p(s)\big)\big],$$

for every proper S-scheme p, every morphism $g : q' \to f \circ p$ of proper T-schemes, and every section $s \in \mathscr{X}(p)$.

For $k = 0, 1$, for $i \in I$, and for $j \in J_i$, let $p_{k,i,j}$ be a proper S-scheme, let $g_{k,i,j} : q_{i,j} \to f \circ p_{k,i,j}$ be a morphism of proper T-schemes, and let $s_{k,i,j} \in \mathscr{X}(p_{k,i,j})$ for which

$$\sigma_{i,j}^*(t_{k,i}) = [(g_{k,i,j}, s_{k,i,j})].$$

Since

$$\big[\big(g_{0,i,j}, \iota_{p_{0,i,j}}(s_{0,i,j})\big)\big] = f_{\#,\mathrm{pre}}(\iota)_{q_{i,j}}\big([(g_{0,i,j}, s_{0,i,j})]\big) = f_{\#,\mathrm{pre}}(\iota)_{q_{i,j}}\big([(g_{1,i,j}, s_{1,i,j})]\big)$$
$$= \big[\big(g_{1,i,j}, \iota_{p_{1,i,j}}(s_{1,i,j})\big)\big],$$

there exist a proper S-scheme $p_{i,j}$, a morphism $g_{i,j} : q_{i,j} \to f \circ p_{i,j}$ of proper T-schemes, and a morphism $h_{k,i,j} : p_{i,j} \to p_{k,i,j}$ of proper S-schemes, such that $g_{k,i,j} = f_\circ(h_{k,i,j}) \circ g_{i,j}$ and

$$\iota_{p_{i,j}}\big(h^*_{0,i,j}(s_{0,i,j})\big) = h^*_{0,i,j}\big(\iota_{p_{0,i,j}}(s_{0,i,j})\big) = h^*_{1,i,j}\big(\iota_{p_{1,i,j}}(s_{1,i,j})\big) = \iota_{p_{i,j}}\big(h^*_{1,i,j}(s_{1,i,j})\big).$$

Since ι is a monomorphisms, one has

$$h^*_{0,i,j}(s_{0,i,j}) = h^*_{1,i,j}(s_{1,i,j}),$$

and hence

$$\sigma^*_{i,j}(t_{0,i}) = [(g_{0,i,j}, s_{0,i,j})] = [(g_{1,i,j}, s_{1,i,j})] = \sigma^*_{i,j}(t_{1,i}).$$

Thus,

$$(\sigma_i \circ \sigma_{i,j})^*(t_0) = \sigma^*_{i,j}(t_{0,i})^{\mathsf{a}} = \sigma^*_{i,j}(t_{1,i})^{\mathsf{a}} = (\sigma_i \circ \sigma_{i,j})^*(t_1).$$

Since $\{\sigma_i \circ \sigma_{i,j} : q_{i,j} \to q \mid i \in I, j \in J_i\}$ is a τ-covering family in Prop/T, and $f_\#(\mathscr{X})$ is a τ-sheaf, one has $t_0 = t_1$. Therefore, $f_\#$ preserves monomorphisms. ☐

LEMMA 4.2.12. Let $f : S \to T$ be a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor $f_\# : \mathscr{C}_\tau(S) \to \mathscr{C}_\tau(T)$ restricts to an exact functor of Waldhausen categories

$$f^\omega_\# : \mathscr{C}^\omega_\tau(S) \to \mathscr{C}^\omega_\tau(T). \tag{79}$$

PROOF. The proof is essentially the same as of the proof of Lemma 4.2.7, utilising Lemma 4.2.11 and (78) instead of Lemma 4.2.6 and (73). ☐

When no confusion arises, we abuse notation, and refer to $f^\omega_\#$ by $f_\#$.

COROLLARY 4.2.13. Assume that $f : S \to T$ is a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor $f_\# : \mathscr{C}^\omega_\tau(S) \to \mathscr{C}^\omega_\tau(T)$ induces a morphism of spectra

$$f_\# : \mathtt{K}\left(\mathscr{C}^\omega_\tau(S)\right) \to \mathtt{K}\left(\mathscr{C}^\omega_\tau(T)\right),$$

that sends a point in the component $[\underline{\mathtt{h}}^\tau_p]$ to a point in the component $[\underline{\mathtt{h}}^\tau_{f_\circ(p)}]$, for every proper S-scheme p.

Also, we may abuse notation and refer to $\pi_0(f_\#)$ by $f_\#$, if no confusion arises.

EXAMPLE 4.2.14. Assume that $f : S \to T$ is a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$, let x be an S-scheme, and let $i : z \hookrightarrow\!\!\!\to p$ be a compactification of x. Since $f_\#$ is exact and $f_\circ$ preserves complementary open immersions,

$$f_\#(\mathtt{h}^{\mathrm{c}_\tau}_x) \cong \mathtt{coker}\left(f_\#(\underline{\mathtt{h}}^\tau_z) \rightarrowtail f_\#(\underline{\mathtt{h}}^\tau_p)\right) \cong \mathtt{coker}\left(\underline{\mathtt{h}}^\tau_{f_\circ(z)} \rightarrowtail \underline{\mathtt{h}}^\tau_{f_\circ(p)}\right) \cong \mathtt{h}^{\mathrm{c}_\tau}_{f\circ x}.$$

Thus, in particular,

$$f_\#\big([\mathtt{h}^{\mathrm{c}_\tau}_x]\big) = [f_\#(\mathtt{h}^{\mathrm{c}_\tau}_x)] = [\mathtt{h}^{\mathrm{c}_\tau}_{f\circ x}]. \tag{80}$$

Suppose that $f : S \to T$ and $g : T \to U$ are proper morphisms in $\mathsf{Noe}^{\mathrm{fd}}$. Then, there exist canonical natural isomorphisms

$$(g \circ f)_{\#} \overset{\sim}{\Rightarrow} g_{\#} \circ f_{\#} \qquad \text{and} \qquad (\mathrm{id}_S)_{\#} \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathscr{C}^{\omega}_{\tau}(S)},$$

which satisfy the cocycle condition.

Corollary 4.2.15. The fibred Waldhausen category $\mathscr{C}^{\omega}_{\tau}$, given in Corollary 4.2.10, is in fact a proper-fibred Waldhausen category, *i.e.* there exists a pseudofunctor

$$\mathscr{C}^{\omega}_{\tau} : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{prop}} \to \mathsf{Wald}_2,$$

where Wald_2 is the 2-category of essentially small Waldhausen categories, exact functors between them, and natural transformations between the latter, which sends a proper morphism $f : S \to T$ in $\mathsf{Noe}^{\mathrm{fd}}$ to the exact functor $f^{\omega}_{\#}$, as in Lemma 4.2.12. Then, the Waldhausen's $\mathtt{K}$-theory 2-functor induces a pseudofunctor

$$\mathtt{K}(\mathscr{C}^{\omega}_{\tau}) : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{prop}} \to \mathsf{Spt}_2.$$

In contract to the inverse image, the proper direct image is not necessarily strong monoidal. That is, for a proper morphism $f : S \to T$ in $\mathsf{Noe}^{\mathrm{fd}}$, one has

$$f_{\#}(\mathbb{1}_S) = f_{\#}(\underline{\mathbf{h}}^{\tau}_{\mathrm{id}_S}) \cong \underline{\mathbf{h}}^{\tau}_{f_{\circ}(\mathrm{id}_S)} \cong \underline{\mathbf{h}}^{\tau}_{f},$$

which is not necessarily isomorphic to $\mathbb{1}_T$ for a proper morphism f. However, since $f_{\#}$ is a left adjoint to the strong monoidal functor f_*, it is oplax monoidal, see [**CD13**, §.1.1.24].

Example 4.2.16. Suppose that p is a proper S-scheme. Then,

$$\left[\underline{\mathbf{h}}^{\tau}_{p}\right] = \left[(p_{\#} \circ p^*)(\mathbb{1}_S)\right] \in \mathtt{K}_0\left(\mathscr{C}^{\omega}_{\tau}(S)\right).$$

Open Direct Image. For a morphism $f : S \to T$ in $\mathsf{Noe}^{\mathrm{fd}}$, the right adjoint direct image functor

$$f_* : \mathscr{C}_{\tau}(S) \to \mathscr{C}_{\tau}(T)$$

is not necessarily an exact functor of Waldhausen categories, as it may not commute with pushout squares. However, when f is an open immersions, the functor f_* is a strong monoidal exact functor, as seen in Corollary 4.2.18.

For an open immersion $j : S \hookrightarrow T$ in $\mathsf{Noe}^{\mathrm{fd}}$, we first show that the functor j^{-1} is almost cocontinuous, as in Definition A.4.17, then we apply Lemma A.4.18 to deduce that j_* is a strong monoidal exact functor.

Lemma 4.2.17. Let $j : S \hookrightarrow T$ be an open immersion in $\mathsf{Noe}^{\mathrm{fd}}$. When τ is the *cdp*-pretopology or the proper pretopology, the functor j^{-1} is continuous and almost cocontinuous with respect to the τ-pretopology.

Proof. The functor j^{-1} is continuous with respect to the τ-pretopology for preserving τ-covering families, see [**SGA73**, §.III.Prop.1.6].

Assume that $q : Q \to T$ is a proper T-scheme, let $q' : Q' \to S$ be a base change of q along j, and let $\mathscr{U} = \{\sigma_i : p_i \to q'\}$ be a τ-covering family in Prop/S. Recall that schemes of finite type over Noetherian schemes are Noetherian, by [**Sta17**, Tag 01T6]. Since Q' is Noetherian, the open immersion $j' \coloneqq q^{-1}(j)$ is quasi-compact, and hence of finite type, see [**Sta17**, Tags 01P0, 01TU, and 01TW]. Thus, for every $i \in I$, the category of compactifications $\mathsf{Comp}_Q(j' \circ \sigma_i)$ is nonempty, by Nagata's Compactification Theorem. Hence, there exists a proper Q-scheme $z_i : Z_i \to Q$ which admits an open immersion $j_i : j' \circ \sigma_i \hookrightarrow z_i$ in $\mathsf{Sch}^{\mathrm{ft}}/Q$. Consider the commutative diagram

$$\begin{tikzcd}
P_i \arrow[dr, "k_i"] \arrow[drr, bend left, "\sigma_i"] \arrow[ddr, bend right, hook, "j_i \circ"'] & & & \\
& Z_i \times_Q Q' \arrow[r, "\underline{z}_i"'] \arrow[d, hook, "\circ\, \underline{j}_i"] & Q' \arrow[r, "q'"] \arrow[d, hook, "\circ\, j'"] & S \arrow[d, hook, "\circ\, j"] \\
& Z_i \arrow[r, "z_i"'] & Q \arrow[r, "q"'] & T
\end{tikzcd}$$

of Noetherian schemes in $\mathsf{Noe}^{\mathrm{fd}}$, where P_i is the underlying scheme of p_i, and k_i is the unique morphism $P_i \to Z_i \times_Q Q'$ of schemes, induced by the universal property of fibre products, that makes the diagram commute.

Since $\underline{j}_i$ is an open immersion, so is k_i. Also, k_i is proper, as σ_i is proper. Then, k_i is a closed open immersion, by [**Gro67**, Cor.18.12.6]. Let $l_i : W_i \hookrightarrow Z_i$ be the scheme-theoretic image of the immersion j_i, and let $j_i' : P_i \to W_i$ be the unique morphism of Q-schemes for which $\underline{j}_i \circ k_i = l_i \circ j_i'$. Then, j_i' is an open immersion, and the square

$$\begin{tikzcd}
P_i \arrow[r, hook, "k_i"] \arrow[d, hook, "j_i' \circ"'] & Z_i \times_Q Q' \arrow[d, hook, "\circ\, \underline{j}_i"] \\
W_i \arrow[r, hook, "l_i"'] & Z_i
\end{tikzcd}$$

is Cartesian, by Lemma 4.1.3.

Let $z : Z \hookrightarrow Q$ be a closed immersion complementary to $j' : Q' \hookrightarrow Q$, one may choose Z to have the reduced induced structure, but such a choice does not affect the argument. Then, we will see that the set of proper morphisms

$$\mathscr{V} \coloneqq \{z_i \circ l_i : W_i \to Q \mid i \in I\} \bigsqcup \{z : Z \hookrightarrow Q\}$$

is a τ-covering family for Q.

- When τ is the proper pretopology, it is evident that $\mathscr{V}$ is a proper covering family.
- On the other hand, when τ is the *cdp*-pretopology, for every field k, every morphism $x : \operatorname{Spec} k \to Q$ in $\mathsf{Noe}^{\mathrm{fd}}$ lifts either through z or j'. When x lifts through j' to a morphism $x' : \operatorname{Spec} k \to Q'$, since $\mathscr{U}$ is a *cdp*-covering, there exists $i_x \in I$, such that x' lifts to through σ_{i_x}. The k-point x lifts through $z_{i_x} \circ l_{i_x}$. Thus, for every field k, every k-point in Q lifts through a morphism in $\mathscr{V}$, and hence $\mathscr{V}$ is a *cdp*-covering family.

For every $i \in I$, the morphism $j^{-1}(z_i \circ l_i)$ is isomorphic to p_i, and hence factorises through it. On the other hand, the empty sieve is a τ-covering sieve for the empty S-scheme $j^{-1}(z) \cong \varnothing_{Q'}$. Therefore, the functor j^{-1} is almost cocontinuous, as in Definition.A.4.17. □

The proof above shows, in particular, that the functor $j^{-1} : \mathsf{Prop}/T \to \mathsf{Prop}/S$ is essentially surjective, when $j : S \hookrightarrow T$ is an open immersion in $\mathsf{Noe}^{\mathrm{fd}}$.

Corollary 4.2.18. Let $j : S \hookrightarrow T$ be an open immersion in $\mathsf{Noe}^{\mathrm{fd}}$. When τ is the *cdp*-pretopology or the proper pretopology, the direct image functor $j_* : \mathscr{C}_\tau(S) \to \mathscr{C}_\tau(T)$ is a strong monoidal exact functor, *cf.* [**GK15**, Prop.4.5].

Proof. A direct result of Lemma 4.2.17 and Lemma A.4.18. □

Remark 4.2.19. Since j_* commutes with pushout squares, by Lemma A.4.18, it suffices to show that it restricts to a functor $j_*^0 : \mathscr{C}_\tau^0(S) \to \mathscr{C}_\tau^\omega(T)$, in order to induce a strong monoidal exact functor $j_*^\omega : \mathscr{C}_\tau^\omega(S) \to \mathscr{C}_\tau^\omega(T)$. However, it is not clear to us that it induces the functor j_*^0. Also, we intended to use the functor j_* to extend the proper direct image to a properly supported direct image for all separated morphisms of finite type between Noetherian schemes of finite Krull dimension, but it does not seem to provide an extension independent from the choice of the compactification. We do not pursue such extension here, and we leave it for a further work.

Proper Base Change. The inverse image and proper direct image functors satisfy the proper-base change property, as in [**CD13**, §.1.1.9].

Lemma 4.2.20. Assume that f is a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$, and let

$$\begin{tikzcd} S' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d, "f"] \\ T' \arrow[r, "g"'] & T \end{tikzcd} \tag{81}$$

be a Cartesian square in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the exchange natural transformation

$$\Theta : f'_{\#} \circ g'^{*} \Rightarrow g^{*} \circ f_{\#} : \mathscr{C}^{\omega}_{\tau}(S) \to \mathscr{C}^{\omega}_{\tau}(T'), \tag{82}$$

induced by the adjunctions $f'_{\#} \dashv f'^{*}$ and $f_{\#} \dashv f^{*}$ is an isomorphism, which induces a canonical homotopy

$$\Theta : f'_{\#} \circ g'^{*} \Rightarrow g^{*} \circ f_{\#} : \mathtt{K}\left(\mathscr{C}^{\omega}_{\tau}(S)\right) \to \mathtt{K}\left(\mathscr{C}^{\omega}_{\tau}(T')\right).$$

Proof. The functor $f'_{\#} \circ g'^{*}$ (resp. $g^{*} \circ f_{\#}$) is given by the τ-sheafification of a left Kan extension along $(f'_{\circ} \circ g'^{-1})^{\mathrm{op}}$ (resp. $(g^{-1} \circ f_{\circ})^{\mathrm{op}}$), and the natural transformation Θ is induced from the canonical natural transformation $f'_{\circ} \circ g'^{-1} \Rightarrow g^{-1} \circ f_{\circ} : \mathsf{Prop}/S \to \mathsf{Prop}/T'$, by the universal property of Kan extensions.

For a proper S-scheme $p : P \to S$, considering the Cartesian diagram

$$\begin{tikzcd}
P' \arrow[r, "g'^{-1}(p)"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & T' \arrow[d, "g"] \\
P \arrow[r, "p"'] & S \arrow[r, "f"'] & T,
\end{tikzcd}$$

in $\mathsf{Noe}^{\mathrm{fd}}$, one sees that the morphism $f'_{\circ}\big(g'^{-1}(p)\big) \to g^{-1}\big(f_{\circ}(p)\big)$ is an isomorphism in Prop/T', and hence the induced natural transformation

$$\Theta : f'_{\#} \circ g'^{*} \Rightarrow g^{*} \circ f_{\#}$$

is a natural isomorphism, *cf.* [**CD13**, Ex.1.1.11]. □

Corollary 4.2.21. Let $i : S \hookrightarrow T$ be a closed immersion in $\mathsf{Noe}^{\mathrm{fd}}$. Then, the functor $i_{\#} : \mathscr{C}^{\omega}_{\tau}(S) \to \mathscr{C}^{\omega}_{\tau}(T)$ is fully faithful.

Proof. Corollary 4.2.10, Lemma 4.2.12, and Lemma 4.2.20 imply that $\mathscr{C}^{\omega}_{\tau}$ is a proper-fibred category over $\mathsf{Noe}^{\mathrm{fd}}$, see [**CD13**, §.1]. Then, the statement of the corollary follows from [**CD13**, Cor.1.1.20]. □

Example 4.2.22. Assume that $i : V \hookrightarrow S$ is a closed immersion in $\mathsf{Noe}^{\mathrm{fd}}$ with complementary open immersion $j : U \hookrightarrow S$. Then, one has the Cartesian squares

$$\begin{tikzcd}
\varnothing \arrow[r, hook, "\varnothing_V"] \arrow[d, hook, "\varnothing_U"'] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, hook, "i"] \\
U \arrow[r, hook, "j"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, hook, "\mathrm{id}_V"] \arrow[d, hook, "\mathrm{id}_V"'] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, hook, "i"] \\
V \arrow[r, hook, "i"'] & S
\end{tikzcd}$$

in $\mathsf{Noe}^{\mathrm{fd}}$. Since $\mathscr{C}_{\tau}(\varnothing)$ is isomorphic to the terminal Waldhausen category with one object and one morphism, one has a natural isomorphism

$$(j^{*} \circ i_{\#})(\mathscr{X}) \cong \underline{\mathbf{h}}^{\tau}_{\varnothing_U} = \mathbf{0}$$

for every $\mathscr{X} \in \mathscr{C}_\tau^\omega(V)$ *i.e.* $j^* \circ i_\#$ is naturally isomorphic to the zero functor $\mathscr{C}_\tau^\omega(V) \to \mathscr{C}_\tau^\omega(U)$. Also, the adjoint unit $\mathrm{id}_{\mathscr{C}_\tau^\omega(V)} \Rightarrow i^* \circ i_\#$ is a natural isomorphism, and hence i^* is essentially surjective.

Proper Projection Formula. The inverse image and proper direct image functors also satisfy the proper-projection formula property, as in [**CD13**, §.1.1.26].

Lemma 4.2.23. Assume that $f : S \to T$ is a proper morphism in $\mathsf{Noe}^{\mathrm{fd}}$. Then, f satisfies the projection formula, *i.e.* for every pointed τ-sheaves $\mathscr{X}$ in $\mathscr{C}_\tau^\omega(S)$ and $\mathscr{Y}$ in $\mathscr{C}_\tau^\omega(T)$, the projection natural transformation

$$f_\#\big(\mathscr{X} \wedge_S f^*(\mathscr{Y})\big) \Rightarrow f_\#(\mathscr{X}) \wedge_T \mathscr{Y}, \tag{83}$$

induced by the adjunction $f_\# \dashv f^*$, is an isomorphism in $\mathscr{C}_\tau^\omega(T)$, *cf.* [**CD13**, Ex.1.1.28] .Hence, there exists a canonical path

$$f_\#\big(\underline{x} \cdot f^*(\underline{y})\big) \to f_\#(\underline{x}) \cdot \underline{y}$$

in $\mathtt{K}\big(\mathscr{C}_\tau^\omega(T)\big)$, for every $\underline{x} \in \mathtt{K}\big(\mathscr{C}_\tau^\omega(S)\big)$ and $\underline{y} \in \mathtt{K}\big(\mathscr{C}_\tau^\omega(T)\big)$. In particular, for an S-scheme x and a T-scheme y, one has

$$f_\#\big([\mathtt{h}_x^{\mathrm{c}_\tau}] \cdot f^*([\mathtt{h}_y^{\mathrm{c}_\tau}])\big) = f_\#([\mathtt{h}_x^{\mathrm{c}_\tau}]) \cdot [\mathtt{h}_y^{\mathrm{c}_\tau}].$$

Proof. Suppose that $p : P \to S$ (resp. $q : Q \to T$) is a proper S-scheme (resp. T-scheme). Similar to the proof of Lemma 4.2.20, considering the Cartesian diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q' \arrow[r] \arrow[d, "f^{-1}(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "q"] \\ P \arrow[r, "p"'] & S \arrow[r, "f"'] & T, \end{tikzcd}$$

in $\mathsf{Noe}^{\mathrm{fd}}$, one sees that the projection morphism $f_\circ\big(p \times_{\mathrm{id}_S} f^{-1}(q)\big) \to f_\circ(p) \times_{\mathrm{id}_T} q$, induced by the adjunction $f_\circ \dashv f^{-1}$, is an isomorphism in Prop/T, and hence the projection natural transformation

$$f_\#\big(\underline{\mathtt{h}}_p^\tau \wedge_S f^*(\underline{\mathtt{h}}_q^\tau)\big) \Rightarrow f_\#(\underline{\mathtt{h}}_p^\tau) \wedge_T \underline{\mathtt{h}}_q^\tau$$

is an isomorphism in $\mathscr{C}_\tau^\omega(T)$. Then, the statement of the proposition follows from the construction of the symmetric monoidal Waldhausen categories $\mathscr{C}_\tau^\omega(S)$ and $\mathscr{C}_\tau^\omega(T)$, the symmetric product $\wedge_S$ and $\wedge_T$ being biexact, and the functors $f_\#$ and f^* being exact. $\square$

4.2.1.3. *Counting Points.* The motivic measure of counting rational points over a finite field is a shadow of a point on the *cdp*-site of proper schemes over the ground field, as seen in Corollary 4.2.26.

Fix a finite field $\mathbb{F}_q$ with q elements. Recall that a point on the *cdp*-site $(\mathsf{Prop}/\mathbb{F}_q, \mathrm{cdp})$ is an adjunction

$$u^* : \mathsf{Shv}_{\mathrm{cdp}}(\mathsf{Prop}/\mathbb{F}_q) \rightleftarrows \mathsf{Set} : u_*,$$

in which u^* is left exact, see §.A.4.2.1. Since both u_* and u^* are left exact, they both preserve final objects; and hence they induce an adjunction

$$u^*_\bullet : \mathscr{C}(\mathbb{F}_q) \rightleftarrows \mathsf{Set}_\bullet : u_{*,\bullet},$$

for having $\mathscr{C}(\mathbb{F}_q) \cong * \downarrow \mathsf{Shv}_{\mathrm{cdp}}(\mathsf{Prop}/\mathbb{F}_q)$ and $\mathsf{Set}_\bullet \cong * \downarrow \mathsf{Set}$. Moreover, $u^*_\bullet$ is also left exact; thus, it is a strong monoidal exact functor.

Recall that if a functor $u : \mathsf{Prop}/\mathbb{F}_q \to \mathsf{Set}$ is flat and continuous with respect to the *cdp*-pretopology, it defines a point (u^*, u_*) in the *cdp*-site, where

$$u^* = - \otimes_{\mathsf{Prop}/\mathbb{F}_q} u \qquad \text{and} \qquad u_* = \mathtt{Hom}^{\mathsf{Prop}/\mathbb{F}_p}(u, -)$$

are the stalks and skyscraper functors associated to u, respectively, see §.A.4.2.1.

Lemma 4.2.24. The functor $\Gamma_\bullet : \mathscr{C}(\mathbb{F}_q) \to \mathsf{Set}_\bullet$, induced by the global section functor $\Gamma : \mathsf{Shv}_{\mathrm{cdp}}(\mathsf{Prop}/\mathbb{F}_q) \to \mathsf{Set}$ is a strong monoidal exact functor. Moreover, for every $\mathbb{F}_q$-scheme X, one has an isomorphism of pointed sets

$$\Gamma_\bullet(\mathsf{h}^{\mathrm{c}}_X) \cong X(\mathbb{F}_q)_+.$$

Proof. The corepresentable functor $u := \mathsf{h}^{\mathsf{Spec}\,\mathbb{F}_q} : \mathsf{Prop}/\mathbb{F}_q \to \mathsf{Set}$ is flat and continuous with respect to the *cdp*-pretopology, as seen below.

- Since $\mathsf{Prop}/\mathbb{F}_q$ is Cartesian, every corepresentable functor is flat, as its category of elements is cofiltered. In particular, u is flat.
- In the light of [**MLM92**, §VII.5.Lem.3], to show that u is continuous, it suffices to show that $\mathtt{Hom}^{\mathsf{Prop}/\mathbb{F}_p}(u, S)$ is a *cdp*-sheaf for every set $S \in \mathsf{Set}$ and that the sheafification morphism $\eta_{\mathscr{P}} : \mathscr{P} \to \mathscr{P}^{\mathrm{a}_{\mathrm{cdp}}}$ is mapped to a bijection by the functor $- \otimes_{\mathsf{Prop}/\mathbb{F}_q} u$, for every presheaf $\mathscr{P} \in \mathsf{PSh}(\mathsf{Prop}/\mathbb{F}_q)$.
  - For a *cdp*-square (91) in $\mathsf{Prop}/\mathbb{F}_q$, a rational point $x : \mathtt{Spec}\,\mathbb{F}_q \to X$ factorises uniquely though either A or Y, or both (in which case it factorises uniquely through B). Thus, the functor u maps every *cdp*-square in $\mathsf{Prop}/\mathbb{F}_q$ to a pushout square in Set. Then, the functor u^{op} maps *cdp*-squares to Cartesian squares in $\mathsf{Set}^{\mathrm{op}}$. Since limits commute with each other and representable functors preserve limits, the presheaf $\mathtt{Hom}^{\mathsf{Prop}/\mathbb{F}_p}(u, S)$ maps *cdp*-squares to pullback squares. Also, it maps the empty $\mathbb{F}_q$-scheme to a terminal set. Hence, $\mathtt{Hom}^{\mathsf{Prop}/\mathbb{F}_p}(u, S)$ is a *cdp*-sheaf, for every set $S \in \mathsf{Set}$, by [**Voe10a**, Lem.2.9] and [**Voe10b**, Th.2.2].
  - For every presheaf $\mathscr{P} \in \mathsf{PSh}(\mathsf{Prop}/\mathbb{F}_q)$, one has

$$\mathscr{P} \otimes_{\mathsf{Prop}/\mathbb{F}_q} u = \int^{P \in \mathsf{Prop}/\mathbb{F}_q} \mathsf{PSh}(\mathsf{Prop}/\mathbb{F}_q)(\mathsf{h}_P, \mathscr{P}) \times u(P) \cong \mathscr{P}(\mathtt{Spec}\,\mathbb{F}_q).$$

Since the *cdp*-pretopology is completely decomposable, every *cdp*-covering family for $\mathsf{Spec}\,\mathbb{F}_q$ splits. That is, every *cdp*-covering family $\mathscr{U} = \{\sigma_i : P_i \to \mathsf{Spec}\,\mathbb{F}_q \mid i \in I\}$ in Prop/S admits a refinement $\mathscr{V} = \{\delta_j : Q_j \to \mathsf{Spec}\,\mathbb{F}_q \mid j \in J\}$ such that $Q_{j_0} = \mathsf{Spec}\,\mathbb{F}_q$ and $\delta_{j_0} = \mathrm{id}_{\mathsf{Spec}\,\mathbb{F}_q}$ for some $j_0 \in J$. Therefore, $\eta_{\mathscr{P},\mathsf{Spec}\,\mathbb{F}_q} : \mathscr{P}(\mathsf{Spec}\,\mathbb{F}_q) \to \mathscr{P}^{\mathrm{a}_{\mathrm{cdp}}}(\mathsf{Spec}\,\mathbb{F}_q)$ is a bijection, and so is $\eta_{\mathscr{P}} \otimes_{\mathsf{Prop}/\mathbb{F}_q} u$.

Then, there exists a *cdp*-point

$$u^* : \mathsf{Shv}_{\mathrm{cdp}}(\mathsf{Prop}/\mathbb{F}_q) \rightleftarrows \mathsf{Set} : u_*,$$

with the stalks and skyscrapers functors

$$u^* = - \otimes_{\mathsf{Prop}/\mathbb{F}_q} u \qquad \text{and} \qquad u_* = \mathtt{Hom}^{\mathsf{Prop}/\mathbb{F}_p}(u, -).$$

In particular, for a *cdp*-sheaf $\mathscr{X}$ on $\mathsf{Prop}/\mathbb{F}_q$, one has

$$u^*(\mathscr{X}) \cong \mathscr{X}(\mathsf{Spec}\,\mathbb{F}_q) \cong \mathsf{Shv}_{\mathrm{cdp}}(\mathsf{Prop}/\mathbb{F}_q)(\mathtt{h}^{\mathrm{a}_{\mathrm{cdp}}}_{\mathsf{Spec}\,\mathbb{F}_q}, \mathscr{X}) \cong \mathsf{Shv}_{\mathrm{cdp}}(*, \mathscr{X}) \cong \Gamma(\mathscr{X}).$$

Therefore, the induced functor

$$\Gamma_\bullet : \mathscr{C}(\mathbb{F}_q) \to \mathsf{Set}_\bullet$$

is a strong monoidal exact functor, for being a left exact left adjoint functor.

For an $\mathbb{F}_q$-scheme X, the category of compactifications $\mathsf{Comp}_{\mathbb{F}_q}(X)$ is nonempty; let $i : Z \hookrightarrow\!\!\!\to P$ be a compactification of X in $\mathsf{Sch}^{\mathrm{ft}}/\mathbb{F}_q$. Since every *cdp*-covering family for $\mathsf{Spec}\,\mathbb{F}_q$ splits, one has

$$\begin{aligned}\Gamma_\bullet(\mathtt{h}^{\mathrm{c}}_X) &\cong \mathtt{coker}\left(\Gamma_\bullet(\underline{\mathtt{h}}_Z) \rightarrowtail \Gamma_\bullet(\underline{\mathtt{h}}_P)\right) \cong \mathtt{coker}\left(\mathtt{h}^{\mathrm{a}_{\mathrm{cdp}}}_{Z,+}(\mathbb{F}_q) \rightarrowtail \mathtt{h}^{\mathrm{a}_{\mathrm{cdp}}}_{P,+}(\mathbb{F}_q)_+\right)\\ &\cong \mathtt{coker}\left(\mathtt{h}_{Z,+}(\mathbb{F}_q) \rightarrowtail \mathtt{h}_{P,+}(\mathbb{F}_q)_+\right) \cong \mathtt{coker}\left(Z(\mathbb{F}_q)_+ \rightarrowtail P(\mathbb{F}_q)_+\right) \cong X(\mathbb{F}_q)_+.\end{aligned} \tag{84}$$

□

Lemma 4.2.25. The strong monoidal exact functor $\Gamma_\bullet : \mathscr{C}(\mathbb{F}_q) \to \mathsf{Set}_\bullet$ restricts to a strong monoidal exact functor

$$\Gamma^\omega_\bullet : \mathscr{C}^\omega(\mathbb{F}_q) \to \mathsf{FSet}_\bullet.$$

Proof. Since $\mathscr{C}^\omega(\mathbb{F}_q)$ is a full symmetric monoidal Waldhausen subcategory in $\mathscr{C}(\mathbb{F}_q)$, the statement of the lemma follows from the existence of the restriction $\Gamma^\omega_\bullet$.

Since $\Gamma_\bullet$ commutes with pushout squares, it suffices to show that $\Gamma_\bullet$ restricts to a functor $\Gamma^0_\bullet : \mathscr{C}^0(\mathbb{F}_q) \to \mathsf{FSet}_\bullet$, in order to induce a strong monoidal exact functor $\Gamma^\omega_\bullet : \mathscr{C}^\omega(\mathbb{F}_q) \to \mathsf{FSet}_\bullet$, which is a result of (84), see §.3.2. □

Corollary 4.2.26. The functor $\Gamma^\omega_\bullet$ induces a morphism of commutative ring spectra

$$\Gamma_\bullet : \mathtt{K}\left(\mathscr{C}^\omega(\mathbb{F}_q)\right) \to \mathbb{S},$$

that sends a point in the component $[\mathsf{h}^{c}_{X}]$ to a point in the component $[X(\mathbb{F}_q)_+]$, for every $\mathbb{F}_q$-scheme X. Hence, it factorises the classical motivic measure of counting points over $\mathbb{F}_q$ through the motivic measure (66) as

$$\mu_{\#} = \pi_0(\Gamma_\bullet) \circ \mu_{\mathrm{cdp}}. \tag{85}$$

The homomorphism $\pi_n(\Gamma_\bullet)$, for $n \geq 1$, might be thought of as higher point counting measures, which we investigate in a future work.

REMARK 4.2.27. This argument applies for every scheme S in $\overset{\mathrm{fd}}{\mathsf{Noe}}$ and for every τ-point (u^*, u_*) on Prop/S, whenever the stalks functor $u^*_\bullet$ restricts to a functor $\mathscr{C}^{\omega}_{\tau}(S) \to \mathsf{FSet}_\bullet$.

COROLLARY 4.2.28 ([**Cam17**, Prop.5.21]). The composition $\Gamma^{\omega}_\bullet \circ \upsilon_{\mathbb{F}_q} : \mathsf{FSet}_\bullet \to \mathsf{FSet}_\bullet$ is an exact equivalence of Waldhausen categories, where $\upsilon_{\mathbb{F}_q}$ is the exact functor in (71). Thus, the map of spectra $\mathtt{K}(\Gamma^{\omega}_\bullet \circ \upsilon_{\mathbb{F}_q}) : \mathbb{S} \to \mathbb{S}$ is a homotopy equivalence, and hence the spectrum $\mathtt{K}\big(\mathscr{C}^{\omega}(\mathbb{F}_q)\big)$ splits through $\mathbb{S}$. That is,

$$\mathtt{K}\big(\mathscr{C}^{\omega}(\mathbb{F}_q)\big) \cong \mathbb{S} \vee \tilde{\mathtt{K}}\big(\mathscr{C}^{\omega}(\mathbb{F}_q)\big),$$

where $\tilde{\mathtt{K}}\big(\mathscr{C}^{\omega}(\mathbb{F}_q)\big)$ is a cofibre of $\mathtt{K}(\upsilon_{\mathbb{F}_q})$.

4.2.2. Motivic Spaces with Proper Support.

Recall that some of the statements in motivic homotopy theory are sensitive to the considered category of schemes and to the topology, like the Gluing Theorem 2.3.1 and the Purity Theorem 2.3.3. Then, to have an analogous of the compactified Yoneda embedding for a motivic category, it is convenient to start with a topology that is both

- as fine as the Nisnevich topology, for the such statements to hold; and
- as fine as the *cdp*-topology, for the Yoneda embedding to define a *cdp*-functor.

The coarsest such topology is the *cdh*-topology, see §.A.4.3. Since $\mathbb{A}^1$-localisation is left Bousfield localisation, it preserves colimits, and hence repeating the argument in §.4.2.1 produces a motivic measure

$$\mathtt{K}_0(\mathsf{h}^{c}_{\mathrm{cdh}}) : \mathtt{K}_0(\overset{\mathrm{ft}}{\mathsf{Sch}}/S) \to \mathtt{K}_0\left(\mathsf{Shv}^{c}_{\bullet,\mathrm{cdh}}(\overset{\mathrm{ft}}{\mathsf{Sch}}/S)_{\mathbb{A}^1}\right), \tag{86}$$

which sends the class of a proper S-scheme x to the class of $\mathsf{h}_{x,+}$, where $\mathsf{Shv}_{\bullet,\mathrm{cdh}}(\overset{\mathrm{ft}}{\mathsf{Sch}}/S)_{\mathbb{A}^1}$ is the $\mathbb{A}^1$-localised model category of pointed simplicial cdh-sheaves over $\overset{\mathrm{ft}}{\mathsf{Sch}}/S$.

For a field k of characteristic zero, the localisation functor

$$\mathsf{Spt}^{\Sigma}_{S^1}\mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/S)_{\mathbb{A}^1,\mathrm{stab}} \to \mathsf{Spt}^{\Sigma}_{S^1}\mathsf{sShv}_{\bullet,\mathrm{cdh}}(\overset{\mathrm{ft}}{\mathsf{Sch}}/S)_{\mathbb{A}^1,\mathtt{inj}}$$

is Quillen equivalence, where $\mathsf{Spt}^{\Sigma}_{S^1}\mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/S)_{\mathbb{A}^1}$ is the $\mathbb{A}^1$-localised stable model category of S^1-symmetric spectra of pointed simplicial Nisnevich sheaves over Sm/k. That is due to k admitting resolutions of singularities, see [**Voe10b**] and [**MV99**, §.3.Rem.2.30]. Then, the motivic measure $\mathtt{K}_0(\mathsf{h}^{c}_{\mathrm{cdh}})$ factorises through the ring

$$\mathtt{K}_0\left(\mathsf{Spt}^{\Sigma}_{S^1}\mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/S)^{c}_{\mathbb{A}^1,\mathrm{stab}}\right).$$

The resulting motivic measure coincides with the motivic measure defined in [**Rön16**, §.5].

4.3. Motivic Measures through Stable Motivic Spaces

Our main motivation to consider functors compactifications is our attempt to factorise the motivic measure of counting points over a finite field through the $\mathtt{K}$-theory of the unstable motivic homotopy category. We only obtain a partial result in this direction, and in this section we introduce our candidate for counting points on motivic spaces, leaving its detailed development to a future work.

For a subcanonical topology τ on the category $\mathsf{Sm}/\mathbb{F}_q$, let $\underline{\Delta}^{\bullet}_{I,s,+} : \Delta \to \mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/\mathbb{F}_q)$ be the functor given by

$$\underline{\Delta}^{n}_{I,s,+} = \Delta^{n}_{\mathbb{A}^n_{\mathbb{F}_q}\times \mathsf{Spec}\,\mathbb{F}_{q^s},+},$$

and consider the diagram

$$\begin{tikzcd} \Delta \arrow[r, "\underline{\Delta}^{\bullet}_{I,s,+}"] \arrow[d, "\mathtt{h}_{-,+}"'] & \mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/\mathbb{F}_q). \\ \mathsf{sSet}_{\bullet} & \end{tikzcd}$$

Then, there exists a $\underline{\Delta}^{\bullet}_{I,s,+}$-tensor and $\underline{\mathtt{Hom}}$ adjunction, see Example A.3.8. For a pointed simplicial sheaf $\mathscr{X}$ one has

$$\begin{aligned}\underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+},\mathscr{X})_n &= \mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/\mathbb{F}_q)\big(\Delta^{n}_{\mathbb{A}^n_{\mathbb{F}_q}\times \mathsf{Spec}\,\mathbb{F}_{q^s},+},\mathscr{X}\big)\\ &\cong \mathsf{Shv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/\mathbb{F}_q)\big(\mathtt{h}_{\mathbb{A}^n_{\mathbb{F}_q},+}\wedge \mathtt{h}_{\mathsf{Spec}\,\mathbb{F}_{q^s},+},\mathscr{X}_n\big).\end{aligned}$$

Whereas, for a pointed simplicial set S,

$$(S\wedge_{\Delta}\underline{\Delta}^{\bullet}_{I,s,+})_n = S_n\wedge \mathtt{h}_{\mathbb{A}^n_{\mathbb{F}_q}}\wedge \mathtt{h}_{\mathsf{Spec}\,\mathbb{F}_{q^s}}.$$

Then, in particular, $\underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+},-)$ is monoidal. The tensor functor $\wedge_{\Delta}\underline{\Delta}^{\bullet}_{I,s,+}$ preserves monomorphisms. Also, it maps weak equivalences of simplicial sets to $\mathbb{A}^1$-weak equivalences of simplicial sheaves, and hence one has a Quillen adjunction

$$-\wedge_{\Delta}\underline{\Delta}^{\bullet}_{I,s,+} : \mathsf{sSet}_{\bullet,\mathrm{KQ}} \rightleftarrows \mathsf{sShv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/\mathbb{F}_q) : \underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+},-).$$

Example 4.3.1. For an $\mathbb{A}^1$-rigid scheme X over $\mathbb{F}_q$, *e.g.* an abelian $\mathbb{F}_q$-scheme, one has canonical isomorphisms

$$\begin{aligned}\underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+},\mathtt{h}_{X,+})_n &\cong \mathsf{Shv}_{\bullet,\mathrm{Nis}}(\mathsf{Sm}/\mathbb{F}_q)(\mathtt{h}_{\mathbb{A}^n_{\mathbb{F}_q,+}}\wedge \mathtt{h}_{\mathsf{Spec}\,\mathbb{F}_{q^s},+},\mathtt{h}_{X,+}) \cong X(\mathbb{A}^n_{\mathbb{F}_q}\times \mathsf{Spec}\,\mathbb{F}_{q^s})_+\\ &\cong X(\mathbb{F}_{q^s})_+.\end{aligned}$$

Hence, $\underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+},\mathtt{h}_{X,+})$ is a discrete pointed simplicial set with $\#X(\mathbb{F}_{q^s})$ elements, in addition to a disjoint base point.

The Quillen adjunction above induces derived functors

$$- \otimes^{\mathtt{L}}_{\Delta} \underline{\Delta}^{\bullet}_{I,s,+} : \mathcal{H}\mathsf{sSet}_{\bullet,\mathrm{KQ}} \rightleftarrows \mathcal{H}_{\bullet}(\mathbb{F}_q) : \mathbf{R}\,\underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+}, -).$$

Also, since both functors preserve cofibrations and weak equivalences, they define morphisms of Waldhausen structures associated with the model categories, and induce group homomorphisms

$$\mathtt{K}_0(- \wedge_{\Delta} \underline{\Delta}^{\bullet}_{I,s,+}) : \mathtt{K}_0(\mathsf{sSet}^{c}_{\bullet,\mathrm{KQ}}) \rightleftarrows \mathtt{K}_0\left(\mathsf{sShv}^{c}_{\mathrm{Nis},\bullet}(\mathsf{Sm}/\mathbb{F}_q)_{\mathbb{A}^1}\right) : \mathtt{K}_0\left(\underline{\mathtt{Hom}}^{\Delta}_{\bullet}(\underline{\Delta}^{\bullet}_{I,s,+}, -)\right),$$

where $\mathtt{K}_0(- \wedge_{\Delta} \underline{\Delta}^{\bullet}_{I,s,+})$ is given by the multiplication by $[\mathtt{h}_{\mathsf{Spec}\,\mathbb{F}_{q^s}}]$.

4.4. Further Research

There are two questions that arise naturally from the constitutions in §.4.2.

On the one hand, we would like to understand the relation between $\mathtt{K}^{\mathrm{uh}}_0(\mathsf{Sch}^{\mathrm{ft}}/S)$ and $\mathtt{K}_0\left(\mathscr{C}^{\omega}(S)\right)$ in (66). In particular, we are examining the validity of Conjecture 4.2.4. While we have partial results pointing in its direction, we do not have a complete proof, yet. Also, since important geometric questions are addressed through the (modified) Grothendieck ring of varieties, it is desirable to understand what geometric information the higher groups of the spectrum $\mathtt{K}\left(\mathscr{C}^{\omega}(S)\right)$ curry. Then, we would investigate which of the known motivic measures arises from exact Waldhausen functor out of $\mathscr{C}^{\omega}(S)$.

On the other hand, the Grothendieck ring of varieties has zero divisors annihilated by the class of the affine line, as in [**Bor15**]. Some of the argument to solve intriguing questions in algebraic geometry that where originally considered in the ring $\mathtt{K}_0(\mathsf{Var}/k)$ are obstructed by the class of the affine line being a zero divisor. The functor $\mathtt{h}^{\mathrm{c}}$ sends the affine line over a field k to the S^1-symmetric suspension of $(\mathbb{P}^1_k, \infty)$, which is inverted in the $\mathbb{P}^1_k$-stable homotopy theory of schemes. That can be used to transport some of the aforementioned arguments to the Grothendieck ring of the $\mathbb{P}^1_k$-stable homotopy theory of schemes. This idea, among others, is due to Vladimir Guletskiĭ.

In order to be able use the machinery available for stable motivic homotopy theory to consider such questions, we would like to investigate if the motivic measures defined in §.4.2.2 exists for the Nisnevich topology over smooth schemes. Similar question for motivic spaces were answered in [**Voe00**, §.4] using resolutions of singularities for a field of characteristic zero, and in [**Kel12**] using alterations of singularities for a perfect field, after inverting its exponential characteristic. We expect that applying De Jong alterations of singularities to [**Voe10b**]'s approach, instead of resolutions of singularities, may allow to establish the desired measures.

APPENDIX A

Categorical Recollections

Assuming the reader's familiarity with the basics of category theory, as in [**ML98**], we briefly recall the main categorical notions used in this thesis.

A.1. Foundations

There are several possible foundations for categories, which affect the resulting theory, see [**Shu08**]. The assumed foundations mainly impact the existence of desired constructions, like the $\mathtt{hom}$-bifunctor and functor categories, and hence all notions depending on them, including the Yoneda embedding and Kan extensions, see [**ML69**]. In this section, we fix the foundations adopted in this thesis.

One possible foundation is Zermelo-Fraenkel set theory with both the axiom of choice and the one universe axiom, as in [**ML69**]. Assume and fix a model for Zermelo-Fraenkel set theory with the axiom of choice (**ZFC**), see [**Jec03**]. Then, a *set* refers to an object of this model; and a *category* refers to a pair of sets - of objects and morphisms - with the source, target, composition, and identity maps, subject to the associativity and unit axioms, as in [**KS06**, Def.1.2.1].

The main advantage, in category theory, of adopting the axiom of choice is allowing universal properties to induce desired functors, as it is the case for the $\mathtt{lim}$ and $\mathtt{colim}$ functors. Otherwise, universal properties only produce anafunctors, see [**Mak96**]. The axiom of choice also implies the equivalence between a functor admitting a quasi-inverse and being fully faithful and essentially surjective. It also allows inducing total derived functors between derived categories.

Following Mac Lane's proposal in [**ML69**], we assume and fix an uncountable *Grothendieck universe* $\mathfrak{U}$, see [**SGA73**, Exposé I.§.0]. Elements of $\mathfrak{U}$ are called *small sets*, whereas subsets of $\mathfrak{U}$ are called *classes*.

The universe $\mathfrak{U}$ defines a model for **ZFC**, as ordinary operations on sets can be carried out internally on small sets. Therefore, one now has two models for **ZFC**, one of which - the Grothendieck universe - is an object of the other. One can consider set-theoretic mathematics internally in $\mathfrak{U}$, *i.e.* on small sets; whereas the remaining sets of the ambient model may be used to study and describe mathematics in $\mathfrak{U}$, see [**ML69**].

In particular, there exists a category Set of all small sets and all maps between them, with $\mathtt{Ob}(\mathsf{Set}) = \mathfrak{U}$.

DEFINITION A.1.1. A category $\mathscr{C}$ is said to be

- *locally small* if $\mathscr{C}(X, Y)$ is a small set, for every pair of objects $X, Y \in \mathtt{Ob}(\mathscr{C})$; otherwise, it is said to be a *big category*;
- *small* if it is a locally small category such that $\mathtt{Ob}(\mathscr{C})$ is a small set;
- *large* if it is a locally small category such that $\mathtt{Ob}(\mathscr{C})$ is a class; and
- *essentially small* if it is a large category that is equivalent to a small category.

Denote the large category of small categories and functors between them by Cat, and denote the category of large categories and functors between them by CAT.

The adopted foundation allows the construction of the functor category $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ for every pair of categories $\mathscr{C}$ and $\mathscr{D}$, which does not hold for all foundations, as explained in [**ML69**, p.193]. For locally small categories $\mathscr{C}$ and $\mathscr{D}$, the functor category $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ does not have to be locally small. However, when $\mathscr{C}$ is also essentially small, the category $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ is locally small.

A.2. Enriched Categories

The $\mathtt{hom}$-sets of some categories admit natural mathematical structures, usually arising from the structures on the objects. For instance, the $\mathtt{hom}$-set $\mathsf{Grp}(G, H)$ has a natural group structure, for every pair of small groups G and H. Also, in topology, if we restrict ourself to compactly generated Hausdorff spaces, then the $\mathtt{hom}$-set $\mathbf{CGHaus}(X, Y)$ admits a canonical topology for every $X, Y \in \mathbf{CGHaus}$, namely the compact-open topology, see [**ML98**, §.VII.8]. In these examples, the composition and the identity maps are compatible with the induced structure on the $\mathtt{hom}$-sets. This observation naturally gives rise to the notion of enriched categories.

DEFINITION A.2.1. Let $\mathscr{M} = (\otimes, \mathbb{1}, \alpha, l, r)$ be a monoidal structure on a category $\mathscr{M}_0$, see [**ML98**, §.VII.1]. An *$\mathscr{M}$-enriched category* $\mathscr{C}$ is given by

- a set $\mathtt{Ob}(\mathscr{C})$, called the set of objects of $\mathscr{C}$;
- for every $X, Y \in \mathtt{Ob}(\mathscr{C})$, an object $\mathtt{Map}_{\mathscr{C}}(X, Y) \in \mathscr{M}_0$, called the $\mathtt{hom}$-*object* from X to Y;
- for every $X, Y, Z \in \mathtt{Ob}(\mathscr{C})$, a morphism
$$\circ_{X,Y,Z} : \mathtt{Map}_{\mathscr{C}}(Y, Z) \otimes \mathtt{Map}_{\mathscr{C}}(X, Y) \to \mathtt{Map}_{\mathscr{C}}(X, Z)$$
in $\mathscr{M}_0$, called the *composition morphism*; and
- for every $X \in \mathtt{Ob}(\mathscr{C})$, a morphism $\mathrm{id}_X : \mathbb{1} \to \mathtt{Map}_{\mathscr{C}}(X, X)$, called the *identity morphism* of X;

which satisfy the associativity and unit axioms, as in [**Kel05**, §.1.2].

Remark A.2.2. When the monoidal category $(\mathscr{M}_0, \mathscr{M})$ admits a faithful strong monoidal functor $U : \mathscr{M}_0 \to \mathsf{Set}$, with respect to the Cartesian monoidal structure on Set, setting $\mathscr{C}(X,Y) \coloneqq U\big(\mathtt{Map}_{\mathscr{C}}(X,Y)\big)$, for every $X, Y \in \mathscr{C}$, defines a category, called the *underlying category of* $\mathscr{C}$ and denoted by $\mathscr{C}_0$. Such a category $\mathscr{C}_0$ is said to have an *$\mathscr{M}$-enriched structure*.

Example A.2.3. Set-enriched categories coincide with locally small categories.

Example A.2.4. A closed symmetric monoidal category is enriched over itself.

Definition A.2.5. Let $(\mathscr{M}_0, \mathscr{M})$ be a monoidal category. An *$\mathscr{M}$-enriched functor* $F : \mathscr{C} \to \mathscr{D}$ between $\mathscr{M}$-enriched categories $\mathscr{C}$ and $\mathscr{D}$ is given by

- a function $F : \mathtt{Ob}(\mathscr{C}) \to \mathtt{Ob}(\mathscr{D})$; and
- for every $X, Y \in \mathtt{Ob}(\mathscr{C})$, a morphism $F_{X,Y} : \mathtt{Map}_{\mathscr{C}}(X,Y) \to \mathtt{Map}_{\mathscr{D}}(F(X), F(Y))$ in $\mathscr{M}_0$ that commutes with the composition and the identity morphisms.

An *$\mathscr{M}$-enriched natural transformation* $\alpha : F \to G$ between $\mathscr{M}$-enriched functors $F, G : \mathscr{C} \to \mathscr{D}$ is given by a morphism $\alpha_X : \mathbb{1}_{\mathscr{D}} \to \mathtt{Map}_{\mathscr{D}}\big(F(X), G(X)\big)$ in $\mathscr{M}_0$ for every $X \in \mathtt{Ob}(\mathscr{C})$ subject to $\mathscr{M}$-naturality, as in [**Kel05**, §.1.2.(1.7)].

A.2.1. Strict 2-Categories. Recall that the category Cat of small categories is Cartesian monoidal. Locally small strict 2-categories and strict 2-functors between them may be defined to be Cat-enriched categories and Cat-enriched functors, respectively. Equivalently, a locally small strict 2-category can be defined by

(1) a set of objects, often called 0-morphisms;
(2) a small set of 1-morphisms for every pair of 0-morphisms, and 1-composition and 1-identity maps that satisfy the strict associativity and unit axioms; and
(3) a small set of 2-morphisms for every pair of 1-morphisms, and 2-composition and 2-identity maps that satisfy the interchange law, in addition to the strict associativity and unit axioms, as in [**Bor94a**, §.7.1].

Example A.2.6. The category Cat gives rise to the locally small strict 2-category Cat_2 whose objects are small categories, 1-morphisms are functors between them, and 2-morphisms are natural transformations between the latter.

More generally, strict 2-categories can be defined similar to locally small strict 2-categories allowing sets (not necessarily small sets) of 1-morphisms and 2-morphisms between pairs of 0-morphisms and 1-morphisms, respectively. Then, in particular, CAT-enriched categories are strict 2-categories, and the category CAT gives rise to the strict 2-category CAT_2.

A.2.1.1. 2-*Universal Morphisms in Strict* 2-*Categories.* Localisations of large categories are given by (initial) 2-universal morphisms in the strict 2-category CAT_2, as seen in §.1.1. To motivate the definition of (initial) 2-universal morphisms, we first recall the notion of (initial) universal morphisms in ordinary categories, and study its generalisation to strict 2-categories.

Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and let $d \in \mathscr{D}$. An (*initial*) *universal morphism* from d to F is defined to be the initial object of the comma category $d{\downarrow}F$, if it exists, see [**ML98**, §.III.1]. When it exists, it is a morphism $\eta_d : d \to F(c_d)$ for $c_d \in \mathscr{C}$, such that for every morphism $f : d \to F(c)$ with $c \in \mathscr{C}$, there exists a unique morphism $g_f : c_d \to c$ in $\mathscr{C}$ that makes the triangle below strictly commute

$$\begin{tikzcd} d \arrow[r, "\eta_d"] \arrow[dr, "f"'] & F(c_d) \arrow[d, dashed, "F(g_f)"] & \overset{F}{\leftarrow} & c_d \arrow[d, "g_f"', "\exists!"] \\ & F(c) & & c. \end{tikzcd}$$

In other words, it is a morphism $\eta_d : d \to F(c_d)$ that induces a bijection of sets

$$\eta_d^* : \mathscr{C}(c_d, c) \to \mathscr{D}(d, F(c))$$

for every $c \in \mathscr{C}$. The surjectivity of η_d^* is equivalent to the existence of the factorisation, whereas its injectivity is equivalent to the uniqueness of the factorisation, when it exists. The (initial) universal morphism η_d is unique up to isomorphisms, if it exists, and the factorisation of f above is unique, for a given choice of the universal morphism.

Definition A.2.7. Let $F : \mathscr{C} \to \mathscr{D}$ be a strict 2-functor between strict 2-categories, and let $d \in \mathscr{D}$. An (*initial*) 2-*universal* 1-*morphism* from d to F is a 1-morphism $\eta_d : d \to F(c_d)$ for an object $c_d \in \mathscr{C}$ that induces an equivalence of categories

$$\eta_d^* : \mathtt{Map}_{\mathscr{C}}(c_d, c) \to \mathtt{Map}_{\mathscr{D}}\big(d, F(c)\big)$$

for every $c \in \mathscr{C}$.

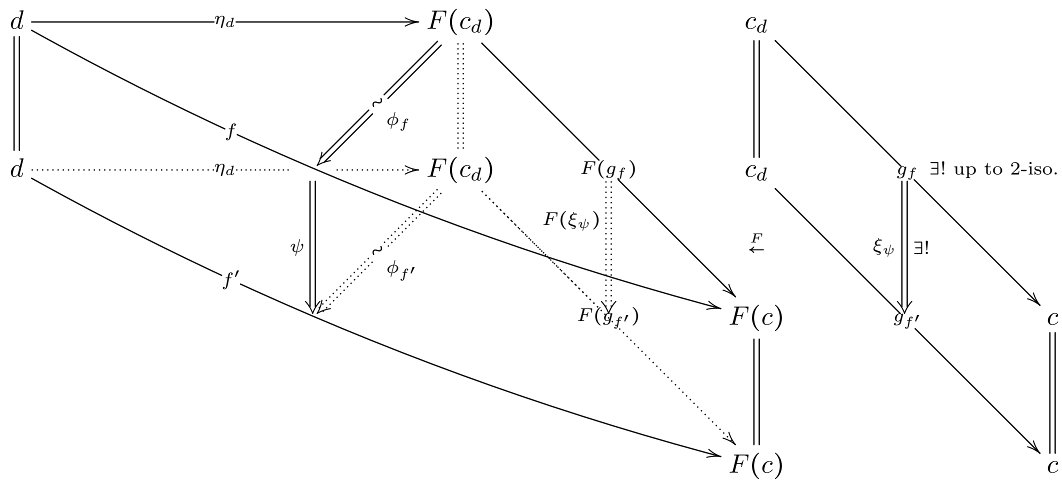
Remark A.2.8. Since we assume the axiom of choice, the 1-morphism η_d is a 2-universal 1-morphism from d to F if and only if the following two conditions hold

Esse. surj.) for every 1-morphism $f : d \to F(c)$ with $c \in \mathscr{C}$ there exists a 1-morphism $g_f : c_d \to c$ in $\mathscr{C}$ and a 2-isomorphism $\phi_f : F(g_f) \circ \eta_d \Rightarrow f$ in $\mathscr{D}$; and

Full. faith.) for every pair of parallel 1-morphisms $f, f' : d \to F(c)$ with $c \in \mathscr{C}$ and a 2-morphism $\psi : f \Rightarrow f'$ there exists a unique 2-morphism $\xi_\psi : g_f \Rightarrow g_{f'}$, for the choice of (g_f, ϕ_f) and $(g_{f'}, \phi_{f'})$, such that

$$\psi = \phi_{f'} \cdot F(\xi_\psi) \circ \eta_d \cdot \phi_f^{-1}.$$

That, in particular, implies the uniqueness of (g_f, ϕ_f) up to 2-isomorphisms.



The 2-universal 1-morphism η_d, when it exists, is unique up to 1-equivalences; and the factorisation (g_f, ϕ_f) of the 1-morphism f is unique up to 2-isomorphisms, for a given choice of the universal morphism η_d; whereas the factorisation ξ_ψ of the 2-morphism ψ is unique, for the choices of the factorisations (g_f, ϕ_f) and $(g_{f'}, \phi_{f'})$ of the 1-morphisms f and f', respectively.

A.2.1.2. *Lax Notions.* Let $F, G : \mathscr{C} \to \mathscr{D}$ be strict 2-functors between strict 2-categories. A *strict* 2*-natural transformation* $\alpha : F \to G$ is given by a 1-morphism $\alpha_X : F(X) \to G(X)$ for every $X \in \mathtt{Ob}(\mathscr{C})$ subject to 2-naturality, as in [**KS74**, §.1.4].

In 2-category theory, it so happens that one needs more relaxed notions of the strict 2-categories, 2-functors, and 2-natural transformations. One obtains a pseudo notion when replacing equalities (strict commutativity) with isomorphisms, and a lax notion when replacing it with mere morphisms. We will restrict ourself to reviewing the notions of pseudofunctors and lax 2-natural transformation, needed for §.4.2.1.2 and §.1.1. The interested reader my consult [**KS74**].

Definition A.2.9. Let $\mathscr{C}$ and $\mathscr{D}$ be strict 2-categories. A *pseudofunctors* $F : \mathscr{C} \to \mathscr{D}$ is given by

- a function $F : \mathtt{Ob}(\mathscr{C}) \to \mathtt{Ob}(\mathscr{D})$; and
- for every $X, Y \in \mathtt{Ob}(\mathscr{C})$, a functor $F_{X,Y} : \mathtt{Map}_{\mathscr{C}}(X, Y) \to \mathtt{Map}_{\mathscr{D}}(F(X), F(Y))$ that commutes with the composition and the identity morphisms only up to isomorphisms, see [**Vis08**, §.3.1.2] and [**CD13**, §.1.1].

Definition A.2.10. Let $F, G : \mathscr{C} \to \mathscr{D}$ be strict 2-functors between strict 2-categories. A (*lax*) 2*-natural transformation* $\alpha : F \to G$ is given by

- a 1-morphism $\alpha_X : F(X) \to G(X)$ for every $X \in \mathtt{Ob}(\mathscr{C})$; and
- a 2-morphism $\alpha_f : \alpha_Y \circ F(f) \Rightarrow G(f) \circ \alpha_X$ for every 1-morphism $f : X \to Y$ in $\mathscr{C}$,

which respect 1-compositions and 1-identities of $\mathscr{C}$, and are natural in 2-morphisms of $\mathscr{C}$. A 2-natural transformations is said to be a *pseudo-natural transformation* when the 2-morphism α_f is invertible for every 1-morphism f in $\mathscr{C}$.

One recovers the notion of a strict 2-natural transformation from a lax 2-natural transformation when the 2-morphism α_f is the identity morphism for every 1-morphism f in $\mathscr{C}$.

Definition A.2.11. Let $\alpha, \beta : F \Rightarrow G : \mathscr{C} \to \mathscr{D}$ be 2-natural transformations. A morphism of 2-natural transformations $\tau : \alpha \to \beta$, called a *modification*, is given by a 2-morphism $\tau_X : \alpha_X \Rightarrow \beta_X$ for every object $X \in \mathtt{Ob}(\mathscr{C})$, which are natural in 1-morphisms of $\mathscr{C}$, *i.e.* $G(f) \circ \tau_X = \tau_Y \circ F(f)$ for every 1-morphism $f : X \to Y$ in $\mathscr{C}$.

A.3. Kan Extensions

The importance of Kan extensions might be best summarised by [**ML98**, §.X.7] title "All Concepts are Kan Extensions". Kan extensions are present in different areas of mathematics, and they encode other universal constructions, see Example A.3.4, Lemma A.3.11, and Example A.3.12.

When they exist, they provide canonical solutions for the easily stated, yet very important, problem of extending a functor $F : \mathscr{C} \to \mathscr{A}$ along a functor $p : \mathscr{C} \to \mathscr{D}$ up to natural transformations, see [**ML98**, §.X].

A.3.1. Weak Kan Extensions.

A.3.1.1. *Global Kan Extensions.* For any category $\mathscr{A}$, every functor $p : \mathscr{C} \to \mathscr{D}$ induces a canonical functor

$$p^* : \mathsf{Fun}(\mathscr{D}, \mathscr{A}) \to \mathsf{Fun}(\mathscr{C}, \mathscr{A}),$$

given by precomposition with p. Extending functors $\mathscr{C} \to \mathscr{A}$ along p can be realised if p^* is weakly inverted, *i.e.* admits an adjoint.

Definition A.3.1. Let $p : \mathscr{C} \to \mathscr{D}$ be a functor, and let $\mathscr{A}$ be a category. A *global left* (resp. *right*) *Kan extension* along p is a left (resp. right) adjoint to p^*. When it exists, denote the global left (resp. right) Kan extension along p by $\mathtt{Lan}_p$ (resp. $\mathtt{Ran}_p$).

A global left Kan extension along p is determined by a functor

$$\mathsf{Lan}_p : \mathsf{Fun}(\mathscr{C}, \mathscr{A}) \to \mathsf{Fun}(\mathscr{D}, \mathscr{A})$$

and a natural transformation $F \Rightarrow \mathsf{Lan}_p(F) \circ p$ for every functor $F \in \mathsf{Fun}(\mathscr{C}, \mathscr{A})$ that forms a universal morphism, which is the unit of the adjunction $\mathsf{Lan}_p \dashv p^*$. Whereas, a global right Kan extension along p is determined by a functor

$$\mathsf{Ran}_p : \mathsf{Fun}(\mathscr{C}, \mathscr{A}) \to \mathsf{Fun}(\mathscr{D}, \mathscr{A})$$

and a natural transformation $\mathtt{Ran}_p(F) \circ p \Rightarrow F$ for every $F \in \mathsf{Fun}(\mathscr{C}, \mathscr{A})$ that forms a universal morphism, which is the counit of the adjunction $p^* \dashv \mathtt{Ran}_p$.

Global Kan extensions do not always exist. However, the uniqueness of the left and right adjunctions up to isomorphisms implies uniqueness of the left and right Kan extensions up to isomorphisms, when they exist.

LEMMA A.3.2. Let $p : \mathscr{C} \to \mathscr{D}$ be a functor, and let $\mathscr{A}$ be a category. Assume that $\mathscr{C}$ is essentially small. Then,

- if $\mathscr{A}$ is cocomplete, then the left Kan extension $\mathtt{Lan}_p$ exists; and
- if $\mathscr{A}$ is complete, then the right Kan extension $\mathtt{Ran}_p$ exists.

PROOF. Since the category $\mathscr{C}$ has a small skeleton and precomposing with cofinal functors preserves colimits, [**ML98**, §.IX.3.Th.1] and Theorem A.3.5 imply the first statement. The second statement holds by duality. □

A.3.1.2. *Local Kan Extensions.* In some occasions, one is interested in extending a particular functor $F : \mathscr{C} \to \mathscr{A}$ along $p : \mathscr{C} \to \mathscr{D}$, even if the global extensions do not exist.

DEFINITION A.3.3 ([**ML98**, §.X.3.Def]). Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors.

- A *local left Kan extension* of F along p, if it exists, is a pair $(\mathtt{Lan}_p F, \eta_F)$ of a functor $\mathtt{Lan}_p F : \mathscr{D} \to \mathscr{A}$ and a natural transformation $\eta_F : F \Rightarrow \mathtt{Lan}_p F \circ p$ that is a universal morphism from F to p^*.
- A *local right Kan extension* of F along p, if it exists, is a pair $(\mathtt{Ran}_p F, \epsilon_F)$ of a functor $\mathtt{Ran}_p F : \mathscr{D} \to \mathscr{A}$ and a natural transformation $\epsilon_F : \mathtt{Ran}_p F \circ p \Rightarrow F$ that is a universal morphism from p^* to F.

EXAMPLE A.3.4 ([**Bor94a**, Prop.3.7.5]). Let $p : \mathscr{C} \to *$ be the terminal functor.

- The local left Kan extension of a functor $F : \mathscr{C} \to \mathscr{A}$ along p exists if and only if the colimit of F exists. When they exist, $\mathtt{Lan}_p F$ is canonically isomorphic to $\mathtt{colim}\, F$.
- The local right Kan extension of a functor $F : \mathscr{C} \to \mathscr{A}$ along p exists if and only if the limit of F exists. When they exist, $\mathtt{Ran}_p F$ is canonically isomorphic to $\mathtt{lim}\, F$.

A.3.2. Point-wise Kan Extensions. Most Kan extensions that arise naturally can be given object-wise by the (co)limit formula recalled in the following theorem.

THEOREM A.3.5. Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors.

- When all the colimits below exist, there exists a local left Kan extension $(\mathtt{Lan}_p F, \eta_F)$, with $\mathtt{Lan}_p F$ given on an object $D \in \mathscr{D}$ by

$$(\mathtt{Lan}_p F)\big(D\big) \coloneqq \mathtt{colim}(F \circ U_D),$$

for the canonical projection $U_D : p{\downarrow}D \to \mathscr{C}$, and on a morphism $f : D \to D'$ in $\mathscr{D}$ by the unique morphism

$$f_* : (\mathtt{Lan}_p\, F)\big(D\big) \to (\mathtt{Lan}_p\, F)\big(D'\big),$$

induced by the universal property of colimits and by the canonical functor $f_* : p{\downarrow}D \to p{\downarrow}D'$; whereas the natural transformation η_F is given component-wise on $C \in \mathscr{C}$ by the unique morphism $F(C) \to (\mathtt{Lan}_p\, F)\big(p(C)\big)$ factorising through the evident colimit cocones; and

- when all the limits below exist, there exists a local right Kan extension $(\mathtt{Ran}_p\, F, \epsilon_F)$, with $\mathtt{Ran}_p\, F$ given on an object $D \in \mathscr{D}$ by

$$(\mathtt{Ran}_p\, F)\big(D\big) \coloneqq \mathtt{lim}(F \circ U^D),$$

for the canonical projection $U^D : D{\downarrow}p \to \mathscr{C}$, and on a morphism $f : D \to D'$ in $\mathscr{D}$ by the unique morphism

$$f_* : (\mathtt{Ran}_p\, F)\big(D\big) \to (\mathtt{Ran}_p\, F)\big(D'\big),$$

induced by the universal property of limits and by the canonical functor $f^* : D' \downarrow p \to D \downarrow p$; whereas the natural transformation ϵ_F is given component-wise on $C \in \mathscr{C}$ by the unique morphism $(\mathtt{Ran}_p\, F)\big(p(C)\big) \to F(C)$ factorising through the evident limit cones.

Proof. See [**ML98**, §.X.3.Th.1]. □

Definition A.3.6. Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors. The *point-wise left* (resp. *right*) *Kan extension* of F along p, if it exists, is the local left (resp. right) Kan extension $(\mathtt{Lan}_p\, F, \eta_F)$ (resp. $(\mathtt{Ran}_p\, F, \epsilon_F)$) given in Theorem A.3.5.

When Kan extensions of $F : \mathscr{C} \to \mathscr{A}$ along $p : \mathscr{C} \to \mathscr{D}$ exist, their morphisms η_F and ϵ_F are not necessarily isomorphisms, *i.e.* Kan extensions are not necessarily extensions in the naive sense. That is, one does not necessarily retrieve the functor F, not even up to isomorphisms, by either of compositions $\mathtt{Lan}_p\, F \circ p$ or $\mathtt{Ran}_p\, F \circ p$. For instance, recall that $\mathtt{colim}\, F$ is a left Kan extension $\mathtt{Lan}_p\, F$ of a functor $F : \mathscr{C} \to \mathscr{A}$ along the terminal functor $p : \mathscr{C} \to *$, and $\mathtt{Lan}_p\, F \circ p$ is not isomorphic to F, unless F is essentially constant, see [**Kel05**, (4.34)].

Lemma A.3.7. Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors, such that $\mathscr{C}$ is essentially small.

- Assume that there exists a point-wise left Kan extension $(\mathtt{Lan}_p\, F, \eta_F)$. Then, $\eta_F : F \Rightarrow \mathtt{Lan}_p\, F \circ p$ is an isomorphism if and only if p is fully faithful.
- Assume that there exists a point-wise right Kan extension $(\mathtt{Ran}_p\, F, \epsilon_F)$. Then, $\epsilon_F : \mathtt{Lan}_p\, F \circ p \Rightarrow F$ is an isomorphism if and only if p is fully faithful.

Moreover, when p is an inclusion of a subcategory, each of the natural isomorphisms η_F and ϵ_F, if it exists, can be chosen to be the identity morphism id_F.

Proof. The statement follows from $(C, \mathrm{id}_{p(C)})$ being a terminal object in the category $p{\downarrow}p(C)$ and an initial object in the category $p(C){\downarrow}p$, due to p being fully faithful, see [**Kel05**, Prop.4.23], [**ML98**, §.X.3.Cor.3 and Cor.4] or [**Bor94a**, Th.3.7.3]. □

Point-wise left (resp. right) Kan extensions along the Yoneda embedding preserve colimits (resp. limits). However, Kan extensions do not preserve (co)limits in general, as it is the case for the nerve functor in the following example. This example is prototypical of Kan extensions. It is due to Kan, and hence the name, see [**Kan58**].

Example A.3.8 (The Tensor–$\underline{\mathtt{Hom}}$ or Realisation–Nerve adjunction). Let $F : \mathscr{C} \to \mathscr{D}$ be a functor from an essentially small category $\mathscr{C}$ to a locally small category $\mathscr{D}$. Then, the left Kan extension $\mathtt{Lan}_F\,\mathtt{h}$ of the Yoneda embedding $\mathtt{h} : \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ along F exists, and it is given by the pullback of the Yoneda embedding $\mathtt{h} : \mathscr{D} \to \mathsf{PSh}(\mathscr{D})$ along the opposite functor $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$, *i.e.*

$$(\mathtt{Lan}_F\,\mathtt{h})\,(-)(--) = \mathscr{D}(F(--), -).$$

The functor $\mathtt{Lan}_F\,\mathtt{h}$ is usually called the F-$\underline{\mathtt{Hom}}$ *functor* (or F-*nerve functor*), and it is denoted by $\underline{\mathtt{Hom}}^{\mathscr{C}}(F, -)$. Moreover, when $\mathscr{D}$ is cocomplete, the functor $\underline{\mathtt{Hom}}^{\mathscr{C}}(F, -)$ admits a left adjoint given by the left Kan extension $\mathtt{Lan}_{\mathtt{h}}\,F$, called the F-*tensor functor* (or F-*realisation functor*), and it is denoted by $- \otimes_{\mathscr{C}} F$, see [**Kan58**, §.2]. Then, for an object $C \in \mathscr{C}$, one has a canonical isomorphism $F(C) \cong \mathtt{h}_C \otimes_{\mathscr{C}} F$. The functor $- \otimes_{\mathscr{C}} F$ is right exact for being a left adjoint, and the functor F is said to be *flat* when $- \otimes_{\mathscr{C}} F$ is also left exact. For instance, if $\mathscr{C}$ is complete and F preserves limits, then F is flat. Cases of particular interests include:

(1) when F is the standard cosimplicial topological space $\Delta^{\bullet}_{\mathrm{top}} : \varDelta \to \mathsf{Top}$, one recovers the geometric realisation and simplicial singular functors as the $\Delta^{\bullet}_{\mathrm{top}}$-tensor and $\underline{\mathtt{Hom}}$ functors, respectively, in which case F is not flat unless restricted to a suitable category of topological spaces, see [**Hov99**, §.3.1];
(2) when F is the standard cosimplicial category $\Delta^{\bullet}_{\mathsf{Cat}} : \varDelta \hookrightarrow \mathsf{Cat}$, *i.e.* $\Delta^{n}_{\mathsf{Cat}}$ is the poset $[n]$ for every non-negative integer n, then $\underline{\mathtt{Hom}}^{\varDelta}(\Delta^{\bullet}_{\mathsf{Cat}}, -)$ is the fully faithful *nerve functor* $\mathtt{N} : \mathsf{Cat} \hookrightarrow \mathsf{sSet}$, and $- \otimes_{\varDelta} \Delta^{\bullet}_{\mathsf{Cat}}$ is the *fundamental category functor* $\mathsf{c} : \mathsf{sSet} \to \mathsf{Cat}$, which is left exact, see [**Joy02**, p.208];
(3) when F is the comma categories functor $\mathscr{C}{\downarrow}- : \mathscr{C} \to \mathsf{Cat}$, for a small category $\mathscr{C}$, the tensor functor $- \otimes_{\mathscr{C}} \mathscr{C}{\downarrow}-$ is the *category of elements* functor El, which is called the *category of simplices* functor when $\mathscr{C} = \varDelta$, see [**LTW79**];
(4) when F is given by the restriction of Moore's *normalized chain functor* $\mathtt{C}_{\mathtt{N}} : \mathsf{sSet} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ to the simplex category $\varDelta$, the functor $\mathtt{K} \coloneqq \underline{\mathtt{Hom}}^{\varDelta}(\mathtt{C}_{\mathtt{N}|_{\varDelta}}, -)$ factorises through the category of simplicial abelian groups $\varDelta^{\mathrm{op}}\mathsf{Ab}$, see [**Kan58**, §.8]; then, the tensor-$\underline{\mathtt{Hom}}$ adjunction gives rise to the Dold-Kan correspondence, that is the equivalence of categories

$$\mathtt{C}_{\mathtt{N}|_{\varDelta^{\mathrm{op}}\mathsf{Ab}}} : \varDelta^{\mathrm{op}}\mathsf{Ab} \rightleftarrows \mathsf{Ch}_{\geq 0}(\mathsf{Ab}) : \mathtt{K},$$

see [**Kan58**, Th.8.1 and Th.8.2]; and

(5) when F is given by the two-folded product of the standard cosimplicial simplicial set $\Delta^\bullet \times \Delta^\bullet : \mathit{\Delta} \times \mathit{\Delta} \to \mathsf{sSet}$, the $\Delta^\bullet \times \Delta^\bullet$-tensor functor coincide with the diagonal functor for bisimplicial sets. On the other hand, every simplicial set K defines bisimplicial set $\underline{\mathtt{Hom}}^{\mathit{\Delta}\times\mathit{\Delta}}(\Delta^\bullet \times \Delta^\bullet, K)$ with

$$\underline{\mathtt{Hom}}^{\mathit{\Delta}\times\mathit{\Delta}}(\Delta^\bullet \times \Delta^\bullet, K)_{p,q} = \mathsf{sSet}(\Delta^p \times \Delta^q, K),$$

for every pair of non-negative integers $p, q \geq 0$.

Dually, for a functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$, the right Kan extension $\mathtt{Ran}_G \,\mathtt{h}^{\mathrm{op}}$ of the opposite of the Yoneda embedding $\mathtt{h} : \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ along G exists, and is given by

$$(\mathtt{Ran}_G \,\mathtt{h}^{\mathrm{op}})(-)(--) = \mathscr{D}(-, F(--)).$$

When $\mathscr{D}$ is complete, the functor $\mathtt{Ran}_G \,\mathtt{h}^{\mathrm{op}}$ admits a right adjoint given by the right Kan extension $\mathtt{Ran}_{\mathtt{h}^{\mathrm{op}}} G$, called the *right* G-$\underline{\mathtt{Hom}}$ *functor*, and it is denoted by $\underline{\mathtt{Hom}}_{\mathscr{C}}(-, G)$, see [**GS09**, p.3097]. Also, for an object $C \in \mathscr{C}$, one has a canonical isomorphism $\underline{\mathtt{Hom}}_{\mathscr{C}}(\mathtt{h}_C^{\mathrm{op}}, G) \cong G(C)$. The functor $\underline{\mathtt{Hom}}_{\mathscr{C}}(-, G)$ is left exact for being a right adjoint, and the functor G is said to be *coflat* when $\underline{\mathtt{Hom}}_{\mathscr{C}}(-, G)$ is also right exact.

Realising colimits as quotients of coproducts allows expressing point-wise left Kan extensions using coends, see [**ML98**, §.IX.6]. That is, for functors $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$, where $\mathscr{C}$ is an essentially small category and $\mathscr{A}$ is a cocomplete category, the left Kan extension $\mathtt{Lan}_p F$ is given on an object $D \in \mathscr{D}$ by the coend

$$\mathtt{Lan}_p F(D) = \int^{C \in \mathscr{C}} \coprod_{\mathscr{D}(p(C), D)} F(C) = \mathsf{Coend}\Bigg(\coprod_{\mathscr{D}(p(-), D)} F(-) \Bigg), \tag{87}$$

see [**ML98**, §.X.4.Th.1]. For instance, the conventional formula (3) for the geometric realisation of a simplicial set $X_\bullet$ is nothing but the coend of the bifunctor $X_{\bullet,\mathrm{dis}} \times \Delta^\bullet_{\mathrm{top}} : \mathit{\Delta}^{\mathrm{op}} \times \mathit{\Delta} \to \mathsf{Top}$, obtained from the discrete simplicial space $X_{\bullet,\mathrm{dis}}$ associated to $X_\bullet$ and the standard cosimplicial topological space $\Delta^\bullet_{\mathrm{top}}$. Coends generalise tensor products, which justifies the notation used for the tensor functor in Example A.3.8, see [**ML98**, §.IX.6]. In fact, for rings R and S, and for an (R, S)-module $M : \mathbf{B}R \to \mathsf{AddFun}(\mathbf{B}S^{\mathrm{op}}, \mathsf{Ab})$, the induced M-tensor-$\underline{\mathtt{Hom}}$ adjunction

$$- \otimes_R M : \mathsf{AddFun}(\mathbf{B}R^{\mathrm{op}}, \mathsf{Ab}) \rightleftarrows \mathsf{AddFun}(\mathbf{B}S^{\mathrm{op}}, \mathsf{Ab}) : \underline{\mathtt{Hom}}^R(M, -) \tag{88}$$

is essentially the conventional tensor-$\underline{\mathtt{Hom}}$ adjunction for modules, as the category of right R-modules is isomorphic to the additive functor category $\mathsf{AddFun}(\mathbf{B}R^{\mathrm{op}}, \mathsf{Ab})$, where $\mathbf{B}$ is the delooping space functor, which sends a ring R to a one-object ringoid whose set of morphisms is R. Moreover, the (R, S)-module M is flat if and only if the functor $- \otimes_R M$ in (88) is left exact.

Example A.3.9. Let $\mathscr{B}$ be an additive category, and let $\underline{\mathbb{N}}$ be the preordered category corresponding to the ordered set $(\mathbb{N}, \leq)$. The category $\mathsf{Ch}\big(\mathsf{Ch}(\mathscr{B})\big)$ (resp.

$\mathsf{Ch}(\mathscr{B}))$ is a full subcategory of $\mathscr{B}^{\underline{\mathbb{N}}^{\mathrm{op}}\times\underline{\mathbb{N}}^{\mathrm{op}}}$ (resp. $\mathscr{B}^{\underline{\mathbb{N}}^{\mathrm{op}}}$). The category $\underline{\mathbb{N}}^{\mathrm{op}}$ admits a symmetric monoidal structure, with a monoidal product $\oplus : \underline{\mathbb{N}}^{\mathrm{op}} \times \underline{\mathbb{N}}^{\mathrm{op}} \to \underline{\mathbb{N}}^{\mathrm{op}}$, given on objects by $n \oplus m = n + m$. Then, the total complex functor $\mathtt{Tot} : \mathsf{Ch}\big(\mathsf{Ch}(\mathscr{B})\big) \to \mathsf{Ch}(\mathscr{B})$ is given by the restriction of the Kan extension $\mathtt{Lan}_{\oplus} \cong \mathtt{Ran}_{\oplus}$, *i.e.* for a double complex $X_{\bullet,\bullet}$ in $\mathscr{B}$, the total complex $\mathtt{Tot}_{\bullet}(X)$ is given for $n \in \mathbb{N}$ by

$$\mathtt{Tot}_n(X) = \int^{(p,q)\in\oplus\downarrow n} X_{p,q} \cong \int_{(p,q)\in n\downarrow\oplus} X_{p,q} \cong \bigoplus_{p+q=n} X_{p,q}.$$

Example A.3.10. Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors, where $\mathscr{C}$ is an essentially small category and $\mathscr{D}$ is a cocomplete category. When $\mathscr{A} = \mathsf{Set}$, there exist canonical isomorphisms

$$\begin{aligned}\mathtt{Lan}_p F(D) &= \int^{C\in\mathscr{C}} \coprod_{\mathscr{D}(p(C),D)} F(C) \cong \int^{C^{\mathrm{op}}\in\mathscr{C}^{\mathrm{op}}} \mathscr{D}^{\mathrm{op}}(D^{\mathrm{op}}, p^{\mathrm{op}}(C^{\mathrm{op}})) \times F(C)\\ &\cong \int^{C^{\mathrm{op}}\in\mathscr{C}^{\mathrm{op}}} \mathscr{D}^{\mathrm{op}}(D^{\mathrm{op}}, p^{\mathrm{op}}(C^{\mathrm{op}})) \times \mathsf{PSh}(\mathscr{C}^{\mathrm{op}})(\mathtt{h}_{C^{\mathrm{op}}}, F),\end{aligned}$$

for every $D \in \mathscr{D}$. Also, when $\mathscr{A} = \mathsf{Set}_{\bullet}$, the category of pointed small sets, one has a canonical isomorphism

$$\mathtt{Lan}_p F(D) \cong \int^{C\in\mathscr{C}} \bigvee_{\mathscr{D}(p(C),D)} F(C) \cong \int^{C^{\mathrm{op}}\in\mathscr{C}^{\mathrm{op}}} \bigvee_{\mathscr{D}^{\mathrm{op}}(D^{\mathrm{op}},p^{\mathrm{op}}(C^{\mathrm{op}}))} \big(\mathsf{PSh}_{\bullet}(\mathscr{C}^{\mathrm{op}})(\mathtt{h}_{C^{\mathrm{op}},+}, F), *\big), \quad (89)$$

for every $D \in \mathscr{D}$.

Point-wise Kan extensions are characterised by the following representability criterion.

Lemma A.3.11. Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors. Then,

- a pair $(L : \mathscr{D} \to \mathscr{A}, \eta : F \Rightarrow L \circ p)$ is a point-wise left Kan extension of F along p if and only if for every $D \in \mathscr{D}$ and $A \in \mathscr{A}$ the morphism
$$\mathscr{A}\big(L(D), A\big) \to \mathsf{Nat}(\mathtt{h}_D \circ p^{\mathrm{op}}, \mathtt{h}_A \circ F^{\mathrm{op}}),$$
sending $g : L(D) \to A$ to the natural transformation with the component
$$\mathtt{h}_D\, p^{\mathrm{op}}(C^{\mathrm{op}}) \xrightarrow{L} \mathtt{h}_{L(D)}\, L^{\mathrm{op}} p^{\mathrm{op}}\big(C^{\mathrm{op}}\big) \xrightarrow{\mathtt{h}_g(\eta^{\mathrm{op}}_{C^{\mathrm{op}}})} \mathtt{h}_A\, F^{\mathrm{op}}\big(C^{\mathrm{op}}\big)$$
for every $C \in \mathscr{C}$, is a bijection; and
- a pair $(R : \mathscr{D} \to \mathscr{A}, \epsilon : R \circ p \Rightarrow F)$ is a point-wise right Kan extension of F along p if and only if for every $D \in \mathscr{D}$ and $A \in \mathscr{A}$ the morphism
$$\mathscr{A}\big(A, R(D)\big) \to \mathsf{Nat}(\mathtt{h}^D \circ p, \mathtt{h}^A \circ F),$$
sending $g : A \to R(D)$ to the natural transformation with the component
$$\mathtt{h}^D\, p(C) \xrightarrow{R} \mathtt{h}^{R(D)}\, Rp\big(C\big) \xrightarrow{\mathtt{h}^g(\epsilon_C)} \mathtt{h}^A\, F\big(C\big)$$
for every $C \in \mathscr{C}$, is a bijection.

PROOF. See [**ML98**, §.X.5.Cor.4]. $\square$

A.3.2.1. *Preserving Kan Extensions.* Among the different types of Kan extensions, point-wise extensions are the most accessible. For they are given by hands-on formulae that are easy to manipulate and work with. However, not all Kan extensions are point-wise, and the latter are distinguished by being preserved by corepresentable functors, as can be seen in Theorem A.3.13.

Let $p : \mathscr{C} \to \mathscr{D}$, $F : \mathscr{C} \to \mathscr{A}$ and $G : \mathscr{A} \to \mathscr{B}$ be functors, and assume that a local left Kan extension of F along p exists. We say that G *preserves* the left Kan extension $(\mathtt{Lan}_p F, \eta_F)$ if $(G \circ \mathtt{Lan}_p F, G \circ \eta_F)$ is a left Kan extension of $G \circ F$ along p. The dual notion is also defined for right Kan extensions.

It is well-known that left (resp. right) adjoints preserve colimits (resp. limits). Moreover, they preserve local left (resp. right) Kan extensions, see [**Bor94a**, Prop.3.7.4].

EXAMPLE A.3.12. Let $F : \mathscr{C} \to \mathscr{A}$ be a functor between essentially small categories. Then, the following conditions are equivalent,

- F admits a right adjoint;
- $\mathtt{Lan}_F \operatorname{id}_{\mathscr{C}}$ exists and is preserved by every functor $\mathscr{C} \to \mathscr{B}$; and
- $\mathtt{Lan}_F \operatorname{id}_{\mathscr{C}}$ exists and is preserved by F.

When any, and hence all, of the three conditions are satisfied, $\mathtt{Lan}_F \operatorname{id}_{\mathscr{C}}$ is a right adjoint of F, see [**Bor94a**, Prop.3.7.6]. Dually, the following conditions are equivalent,

- F admits a left adjoint;
- $\mathtt{Ran}_F \operatorname{id}_{\mathscr{C}}$ exists and is preserved by every functor $\mathscr{C} \to \mathscr{B}$; and
- $\mathtt{Ran}_F \operatorname{id}_{\mathscr{C}}$ exists and is preserved by F.

When any, and hence all, of the three conditions are satisfied, $\mathtt{Ran}_F \operatorname{id}_{\mathscr{C}}$ is a left adjoint of F.

THEOREM A.3.13. Let $p : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{C} \to \mathscr{A}$ be functors between locally small categories, and assume that there exists a local left (resp. right) Kan extension of F along p. Then, the local Kan extension is a point-wise Kan extension if it is preserved by the corepresentable functor $\mathtt{h}^A : \mathscr{A} \to \mathsf{Set}$ for every object $A \in \mathscr{A}$.

PROOF. See [**ML98**, §.X.5.Defintion and Theorem.3]. $\square$

A.3.3. Density. Some structures on locally small categories, like model structures, require the underlying category to be bicomplete. Given a locally small category that is not bicomplete, one may consider a locally small bicompletion of the given category, if it exists, and study such structures on the bicompletion. For an essentially small category $\mathscr{D}$, one may consider its bicompletion by the locally small category of presheaves $\mathsf{PSh}(\mathscr{D})$, or by a subcategory of τ-sheaves $\mathsf{Shv}_\tau(\mathscr{D})$ for a subcanonical topology τ on $\mathscr{D}$, depending on the sought behaviour of colimits, see Remark A.4.7. However, when the category $\mathsf{PSh}(\mathscr{D})$ is not locally small, it is desired to look for different methods of bicompletion. For a functor $F : \mathscr{C} \to \mathscr{D}$ from an essentially small category $\mathscr{C}$, the F-nerve functor $\underline{\mathtt{Hom}}^{\mathscr{C}}(F, -) : \mathscr{D} \to \mathsf{PSh}(\mathscr{C})$, as in Example A.3.8, may be used as a bicompletion of $\mathscr{D}$ when it is fully faithful.

Lemma A.3.14. Let $p : \mathscr{C} \to \mathscr{D}$ be a functor from an essentially small category $\mathscr{C}$ to a locally small category $\mathscr{D}$. Then, the following conditions are equivalent,

- the functor $\underline{\mathtt{Hom}}^{\mathscr{C}}(F, -)$ is fully faithful; and
- the pair $(\mathrm{id}_{\mathscr{D}}, \mathrm{id}_{\mathrm{id}_{\mathscr{D}}})$ is a point-wise left Kan extension of p along itself.

Proof. See [**Kel05**, Th.5.1]. □

Definition A.3.15. Let $p : \mathscr{C} \to \mathscr{D}$ be a functor. The functor p is said to be *dense* (resp. *codense*) if $(\mathrm{id}_{\mathscr{D}}, \mathrm{id}_{\mathrm{id}_{\mathscr{D}}})$ is a point-wise left (resp. right) Kan extension of p along itself, *i.e.* p is dense if for every object $D \in \mathscr{D}$, one has an isomorphism $D \cong \mathtt{colim}\, p \circ U_D$, where $U_D : p \downarrow D \to \mathscr{C}$ is the canonical such projection functor. A subcategory $\mathscr{C} \hookrightarrow \mathscr{D}$ is said to be dense (resp. codense) if the inclusion functor is dense (resp. codense).

For a dense subcategory $i : \mathscr{C} \hookrightarrow \mathscr{D}$, since point-wise Kan extensions are preserved by corepresentable functors, the left Kan extension $\mathtt{Lan}_i\, \mathtt{h}^D_{|\mathscr{C}}$ of the restriction of $\mathtt{h}^D$ to $\mathscr{C}$ along i is canonically isomorphic to $\mathtt{h}^D$, for every $D \in \mathscr{D}$.

Example A.3.16. Let $\mathscr{C}$ be a locally small category. Then, the Yoneda embedding $\mathsf{h}_- : \mathscr{C} \hookrightarrow \mathsf{PSh}(\mathscr{C})$ (resp. $\mathtt{h}^- : \mathscr{C} \hookrightarrow \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$) is dense (resp. codense), see [**ML98**, §.X.6.Cor.3].

A.4. Grothendieck Sites

A Grothendieck topology is a generalisation of topological coverings to abstract categories, which enables the development of cohomology theories on abstract categories.

Throughout this section, let $\mathscr{C}$ be a locally small category. A *sieve* S on an object $U \in \mathscr{C}$ is an inclusion $S \subset \mathsf{h}_U : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$, and a *refinement* of a sieve S on U is an inclusion $S' \subset S$. The map that sends a sieve S on $U \in \mathscr{C}$ to the set $\mathtt{Ob}\big(\mathsf{El}(S)\big)$ of objects of its category of elements defines a bijection

$$\{\text{sieves on } U\} \longleftrightarrow \{\text{right ideals in } \mathscr{C}, \text{ with a common codomain } U\},$$

where a set of morphism in $\mathscr{C}$ is called a *right ideal* if it is closed with respect to precompositions with morphisms in $\mathscr{C}$, see [**Joh02**, p.538]. For a set $\mathscr{U}$ of morphisms with a common codomain in $\mathscr{C}$, let $S_{\mathscr{U}}$ denote the sieve corresponding to the right ideal in $\mathscr{C}$ generated by $\mathscr{U}$. The sieve $S_{\mathscr{U}}$ is said to be *generated* by the set $\mathscr{U}$.

Definition A.4.1. Let S be a sieve on $U \in \mathscr{C}$, let T be a sieve on $V \in \mathscr{C}$ and let $f : T \to \mathsf{h}_U$ be a morphism in the functor category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$, *i.e.* a natural transformation. The *restriction* of S to V along f is defined to be the sieve on V that is the image of the pullback projection $S \times_{\mathsf{h}_U} T \to T$ in $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$, it is denoted by f^*S. For a morphism $\varphi : V \to U$ in $\mathscr{C}$, we abuse notation and write φ^*S for $\mathsf{h}_\varphi^* S$.

For a morphism $\varphi : V \to U$ in $\mathscr{C}$, the sieve φ^*S corresponds to the right ideal

$$\{\psi : W \to V \text{ in } \mathscr{C} \mid \varphi \circ \psi \in S(W)\}.$$

Definition A.4.2. A (*Grothendieck*) *topology* τ on the category $\mathscr{C}$ is a set

$$\tau = \{\mathsf{Cov}_\tau(U) \mid U \in \mathscr{C}\},$$

in which $\mathsf{Cov}_\tau(U)$ is a set of sieves on U, for every object $U \in \mathscr{C}$, such that

- (Stability) the restriction φ^*S of S along φ belongs to $\mathsf{Cov}_\tau(V)$, for every sieve $S \in \mathsf{Cov}_\tau(U)$ and for every morphism $\varphi : V \to U$ in $\mathscr{C}$;
- (Maximal Sieve) the sieve h_U belongs to $\mathsf{Cov}_\tau(U)$, for every object $U \in \mathscr{C}$; and
- (Local Character) a sieve R on an object $U \in \mathscr{C}$ belongs to $\mathsf{Cov}_\tau(U)$ whenever φ^*R belongs to $\mathsf{Cov}_\tau(V)$ for every morphism $\varphi : V \to U$ in a sieve $S \in \mathsf{Cov}_\tau(U)$.

A sieve S on $U \in \mathscr{C}$ is called a τ-*covering sieves* if it belongs to $\mathsf{Cov}_\tau(U)$, and the pair $(\mathscr{C}, \tau)$ is called a (*Grothendieck*) *site*, usually denoted by $\mathscr{C}_\tau$. Moreover, when $\mathscr{C}$ is essentially small, we say that $\mathscr{C}_\tau$ is an *essentially small site*, not be confused with the notion of essentially small sites in [**Joh02**].

Example A.4.3. A sieve is said to be *effective epimorphic* if it forms a colimit cocone, and it is said to be *universally effective epimorphic* if all its restrictions are *effective epimorphic*. Every category $\mathscr{C}$ admits a topology whose covering sieves are universally effective epimorphic sieves, called the *canonical topology* on $\mathscr{C}$, see [**Joh02**, p.542-543]. A topology that is contained in the canonical topology is said to be *subcanonical*.

The intersection of topologies on the category $\mathscr{C}$ is a topology. Hence, given a set $\mathscr{S}$ of sets of sieves on a category $\mathscr{C}$, the intersection of all topologies on $\mathscr{C}$ that contain $\mathscr{S}$ is a topology on $\mathscr{C}$, called the *topology generated by* $\mathscr{S}$. In some occasions, it may be simpler to specify sieves, and hence topologies, in terms of sets of morphisms that generate them, as in §.A.4.3.

A *Grothendieck pretopology* on a category $\mathscr{C}$ with pullbacks is a set

$$\tau = \{\mathsf{Cov}_\tau(U) \mid U \in \mathscr{C}\},$$

in which $\mathsf{Cov}_\tau(U)$ is a set of families of morphisms with a common codomain that satisfies closure conditions similar to those of a topology, namely

- the set $\{\underline{\sigma}_\alpha : U_\alpha \times_U V \to V \mid \alpha \in A\}$ belongs to $\mathsf{Cov}_\tau(V)$, for every family $\{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ in $\mathsf{Cov}_\tau(U)$ and for every morphism $\varphi : V \to U$ in $\mathscr{C}$;
- the family $\{\mathrm{id}_U\}$ belongs to $\mathsf{Cov}_\tau(U)$, for every object $U \in \mathscr{C}$; and
- the family $\{\sigma_\alpha \circ \delta_{\alpha,\beta} : U_{\alpha,\beta} \to U \mid \alpha \in A, \beta \in B_\alpha\}$ belongs to $\mathsf{Cov}_\tau(U)$, for every family $\{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ in $\mathsf{Cov}_\tau(U)$ and every family $\{\delta_{\alpha,\beta} : U_{\alpha,\beta} \to U_\alpha \mid \beta \in B_\alpha\}$ in $\mathsf{Cov}_\tau(U_\alpha)$ for $\alpha \in A$,

see [**MLM92**, §.III.2.Def.2]. A family $\mathscr{U}$ in $\mathsf{Cov}_\tau(U)$ is called a τ-*covering family* of U. The unique element of a singleton τ-covering family of $U \in \mathscr{C}$ is called a τ-*cover* of U.

A *refinement* of a family of morphisms $\mathscr{U} = \{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ is a pair $(f, \mathscr{U}')$ of a map $f : A' \to A$ and a family of morphisms $\mathscr{U}' = \{\sigma'_{\alpha'} : U'_{\alpha'} \to U \mid \alpha' \in A'\}$ such that $\sigma'_{\alpha'}$ factorises through $\sigma_{f(\alpha)}$, for every $\alpha' \in A'$.

Definition A.4.4. Let $\mathscr{C}$ be a locally small category with pullbacks. A pretopology τ on $\mathscr{C}$ is said to be *saturated* if every family of morphisms in $\mathscr{C}$ with a common codomain that admits a refinement by a τ-covering family is a τ-covering family. Assume that $\mathscr{C}$ admits finite coproducts, the pretopology τ is said to be *additively-saturated* if for every τ-covering family $\mathscr{U} = \{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$, the set A is finite and the singleton

$$\left\{ \coprod_{\alpha \in A} \sigma_\alpha : \coprod_{\alpha \in A} U_\alpha \to U \right\}$$

is a τ-covering family.

In particular, a saturated pretopology on a category that admits finite coproducts is additively-saturated.

Every pretopology admits a *saturation*, that is a pretopology in which covering families are precisely families that admit refinements in the given pretopology, see [**Vis08**, Def.2.52 and Prop.2.53]. Also, *additive-saturations* are defined similarly.

In practice, the pretopologies one considers are almost never saturated, and their saturations allow redundant covering families that does not reflect the intended properties of the topology. For example, for a pretopology τ and a τ-covering family $\mathscr{U} = \{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$, the set $\mathscr{U} \coprod \{f\}$ is a covering family in the saturation of τ, for any morphism $f : V \to U$. It is often more practical to consider additively-saturated pretopologies, as in Example A.4.35.

A pretopology defines a topology, generated by its covering families, and different pretopologies may define the same topology. In particular, a pretopology and its (additive-)saturation define the same topology. For a pretopology τ, we may abuse notation and use τ to also denote the topology defined by it.

A.4.1. The Category of Sheaves. A functor $P : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$ is called a *presheaf of* (*small*) sets on $\mathscr{C}$, the functor category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$ is called the *category of presheaves* on $\mathscr{C}$, and it is usually denoted by $\mathsf{PSh}(\mathscr{C})$.

Axiomatising the properties of sheaves of sections of étale spaces yields the definition for a sheaf on a site, see [**MLM92**, §.II.6].

Definition A.4.5. Let $\mathscr{C}_\tau$ be a site. A presheaf $P : \mathscr{C}^{op} \to \mathsf{Set}$ is said to be a *τ-separated presheaf*, a *τ-weak sheaf*, or a *τ-sheaf* if for every object $U \in \mathscr{C}$ and for every τ-covering sieve S on U, the canonical map

$$\iota_S^* : \mathsf{PSh}(\mathscr{C})\big(\mathfrak{h}_U, P\big) \to \mathsf{PSh}(\mathscr{C})\big(S, P\big), \tag{90}$$

induced by the inclusion $\iota_S : S \hookrightarrow \mathfrak{h}_U$, is injective, surjective, or bijective, respectively. Denote the full subcategory in $\mathsf{PSh}(\mathscr{C})$ of τ-separated presheaves, τ-weak sheaves, and τ-sheaves on $\mathscr{C}$ by $\mathsf{Sep}_\tau(\mathscr{C})$, $\mathsf{WShv}_\tau(\mathscr{C})$, and $\mathsf{Shv}_\tau(\mathscr{C})$, respectively.

Evidently the maximum sieve axiom does not affect the sheaf condition (90), neither does the local character axiom, see [**Joh02**, §.C.Lem.2.1.7]. They are merely closure conditions that are particularly useful in the double plus construction of the τ-sheafification functor, see [**Bor94b**, p.205]. Also, as a direct consequence of Definition A.4.5, one finds that the subcategory of τ-sheaves is closed under limits in the category of presheaves, and hence it is complete, with limits given object-wise.

When the site is $\mathscr{C}_\tau$ essentially small, the set $\mathtt{Cov}_\tau(U)$ forms an essentially small cofiltered[1] sublattice in the lattice of subobjects of $\mathfrak{h}_U$, for every object $U \in \mathscr{C}$, see [**Bor94b**, Prop.3.2.5]. That results in the subcategory of τ-sheaves being a Cartesian reflective subcategory in the category of presheaves, *i.e.* the inclusion $\mathsf{Shv}_\tau(\mathscr{C}) \hookrightarrow \mathsf{PSh}(\mathscr{C})$ admits a left adjoint[2] which is left exact, called the *τ-sheafification functor* or *associated τ-sheaf functor*, and it is denoted by $-^{\mathsf{a}_\tau}$, see [**Bor94b**, Th.3.3.12]. For a presheaf P on $\mathscr{C}$, there exists a τ-separated presheaf P^{+_τ}, given for an object $U \in \mathscr{C}$ by the filtered colimit

$$P^{+_\tau}(U) \coloneqq \mathtt{colim}\ \mathsf{PSh}(\mathscr{C})\big(i_U(-), P\big),$$

where $i_U : \mathtt{Cov}_\tau(U) \hookrightarrow \mathsf{PSh}(\mathscr{C})$ is the canonical such inclusion, and $\mathtt{Cov}_\tau(U)$ is consider as a preordered category. Then, the associated τ-sheaf P^{a_τ} can be given by

$$P^{\mathsf{a}_\tau} \coloneqq P^{+_\tau +_\tau},$$

see [**Bor94b**, §.3.3]. Alternatively, to a presheaf P one defines a τ-separated presheaf P^{s_τ}, given on an object $U \in \mathscr{C}$ by the quotient

$$P^{\mathsf{s}_\tau}(U) \coloneqq P(U)/\sim,$$

[1]We adopt the terminology of [**ML98**, §.IX.1], where *filtered categories* generalise directed sets, and *cofiltered categories* refer to what is called ‘filtering categories’ in [**MLM92**, §.VII.6].

[2]A left adjoint of an inclusion functor is called a *reflector*.

where $\sim$ is the relation on $P(U)$, with $p \sim p'$ for $p, p' \in P(U)$ if and only if there exists a τ-covering sieve on which the restrictions of p and p' coincide, and given on morphisms by the universal property of the quotient maps. Then, the associated τ-sheaf P^{a_τ} can be given by

$$P^{\mathsf{a}_\tau} \coloneqq (P^{\mathsf{s}_\tau})^{+_\tau},$$

see the proof of [**Vis08**, Th.2.64.(ii)]. Since P^{s_τ} is τ-separated, the canonical morphism $P^{\mathsf{s}_\tau} \to P^{\mathsf{a}_\tau}$ is a monomorphism. For a section $p \in P(U)$, we denote its image in $P^{\mathsf{s}_\tau}(U)$ (resp. $P^{\mathsf{a}_\tau}(U)$) by p^{s} (resp. p^{a}).

For an essentially small $\mathscr{C}_\tau$, the category of τ-sheaves on $\mathscr{C}$ is bicomplete, with colimits given by the τ-sheafification of colimits in the category of presheaves, that the τ-sheafification functor preserves colimits. The category of presheaves $\mathsf{PSh}(\mathscr{C})$ is Cartesian closed, with internal $\underline{\mathtt{Hom}}$ given by

$$\underline{\mathtt{Hom}}_{\mathsf{PSh}(\mathscr{C})}(P, Q) = \mathsf{PSh}(\mathscr{C})(P \times \mathtt{h}_-, Q),$$

for a pair of presheaves $P, Q \in \mathsf{PSh}(\mathscr{C})$. Also, the category $\mathsf{Shv}_\tau(\mathscr{C})$ is also Cartesian closed, with internal $\underline{\mathtt{Hom}}$ given by

$$\underline{\mathtt{Hom}}_{\mathsf{Shv}_\tau(\mathscr{C})} = -^{\mathsf{a}_\tau} \circ \underline{\mathtt{Hom}}_{\mathsf{PSh}(\mathscr{C})} \, .$$

In fact, for an essentially small category $\mathscr{C}$, the map that sends each topology on $\mathscr{C}$ to the Cartesian reflective subcategory in $\mathsf{PSh}(\mathscr{C})$ of its sheaves defines a bijection

$$\left\{ \text{ topologies on } \mathscr{C} \ \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{relective subcategories in the category of presheaves} \\ \mathsf{PSh}(\mathscr{C}) \text{ with left exact reflectors} \end{array} \right\},$$

see [**Joh02**, C.Cor.2.1.11].

For an essentially small site $\mathscr{C}_\tau$, a morphism of presheaves on $\mathscr{C}$ is said to be a *τ-local isomorphism* if its τ-sheafification is an isomorphism. Then, the category $\mathsf{Shv}_\tau(\mathscr{C})$ is a reflective localisation of $\mathsf{PSh}(\mathscr{C})$ with respect to τ-local isomorphisms, and hence τ-sheaves coincide with τ-local objects in $\mathsf{PSh}(\mathscr{C})$.

Example A.4.6. The component of the τ-sheafification adjunction unit is given for a presheaf P by a morphism $\eta^\tau_P : P \to P^{\mathsf{a}_\tau}$ in $\mathsf{PSh}(\mathscr{C})$ for which there exists a bijection

$$\eta^{\tau\,*}_P : \mathsf{PSh}(\mathscr{C})(P^{\mathsf{a}_\tau}, S) \to \mathsf{PSh}(\mathscr{C})(P, S),$$

for every τ-sheaf $S \in \mathsf{Shv}_\tau(\mathscr{C}) \subset \mathsf{PSh}(\mathscr{C})$, and hence η^τ_P is a τ-local isomorphism.

Remark A.4.7. Giving a topology on an essentially small category $\mathscr{C}$ is a way of formally declaring specific cocones to be colimit cocones in the resulting category of sheaves. Recall that the category of presheaves $\mathsf{PSh}(\mathscr{C})$ is the free cocompletion for $\mathscr{C}$, and hence it formally adds all small colimits, forgetting the colimits that already exist in $\mathscr{C}$. The sheaf condition (90) shows that the cocone of a covering sieve is mapped into a colimit cocone in the category of sheaves. That is, for a topology τ on $\mathscr{C}$, for every τ-sheaf P on $\mathscr{C}$, and for every τ-covering sieve S on $U \in \mathscr{C}$, one has

$$\begin{aligned}\mathsf{Shv}_\tau(\mathscr{C})\big(\mathtt{colim}(\mathtt{h}^{\mathsf{a}_\tau}_{-}\,\pi_S), P\big) &\cong \mathsf{PSh}(\mathscr{C})\big(\mathtt{colim}(\mathtt{h}_{-}\,\pi_S), P\big) \cong \mathsf{PSh}(\mathscr{C})\big(S, P\big)\\ &\cong \mathsf{PSh}(\mathscr{C})\big(\mathtt{h}_U, P\big) \cong \mathsf{Shv}_\tau(\mathscr{C})\big(\mathtt{h}^{\mathsf{a}_\tau}_U, P\big),\end{aligned}$$

where $\pi_S : \mathsf{El}(S) \to \mathscr{C}$ is the canonical such projection functor, and hence $\mathtt{colim}(\mathtt{h}^{\mathsf{a}_\tau}_{-}\,\pi_S) \cong \mathtt{h}^{\mathsf{a}_\tau}_U$ in $\mathsf{Shv}_\tau(\mathscr{C})$. In particular, the canonical topology is the coarsest topology that retrieves universal colimits cocones that exist in $\mathscr{C}$.

Example A.4.8. For an essentially small site $\mathscr{C}_\tau$, representable presheaves are τ-sheaves if and only if the composition $-^{\mathsf{a}_\tau} \circ \mathtt{h}_{-} : \mathscr{C} \to \mathsf{Shv}_\tau(\mathscr{C})$ is fully faithful, which occurs only when the topology τ is subcanonical, see [**Joh02**, p.542-543]. More generally, for any essentially small site $\mathscr{C}_\tau$ one has a canonical equivalence of categories

$$\mathsf{Shv}_\tau(\mathscr{C}) \cong \mathsf{Shv}_{\mathrm{can}}\big(\mathsf{Shv}_\tau(\mathscr{C})\big),$$

see [**Joh02**, §.C.2.2].

A.4.1.1. *Local Epimorphisms, Monomorphisms and Isomorphisms.* For an essentially small site $\mathscr{C}_\tau$, a morphism of presheaves is said to be a *τ-local epimorphism* (resp. *τ-local monomorphism*) if its τ-sheafification is an epimorphism (resp. a monomorphism), and hence a morphism of presheaves is a τ-local isomorphism if and only if it is both τ-local epimorphism and τ-local monomorphism. In particular, epimorphisms (resp. monomorphisms) are τ-local epimorphisms (resp. τ-local monomorphisms), that the τ-sheafification functor preserves epimorphisms for being a reflector and preserves monomorphisms for being left exact. We recall below the characterisation of τ-local epimorphisms and monomorphisms.

Local Epimorphisms. Every morphism $f : P \to \mathtt{h}_U$ of presheaves on $\mathscr{C}$ admits a canonical factorisation as $f = \iota_f \circ \overline{f}$, where $\iota_f : \mathtt{im}\, f \subset \mathtt{h}_U$ is a sieve on U and $\overline{f}$ is the canonical epimorphism $P \to \mathtt{im}\, f$. The sheaf condition (90) implies that the inclusion $S \subset \mathtt{h}_U$ is a τ-local isomorphism if and only if S is a τ-covering sieve on $U \in \mathscr{C}$, see [**Bor94b**, Lem.3.5.1]. Thus, f is a τ-local epimorphism if and only if $\mathtt{im}\, f \subset \mathtt{h}_U$ is a τ-covering sieve.

For every presheaf $Q : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$, the Yoneda lemma gives a canonical isomorphism $Q(U) \cong \mathsf{PSh}(\mathscr{C})(\mathtt{h}_U, Q)$ for every $U \in \mathscr{C}$. Then, a morphism $f : P \to Q$ of presheaves on $\mathscr{C}$ is a (τ-local) epimorphism of presheaves if and only if the projection $\varphi^* P \to \mathtt{h}_U$ is a (τ-local) epimorphism for every morphism of presheaves $\varphi : \mathtt{h}_U \to Q$.

τ-local epimorphisms retain the essential properties of epimorphisms of presheaves of sets, as they contain all epimorphisms, stable under composition, left decomposition, and pullbacks. Also, they are determined by pullbacks along elements of their codomains, see [**KS06**, p.390-391 and Prop.16.1.11]. A set of morphisms of presheaves that satisfies these properties is called a *systems of local epimorphisms*. They may be though of as generalised covering sieves. In fact, the map that sends a topology to the set of its local epimorphisms defines a bijection

$$\{\text{topologies on } \mathscr{C}\} \longleftrightarrow \{\text{systems of local epimorphisms on } \mathscr{C}\},$$

with an inverse sending a system of local epimorphisms on $\mathscr{C}$ to the topology whose covering sieves are the sieves that are local epimorphisms.

Example A.4.9. The *initial* (or *discrete*) *topology* and *terminal* (or *indiscrete*) *topology* on an essentially small category $\mathscr{C}$ is defined to be the topology whose local epimorphisms are all morphisms and epimorphisms, respectively, see example [**KS06**, Ex.16.1.9].

Lemma A.4.10. Let $\mathscr{C}_\tau$ be an essentially small site, and let $f : P \to Q$ be a morphism of presheaves on $\mathscr{C}$. Then, f is a τ-local epimorphism if and only if for every $U \in \mathscr{C}$ and for every morphism of presheaves $q : \mathtt{h}_U \to Q$, there exists a τ-local epimorphism $S \to \mathtt{h}_U$ that fists into a commutative diagram

$$\begin{tikzcd} & & P \arrow[d, "f"] \\ S \arrow[r] \arrow[urr, dashed] & \mathtt{h}_U \arrow[r, "q"'] & Q \end{tikzcd}$$

of morphisms of presheaves.

Proof. See [**KS06**, Lem.16.1.6]. □

While morphisms of τ-sheaves that are surjective object-wise are epimorphisms of τ-sheaves, the inverse does not hold as can be deduced from the following corollary.

Corollary A.4.11. Let $\mathscr{C}$ be an essentially small category with pullbacks, let τ be a pretopology on $\mathscr{C}$, and let $f : P \to Q$ be a morphism of presheaves on $\mathscr{C}$. Then, f is a τ-local epimorphism if and only if for every object $U \in \mathscr{C}$ and for every section $q \in Q(U)$, there exists a τ-covering family $\mathscr{U} = \{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ and a section $p_\alpha \in P(U_\alpha)$ such that $\sigma_\alpha^*(q) = f_{U_\alpha}(p_\alpha)$, for every $\alpha \in A$.

Proof. See [**Jar15**, Lem.3.16]. □

Proposition A.4.12. Let $\mathscr{C}_\tau$ be an essentially small site, let I be an essentially small category, and let $F : I \to \mathsf{Mor}\big(\mathsf{PSh}(\mathscr{C})\big)$ be a functor. Assume that the morphism $F(i)$ is a τ-local epimorphism for every $i \in I$. Then, the morphism $\mathtt{colim}\, F$ is a τ-local epimorphism.

Proof. See [**KS06**, Prop.16.1.12]. □

Local Monomorphisms. Recall that in a Cartesian category $\mathscr{D}$, the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a monomorphism for every morphism $f : X \to Y$ in $\mathscr{D}$, and a formal diagram chase shows that f is a monomorphism if and only if Δ_f is an epimorphism. Hence, for an essentially small site $\mathscr{C}_\tau$, a morphism of presheaves is a τ-local monomorphism if and only if its diagonal is τ-local epimorphism.

Lemma A.4.13. Let $\mathscr{C}_\tau$ be an essentially small site, and let $f : P \to Q$ be a morphism of presheaves on $\mathscr{C}$. Then, f is a τ-local monomorphism if and only if for every $U \in \mathscr{C}$ and for every commutative diagram $\mathtt{h}_U \rightrightarrows P \to Q$, there exists a τ-covering sieve $S \subset \mathtt{h}_U$ that makes the digram $S \subset \mathtt{h}_U \rightrightarrows P$ commute.

Proof. See [**KS06**, Lem.16.2.3.(iii)]. □

Corollary A.4.14. Let $\mathscr{C}$ be an essentially small category with pullbacks, let τ be a pretopology on $\mathscr{C}$, and let $f : P \to Q$ be a morphism of presheaves on $\mathscr{C}$. Then, f is a τ-local monomorphism if and only if for every object $U \in \mathscr{C}$ and for every pair of sections $p, p' \in P(U)$ for which $f_U(p) = f_U(p') \in Q(U)$ there exists a τ-covering family $\mathscr{U} = \{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ such that $\sigma_\alpha^*(p) = \sigma_\alpha^*(p')$, for every $\alpha \in A$.

Proof. See [**Jar15**, Lem.3.16]. □

Local Isomorphisms. For an essentially small site $\mathscr{C}_\tau$, the set of τ-local isomorphisms is closed under pullback and satisfies the two-out-of-three property, see [**KS06**, Lem.16.2.4.(i) and (vii)]. A set of morphisms of presheaves that satisfies these properties is called a *system of local isomorphisms*. In fact, the map that sends a topology to the set of its local isomorphisms defines a bijection between topologies on an essentially small category and systems of local isomorphisms on it, see [**Bor94a**, Prop.5.6.2].

Proposition A.4.15. Let $\mathscr{C}_\tau$ be an essentially small site, let I be an essentially small category, and let $F : I \to \mathsf{Mor}\big(\mathsf{PSh}(\mathscr{C})\big)$ be a functor. Assume that the morphism $F(i)$ is a τ-local isomorphism for every $i \in I$. Then, the morphism $\mathtt{colim}\, F$ is a τ-local isomorphism.

Proof. See [**KS06**, Prop.16.3.4]. □

A.4.1.2. *Sheaves on Larger Sites.* For an essentially small site $\mathscr{C}_\tau$, the existence of a left exact τ-sheafification functor is due to having essentially small filtered categories of coverings for objects of $\mathscr{C}$, which does not necessarily hold for larger sites. For a site $\mathscr{C}_\tau$ that is not essentially small, the categories of presheaves and τ-sheaves are not necessarily locally small, and the τ-sheafification functor does not necessarily exist, and hence $\mathsf{Shv}_\tau(\mathscr{C})$ is not necessarily cocomplete, and its Cartesian structure is not necessarily closed. Although one can define τ-local epimorphism, and hence τ-local isomorphism, by their characteristic properties, the localisation of the category of presheaves $\mathsf{PSh}(\mathscr{C})$ with respect to τ-local isomorphisms does not have to be reflective on $\mathsf{Shv}_\tau(\mathscr{C})$. Yet, the situation may be remedied when $\mathscr{C}$ admits a τ-dense subcategory, see the Comparison Lemma [**Joh02**, §.C.Th.2.2.3]. Since all sites we consider are essentially small, we do not pursue the theory of sheaves on larger sites, and we refer the interested reader to [**Joh02**, §.C.2.2].

A.4.2. Continuous Maps of Sites. Let $\mathscr{C}_\tau$ and $\mathscr{D}_\varsigma$ be essentially small sites. A functor $f^{-1} : \mathscr{D} \to \mathscr{C}$ induces a functor

$$f_* : \mathsf{PSh}(\mathscr{C}) \to \mathsf{PSh}(\mathscr{D}),$$

given by precomposition with $(f^{-1})^{\mathrm{op}}$, which is called the *direct image* functor. The direct image functor admits a left adjoint $f^*_{\mathrm{pre}} : \mathsf{PSh}(\mathscr{D}) \to \mathsf{PSh}(\mathscr{D})$, given by the left Kan extension $f^*_{\mathrm{pre}} = \mathsf{Lan}_{(f^{-1})^{\mathrm{op}}}$, and it is called the *inverse image* functor.

Definition A.4.16. Let $\mathscr{C}_\tau$ and $\mathscr{D}_\varsigma$ be essentially small sites. A functor $f^{-1} : \mathscr{D} \to \mathscr{C}$ is said to be *continuous* with respect to the topologies τ and ς if f_* sends τ-sheaves to ς-sheaves. A *continuous map of sites* $f : \mathscr{C}_\tau \to \mathscr{D}_\varsigma$ is a continuous functor $f^{-1} : \mathscr{D} \to \mathscr{C}$.

Since the category of sheaves on an essentially small site is a reflective localisation of the category of presheaves with left exact reflector, f^{-1} is continuous if and only if f^*_{pre} preserves local isomorphisms, recall Remark 1.1.6. Also, the functor f^{-1} is continuous if and only if for every ς-sieve S in $\mathscr{D}$, the sieve generated by $f^{-1}(S)$ is a τ-sieve in $\mathscr{C}$, see [**Joh02**, §.C.2.3]. In particular, when f^{-1} is Cartesian between Cartesian categories and the topologies τ and ς are defined by pretopologies, the functor f^{-1} is continuous if it preserves covering families.

For a continuous map of sites $f : \mathscr{C}_\tau \to \mathscr{D}_\varsigma$, there exists an adjunction

$$f^* : \mathsf{Shv}_\varsigma(\mathscr{D}) \rightleftarrows \mathsf{Shv}_\tau(\mathscr{C}) : f_*,$$

where f^* is given by the composition of f^*_{pre} with the associated τ-sheaf functor $-^{\mathsf{a}_\tau}$, and it is called the *sheaf inverse image* functor. Since f^* is a left adjoint, it preserves colimits. When, in addition, f^* is left exact, the continuous map f is called a *morphism of sites*.

In addition to the notion of continuous functors, we need to recall the notion of almost cocontinuous functors, which admits a well-behaved direct image, as recalled below.

Definition A.4.17 ([**Sta17**, Tag 04B7]). Let $\mathscr{C}$ and $\mathscr{D}$ be essentially small categories with pullbacks, and let τ and ς be pretopologies on $\mathscr{C}$ and $\mathscr{D}$, respectively. A functor $f^{-1} : \mathscr{D} \to \mathscr{C}$ is said to be *almost cocontinuous* if for every object $V \in \mathscr{D}$ and for every τ-covering family $\mathscr{U} = \{\sigma_\alpha : U_\alpha \to f^{-1}(V) \mid \alpha \in A\}$ there exists a ς-covering family $\mathscr{V} = \{\delta_\beta : V_\beta \to V \mid \beta \in B\}$ such that for every $\beta \in B$ either

(1) the morphisms $f^{-1}(\delta_\beta)$ factorises through σ_α, for some $\alpha \in A$; or

(2) the empty sieve is a τ-covering sieve for $f^{-1}(V_\beta)$.

LEMMA A.4.18. Assume that $\mathscr{C}$ and $\mathscr{D}$ be essentially small categories, let τ and ς be pretopologies on $\mathscr{C}$ and $\mathscr{D}$, respectively, and let $f^{-1} : \mathscr{D} \to \mathscr{C}$ be a continuous and an almost cocontinuous functor. Then, the direct image functor

$$f_* : \mathsf{Shv}_\tau(\mathscr{C}) \to \mathsf{Shv}_\varsigma(\mathscr{D})$$

commutes with pushout squares.

PROOF. See [**Sta17**, Tag 04B9]. □

DEFINITION A.4.19. A category that is equivalent to the category of sheaves on a small Grothendieck site is called a *Grothendieck topos*. A *geometric morphism* $f : \mathscr{E} \to \mathscr{F}$ between Grothendieck topoi is an adjunction

$$f^* : \mathscr{F} \rightleftarrows \mathscr{E} : f_*,$$

in which f^* is left exact.

A.4.2.1. *Points of Sites.* Similar to sheaves on topological spaces, isomorphisms can be detected on the level of stalks, for sites that have enough points.

DEFINITION A.4.20. Let $\mathscr{C}_\tau$ be a site. A *point of the site* $\mathscr{C}_\tau$ is a geometric morphism $p : \mathsf{Set} \cong \mathsf{Shv}(*) \rightleftarrows \mathsf{Shv}_\tau(\mathscr{C})$, where $*$ is the terminal site. The inverse image $p^* : \mathsf{Shv}_\tau(\mathscr{C}) \to \mathsf{Set}$ is called the *stalks functor* at p, whereas the direct image p_* is called the *τ-skyscraper sheaf functor* at p.

EXAMPLE A.4.21. Let X be a small topological spaces, let $\mathscr{C}$ be the category of open sets in X, and let τ be the topology generated by open covers in X. Then, every set-theoretic point $x \in X$ defines a point p_x of $\mathscr{C}_\tau$, for which p_x^* and p_{x*} are the conventional stalks and skyscraper sheaf functors, respectively.

For every point p of a subcanonical site $\mathscr{C}_\tau$, the composition of the stalks functor at p with the Yoneda embedding yields a functor

$$p^* \circ \mathtt{h}_- : \mathscr{C} \to \mathsf{Shv}_\tau(\mathscr{C}) \to \mathsf{Set}.$$

In facts, points of an essentially small site $\mathscr{C}_\tau$ correspond to flat functors $\mathscr{C} \to \mathsf{Set}$ that are continuous, with respect to τ and the canonical topology on Set, see [**MLM92**, §VII.5.Cor.4]. Since the category Set is cocomplete, every functor $u : \mathscr{C} \to \mathsf{Set}$ induces the u-tensor-$\underline{\mathtt{Hom}}$ adjunction

$$- \otimes_{\mathscr{C}} u : \mathsf{PSh}(\mathscr{C}) \rightleftarrows \mathsf{Set} : \mathtt{Hom}^{\mathscr{C}}(u, -),$$

as in Example A.3.8. When u is taken to be the composition $p^*_{\mathrm{pre}} \circ \mathtt{h}_-$ for a point p on $\mathscr{C}_\tau$, there exist canonical isomorphisms

$$p^* \cong - \otimes_{\mathscr{C}} u \qquad \text{and} \qquad p_* \cong \mathtt{Hom}^{\mathscr{C}}(u, -),$$

see [**MLM92**, p.381.(11)], so in particular $- \otimes_{\mathscr{C}} u$ is left exact.

Let $\mathscr{C}$ be an essentially small category, recall that a functor $u : \mathscr{C} \to \mathsf{Set}$ is flat if and only if its category of elements $\mathsf{El}(u)$ is cofiltered, see [**MLM92**, §.VII.6.Th.3]. On the other hand, since there exists a canonical equivalence of categories $\mathsf{Set} \cong \mathsf{Shv}_{\mathrm{can}}(\mathsf{Set})$, a functor $u : \mathscr{C} \to \mathsf{Set}$ is continuous, with respect to τ and the canonical topology on Set, if the geometric morphism $- \otimes_{\mathscr{C}} u : \mathsf{PSh}(\mathscr{C}) \rightleftarrows \mathsf{Set} : \mathtt{Hom}^{\mathscr{C}}(u, -)$ factorises through the τ-sheafification geometric morphism $\mathsf{PSh}(\mathscr{C}) \rightleftarrows \mathsf{Shv}_{\tau}(\mathscr{C})$, *i.e.* if $\mathtt{Hom}^{\mathscr{C}}(u, S)$ is a τ-sheaf for every set $S \in \mathsf{Set}$ and the τ-sheafification morphism $\eta_P : P \to P^{\mathsf{a}_\tau}$ is mapped to an isomorphism by $- \otimes_{\mathscr{C}} u$, for every presheaf $P \in \mathsf{PSh}(\mathscr{C})$, see [**MLM92**, §VII.5.Lem.3] and [**Joh02**, §.C.Lem.2.3.8].

Definition A.4.22. Let $\mathscr{C}_\tau$ be a site. Then, a *conservative set of points* of $\mathscr{C}_\tau$ is a set $C = \{p_i \mid i \in I\}$ of points of $\mathscr{C}_\tau$, such that a morphism $f : P \to Q$ of τ-sheaves on $\mathscr{C}$ is an isomorphism if and only if the morphism of stalks $p_i^*(f)$ is a bijection for every $p_i \in C$. The site $\mathscr{C}_\tau$ is said to *have enough points* if it admits a conservative set of points.

A.4.3. Grothendieck Topologies in Algebraic Geometry. In some situations, one may establish a notion that is well-behaved on stalks at points for a certain site, in which case, it is convenient to consider sheaves on that site, see the proof of [**MV99**, §.3.Th.2.21 and §.3.Th.2.23]. Also, one may have a well-behaved notion, when certain (homotopy) colimits exist and are represented; which may be forced to hold by considering sheaves with respect to the topology whose coving sieves are generated by the desired colimit cocones, as in §.4.2.

In addition to desired behaviours, the choice of the topology may also be influenced by the available machinery. For example, Voevodsky utilised the *cdh*-topology to construct (properly supported) geometric motives for singular schemes over fields of characteristic zero, as the latter admit resolutions of singularities. Whereas, in the absence of resolutions of singularities, geometric motives for singular schemes were extended to perfect fields in [**Kel12**] using the ℓdh-topology, which is an extension of the *cdh*-topology that riles on Gabber's Local Uniformisation Theorem [**ILO16**, Th.3.2.1].

We conclude this section by recalling the topologies used in this thesis. For an elaborative treatment for topologies used in algebraic geometry, see [**GK15**].

Fix a Noetherian scheme S of finite Krull dimension, and recall the conventions and notations in §.0.2. In particular, the category of Noetherian schemes of finite Krull dimensions is denoted by $\overset{\mathrm{fd}}{\mathsf{Noe}}$, whereas the category of schemes of finite type over S is denoted by $\overset{\mathrm{ft}}{\mathsf{Sch}}/S$, and an S-scheme refers to an object in $\overset{\mathrm{ft}}{\mathsf{Sch}}/S$. Also, the full subcategory in $\mathsf{Sch}^{\mathrm{ft}}/S$ of smooth (resp. proper) S-schemes is denoted by Sm/S (resp. Prop/S).

Definition A.4.23. Assume that U is a Noetherian scheme in $\mathsf{Noe}^{\mathrm{fd}}$. A finite family of morphisms $\{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ is said to be an *étale* (resp. *proper*) *covering* family of U, if

- the morphism σ_α is étale (resp. is proper) for every $\alpha \in A$; and
- it is jointly surjective, *i.e.* the underlying map of coproduct morphism $\coprod_{\alpha\in A} \sigma_\alpha : \coprod_{\alpha\in A} U_\alpha \to U$ is a surjection of sets.

The *étale* (resp. *proper*) *pretopology* on the category $\mathsf{Noe}^{\mathrm{fd}}$ is the pretopology whose covering families are

- étale (resp. proper) finite covering families in $\mathsf{Noe}^{\mathrm{fd}}$; and
- the empty covering family of the empty scheme.

For more general scheme, one needs to consider all such families (not necessarily finite ones). However, for Noetherian schemes, such coverings always admit finite refinements.

Definition A.4.24. Assume that U is a Noetherian scheme in $\mathsf{Noe}^{\mathrm{fd}}$. An étale (resp. a proper) covering family $\{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ is said to be a *Nisnevich* (resp. *cdp*[3]) covering family if it is *completely decomposed*, *i.e.* for every $u \in U$ there exists $\alpha \in A$ and $u_\alpha \in U_\alpha$ such that $\sigma_\alpha(u_\alpha) = u$ and the induced morphism of residue fields $\kappa(u) \to \kappa(u_\alpha)$ is an isomorphism[4].

The *Nisnevich* (resp. *cdp*[5]) *pretopology* on the category $\mathsf{Noe}^{\mathrm{fd}}$ is the pretopology whose covering families are

- Nisnevich (resp. *cdp*) finite covering families in $\mathsf{Noe}^{\mathrm{fd}}$; and
- the empty covering family of the empty scheme.

Whereas, the *cdh-pretopology* on the category $\mathsf{Noe}^{\mathrm{fd}}$ is the pretopology generated by the Nisnevich and the *cdp*-pretopologies.

Thus, the Nisnevich (resp. *cdp*) pretopology is coarser than the étale (resp. proper) pretopology, and finer than the Zariski (resp. closed) pretopology in which nonempty covering families consist of open (resp. closed) immersions.

Remark A.4.25. Assume that U is a Noetherian scheme in $\mathsf{Noe}^{\mathrm{fd}}$, and let $\mathscr{U} \coloneqq \{\sigma_\alpha : U_\alpha \to U \mid \alpha \in A\}$ be an étale (resp. a proper) covering family of U. Then, $\mathscr{U}$ is a Nisnevich (resp. *cdp*) covering family, if and only if the map

$$\Big(\coprod_{\alpha\in A} \sigma_\alpha\Big)_* : \mathsf{Noe}^{\mathrm{fd}}\Big(\operatorname{Spec} k, \coprod_{\alpha\in A} U_\alpha\Big) \to \mathsf{Noe}^{\mathrm{fd}}\Big(\operatorname{Spec} k, U\Big)$$

[3]Remark A.4.25 shows that the *cdp*-topology coincides with the *envelop topology*, used in [GS09].
[4]This condition is also referred to by the *Nisnevich condition*, as it first appeared in [Nis89].
[5]The notation *cdp* appears in [GK15], but some authors use *pro cdh* instead; others use *abs* in reference to abstract blow up squares.

is surjective for every field k.

REMARK A.4.26. All the morphisms in the covering families defined above are of finite type, and hence all the pretopologies defined above restrict to the category $\mathsf{Sch}^{\mathrm{ft}}/S$. In fact, for every pretopology τ defined above on the large category category $\mathsf{Noe}^{\mathrm{fd}}$, the canonical monoidal finite type-fibred essentially small category $\mathsf{Sch}^{\mathrm{ft}}/- : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{op}} \to \mathsf{CAT}_2^{\otimes}$, as in [**CD13**, §.1.1.Ex.4, 11, 23, and 28], induces a monoidal finite type-fibred essentially small site

$$\mathsf{Sch}^{\mathrm{ft}}/-_{\tau} : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{op}} \to \mathsf{Site}_2^{\otimes},$$

where $\mathsf{Site}_2^{\otimes}$ is the 2-category of symmetric monoidal essentially small sites, weak monoidal continuous functors between them, and monoidal natural transformations between the latter. When τ is coarser than the étale (resp. proper) pretopology, there exists a monoidal smooth (resp. proper)-fibred essentially small site

$$\mathsf{Sm}/-_{\tau} : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{op}} \to \mathsf{Site}_2^{\otimes} \qquad (\text{resp. } \mathsf{Prop}/-_{\tau} : \mathsf{Noe}^{\mathrm{fd}\,\mathrm{op}} \to \mathsf{Site}_2^{\otimes})$$

The pseudofunctor $\mathsf{Prop}/-_{\tau}$ is explored further in §.4.2.1.2.

The étale pretopology is subcanonical on $\mathsf{Sch}^{\mathrm{ft}}/S$ (resp. Sm/S), and hence the Nisnevich pretopology is subcanonical on the category $\mathsf{Sch}^{\mathrm{ft}}/S$ (resp. Sm/S), see [**SGA73**, Exposé VII.§.2]. On the other hand, the closed pretopology is not subcanonical on $\mathsf{Sch}^{\mathrm{ft}}/S$, and hence the *cdp*-pretopology, *cdh*-pretopology, and the proper pretopology are not subcanonical. That is, a surjective closed immersion $i : z \hookrightarrow\!\!\!\!\!\rightarrow x$ is a closed cover of $x \in \mathsf{Sch}^{\mathrm{ft}}/S$. However, $i^* : \mathsf{h}_z(x) \to \mathsf{h}_z(z)$ is not always a bijection. For example, let $S = \operatorname{\mathtt{Spec}} k$ for a field k, and let i be the surjective closed immersion $\operatorname{\mathtt{Spec}} k[t]/(t^2) \hookrightarrow\!\!\!\!\!\rightarrow \operatorname{\mathtt{Spec}} k[t]/(t^3)$ in $\mathsf{Sch}^{\mathrm{ft}}/S$.

REMARK A.4.27. Assume that $\sigma : Y \to X$ is a *cdp*-cover and that X is reduced. Then, σ admits a refinement by a birational *cdp*-cover $\sigma' : Y' \to X$, *i.e.* there exists an open dense immersion $j : U \hookrightarrow X$ such that the base change of σ' along j is an isomorphism, see [**MVW06**, Ex.12.25].

A.4.3.1. *Completely Decomposed Structures.*

DEFINITION A.4.28. Suppose that x is an S-scheme. Then, a Cartesian square

$$\begin{tikzcd} b \arrow[r, hook, "e'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\ulcorner", very near start] & y \arrow[d, "p"] \\ a \arrow[r, hook, "e"'] & x \end{tikzcd} \tag{91}$$

in $\mathsf{Sch}^{\mathrm{ft}}/S$ is called

- a *Nisnevich square* over x if p is an étale morphism and e is an open immersion, such that the base change

$$(y \smallsetminus e')_{\mathrm{red}} \to (x \smallsetminus e)_{\mathrm{red}}$$

is an isomorphism; and

- a *cdp-square*[6] over x if p is a proper morphism and e is a closed immersion, such that the base change

$$\big(y \smallsetminus e'\big) \to \big(x \smallsetminus e\big)$$

is an isomorphism.

LEMMA A.4.29. The Nisnevich (resp. *cdp*) topology on the category of S-schemes coincides with the Grothendieck topology generated by the covering families

- $\{p : y \to x, e : a \hookrightarrow x\}$, for every pair of morphisms $p : y \to x$ and $e : a \hookrightarrow x$ that fit into a Nisnevich (resp. *cdp*) square in $\mathsf{Sch}^{\mathrm{ft}}/S$; and
- the empty covering family of the empty S-scheme.

Equivalently, a presheaf of sets $P \in \mathsf{PSh}(\mathsf{Sch}^{\mathrm{ft}}/S)$ is a Nisnevich (resp. *cdp*) sheaf if and only if

- P sends every Nisnevich (resp. *cdp*) square to a Cartesian square; and
- $P(\varnothing_S) \cong *$.

PROOF. See [**Voe10a**, Cor.2.17] and [**Voe10b**, Th.2.2]. □

PROPOSITION A.4.30. The Nisnevich (resp. *cdp*) sheafification of the Yoneda embedding takes every Nisnevich (resp. *cdp*) square of S-schemes to a cocartesian square of Nisnevich (resp. *cdp*) sheaves on $\mathsf{Sch}^{\mathrm{ft}}/S$.

PROOF. See [**Voe10a**, Cor.2.16] and [**Voe10b**, Th.2.2]. □

The analogue of Lemma A.4.29 and Proposition A.4.30 hold for the Nisnevich (resp. *cdp*) pretopology on Sm/S (resp. Prop/S), see [**Voe10b**, Lem.2.3].

A.4.3.2. *Splitting Sequences.* Assume that $f : Y \to X$ is a morphism of schemes. A *splitting sequence* for f is a finite sequence of closed embeddings

$$\varnothing = Z_{n+1} \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_0 = X,$$

such that the base change $f^{-1}(Z_i - Z_{i-1}) \to (Z_i - Z_{i-1})$ splits, *i.e.* it admits a section.

LEMMA A.4.31. Let $\mathscr{U} = \{\sigma_\alpha : u_\alpha \to u \mid \alpha \in A\}$ be a Nisnevich (or *cdp*) covering family of an S-scheme u. Then, the morphism $\coprod_{\alpha\in A} f_\alpha : \coprod_{\alpha\in A} u_\alpha \to u$ has a splitting sequence.

PROOF. See [**Voe10b**, Lem.2.16, Prop.2.17, and Prop.2.18]. □

[6] Some sources refer to the square (91) by an *abstract blow up* square, as in [**MVW06**, Def.12.21]; others reserve the term for a square that satisfies additional properties, as in [**SV00**, Def.2.2.4].

τ	P_τ
Zariski	is a local ring
Closed	is a integral domain
Nisnevich	is a Henselian local ring
cdh	is a Henselian valuation ring
Étale	is a Henselian local ring with a separably closed residue field

TABLE 1. A conservative set of points for the site $\mathsf{Sch}^{\mathrm{ft}}/S_\tau$.

A.4.3.3. *Points.* Some of the aforementioned sites have enough points. That is, they admit conservative sets of points, such that isomorphisms between sheaves are determined on the stalks at those points, see §.A.4.22.

The general definition of a point of a topos is rather abstract, and does offer a scheme-theoretic description that fits with the geometric intuition of points of schemes. However, on sites that admits some finiteness conditions, like $\mathsf{Sch}^{\mathrm{ft}}/S$ and its subcategories, the points of the topos $\mathsf{Shv}_\tau(\mathsf{Sch}^{\mathrm{ft}}/S)$, for a pretopology τ on $\mathsf{Sch}^{\mathrm{ft}}/S$, might be given by some S-schemes. Recall that, for every S-scheme x, since the category $\mathsf{Sch}^{\mathrm{ft}}/S$ is Cartesian and the corepresentable functor h^x commutes with limits, the scheme x gives rise to a point $(\mathsf{h}^{x,*}, \mathsf{h}^x_*)$ if h^x is continuous, which in particular requires the canonical morphism

$$\coprod_{\alpha\in A} \mathsf{Sch}^{\mathrm{ft}}/S(x, u_\alpha) \to \mathsf{Sch}^{\mathrm{ft}}/S(x, u) \tag{92}$$

to be surjective, for every τ-covering family $\{\sigma_\alpha : u_\alpha \to u \mid \alpha \in A\}$ of an S-scheme u, *cf.* [**GK15**, Def.0.1] and [**GL01**, §.2]. A scheme x, for which (92) is an surjective, does not, a priori, define a point, as that requires sending all τ-covering sieves to colimits cocones. However, for most the pretopologies that we are interested in, there exist conservative sets of points that admits such a scheme-theoretic description, see [**GK15**, Th.0.2].

LEMMA A.4.32. Assume that $\varphi : \mathscr{X} \to \mathscr{Y}$ is a morphism of τ-sheaves on $\mathsf{Sch}^{\mathrm{ft}}/S$, for a pretopology τ, in the Table 1, on $\mathsf{Sch}^{\mathrm{ft}}/S$. Then φ is an isomorphism if and only if the morphism of stalks $\varphi_{\mathsf{Spec}\,R}$ is a bijection, for every ring R (not necessarily of finite type over S) that satisfies the property P_τ, in the Table 1.

PROOF. See [**GK15**, Th.2.6]. □

REMARK A.4.33. The proper and *cdp*-pretopologies on $\mathsf{Sch}^{\mathrm{ft}}/S$ do not admits a conservative set of points given by affine schemes, see [**GK15**, p.4673]. However, points of a site are points of a coarser site. In particular, valuation rings (resp. valuation rings with algebraically closed fraction fields) define points for the *cdp*-pretopology (resp. proper pretopology) on $\mathsf{Sch}^{\mathrm{ft}}/S$, see [**GL01**, Prop.2.1 and Prop.2.2].

A.4.3.4. *Representable Sheaves.* For canonical sites, representable presheaves are sheaves, and morphisms between them correspond to morphisms in the original category. Since some of the pretopologies we are interested in are not subcanonical, we devote this section to understanding morphisms between their representable sheaves on the categories $\mathsf{Sch}^{\mathrm{ft}}/S$ and Prop/S. The argument below essentially follows [**Voe96**, §.3.2], and the statements in the rest of this section are applied in §.4.2.1.

Remark A.4.34. For every (proper) S-scheme p, the representable presheaf h_p is additive, by the very definition of colimits, *i.e.* for (proper) S-schemes z and w, the canonical morphism

$$\mathsf{h}_p(z \coprod w) \to \mathsf{h}_p(z) \times \mathsf{h}_p(w)$$

is an isomorphism. Also, the τ-sheaf $\mathsf{h}_p^{\mathsf{a}\tau}$ is additive for every pretopology τ on (proper) S-schemes that is finer than the closed pretopology. For a (additively-)saturated pretopology τ on the category of (proper) S-schemes that is finer than the closed pretopology, and for a τ-covering family $\mathscr{U} = \{\sigma_\alpha : z_\alpha \to z \mid \alpha \in A\}$ in Prop/S (resp. $\mathsf{Sch}^{\mathrm{ft}}/S$), one has a τ-covering family

$$\mathscr{U}' \coloneqq \left\{ \coprod_{\alpha \in A} \sigma_\alpha : \coprod_{\alpha \in A} z_\alpha \to z \right\}$$

in Prop/S (resp. $\mathsf{Sch}^{\mathrm{ft}}/S$). The additivity of h_p and $\mathsf{h}_p^{\mathsf{a}\tau}$ implies that sections of h_p and $\mathsf{h}_p^{\mathsf{a}\tau}$ on $\mathscr{U}$ correspond to their sections on $\mathscr{U}'$. Thus, without loss of generality, when considering h_p and $\mathsf{h}_p^{\mathsf{a}\tau}$, one may assume the involved τ-covering families are singletons.

Example A.4.35. Additively-saturated pretopologies on the category of (proper) S-schemes, that are finer than the closed pretopology, include:

(1) the proper pretopology, see [**Sta17**, Tags 01T1, 01KH, and 0BX5];
(2) the *cdp*-pretopology, see Remark A.4.25;
(3) the *finite pretopology* (resp. *cdf-pretopology*), which is coarser than the proper pretopology (resp. *cdp*-pretopology), whose nonempty covering families consist of finite morphisms, see [**Sta17**, Tag 0CYI]; and
(4) the *unramified proper pretopology* (resp. *unramified cdp-pretopology*), which is coarser than the proper pretopology (resp. *cdp*-pretopology), whose nonempty covering families consist of unramified morphisms, see [**Sta17**, Tag 02G4].

While the proper pretopology (resp. *cdp*-pretopology) on the category Prop/S is saturated, as morphisms between proper S-schemes are proper, its counterpart on the category $\mathsf{Sch}^{\mathrm{ft}}/S$ is not saturated. For instance, let $S = \mathsf{Spec}\, k$, for a field k. Then, for every S-scheme X that admits an k-rational point, the structure morphism $X \to S$ is a cover in the saturation of the proper pretopology (resp. *cdp*-pretopology) on $\mathsf{Sch}^{\mathrm{ft}}/S$. Also, the finite pretopology, the *cdf*-pretopology, the unramified proper pretopology, and the unramified *cdp*-pretopology are not saturated on the category of (proper) S-schemes.

Lemma A.4.36 ([**Voe96**, Lem.3.2.2]). Assume that τ is a cover-saturated pretopology on (proper) S-schemes that is finer than the closed pretopology, such that τ-covers are surjective, and let p and q be S-schemes, such that p is reduced. Then,

- the canonical map $\mathsf{Sch}^{\mathrm{ft}}/S(p,q) \to \mathsf{Shv}_\tau(\mathsf{Sch}^{\mathrm{ft}}/S)(\mathtt{h}_p^{\mathsf{a}\tau}, \mathtt{h}_q^{\mathsf{a}\tau})$ is an injection; and
- when p and q are proper S-schemes, the canonical map

$$\mathsf{Prop}/S(p,q) \to \mathsf{Shv}_\tau(\mathsf{Prop}/S)(\mathtt{h}_p^{\mathsf{a}\tau}, \mathtt{h}_q^{\mathsf{a}\tau})$$

  is an injection.

Proof. Assume that $f_0, f_1 : p \to q$ are morphisms of (proper) S-schemes, such that p is reduced and suppose that $f_{0,*} = f_{1,*} : \mathtt{h}_p^{\mathsf{a}\tau} \to \mathtt{h}_q^{\mathsf{a}\tau}$. Then, in particular, for the section $\mathrm{id}_p^{\mathsf{a}} \in \mathtt{h}_p^{\mathsf{a}\tau}(p)$, one has $\big(f_{0,*}(\mathrm{id}_p)\big)^{\mathsf{a}} = \big(f_{1,*}(\mathrm{id}_p)\big)^{\mathsf{a}}$, and hence there exists a τ-cover $\sigma : z \to p$ such that

$$f_0 \circ \sigma = \sigma^*\big(f_{0,*}(\mathrm{id}_p)\big) = \sigma^*\big(f_{1,*}(\mathrm{id}_p)\big) = f_1 \circ \sigma \in \mathtt{h}_q(z).$$

Since p is reduced and τ-covers are surjective, a diagram chase shows that σ is an epimorphism in the category of (proper) S-schemes, and hence $f_0 = f_1$. $\square$

Proposition A.4.37 ([**Voe96**, Prop.3.2.5]). Assume that τ is a additively-saturated pretopology on (proper) S-schemes, and let $f : p \to q$ be a morphism of (proper) S-schemes. Then,

(1) the morphism $f_* : \mathtt{h}_p^{\mathsf{a}\tau} \to \mathtt{h}_q^{\mathsf{a}\tau}$ is an epimorphism if and only if f is a cover in the saturation of τ; and
(2) assuming that τ is finer than the closed pretopology, such that τ-covers are surjective, the morphism $f_* : \mathtt{h}_p^{\mathsf{a}\tau} \to \mathtt{h}_q^{\mathsf{a}\tau}$ is a monomorphism if and only if f is universally injective.

Proof.

(1) Assume that f is a cover in the saturation of τ, *i.e.* there exists a morphism $\sigma' : p' \to p$ of (proper) S-schemes such that $\sigma \coloneqq f \circ \sigma'$ is a τ-cover. The morphism $\sigma_* : \mathtt{h}_{p'} \to \mathtt{h}_q$ is a τ-local epimorphism because it factorises as an epimorphism $\mathtt{h}_{p'} \to \mathtt{im}\,\sigma_*$ followed by the inclusion $\mathtt{im}\,\sigma_* \subset \mathtt{h}_q$ of the τ-covering sieve generated by σ. Thus, the morphism $f_* : \mathtt{h}_p \to \mathtt{h}_q$ is a τ-local epimorphism, and hence the morphism $f_* : \mathtt{h}_p^{\mathsf{a}\tau} \to \mathtt{h}_q^{\mathsf{a}\tau}$ is an epimorphism of τ-sheaves.

    On the other hand, assume that $f_* : \mathtt{h}_p^{\mathsf{a}\tau} \to \mathtt{h}_q^{\mathsf{a}\tau}$ is an epimorphism of τ-sheaves, *i.e.* $f_* : \mathtt{h}_p \to \mathtt{h}_q$ is a τ-local epimorphism. For $\mathrm{id}_q \in \mathtt{h}_q(q)$, there exists a τ-cover $\sigma : w \to q$ and a section $a \in \mathtt{h}_p(w)$ such that

$$f \circ a = f_*(a) = \sigma^*(\mathrm{id}_q) = \sigma,$$

    by Corollary A.4.11. Thus, the morphism f is a cover in the saturation of τ.

(2) The proof of the *if* implication essentially follows [**And17**], which corrects a mistake in the proof of [**Voe96**, Prop.3.2.5.(i)].

Assume that $f : p \to q$ is universally injective, let z be a (proper) S-scheme, and let $a_0, a_1 \in \mathtt{h}_p(z)$ such that $f \circ a_0 = f_*(a_0) = f_*(a_1) = f \circ a_1$. Consider the commutative solid diagram

$$\begin{tikzcd}
z \arrow[drr, bend left, "a_0"] \arrow[dr, dotted, "\lambda"] \arrow[ddr, bend right, "a_1"'] & & \\
& p \times_q p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & p \arrow[d, "f"] \\
& p \arrow[r, "f"'] & q,
\end{tikzcd}$$

of (proper) S-schemes, and let $\lambda : z \to p \times_q p$ be the unique such morphism that makes the whole diagram commute. Since $f : p \to q$ is universally injective, the diagonal morphism $\Delta_f : p \to p \times_q p$ is a surjective closed immersion, by [**Sta17**, Tag 01S4]. Let $i : z_{\mathrm{red}} \hookrightarrow\!\!\!\to z$ be the close immersion of the maximal reduced closed subscheme in z. The morphism i is a τ-cover, as the τ-pretopology is finer than the closed pretopology. The morphism $\lambda \circ i$ factories through every surjective closed immersion of $p \times_q p$, in particular, it factorises through the diagonal morphism Δ_f, which implies that

$$a_0 \circ i = a_1 \circ i.$$

Thus, the morphism $f_* : \mathtt{h}_p \to \mathtt{h}_q$ is a τ-local monomorphism, by Corollary A.4.14, and hence the morphism $f_* : \mathtt{h}_p^{\mathtt{a}\tau} \to \mathtt{h}_q^{\mathtt{a}\tau}$ is a monomorphism of τ-sheaves.

On the other hand, assume that $f_* : \mathtt{h}_p^{\mathtt{a}\tau} \to \mathtt{h}_q^{\mathtt{a}\tau}$ is a monomorphism of τ-sheaves, and consider the commutative diagram

$$\begin{tikzcd}
p \arrow[drr, bend left, "\mathrm{id}_p"] \arrow[dr, "\Delta_f"] \arrow[ddr, bend right, "\mathrm{id}_p"'] & & \\
& p \times_q p \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & p \arrow[d, "f"] \\
& p \arrow[r, "f"'] & q
\end{tikzcd}$$

of (proper) S-schemes. Recall that both the Yoneda embedding and the τ-sheafification functor preserve finite limits. In particular, the morphisms $\pi_{0,*}$ and $\pi_{1,*}$ are base changes of f_* along itself, and hence they are monomorphisms of τ-sheaves. In fact, the morphisms $\pi_{0,*}$ and $\pi_{1,*}$ are isomorphisms of τ-sheaves, as $\mathrm{id}_{p,*}$ is an epimorphism of τ-sheaves and the category of τ-sheaves of sets is a balanced category. Thus, $\Delta_{f,*}$ is an epimorphism of τ-sheaves, and hence Δ_f is a cover in the saturation of τ, by (1). In particular, Δ_f is surjective, and f is universally injective, by [**Sta17**, Tag 01S4].

□

Corollary A.4.38. Let τ be a additively-saturated pretopology on the category of (proper) S-schemes that is finer than the closed pretopology and coarser than the proper pretopology, and let $f : p \to q$ be a morphism of (proper) S-schemes. Then, the morphism $f_* : \mathsf{h}_p^{\mathsf{a}\tau} \to \mathsf{h}_q^{\mathsf{a}\tau}$ is an isomorphism only if the morphism f is a universal homeomorphism.

Proof. Assume that $f_* : \mathsf{h}_p^{\mathsf{a}\tau} \to \mathsf{h}_q^{\mathsf{a}\tau}$ is an isomorphism, then f is a universally injective cover in the saturation of τ, by Proposition A.4.37. In particular, there exists a morphism $\sigma' : p' \to p$ of (proper) S-schemes such that $\sigma \coloneqq f \circ \sigma'$ is a τ-cover. Since τ is coarser than the proper pretopology, the morphism σ is surjective and universally closed, and hence a universal topological epimorphism[7]. This implies that f is also a universal topological epimorphism, and hence every base change in Sch of f is both an injection and a topological epimorphism. That is the underlying continuous map of every base change in Sch of f is a monomorphism and an extremal epimorphism in the category of topological spaces, and hence a homeomorphism, see [**Nak89**, §.2.6-§.2.9]. Therefore, f is a universal homeomorphism. □

Remark A.4.39. In the sequel, we restrict our attention to additively-saturated pretopologies on the category of proper S-schemes that are finer than the *cdf*-pretopology and coarser than the proper pretopology.

Example A.4.40. Pretopologies on the category of proper S-schemes that satisfy the assumptions of Remark A.4.39 include the finite pretopology, the *cdf*-pretopology, the proper pretopology, and the *cdp*-pretopology.

Proposition A.4.41. Let τ be a pretopology on Prop/S as in Remark A.4.39, and let $f : p \to q$ be a morphism of proper S-schemes. Then, the morphism $f_* : \mathsf{h}_p^{\mathsf{a}\tau} \to \mathsf{h}_q^{\mathsf{a}\tau}$ is an isomorphism if and only if f is a universal homeomorphism.

Proof. Since the *cdf*-pretopology is finer than the closed pretopology, the *only if* implication is the statement of Corollary A.4.38.

[7]A (*universal*) *topological epimorphism* f is a morphism of schemes for which the underlying continuous map (of every base change in Sch) of f is a quotient map, see [**Voe96**, §.3.1].

Assume that f is a universal homeomorphism. Then, f is a surjective universally injective finite morphism, by [**Gro65**, Prop.2.4.5]. In particular, for every field k, the induced map

$$f_* : \overset{\mathrm{fd}}{\mathsf{Noe}}\big(\,\mathsf{Spec}\,k, P\big) \to \overset{\mathrm{fd}}{\mathsf{Noe}}\big(\,\mathsf{Spec}\,k, Q\big)$$

is an injection, by [**Sta17**, Tag 01S4], where P and Q are the underlying schemes for p and q, respectively. Also, since f is of finite type, the map f_* is surjective for every algebraically closed field k. However, we need to show that f_* is surjective for every field k.

For a field k, let $y : \mathsf{Spec}\,k \to Q$ be a morphism of schemes, and consider the Cartesian square

$$\begin{tikzcd} Z \arrow[r, "\underline{y}"] \arrow[d, "\underline{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & P \arrow[d, "f"] \\ \mathsf{Spec}\,k \arrow[r, "y"'] & Q \end{tikzcd}$$

in the category $\overset{\mathrm{fd}}{\mathsf{Noe}}$. The morphism f is a finite universal homeomorphism, and hence Z is a one-point scheme $\mathsf{Spec}\,R$ and $\underline{f}$ is induced by a finite ring homomorphism $\psi : k \hookrightarrow R$, to a local ring R of Krull dimension zero. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa \coloneqq {}^{R}\!/\!_{\mathfrak{m}}$. Then, the induced homomorphism $k \hookrightarrow \kappa$ is a finite field extension. Assuming that $[\kappa : k] \neq 1$, there exist distinct ring homeomorphisms $\kappa \to \kappa$ over k, which contradicts with $\underline{f}$ being universally injective. Thus, one has $[\kappa : k] = 1$, *i.e.* the residue field of Z at its unique point is isomorphic to k. Hence, y lifts along f, and f_* is surjective for every field k. Therefore, f is a *cdf*-cover that is universally injective, and hence a universally injective τ-cover. Therefore, the morphism $f_* : \mathsf{h}_p^{\mathsf{a}\tau} \to \mathsf{h}_q^{\mathsf{a}\tau}$ is an isomorphism, by Proposition A.4.37. $\square$

Corollary A.4.42 ([**Voe96**, Lem.3.2.1]). Let τ be a pretopology on Prop/S as in Remark A.4.39, and let $i : z \to p$ be a surjective closed immersion of proper S-schemes. Then, the morphism $i_* : \mathsf{h}_z^{\mathsf{a}\tau} \to \mathsf{h}_p^{\mathsf{a}\tau}$ is an isomorphism. In particular, for the closed immersion of the maximal reduced closed subscheme $i : p_{\mathrm{red}} \hookrightarrow\!\!\!\!\!\rightarrow p$, the morphism $i_* : \mathsf{h}_{p_{\mathrm{red}}}^{\mathsf{a}\tau} \to \mathsf{h}_p^{\mathsf{a}\tau}$ is an isomorphism.

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