Polynomial invariants of Legendrian links and their fronts

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The present paper is a survey of recent results on restrictions of the classical polynomial link invariants to Legendrian curves in the standard contact 3-space and solid torus. We point out the sets of the rules which completely define these restrictions in terms of the underlying plane fronts. Unlike the case of arbitrary framed links when the framed versions of the polynomials are Laurent in the framing variable x, the polynomials of Legendrian links do not contain any negative powers of x. We give a series of estimates of the Bennequin-Tabachnikov numbers implied by this basic property. We also show how Vassiliev type invariants appear in the polynomials of plane curves.

The entire activity around invariants of plane fronts and regular curves has been inspired by V. I. Arnold. His investigations over the last few years, strongly motivated by his attempt to prove the Last Geometrical Theorem of Jacobi [2], gave a second birth to the topic that goes back to Gauss and Whitney. The central part of it is Arnold's J^+-theory of fronts which is in fact the theory of invariants of Legendrian knots.

1 Legendrian links and plane fronts

1.1 Standard contact spaces

We recall a few basic notions.

A contact element at a point of a plane is a line in the tangent plane. Its *coorientation* is a choice of one of two half-planes into which it divides the tangent plane. The manifold M of all cooriented contact elements of the plane is the spherisation $ST^*\mathbf{R}^2$ of the cotangent bundle of the plane. It is diffeomorphic to the solid torus $\mathbf{R}^2 \times S^1$: the coorienting normal vector of a contact element is defined by the angle $\varphi \bmod 2\pi$ which it makes with a fixed direction on the plane. Manifold M has the standard contact structure defined as zeros of the 1-form $\alpha = (\cos\varphi)dx + (\sin\varphi)dy$, where (x, y) are coordinates on $\mathbf{R}^2$ with the positive direction of the x-axis being that fixed above.

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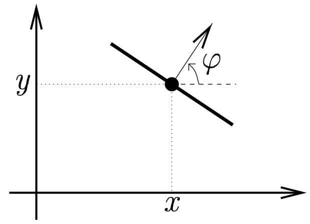


Figure 1: *Coordinates in the solid torus $ST^*\mathbf{R}^2$.*

We equip M with the orientation $dx \wedge dy \wedge d\varphi = -\alpha \wedge d\alpha$. It is opposite to the orientation usually taken in the contact geometry.

Along with the solid torus M we will also be considering its universal cover $\widetilde{M} \simeq \mathbf{R}^3$, with the orientation induced from that of M. Its standard contact form is given by the same formula as α with the only difference that now the angular coordinate φ is not reduced $\bmod 2\pi$.

1.2 Fronts

Definition 1.1 A *Legendrian curve* in a contact 3-manifold is a mapping of a disjoint union of a finite number of circles for which the pull-back of the contact form vanishes. A *Legendrian link* is an embedded Legendrian curve.

The image of the canonical projection of a Legendrian link L from M or $\widetilde{M}$ to the plane is called *the front of* L. An arbitrary small perturbation in the class of Legendrian links is able to bring a link in general position with respect to the canonical projection. The front of such a generic Legendrian link has only transverse double points and semi-cubical cusps as its singularities.

At any point of a front there is a natural coorientin by the coorienting normal of the contact element $a \in L$ whose projection this point is.

A cooriented multi-component plane curve is a front of a unique Legendrian curve in M. So such a curve will be called *a front*, with no reference to the corresponding Legendrian curve.

A necessary and sufficient condition for an above plane curve to be the front of a Legendrian curve in $\widetilde{M}$ is vanishing of the winding numbers of all of its components. The winding number is the number of full rotations made by the coorienting normal as we trace the component once. The Legendrian link in $\widetilde{M}$ is well-defined by its front only if there is chosen a point on each component of the front and there is an indication to which φ-level of $\mathbf{Z}$-many possible ones this point should be rised. We call such fronts *marked winding-free.*

A generic homotopy in the class of Legendrian *immersions* produces generic perestroikas of the front (Fig.2). Only *dangerous self-tangencies* of fronts, when the coorientations of the two tangent branches coincide, correspond to topological changes in the links. A Legendrian link in the solid torus $ST^*\mathbf{R}^2$ gets the double point and experiences the change-crossing at each of these instants (Fig.3).

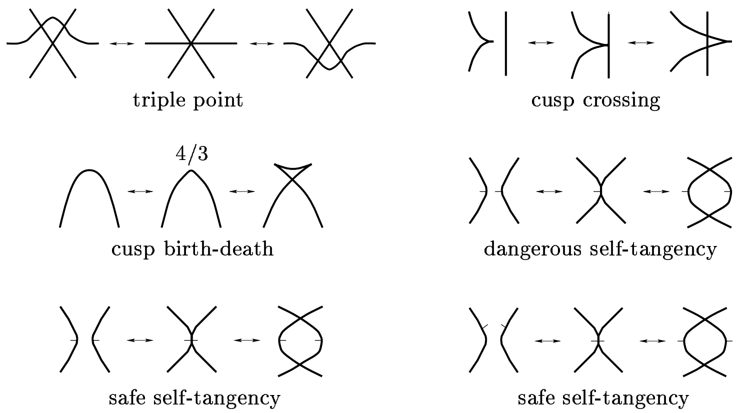


Figure 2: *Perestroikas of generic fronts.*

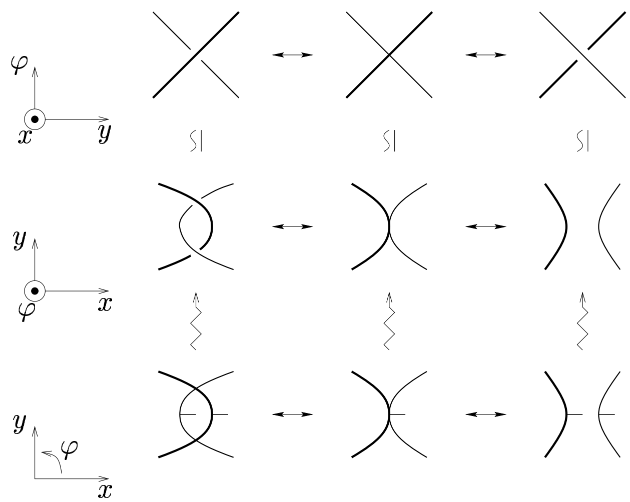


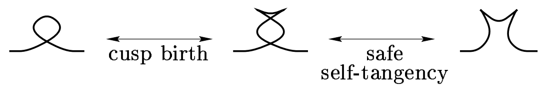
Figure 3: *A dangerous self-tangency of a front rises to the change-crossing of the Legendrian link.*

Definition 1.2 An invariant of fronts is called a J^+*-type invariant* if it does not change under homotopies which involve no dangerous self-tangencies.

Our terminology follows the name of the first invariant of this type introduced by Arnold in [1, 2].

From the above discussion, we see that the theory of invariants of Legendrian links in $ST^*\mathbf{R}^2$ is isomorphic to that of J^+-type invariants of fronts. So in what follows we will make no distinction between an invariant of Legendrian links in the standard contact solid torus and its lowering to the J^+-type invariant of fronts.

Example 1.3 The following local homotopy with no dangerous self-tangencies does not change J^+-type invariants:



Definition 1.4 An invariant of marked winding-free fronts is called a J_0^+*-type invariant* if it does not change under homotopies which involve no direct self-tangencies with coinciding phases $\varphi \in \mathbf{R}$ of the branches at the points of tangency.

The theory of such invariants coincides with the theory of invariants of Legendrian knots in $\widetilde{M} \simeq \mathbf{R}^3$.

1.3 The Bennequin-Tabachnikov number

Any unframed link type in M and $\widetilde{M}$ has a Legendrian representative (see, e.g. [11]). This is not the case for the framed setting.

A Legendrian link in a 3-manifold with a cooriented contact structure has a canonical framing by the coorienting vectors of the contact planes. In the cases of $ST^*\mathbf{R}^2$ and its universal cover, this is isomorphic to the framing by the Legendrian lift of the front of the original link slightly shifted in the direction of its coorientation (Fig.4).

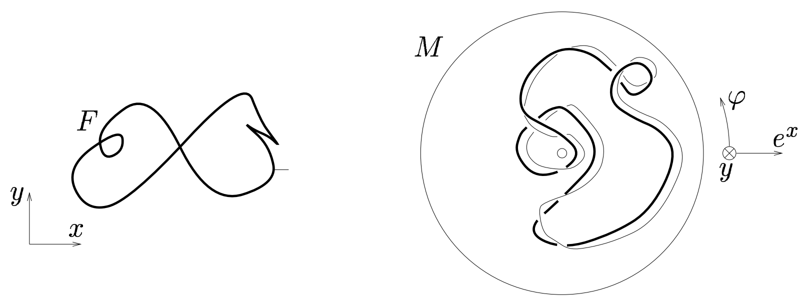


Figure 4: *Legendrian lifting of a plane front to a canonically framed knot in the solid torus* $M = ST^*\mathbf{R}^2$.

Definition 1.5 The writhe β of the canonical framing of a Legendrian knot in $\widetilde{M} \simeq \mathbf{R}^3$ is called the *Bennequin number* of the knot.

The Bennequin number can be calculated as follows. The front, with its double points correctly resolved, is in fact a knot diagram of its Legendrian knot. This assigns "+" or "−" to each of the double points, as if we were calculating the writhe of the blackboard framing. Now β is the sum of all these signs plus half the number of the cusps.

One of the main results of [4] tells that the Bennequin number is bounded from one side on the set of all Legendrian knots representing the same unframed knot type in $\mathbf{R}^3$. For our choice of orientation the numbers are bounded from below. Insertion of a two-cusp zigzag (as in Fig.4) increases β by 1.

Example 1.6 a) For an unknot $\beta \geq 1$ [4].

b) The original estimate of [4] provides the same bounds for a knot and its mirror image. For example, for both the right- and left-handed trefoils it gives $\beta \geq -1$. This bound is exact for the left-handed trefoil. In [14, 10] it was shown that for the right-handed trefoil the exact lower bound is 6.

An analog of the Bennequin number for knots in $ST^*\mathbf{R}^2$ was introduced by Tabachnikov in [18]. He set it to be the index of intersection of the knot shifted in the direction of the framing and a 2-film realising homology between the original knot and a multiple of the fibre of the projection $ST^*\mathbf{R}^2 \to \mathbf{R}^2$ over a sufficiently distant point. According to one of equivalent definitions, this is also the writhe of the canonically framed Legendrian knot with respect to the projection $(x, y, \varphi) \mapsto (e^x, \varphi)$ to the annulus $\mathbf{R}^2 \setminus 0$ with polar coordinates (ρ, φ) (see Fig.4 in which $\beta = 4$).

Via inclusion of the standard solid torus into the standard 3-space, the mentioned boundedness of the Bennequin numbers implies similar boundedness of the Tabachnikov numbers.

Remark 1.7 By the Bennequin and Tabachnikov numbers of oriented Legendrian links we will mean the corresponding writhes of the canonically framed links.

2 Kauffman polynomials of fronts

2.1 The polynomial of Legendrian links in the standard solid torus

In [21] Turaev introduced the Kauffman polynomial of a framed non-oriented link in a solid torus. This is an element of $\mathbf{Z}[x^{\pm1}, y^{\pm1}, \xi_1, \xi_2, \ldots]$ uniquely defined by the relations and initial data of Fig.5. The links L_1 and L_2 there are mutually unlinked. All the links are equipped with the framing which is blackboard with respect to a fixed projection of the solid torus to the annulus. The knot Ξ_3 is a pattern for the whole series Ξ_i.

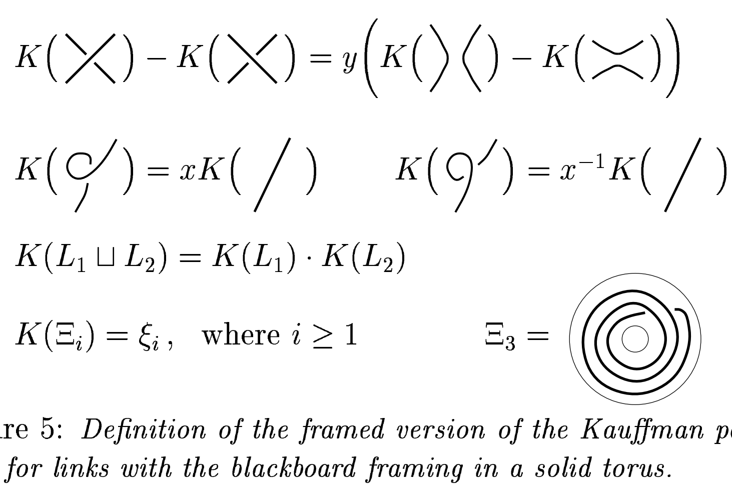


Figure 5: *Definition of the framed version of the Kauffman polynomial for links with the blackboard framing in a solid torus.*

Example 2.1 On an unknot with the trivial framing $K = \frac{x-x^{-1}}{y} + 1$.

The Legendrian lifting lowers the polynomial to generic plane fronts. Translation of the rules of Fig.5 to fronts gives rise to the rules of Fig.6. The fronts F_1 and F_2 of the third line are lying in disjoint half-planes. The relation between the Legendrian generators z_i and the blackboard generators ξ_i is easily seen to be $z_i = x^i \xi_i$ [8].

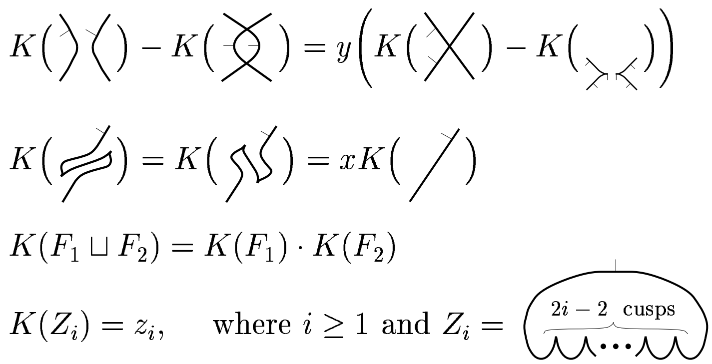
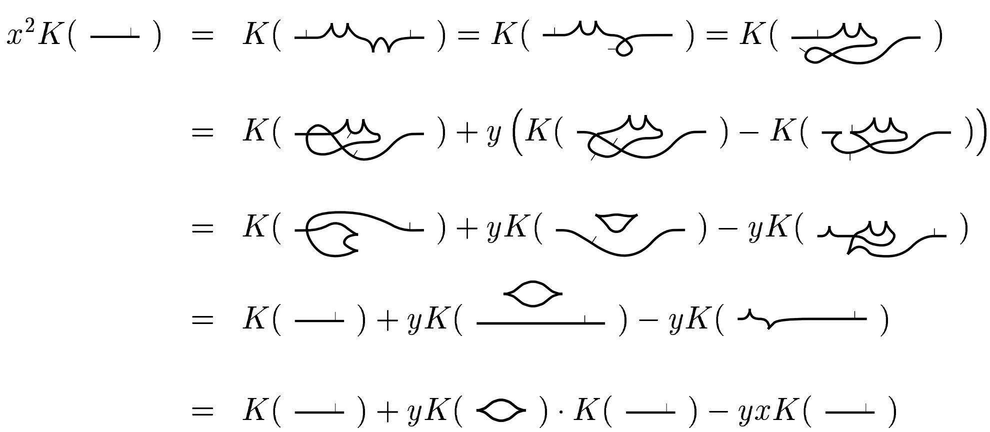


Figure 6: *Definition of the Kauffman polynomial for fronts.*

Theorem 2.2 ([7]) *There exists a unique J^+-type invariant $K(F) \in \mathbf{Z}[x, y^{\pm 1}, z_1, z_2, \ldots]$ of a generic front F satisfying the relations and initial data of Fig.6.*

Note that there are no negative powers of the framing variable x now.

Example 2.3 The lips front, with two cusps and no double points, is the simplest possible winding-free. To calculate its Kauffman polynomial one can proceed as follows (making use of Example 1.3):



So, $K($  $) = \frac{x^2-1}{y} + x$. Indeed, the lips front lifts to the Legendrian unknot in M with $\beta = 1$, so its Kauffman polynomial should be that of an unknot with the trivial framing times x.

2.2 The polynomial of Legendrain links in the standard $\mathbf{R}^3$

The Kauffman polynomial $K_0 \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ of framed links in $\mathbf{R}^3$ is defined by the rules of Fig.5 with all the information about the curves Ξ_i omitted [15]. Its Legendrian version in terms of fronts is respectively given by Fig.6 without mentioning the fronts Z_i. Only generic marked winding-free fronts are now considered. The phases of the two interacting branches in the main skein relation must coincide. We call this modification of Fig.6 its J_0^+*-version.*

Theorem 2.4 ([7]) *There exists a unique J_0^+-type invariant $K_0(F_0) \in \mathbf{Z}[x, y^{\pm 1}]$ of a generic marked winding-free plane front F_0 satisfying the relations and initial data of the J_0^+-version of Fig.6.*

2.3 The Bennequin-Tabachnikov number estimates

Due to Theorems 2.2 and 2.4, the Kauffman polynomial of a (marked winding-free) plane front is a genuine polynomial in x, not a Laurent one. This implies the following restriction on the values of the Bennequin-Tabachnikov numbers of oriented Legendrian links in the solid torus and 3-space.

Let $wr(L)$ be the writhe of an oriented link L either in the solid torus or $\mathbf{R}^3$. Define the unframed versions of the Kauffman polynomial as

$$K_u(L) = x^{-wr(L)} K(L) \quad \text{and} \quad K_{0,u}(L) = x^{-wr(L)} K_0(L)\,.$$

Following [21, 15], these polynomials depend only on unframed topological type of L. Thus we can speak about the polynomials K_u and $K_{0,u}$ of unframed oriented links. In the case of knots the orientation does not matter.

Theorem 2.5 *Let $\mathcal{L}$ be an unframed oriented link in the standard contact manifold $\widetilde{M} \simeq \mathbf{R}^3$ or $M = ST^*\mathbf{R}^2$. Let x^k be the minimal power of the framing variable x in the corresponding unframed version of the Kauffman polynomial of $\mathcal{L}$. Then the Bennequin-Tabachnikov number of any Legendrian representative of $\mathcal{L}$ is at least $-k$.*

Example 2.6 For an unknot (see Example 2.1) this coincides with the classical bound $\beta \geq 1$ [4].

Example 2.7 The Theorem implies that the minimal Bennequin-Tabachnikov number of a Legendrian representative of the basic knot Ξ_i in the solid torus (Fig.5) is that of the Legendrian lifting of the front Z_i, which is $2i-1$. Note that inclusion of the standard contact solid torus into the standard contact 3-sphere gives only $\beta \geq 1$ for any i: all the Ξ_i get unknotted in S^3.

Example 2.8 For the left- and right-handed $(2, q)$-torus links in $\widetilde{M} \simeq \mathbf{R}^3$, Theorem 2.5 gives $\beta \geq 2 - q$ and $\beta \geq 2q$ respectively, $q \geq 2$. The exactness of these estimates in all these cases follows from the examples of Figs. 7 and 8. The double points of the (marked) winding-free fronts in these Figures are resolved respecting the phases φ of the branches.

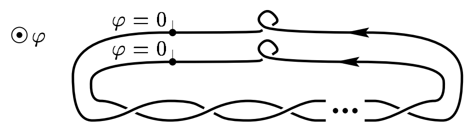


Figure 7: *Legendrian representative of the left-handed* $(2, q)$*-torus link in* $\mathbf{R}^3$ *with the minimal possible Bennequin number* $2 - q$.

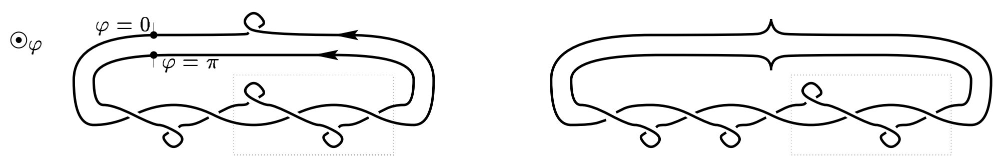


Figure 8: *Legendrian representatives of the right-handed* $(2, 4)$*-torus link and* $(2, 5)$*-torus knot in* $\mathbf{R}^3$ *with the minimal possible Bennequin numbers 8 and 10. Minimal representatives, with* $\beta = 2q$*, of all the other right-handed* $(2, q)$*-torus links are obtained by either omitting the distinguished fragments (for* $q = 2$ *and* $q = 3$*) or by their consecutive repetition (for* $q \geq 6$*).*

Remark 2.9 A similar to ours estimate of the Bennequin number for knots in $\mathbf{R}^3$ by the lowest degree of the framing variable in the mod 2 Kauffman polynomial was derived in [10] from the results of [19]. It is not known if the lowerest degrees in the integer and mod 2 Kauffman polynomials for $\mathbf{R}^3$ may differ. In all the examples we know they coincide. The work [20] implies that for alternating knots they are equal. See also [17].

3 HOMFLY polynomials

3.1 The Legendrian versions

Fig.9 recalls the definition of the framed version of the HOMFLY polynomial of oriented links in the solid torus [21]. This relates an element of $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$ to such a link.

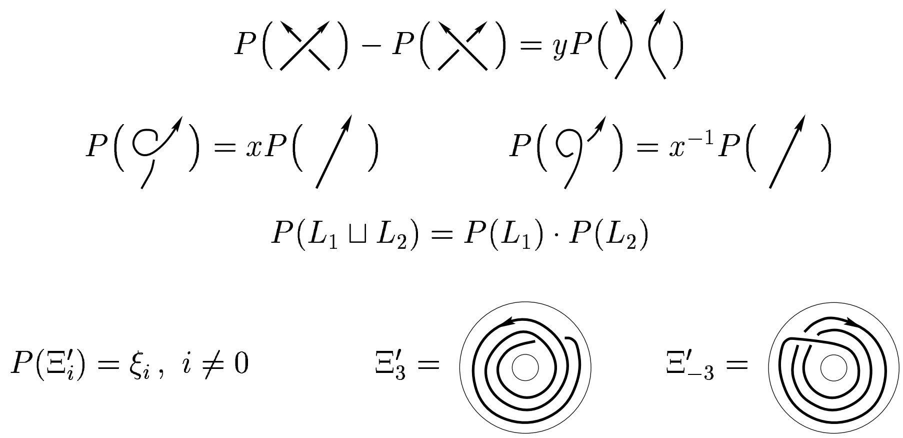


Figure 9: *The definition of the HOMFLY polynomial for oriented links with the blackboard framing in the solid torus.*

Omitting in Fig.9 all the information about the knots Ξ'_i and corresponding variables one gets the definition of the HOMFLY polynomial for knots in the 3-space [13, 15].

Fig.10 translates the rules of Fig.9 to fronts. Relations of its first three lines are also valid for the fragments with all the orientations reversed; $F_1 \sqcup F_2$ is the disjoint union of the two fronts on different sides of a certain straight line.

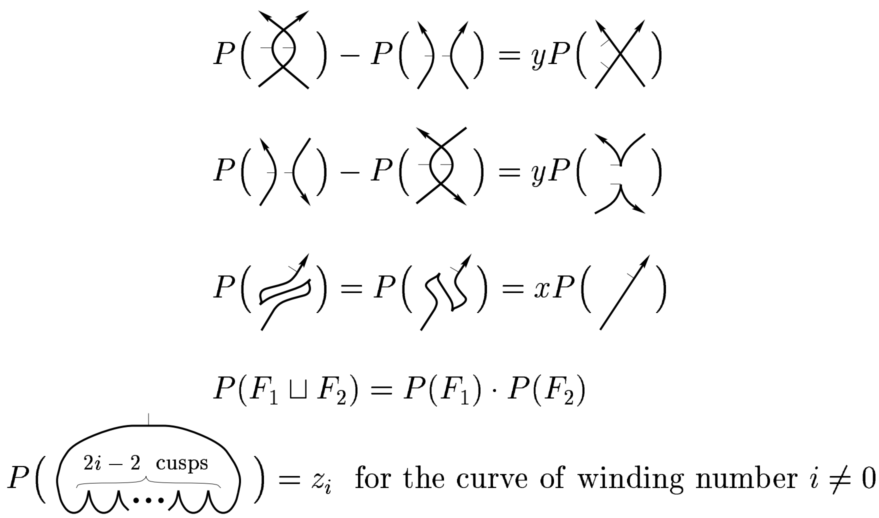


Figure 10: *The definition of the HOMFLY polynomial for oriented plane fronts.*

The definition of the J_0^+-version of Fig.10 is obvious (cf. Section 2.2).

Theorem 3.1 ([7]) *There exist*

1) a unique J^+-type invariant $P(F) \in \mathbf{Z}[x, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \dots]$ of a generic plane front F;

2) a unique J_0^+-type invariant $P_0(F_0) \in \mathbf{Z}[x, y^{\pm 1}]$ of a generic marked winding-free plane front F_0

satisfying the relations and initial data of Fig.10 and of its J_0^+-version respectively.

Thus again, unlike the case of arbitrary framed links, the polynomials of canonically framed Legendrian links do not contain negative powers of x.

3.2 Maslov index

It is easy to strengthen the last theorem and establish divisibility of the HOMFLY polynomials by certain powers of the framing variable.

Consider an oriented and cooriented plane front. A cusp of such a front is called *positive* if the velocity vectors of its outgoing branch have positive projections to the normal of the coorientation at the cusp point. Otherwise the cusp is called *negative*.

Definition 3.2 Half a difference $\mu = \frac{1}{2}(\mu_+ - \mu_-)$ between the numbers of positive and negative cusps is called *the Maslov index* of the front or of the corresponding Legendrian link.

The Maslov index is easily seen to be integer.

All the basic fronts of Fig.10 have zero Maslov index. The Maslov indices of all the three fronts participating in both versions of the main skein relation coincide. The zigzag skeins relate the change of the Maslov index by ± 1 to the divisibility of the polynomial by x. In the chain of calculations of the polynomial of a particular front, the zigzag skeins may be used only to reduce the number of cusps [7]. Thus, for both the theories of plane fronts, we get

Corollary 3.3 ([7]) *In the ring of genuine polynomials in the framing variable, the HOMFLY polynomial of a front is divisible by $x^{|\mu|}$, where μ is the Maslov index of the front.*

3.3 The Bennequin-Tabachnikov number estimates in terms of the HOMFLY polynomial

The unframed analogs of the HOMFLY polynomials are

$$P_u(L) = x^{-wr(L)} P(L) \quad \text{and} \quad P_{0,u}(L) = x^{-wr(L)} P_0(L)\,,$$

where, as in Section 2.3, $wr(L)$ is the writhe of a framed link L either in the solid torus or in $\mathbf{R}^3$.

The non-Laurent polynomiality of the framed versions of the polynomials, in the strengthening of Corollary 3.3, implies

Theorem 3.4 ([7]) *Let $\mathcal{L}$ be an oriented unframed link in the standard contact manifold $\widetilde{M} \simeq \mathbf{R}^3$ or $M = ST^*\mathbf{R}^2$. Let x^r be the minimal power of the framing variable x in the corresponding unframed version of the HOMFLY polynomial of $\mathcal{L}$. Then, for any Legendrian representative L of $\mathcal{L}$,*

$$\beta + |\mu| \geq -r\,,$$

where β and μ are the Bennequin-Tabachnikov number and Maslov index of L.

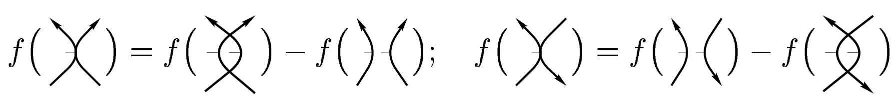
For $\mathbf{R}^3$ this is the theorem of Fuchs-Tabachnikov [10] obtained by a direct comparison of the results of [4, 9, 16].

Example 3.5 (cf. Example 2.8) With no information on μ, for the left-handed $(2,q)$-torus links in $\mathbf{R}^3$ the estimates of Theorems 2.5 and 3.4 are the same: $\beta \geq 2 - q$. For the right-handed series the estimate of Theorem 2.5 is stronger than that of Theorem 3.4: $\beta \geq 2q$ instead of $\beta \geq 2 + q$.

For a generic non-oriented Legendrian knot, the number $|\mu|$ is well-defined. Nevertheless, one cannot sthrengthen Theorem 2.5 on the Kauffman polynomial estimate to include $|\mu|$ similarly to Theorem 3.4: the $(2,5)$-torus knot of Fig.8 has $|\mu| = 1$ and its Bennequin number is equal to the negative of the lowest power of the framing variable in the unframed version of the Kauffman polynomial.

4 Finite order J^+–type invariants

Vassiliev theory of finite order knot invariants is based on the concept of extension of a knot invariant to degenerate knots with double points. In a similar way any J^+–type invariant f recursively extends to fronts with a finite number of dangerous self-tangencies:



These rules are due to the natural coorientation of the strata of dangerous self-tangencies from [2]. When lifted to $ST^*\mathbf{R}^2$ both rules are in fact the definition of an extended invariant of the original Vassiliev theory.

Definition 4.1 A J^+–type invariant f has *order* n *in Vassiliev sense* if n is the maximal number of dangerous self-tangencies of a front on which the extension of f does not vanish. *The symbol* of such f is the restriction of f to the set of fronts with precisely n dangerous self-tangencies.

By Gromov's theorem [11] the winding numbers and Maslov indices of the components are the only invariants of order zero.

The difference of two invariants of order n with the same symbol is an invariant of order less than n.

Theorem 4.2 (cf. [5, 6]) *Set $y = e^{t/2} - e^{-t/2}$ in the Kauffman or HOMFLY polynomials of a plane front C and expand the result in a power series in t. Then the coefficient at t^n in the obtained series is a J^+–type invariant of order at most n in Vassiliev sense.*

There is an obvious analog of this theorem for J_0^+-type invariants.

5 Regular Legendrian knots and immersed plane curves

5.1 Regular Legendrian representatives

Along with the consideration of invariants of arbitrary Legendrian links in $ST^*\mathbf{R}^2$ and its universal cover, that is J^+- and J_0^+-type invariants of plane fronts, one can study invariants of immersed plane curves without dangerous self-tangencies [1, 2]. Now cusps are prohibited, and we find ourselves in a situation which is appriori more poor. But from what follows we see that the difference is basicly only in the vanishing of the Maslov index.

Definition 5.1 A Legendrian link in $M = ST^*\mathbf{R}^2$ or $\widetilde{M} \simeq \mathbf{R}^3$ is called *regular* if its front is an immersed plane curve.

We assume here that the orientation of an immersed plane curve is automatically defined by its coorientation so that the frame {coorienting normal, orienting vector} gives positive orientation of the plane.

Theorem 5.2 ([8]) *Any unframed oriented link type in M and $\widetilde{M}$ has a regular Legendrian representative.*

Example 5.3 The Legendrian representative of Fig.7 of the left-handed $(2, q)$-torus link in $\mathbf{R}^3$ is regular. Fig.11 shows a regular Legendrian right-handed trefoil.

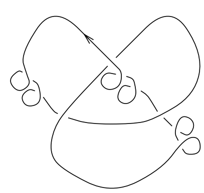


Figure 11: *A regular Legendrian right-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$ with $\beta = 9$.*

On the level of canonically framed knots one of the differences between arbitrary Legendrian and regular knots is the following.

Theorem 5.4 ([2, 3]) *The Bennequin-Tabachnikov number of a regular Legendrian knot is odd.*

One of reflections of this statement is that the analog of the zigzag surgery for fronts is now the insertion of a small fragment containing two curls with opposite directions of rotation. The latter operation increases β by 2.

Possibilities of arbitrary and regular Legendrian representations provides us with two apriori different characteristics of an unframed knot type K in M or $\widetilde{M}$. Those are the minimal Bennequin-Tabachnikov numbers $\beta_{min,reg}(K)$ and $\beta_{min}(K)$ of corresponding Legendrian realisations. Of course, $\beta_{min,reg}(K) \geq \beta_{min}(K)$. For the left-handed trefoil both the numbers coincide. But it is not the case in general.

The difference between the two numbers does not seem to be due only to the parity restriction on $\beta_{min,reg}(K)$. While for the left-handed trefoil we have $\beta_{min} = 6$ (Fig.7), the best regular realisation we know (Fig.11) suggests

Conjecture 5.5 *For the right-handed trefoil* $\beta_{min,reg} = 9$.

5.2 The HOMFLY polynomial

There is no straightforward lowering of the Kauffman polynomial to immersed plane curves: the main skein relation of Fig.6 requires cusps. On the other hand, for the HOMFLY polynomial we have

Theorem 5.6 ([8]) *There exists a unique* J^+*–type invariant* $P(C) \in \mathbf{Z}[x^2, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$ *of a generic oriented plane curve* C *satisfying the relations and initial data of Fig.12.*

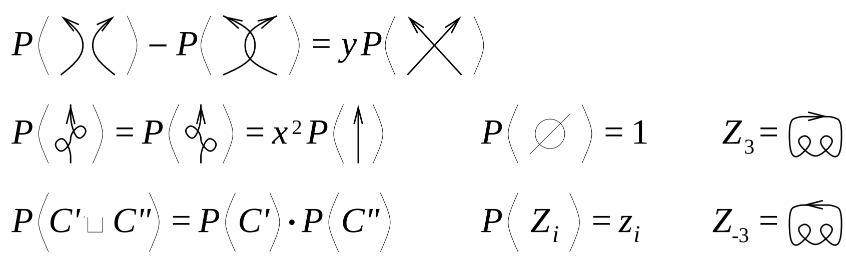


Figure 12: *Legendrian lowering of the definition of Fig.10 to generic collections of regular oriented plane curves.*

Thus, in addition to the genuine polynomiality in the framing variable x which is a common Legendrian property, the regularity implies the evenness in x.

There is an obvious version of the above theorem for marked winding-free generic plane curves.

5.3 Regular curves with few double points

In Fig.13 we give the results of calculations of the polynomial P for Arnold's list [1, 2] of all the regular one-component plane curves with at most 3 double points. We set there

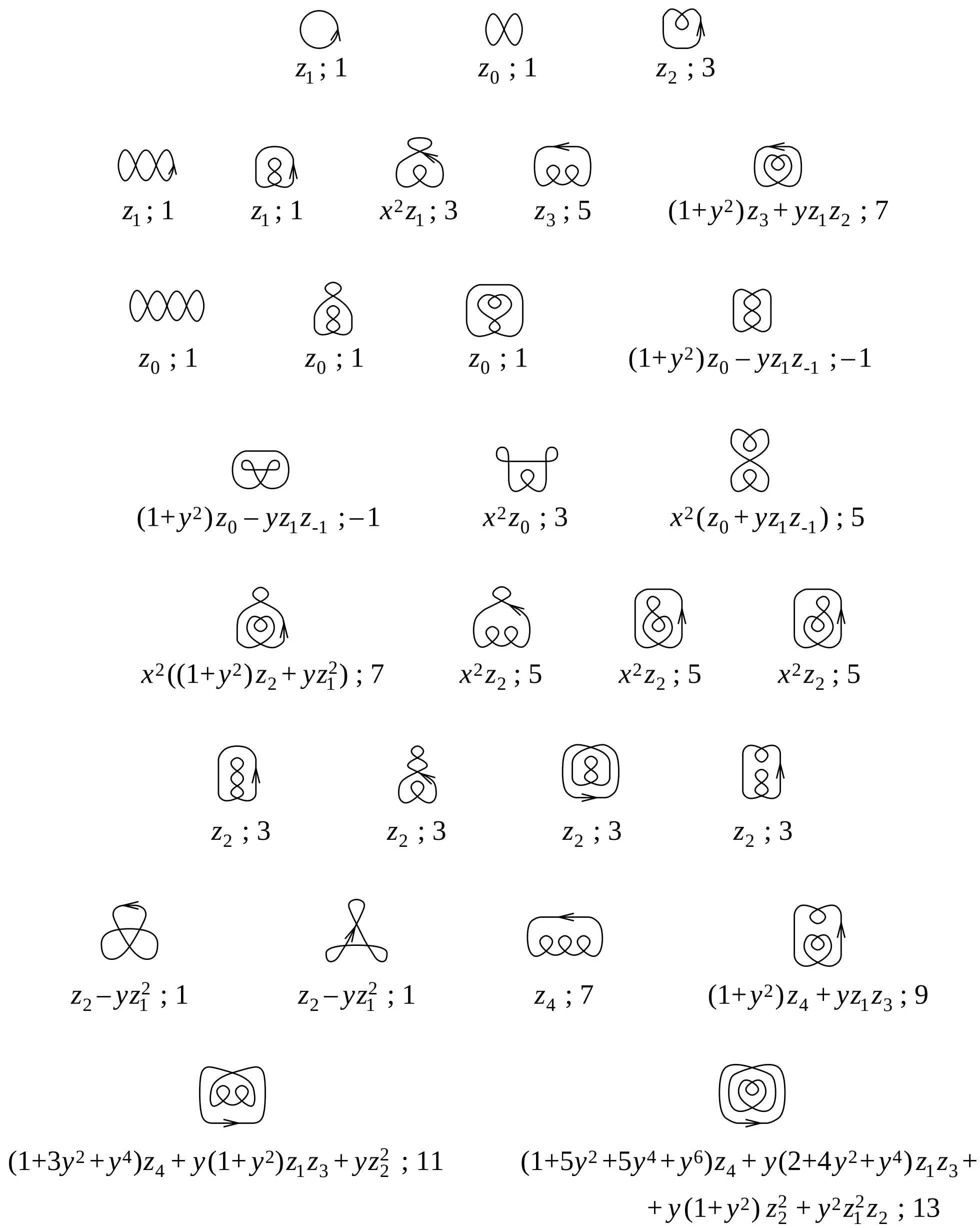


Figure 13: *The HOMFLY polynomials and Tabachnikov numbers of plane curves with at most 3 double points.*

$z_0 = (x^2-1)/y$. The orientations of the curves with non-zero winding numbers are chosen so that these numbers are positive. The change of the orientation implies the change $z_i \to z_{-i}$ for all the indices i.

Most of the polynomials of Fig.13 which have no obvious reason to be divisible by x^2 (those are polynomials of the curves with no pairs of small curls of opposite orientation) are not divisible by it. The non-divisibility of $P(C)$ by x^2 means that the Bennequin-Tabachnikov number of the corresponding Legendrian knot $L_C \subset ST^*\mathbf{R}^2$ is the minimal possible among all the regular knots of the same topological type: $\beta(L_C) = \beta_{min,reg}(L_C)$ (in fact, even stronger: $\beta(L_C) = \beta_{min}(L_C)$).

The inverse does not seem to be true. For example, for the last curve in the 4th line, $P = x^2(\frac{x^2-1}{y} + yz_{-1}z_1)$, but there seems to exist no regular plane curve whose polynomial is that in the brackets of this formula. Another similar example is the first curve of the 5th line. Arnold's tables in [1] contain some other curves of the same nature. All of them are certain modifications of those two of Fig.13. This indicates that the estimate of Theorem 3.4 may not be exact in all the cases. Perhaps, there are some special bounds for powers of x in the coefficients of various products of z-variables in the HOMFLY polynomials of (regular) plane curves.

Appendix A: Another approach to Legendrian links in $\mathbf{R}^3$

The standard contact 3-space can also be treated as the space $J^1(\mathbf{R},\mathbf{R})$ of 1-jets of functions on a line. The contact form α is then $dy - p\,dx$, where y corresponds to values of a function, x to its argument, and p to its derivative. We again orient $\mathbf{R}^3$ with the form $-\alpha\wedge d\alpha = dx\wedge dy\wedge dp$.

Now Legendrian links are represented by their projections to the (x,y)-plane. A generic front in this plane is a curve whose only singularities are cusps and transverse double points and which has no tangents parallel to the y-axis. We call such a curve *a front with no vertical tangents*. In order to restore the Legendrian link in $J^1(\mathbf{R},\mathbf{R})$ from a generic front one resolves each double point putting the branch with the greater slope $\partial y/\partial x$ to the higher p-level (Fig.14). The canonical Legendrian framing now is that by the positive y-direction.

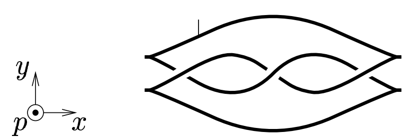


Figure 14: *Lifting of a front with no vertical tangents to the left-handed trefoil in* $J^1(\mathbf{R},\mathbf{R})$.

Generic homotopies in the class of Legendrian immersions in $J^1(\mathbf{R},\mathbf{R})$ provide the same list of generic perestroikas of fronts in the (x,y)-plane as earlier (see Fig.2), except for safe self-tangencies. Of course, no vertical tangents are allowed in any of these perestroikas.

Definition A.1 An invariant of generic fronts with no vertical tangents is called a J^+_{jet}*-type invariant* if it does not change under homotopies which involve no self-tangencies.

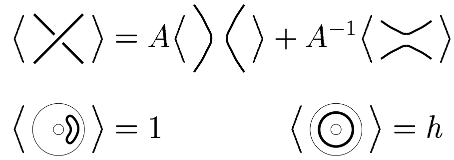
The theory of J^+_{jet}-type invariants is that of invariants of Legendrian knots in $J^1(\mathbf{R},\mathbf{R})$ (and, thus, is isomorphic to the theory of J^+_0-type invariants).

In terms of fronts with no vertical tangents, the rules of the Kauffman polynomial in $\mathbf{R}^3$ are those of Fig.6 with the curves Z_i omitted and all the front fragments rotated by 90 degrees clockwise to avoid vertical tangents. Such a modification of Fig.6 will be called its J^+_{jet}*-version.* In the similar way one defines the J^+_{jet}-version of the HOMFLY rules of Fig.10.

Theorem A.2 ([7]) *There exist unique* J^+_{jet}*-type invariants* $K(F_{jet})$ *and* $P(F_{jet}) \in \mathbf{Z}[x, y^{\pm 1}]$ *of a generic front* F_{jet} *with no vertical tangents satisfying the relations and initial data of the* J^+_{jet}*-version of Figs. 6 and 10 respectively.*

Appendix B: Kauffman bracket of fronts

For a framed link in a solid torus the Kauffman bracket was defined in [12]. Its values belong to $\mathbf{Z}[A^{\pm 1}, h]$.

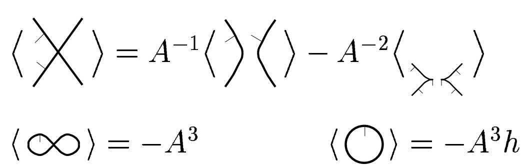


$$< L_1 \sqcup L_2 >= -(A^2 + A^{-2}) < L_1 > \cdot < L_2 >$$

Figure 15: *Definition of the Kauffman bracket for framed links with the blackboard framing in a solid torus.*

The analog for plane fronts and Legendrian links in $ST^*\mathbf{R}^2$ is as follows.

Theorem B.1 ([6]) *There exists a unique* J^+*–type invariant* $< C > \in \mathbf{Z}[A^{\pm 1}, h]$ *of a normal front* C *satisfying the properties:*



$$< C_1 \sqcup C_2 >= -(A^2 + A^{-2}) < C_1 > \cdot < C_2 >, \quad \textit{for } C_1 \neq \emptyset, C_2 \neq \emptyset.$$

Here $C_1 \sqcup C_2$ is a union of two fronts C_1 and C_2 which lie in different half-planes with respect to a certain line in $\mathbf{R}^2$.

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