VECTOR FIELDS AND FUNCTIONS ON DISCRIMINANTS OF COMPLETE INTERSECTIONS AND ON BIFURCATION DIAGRAMS OF PROJECTIONS

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The paper studies vector fields that preserve the discriminants of isolated singularities of complete intersections and bifurcation diagrams of projections to the straight line. The results are applied to find stable functions on discriminants of simple complete intersections and normal forms of functions of general position on bifurcation diagrams of projections of low codimension.

Arnol'd [12] and Zakalyukin [9, 10], in their studies of the evolution of wave fronts, lay the foundation for the analysis of vector fields and functions on spaces containing discriminant varieties. It was shown, in particular, that the algebra of holomorphic vector fields tangent to the discriminant of a function is a free module over the ring of functions on the enclosing space. In this case, the discriminant is a free divisor in Saito's sense [25]. An explicit form of the generators of the module of tangent fields was given in [9], and the normal forms relative to the group of discriminant-preserving biholomorphisms were derived for the functions of general position on the enclosing space.

The research in this direction was subsequently continued by Terao, Bruce, and Looijenga. Thus, Terao [30] proved that another object which naturally arises in the singularity theory of holomorphic functions — the bifurcation diagram of a function — is also a free divisor. He found the generators of the module of diagram-preserving vector fields. Terao's result was proved in a more compact form by Bruce, who studied stable functions on bifurcation diagrams [13].

Looijenga [21] showed that the discriminant of an isolated singularity of a complete intersection is also a Saito divisor. An algorithm for constructing the generators of the corresponding module of vector fields was proposed in [8]. It relies on a number of properties of projections of (singular) manifolds on the straight line, i.e., diagrams of the form $Y \hookrightarrow C^{n+1} \to C$ (the first arrow is embedding of a submanifold, the second arrow is nondegenerate linear projection). These properties coincide with the properties of functions on smooth manifolds and will be reviewed in Sec. 1.

Yet another free divisor was suggested in [8]: the bifurcation diagram of a projection. It may be interpreted as the bifurcation diagram of a complete intersection, because it is obtained from the discriminant of a complete intersection (possibly multiplied by some complex linear space) similarly to the way in which the diagram of a smooth function is obtained from the discriminant of the smooth function — as a bifurcation variety of the stable projection of the discriminant along the straight line.

In this paper, we propose a formula for the generators of the module of vector fields preserving the discriminant of a quasihomogeneous complete intersection of positive dimension (Theorem 2.4). Our formula is more convenient than the general formula. We also apply the results of [8] to find stable functions on the discriminants of a number of simple multiple points (Theorem 3.5) and normal forms of functions of general position on bifurcation diagrams of projections of codimension 2 and 3 (4.6). We determine the number of basis tangent vector fields that are independent at the given point of the discriminant of a complete intersection (Proposition 2.2) or of the bifurcation diagram of a projection (Proposition 4.2). We indicate some modifications of the formulas for computing the generators of the modules of the tangent fields for edge and line singularities (4.8).

All the definitions are presented in Sec. 1.

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1.1. Definitions and Notation. Take a nondegenerate linear projection from $\mathbf{C}^{n+1}$ to the straight line, p: $\mathbf{C}^{n+1} \to \mathbf{C}$.

1.1.1. *Definition.* The projection of the submanifold $Y \hookrightarrow \mathbf{C}^{n+1}$ to the straight line is the diagram $Y \hookrightarrow \mathbf{C}^{n+1} \xrightarrow{p} \mathbf{C}$.

Definition. The projections of the submanifolds $Y_1, Y_2 \subset \mathbf{C}^{n+1}$ are said to be R-, R^+-, or RL-equivalent if there exists a biholomorphism of the enclosing space that takes Y_1 to Y_2, preserves the linear projection p, and induces on the base of p resp. an identity map, a translation, or an arbitrary biholomorphism.

Let u be a coordinate function on the base of p. Taking as Y the graph of a function on $\mathbf{C}^n$ (u is the value of the function), we see that the equivalences introduced in the above definition are generalizations of the corresponding equivalences of smooth functions [3].

Definition. The height function is the restriction of u to Y.

In what follows, the projected submanifold is assumed to be a complete intersection of positive dimension, $Y=f^{-1}(0)$, $f:(\mathbf{C}^{n+1}, 0)\to(\mathbf{C}^m, 0)$, $m=\operatorname{codim} Y\leqslant n$. The projection to the straight line of the manifold f = 0 is called the projection f.

1.1.2. We use the following notation: $\mathcal{O}_Z^m$ is the bundle of holomorphic maps from Z to $\mathbf{C}^m$; $\mathcal{O}_{Z,z}^m$ is its fiber at the point $z \in Z$; $z \in Z$; $\mathcal{O}_Z=\mathcal{O}_Z^1$, $\mathcal{O}^m(k)_z=\mathcal{O}^m_{\mathbf{C}^k,z}$, $\mathcal{O}^m(k)=\mathcal{O}^m(k)_0$; $\mathfrak{m}(m)_z\subset\mathcal{O}(m)_z$ and $\mathfrak{m}(m)\subset\mathcal{O}(m)$ is the maximal ideal; $(x, u)=(x_1, \ldots, x_n, u)\in\mathbf{C}^{n+1}$, $p:(x, u)\mapsto u$; $f_0=f|_{u=0}$, where f = f(x, u); $\mathfrak{A}_W$ is the algebra of the germs at zero of the holomorphic vector fields on $\mathbf{C}^r$ tangent to the set $W \subset \mathbf{C}^r$.

If there is no danger of confusion, we will use the same symbol for the germ of a mapping or a set and for a representative of the germ.

1.1.3. For $f\in\mathcal{O}^m(n+1)$ let

$$T_f=f^*(\mathfrak{m}(m))\,\mathcal{O}^m(n+1)+\mathcal{O}(n+1)\,\langle\partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle$$

and let $T_f^+=T_f+\mathbf{C}\partial f/\partial u$ be the tangent spaces to the germs of R- and R^+-equivalence classes of the projection f. $Q_f=\mathcal{O}^m(n+1)/T_f$, $Q_f^+=\mathcal{O}^m(n+1)/T_f^+$.

Definition. The R- and R^+-codimensions τ and τ^+ of the projection f are the dimensions of the linear spaces Q_f and Q_f^+.

Remark. It is easy to see that

a) for $\tau < \infty$ the complete intersections f(x, u) = 0 and $f_0(x) = 0$ have isolated singularities;

b) for $0 < \tau < \infty$, $\partial f/\partial u \notin T_f$ and $\tau = \tau^+ + 1$.

1.1.4. Consider a k-parameter deformation of the projection f: $F\in\mathcal{O}^m(n+1+k)$; $\lambda \in \mathbf{C}^k$ is the deformation parameter, $F|_{\lambda=0} = f$.

Definition. The deformation F is infinitesimally R^+-versal if its initial velocities $\partial F/\partial\lambda_1|_{\lambda=0}, \ldots, \partial F/\partial\lambda_k|_{\lambda=0}$ generate the linear space Q_f^+.

It is easy to show that the R^+-versal deformation defined in the obvious way [3] is such if and only if it is infinitesimally R^+-versal.

1.1.5. The k-parameter R^+-versal deformation of the projection f is also a k-parameter versal deformation of the complete intersection f(x, u) = 0 and a (k + 1)-parameter versal deformation of the complete intersection $f_0(x) = 0$ (the additional parameter is u). The space $\mathbf{C}^{1+k}$, $(u, \lambda) \in \mathbf{C}^{1+k}$, will be called the extended parameter space.

Definition. The discriminant Δ of the projection f is the discriminant of the complete intersection $f_0(x) = 0$ lying in the extended parameter space of the R^+-versal deformation F.

$\Delta \subset \mathbf{C}^{1+k}$ is the set of critical values of the projection $\pi' : (x, u, \lambda)\mapsto(u, \lambda)$, restricted to F = 0.

1.1.6. Let the height function u have an isolated critical point (x, u) = (0, 0) on the set f(x, u) = 0 (the critical points of the height function are the singular points of the projected set). Let $F_\lambda=F|_{\lambda=\mathrm{const}}$, $Y_\lambda=\{F_\lambda=0\}$.

For almost all sufficiently small values of the parameter λ, the function u has on Y_λ the same number μ of critical points close to $0 \in \mathbf{C}^{n+1}$.

Definition. The number μ is called the multiplicity of the critical point (x, u) = (0, 0) of the height function on the germ f = 0.

Definition. The bifurcation diagram $\Sigma \subset \mathbf{C}^k$ of the projection f is the germ at zero of the set of values of the parameter λ such that the height function has on Y_λ no fewer than μ critical values occurring near the point (x, u) = (0, 0).

Σ is the bifurcation variety of the μ-sheeted covering $\Delta\to\mathbf{C}^k$, $(u,\lambda)\mapsto\lambda$, and in general consists of three components (the notation follows [5, 7]):

A_2 corresponds to degeneration of the critical point of the height function on a smooth Y_λ;

$A_1{}^2$ is the Maxwell stratum (the values of the height function coincide at different critical points on a smooth Y_λ);

B_2 corresponds to nonsmooth sets Y_λ.

1.2. The Numbers μ and τ.

1.2.1. μ is the intersection index in the extended parameter space C^{1+k} of the discriminant with the straight line $\lambda = 0$. Let $I_C \subset \mathcal{O}(n+1)$ be the ideal generated by the coordinate functions $f_1, \dots, f_m$ of the map f and all m-minors of the matrix $(\partial f/\partial x)$, i.e., by the equations determining the critical points of the height function on the manifold $f = 0$. Let $\mathcal{O}_C = \mathcal{O}(n+1)/I_C$ be the space of functions on the critical set. From [21, Sec. 4] we obtain

Proposition 1.1. $\mu = \dim_C \mathcal{O}_C$.

1.2.2. The multiplicity of a critical point of a function on a smooth manifold is equal to the dimension of the base of the R-miniversal deformation of the function. This classical result is extended in its entirety to singularities of the height function.

THEOREM 1.2. [8]. For the projection to the straight line, $\mu = \tau$.

COROLLARY 1.3. Let f be a qusihomogeneous projection and q', q'' the Tyurina numbers of the complete intersections $f(x, u) = 0$ and $f_0(x) = 0$. If $n > m$, then $\tau = q' + q''$.

The proposition follows from [11, 20] and the fact that for a quasihomogeneous complete intersection of positive dimension the Tyurina and Milnor numbers coincide [19].

1.2.3. Let F be a representative of the deformation (not necessarily R^+-versal) of a projection having a finite R^+-codimension. Consider on $F = 0$ a coherent bundle of $\mathcal{O}_\Lambda$ modules, $0 \in \Lambda \subset C^k$:

$$\mathcal{F} = \mathcal{O}^m_{F^{-1}(0)} / \mathcal{O}_{F^{-1}(0)} \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle .$$

The support of $\mathcal{F}$ is the set of critical points of the map $\pi' |_{F=0}$.

Consider the direct image $\pi_* \mathcal{F}$ where $\pi : (x, u, \lambda) \mapsto \lambda$. From [8, Sec. 2] we have

Proposition 1.4. There exists a neighborhood Λ' of the point $0 \in C^k$ such that $(\pi_*\mathcal{F})_{\Lambda'}$ is a free bundle of $\mathcal{O}_{\Lambda'}$-modules of rank τ. If the deformation F is R^+-miniversal ($k = \tau^+$), then $\pi_*(\partial F/\partial\lambda), \pi_*(\partial F/\partial\lambda_1), \dots, \pi_*(\partial F/\partial\lambda_k)$ are free generators of $(\pi_*\mathcal{F})_{\Lambda'}$.

2. VECTOR FIELDS PRESERVING THE DISCRIMINANT OF A COMPLETE INTERSECTION

2.1. The General Case. Let the map $G: (C^{n+1+k}, 0) \to (C^m, 0)$ define a versal deformation of the complete intersection $g_0 = 0$, and let $\lambda = (\lambda_0, \dots, \lambda_k)$ be the deformation parameters, $x \in C^n$. The discriminant of g_0 lies in the parameter space: $\Delta \subset C^{k+1}$.

Assume that the deformation parameters have been chosen so that the axis $0\lambda_0$ has a finite intersection index μ with Δ. Denote by $g = g(x, \lambda_0)$ the restriction of G to $\lambda_1 = \dots = \lambda_k = 0$. Consider the projection $g^{-1}(0) \hookrightarrow C^{n+1} \to C$, $(x, \lambda_0) \mapsto \lambda_0$. By Theorem 1.2, $\mu = \dim_C Q_g$, where

$$Q_g = \mathcal{O}^m(n+1) / \{g^*(\mathfrak{m}(m))\, \mathcal{O}^m(n+1) + \mathcal{O}(n+1) \langle \partial g/\partial x_1, \dots, \partial g/\partial x_n \rangle\}. \tag{1}$$

Since G is a versal deformation of $g_0 = 0$, the preparation theorem implies that as the basis of the μ-dimensional space Q_g we can take the restrictions to $\lambda_1 = \dots = \lambda_k = 0$ of the elements

$$\begin{gathered} \partial G/\partial\lambda_0, \dots, \lambda_0^{\mu_0-1}\partial G/\partial\lambda_0, \quad \partial G/\partial\lambda_1, \dots, \lambda_0^{\mu_1-1}\partial G/\partial\lambda_1, \dots, \\ \partial G/\partial\lambda_k, \dots, \lambda_0^{\mu_k-1}\partial G/\partial\lambda_k, \end{gathered} \tag{2}$$

where all $\mu_j \geq 0$ and $\mu_0 + \mu_1 + \dots + \mu_k = \mu$.

The preparation theorem also implies the existence of the expansion

$$\lambda_0^{\mu_j}\partial G/\partial\lambda_j \equiv \sum_{i=0}^{k} v_{ij}\partial G/\partial\lambda_i + \sum_{s=1}^{n} h_{sj}\partial G/\partial x_s \bmod G^*(\mathfrak{m}(\tilde{m}))\, \mathcal{O}^m(n+1+k), \quad j = 0, \dots, k.$$

Here $h_{sj}(x, \lambda)$ are germs of holomorphic functions, $v_{ij}(\lambda)$ are polynomials in the variable λ_0 of degree strictly less than μ_i (if $\mu_i = 0$, then $v_{ij} = 0$).

THEOREM 2.1 [8]. The algebra $\mathfrak{A}_\Delta$ of germs at $0 \in C^{1+k}$ of holomorphic vector fields tangent to the discriminant Δ of the complete intersection $g_0 = 0$ is generated by the fields $v_j = \sum_{i=0}^{k} (v_{ij} - \delta_{ij}\lambda_0^{\mu_j})\, \partial_{\lambda_i}$, $j = 0, \dots, k$, as a free $\mathcal{O}(1 + k)$-module (δ_{ij} is the Kronecker symbol).

Remark. For $m = 1$ and $g = g_0 + \lambda_0$, this proposition coincides with the theorem of Zakalyukin on vector fields tangent to a front [9].

2.2. The Number of Basis Field Vectors Linearly Independent at a Given Point of the Discriminant. Let $G_\lambda = G\,|_{\lambda=\text{const}}$ and let the manifold $G_\lambda = 0$ have singularities only at the points $x^1, \ldots, x^r$. The Tyurina numbers of these singularities are

$$q^i = \dim_{\mathbf{C}} \mathcal{O}^m(n)_{x^i} / \{G^*_{\lambda,x^i}(\mathfrak{m}(m))\,\mathcal{O}^m(n)_{x^i} + \mathcal{O}(n)_{x^i}\langle \partial G_{\lambda,x^i}/\partial x_1, \ldots, \partial G_{\lambda,x^i}/\partial x_n\rangle\}.$$

Here G_{λ,x^i} is the germ of the map G_λ at the point $x^i \in \mathbf{C}^n$.

If we consider the multigerm space $\bigoplus_{i=1}^{r} \mathcal{O}^m(n)_{x^i}$, then $q(\lambda) = q^1 + \ldots + q^r$ is the codimension in this space of the orbit of the multigerm of the map G_λ relative to the contact group. We will show that $q(\lambda)$ determines the number of basis field vectors $v_0, \ldots, v_k \in \mathfrak{A}_\Delta$, that are linearly independent at a given point $\lambda \in \mathbf{C}^{1+k}$.

Proposition 2.2. $\operatorname{Cork}(v_{ij} - \delta_{ij}\lambda_0^{\mu_j})|_{\lambda=\text{const}} = \tilde{q}(\lambda)$.

Proof. It suffices to take G as a miniversal deformation, i.e., $1 + k = q(0)$. Then in (2) all $\mu_j > 0$.

Assume that $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$. In a sufficiently small neighborhood of zero $\bar{\Lambda} \subset \mathbf{C}^k$ consider the bundle

$$\mathcal{G} = \pi_*(\mathcal{O}^m_{G^{-1}(0)}/\mathcal{O}_{G^{-1}(0)}\langle \partial G/\partial x_1, \ldots, \partial G/\partial x_n\rangle),$$

where $\pi(x, \lambda_0, \bar{\lambda}) = \bar{\lambda}$.

By Proposition 1.4, $\mathcal{G}$ is a free bundle of $\mathcal{O}_{\bar{\Lambda}}$-modules of rank μ. The direct π-images of (2) are the generators of $\mathcal{G}$. The expansion coefficients of the polynomials v_{ij} in powers of λ_0 (holomorphic functions of $\bar{\lambda}$) define the matrix Λ_0 of multiplication by the function λ_0 in $\mathcal{G}$ in this basis. The value of Λ_0 at a particular point $\bar{\lambda}$ is the matrix of multiplication by λ_0 in the μ-dimensional space $\mathcal{G}/\mathfrak{m}(k)_{\bar{\lambda}}\mathcal{G}$. For example, the operator $\Lambda_0(0)$ acts on Q_g. It is nilpotent and its corank equals $\operatorname{codim}_{Q_g} \operatorname{Im} \Lambda_0(0) = 1 + k = q(0)$. Here, 0 is the only critical value of the function λ_0 on the set $\{G|_{\bar{\lambda}=0} = 0\} \subset \mathbf{C}^{n+1}$.

For an arbitrary $\bar{\lambda} \subset \mathbf{C}^k$, everything remains the same. Let $\lambda_0^{(1)}, \ldots, \lambda_0^{(s)}$ be all possible critical values of the function λ_0 on the set $\{G|_{\bar{\lambda}=\text{const}} = 0\} \subset \mathbf{C}^{n+1}$. For each $\lambda_0^{(t)}$ there is a subspace V_t in $\mathcal{G}/\mathfrak{m}(k)_{\bar{\lambda}}\mathcal{G}$ on which the operator $\Lambda_0(\bar{\lambda}) - \lambda_0^{(t)}E$ is nilpotent. The dimension of the subspace V_t is equal to the sum of the multiplicities of the singular points of the height function λ_0 on $G|_{\bar{\lambda}=\text{const}} = 0$ on the critical level $\lambda_0^{(t)}$. The number of eigenvectors of the operator $\Lambda_0(\bar{\lambda})$, contained in V_t is $q(\lambda_0^{(t)}, \bar{\lambda})$. $\sum_1^s \dim V_t = \mu$, therefore $\{\lambda_1^{(1)}, \ldots, \lambda_0^{(s)}\}$ is the spectrum of $\Lambda_0(\bar{\lambda})$ and $\operatorname{cork}(\Lambda_0(\bar{\lambda}) - \lambda_0^{(t)}E) = q(\lambda_0^{(t)}, \bar{\lambda})$.

This concludes the proof, because we can now easily note that the coranks of the matrices $\Lambda_0(\bar{\lambda}) - \lambda_0^{(t)}E$ and $(v_{ij} - \delta_{ij}\lambda_0^{\mu_j})|_{\lambda=(\lambda_0^{(t)},\bar{\lambda})}$ are equal.

2.3. Quasihomogeneous Complete Intersections. Consider an isolated singularity of the quasihomogeneous complete intersection $g_0(x) = 0$ of positive dimension. Let $G \in \mathcal{O}^m(n+q)$, $n > m$, be its q-parameter quasihomogeneous miniversal deformation, and $w_1, \ldots, w_q \in \mathbf{Z}$ the weights of the parameters $\lambda_1, \ldots, \lambda_q$; $\bar{\lambda} = (\lambda_1, \ldots, \lambda_q)$. The Euler field $e = \sum_{i=1}^{q} w_i\lambda_i\partial_{\lambda_i}$ preserves the discriminant $\Delta \subset \mathbf{C}^q$. We will show how the remaining generators of $\mathfrak{A}_\Delta$ are expressed in terms of e.

2.3.1. Let $\Phi = (\Phi_{ij}(\bar{\lambda}))$ be the matrix of multiplication by some function $\varphi \in \mathcal{O}(n)$ in the $\mathcal{O}$-module

$$\mathcal{H} = \mathcal{O}^m(n+q)/\{G^*(\mathfrak{m}(m))\,\mathcal{O}^m(n+q) + \mathcal{O}(n+q)\langle \partial G/\partial x_1, \ldots, \partial G/\partial x_n\rangle\}$$

in the generators $\partial G/\partial\lambda_1, \ldots, \partial G/\partial\lambda_q$:

$$\varphi\,\partial G/\partial\lambda_j = \sum_{i=1}^{q} \Phi_{ij}\,\partial G/\partial\lambda_i. \tag{3}$$

Φ exists because of versality of G, but it is far from being unique: it is defined up to the addition to its columns of other columns formed from the components of any fields from $\mathfrak{A}_\Delta$.

We identify a vector field on $\mathbf{C}^q$ with the column of height q formed from its components.

LEMMA 2.3. $\Phi e \in \mathfrak{A}_\Delta$.

Proof. It suffices to show that $(\Phi e)G = 0$ in $\mathcal{H}$. Indeed, by (3), we have in $\mathcal{H}$

$$(\Phi e)\, G=\sum_{i=1}^{q}\left(\sum_{j=1}^{q}\Phi_{ij}w_j\lambda_j\right)\partial G/\partial\lambda_i=\sum_{j=1}^{q}w_j\lambda_j\left(\sum_{i=1}^{q}\Phi_{ij}\partial G/\partial\lambda_i\right)=\varphi\sum_{j=1}^{q}w_j\lambda_j\partial G/\partial\lambda_j=0,$$

because G is quasihomogeneous.

2.3.2. Consider the ideal $I\subset\mathcal{O}(n)$ generated by the coordinate functions of the map g_0 and all the m-minors of its Jacobi matrix $(\partial g_0/\partial x)$. The dimension of the linear space $\mathcal{O}(n)/I$ is equal to the Milnor number of the singularity $g_0 = 0$. We have noted in 1.2.2 that for a quasihomogeneous complete intersection of positive dimension it is equal to the Tyurina number, i.e., q in our notation. Let $\varphi_1,\dots,\varphi_q\in\mathcal{O}(n)$ be the representatives of the basis of $\mathcal{O}(n)/I$ and $\Phi_1,\dots,\Phi_q$ the corresponding multiplication matrices in $\mathcal{H}$.

THEOREM 2.4. The fields $\Phi_1 e,\dots,\Phi_q e$ are the free generators of the $\mathcal{O}$-module of the vector fields tangent to the discriminant of the isolated singularity of the quasihomogeneous complete intersection $g_0 = 0$ of positive dimension.

2.3.3. Before proceeding with the proof, consider the following example of the curve S_5 in C^3: $g_0 = (x^2 + y^2 + z^2, yz) = 0$ [14-17]. Its miniversal deformation is

$$G=g_0+(\lambda_3y+\lambda_4z+\lambda_5,\ \lambda_1+\lambda_2x);$$
$$\mathcal{O}(3)/I=\mathbf{C}\langle 1,\ x,\ y,\ z,\ x^2\rangle.$$

The fields $\Phi_i e$ are written out by rows:

$$1\cdot e=e=(2\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ 2\lambda_5),$$
$$Xe=(2\lambda_2^3-2\lambda_2\lambda_5,\ 2\lambda_1,\ 4\lambda_2\lambda_4,\ 4\lambda_2\lambda_3,\ 3\lambda_2\lambda_3\lambda_4-4\lambda_1\lambda_2),$$
$$Ye=\Big(\lambda_2^2\lambda_4+\frac{1}{2}\lambda_1\lambda_3,\ \frac{1}{2}\lambda_2\lambda_3,\ -2\lambda_2^2-\frac{1}{2}\lambda_3^2+2\lambda_5,\ -4\lambda_1,\ -3\lambda_1\lambda_4-\lambda_2^2\lambda_3\Big),$$
$$Ze=\Big(\lambda_2^2\lambda_3+\frac{1}{2}\lambda_1\lambda_4,\ \frac{1}{2}\lambda_2\lambda_4,\ -4\lambda_1,\ -2\lambda_2^2-\frac{1}{2}\lambda_4^2+2\lambda_5,\ -3\lambda_1\lambda_3-\lambda_2^2\lambda_4\Big),$$
$$X^2e=\Big(6\lambda_1\lambda_2^2-\frac{3}{2}\lambda_2^2\lambda_3\lambda_4-2\lambda_1\lambda_5,\ 2\lambda_2^3-2\lambda_2\lambda_5,\ 6\lambda_1\lambda_4+4\lambda_2^2\lambda_3,\ 6\lambda_1\lambda_3+4\lambda_2^2\lambda_4,\ -8\lambda_1^2+4\lambda_1\lambda_3\lambda_4+2\lambda_2^2(\lambda_3^2+\lambda_4^2)\Big).$$

The intersection index of the axis $0\lambda_1$ with $\Delta\subset\mathbf{C}^5$ is 6 (by Proposition 1.1, the index is the multiplicity of the critical point $0\in\mathbf{C}^4$ of the function λ_1 on the surface $g_0 + (0, \lambda_1) = (0, 0)$). On the same axis, $\det((\Phi_i e)_j) = 512\lambda_1^6$. Hence, as in [8, Sec. 3], we easily obtain Theorem 2.4 in this particular case.

2.3.4. *Proof of Theorem 2.4.* Consider an auxiliary $(1 + q)$-parameter quasihomogeneous deformation of the complete intersection $g_0 = 0$:

$$\tilde G(x,\ \lambda_0,\ \lambda_1,\dots,\lambda_q)=G(x,\ \lambda_1+\alpha_1\lambda_0^{w_1},\dots,\lambda_q+\alpha_q\lambda_0^{w_q}),$$

where $\alpha_i = 0$ for $w_i \le 0$. For a general choice of the remaining constants α_i, the axis $0\lambda_0$ has a finite intersection index with the discriminant $\tilde\Delta\subset\mathbf{C}^{1+q}$, $\tilde\Delta\cap\{\lambda_0=0\}=\Delta$. Theorem 2.1 gives the generators $v_0,\dots,v_q$ of the module $\mathfrak{A}_{\tilde\Delta}$. For example,

$$v_0=-\partial_{\lambda_0}+\sum_{i=1}^{q}\alpha_iw_i\lambda_0^{w_i-1}\partial_{\lambda_i}\quad(\text{for } \alpha_i\neq 0\ w_i\geqslant 1).$$

For the fields $v_1,\dots,v_q$ the coefficient of ∂_{λ_0} is zero. We call these fields vertical.

$v_1|_{\lambda_0=0},\dots,v_q|_{\lambda_0=0}$ generate $\mathfrak{A}_\Delta$. Our aim is to show that they are expressible in terms of the fields of Theorem 2.4.

a) Let $g=G|_{\bar\lambda=0}$ and $\psi_i=\partial G/\partial\lambda_i|_{\bar\lambda=0}$. Any element from Q_g (see (1)) is uniquely expressible in the form $\sum_{i=1}^{q}p_i(\lambda_0)\psi_i$, where p_i is a polynomial of degree not lower than $\mu_i > 0$ (see 2.1).

As in 2.2, consider the action on Q_g of the operator $\Lambda_0(0)$ (multiplication by λ_0). Let

$$\rho:\mathrm{Ker}\,\Lambda_0(0)\to(\mathfrak{A}_{\tilde\Delta,\text{верт}}/\mathfrak{m}(1+q)\,\mathfrak{A}_{\tilde\Delta,\text{верт}})|_{\bar\lambda=0}\simeq\mathbf{C}\langle v_1|_{\bar\lambda=0},\dots,v_q|_{\bar\lambda=0}\rangle,$$
$$\rho:\sum_{i=1}^{q}p_i(\lambda_0)\psi_i\mapsto\sum_{i=1}^{q}\lambda_0p_i(\lambda_0)\partial_{\lambda_i}.$$

The construction of the basis fields $\mathfrak{A}_{\tilde{\Delta}}$ in 2.1 leads to

LEMMA 2.5. ρ is an isomorphism.

Ker $\Lambda_0(0) \subset Q_g$ also has an alternative description.

LEMMA 2.6. Ker $\Lambda_0(0) = (\mathcal{O}(n)/I)\partial g/\partial\lambda_0$.

Proof. The quasihomogeneous complete intersection of positive dimension $g^{-1}(0) \subset C^{n+1}$ has an isolated singularity. Therefore by [1, p. 499], $h\partial g/\partial\lambda_0 = 0$, $h \in \mathcal{O}(n+1)$ in Q_g if and only if h is contained in the ideal generated by the coordinate functions of the map g, all the m-minors of the matrix $(\partial g/\partial x)$, and the element λ_0. Therefore, all the elements from $(\mathcal{O}(n)/I)\partial g/\partial\lambda_0 \subset \mathcal{O}_g$ are nonzero and lie in Ker $\Lambda_0(0)$. The lemma follows from equality of the dimensions of $\mathcal{O}(n)/I$ and Ker $\Lambda_0(0)$.

COROLLARY 2.7. In Q_g, $\varphi_j\partial g/\partial\lambda_0 = \sum_{i=1}^{q} \gamma_{ij}\rho^{-1}(v_i|_{\bar{\lambda}=0})$, $j = 1, \ldots, q$, $\gamma_{ij} \in \mathbf{C}$. Also $\det(\gamma_{ij}) \neq 0$.

b) The Euler field $e \in \mathfrak{A}_\Delta$ is extended to the Euler field $e + \lambda_0\partial_{\lambda_0} \in \mathfrak{A}_{\tilde{\Delta}}$. Adding $\lambda_0 v_0$, we obtain the extension of e to the vertical field $e + \delta$, where $\delta = \sum_{i=1}^{q} \alpha_i w_i \lambda_0^{w_i}\partial_{\lambda_i}$.

Similarly to Lemma 2.3, we can easily show that $\Phi(e + \delta)$ is the extension of any field $\Phi e \in \mathfrak{A}_\Delta$ to a vertical field from $\mathfrak{A}_{\tilde{\Delta}}$, where the matrix $\tilde{\Phi}(e+\delta)$, is obtained from $\Phi(\bar{\lambda})$ like the mapping $\tilde{G}$ is obtained from G.

We have $\tilde{\Phi}(e+\delta) = \sum_{i=1}^{q} \beta_i v_i$, $\beta_i \in \mathcal{O}(1+q)$.

Let us find $\beta_i(0)$. For $\bar{\lambda} = 0$,

$$\tilde{\Phi}(e+\delta) = \sum_{i,j=1}^{q} \tilde{\Phi}_{ij}|_{\bar{\lambda}=0}\alpha_j w_j \lambda_0^{w_j}\partial\lambda_i.$$

In Q_g,

$$\sum_{i,j=1}^{q} \tilde{\Phi}_{ij}|_{\bar{\lambda}=0}\alpha_j w_j \lambda_0^{w_j-1}\partial G/\partial\lambda_i|_{\bar{\lambda}=0} = \varphi\sum_{j=1}^{q} \alpha_j w_j \lambda_0^{w_j-1}\partial G/\partial\lambda_j|_{\bar{\lambda}=0} = \varphi\partial g/\partial\lambda_0.$$

The values of $\beta_i(0)$ are obtained from the expansion in Q_g

$$\varphi\partial g/\partial\lambda_0 = \sum_{i=1}^{q} \beta_i(0)\rho^{-1}(v_i|_{\bar{\lambda}=0}).$$

The theorem now follows from the fact that by Corollary 2.7 the matrix $(\Gamma_{ij}(0))$ the expansions $\tilde{\Phi}_j(e+\delta) = \sum_{i=1}^{q} \Gamma_{ij}v_i$, $j = 1, \ldots, q$, $\Gamma_{ij} \in \mathcal{O}(1+q)$ is nonsingular.

2.3.5. The generators $\Phi_j e$ may be taken quasihomogeneous. The Poincaré polynomial of the space $\mathcal{O}(n)/I$ is calculated in [4]. The results of [4] lead to

COROLLARY 2.8. Let $A_1, \ldots, A_n$ and $D_1, \ldots, D_m$ be the weights of the arguments and of the coordinate functions of the map g_0, $B = \sum_{i=1}^{m} D_i - \sum_{j=1}^{n} A_j$. The Poincaré polynomial of the set of quasihomogeneous generators of $\mathfrak{A}_\Delta$ is given by

$$P(\mathfrak{A}_\Delta/\mathfrak{m}(q)\mathfrak{A}_\Delta, t) = \operatorname{res}_{s=0}\frac{s^{m-n}}{1+s}\left[s^{n-m-1}\prod_{i=1}^{m}(1-t^{D_i}) \Big/ \prod_{j=1}^{n}(1-t^{A_j}) - \right.$$
$$\left. - t^B\prod_{j=1}^{n}\frac{1+st^{A_j}}{1-t^{A_j}}\prod_{i=1}^{m}\frac{1-t^{D_i}}{1+st^{D_i}} + s^{-2}t^B\right].$$

Remark. The same result can be obtained without Theorem 2.4, using only Lemma 2.5. If w_0 is the weight of λ_0 (previously we had $w_0 = 1$), then

$$P(\mathfrak{A}_\Delta/\mathfrak{m}(q)\mathfrak{A}_\Delta, t) = t^{w_0}P(\operatorname{Ker}\Lambda_0(0), t) = t^{w_0}[(t^{w_0}-1)P(T', t) + P(T'', t)],$$

TABLE 1

u	$\frac{p-1}{p}\lambda_1$	$\frac{p-2}{p}\lambda_2$	$\dots$	$\frac{2}{p}\lambda_{p-2}$	$\frac{1}{p}\lambda_{p-1}$	$\frac{q-1}{q}\lambda_p$	$\frac{q-2}{q}\lambda_{p+1}$	$\dots$	$\frac{2}{q}\lambda_{p+q-3}$	$\frac{1}{q}\lambda_{p+q-2}$	$\frac{p+q}{pq}\lambda_{p+q-1}$
0	u	$\frac{p-1}{p}\lambda_1$	$\dots$	$\frac{3}{p}\lambda_{p-3}$	$\frac{2}{p}\lambda_{p-2}$	0	0	$\dots$	0	$-\frac{p+q}{p}\lambda_{p+q-1}$	0
0	0	u	$\dots$	$\frac{4}{p}\lambda_{p-4}$	$\frac{3}{p}\lambda_{p-3}$	0	0	$\dots$	0	0	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
0	0	0	$\dots$	0	u	0	0	$\dots$	0	0	0
0	0	0	$\dots$	0	$-\frac{p+q}{q}\lambda_{p+q-1}$	u	$\frac{q-1}{q}\lambda_p$	$\dots$	$\frac{3}{q}\lambda_{p+q-4}$	$\frac{2}{q}\lambda_{p+q-3}$	0
0	0	0	$\dots$	0	0	0	u	$\dots$	$\frac{4}{q}\lambda_{p+q-5}$	$\frac{3}{q}\lambda_{p+q-4}$	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
0	0	0	$\dots$	0	0	0	0	$\dots$	0	u	0
0	0	0	$\dots$	$\delta_{q2}(p+2)\lambda_{p+1}$	$-(p+1)\lambda_p$	0	0	$\dots$	0	$-(q+1)\lambda_1$	u

where

$$T'=\mathcal{O}^m(n+1)/\{g^*(\mathfrak{m}(m))\,\mathcal{O}^m(n+1)+\mathcal{O}(n+1)\langle \partial g/\partial x_1,\dots,\partial g/\partial x_n,\ \partial g/\partial\lambda_0\rangle\}$$

and T″ is the corresponding space for g_0. The polynomials P(T′, t) and P(T″, t) are described in [1, p. 479].

3. DISCRIMINANTS OF PROJECTIONS

3.1. Vector Fields Preserving the Discriminant of a Projection. Let $F\in\mathcal{O}^m(n+\tau)$ be a R^+-miniversal deformation of the projection f (recall that $\tau = 1 + \tau^+$, see 1.1.3). By R^+-versality, we have the expansions

$$u\partial F/\partial\lambda_j\equiv\sum_{i=0}^{\tau-1} v_{ij}\partial F/\partial\lambda_i+$$

$$+\sum_{s=1}^{n} h_{sj}\partial F/\partial x_s \bmod F^*(\mathfrak{m}(m))\,\mathcal{O}^m(n+\tau),\quad j=0,\dots,\tau-1.$$

Here $v_{ij}(\lambda)$ and $h_{sj}(x, u, \lambda)$ are germs of holomorphic functions; $\lambda_0 = u$, but $\lambda = (\lambda_1, \dots, \lambda_{\tau-1})$.

Let δ_{ij} be the Kronecker symbol. Using Theorem 1.2, we can prove

THEOREM 3.1 [8]. The algebra $\mathfrak{A}_\Delta$ of the germs at $0 \in C^\tau$ of holomorphic vector fields tangent to the discriminant Δ of the projection f is generated by the fields $v_j=\sum_{i=0}^{\tau-1}(v_{ij}-\delta_{ij}u)\,\partial_{\lambda_i},\quad j=0,\dots,\tau-1,$ as a free $\mathcal{O}(\tau)$module.

3.2. Example. Consider the miniversal deformation of the simple projection $C_{p,q}$, $p \geq q \geq 2$, of a three-dimensional curve on the straight line [5, 7],

$$F(x,u,\lambda)=(x_1^p+\lambda_{p-1}x_1^{p-1}+\dots+\lambda_1x_1+u+x_2^q+\lambda_{p+q-2}x_2^{q-1}+\dots+\lambda_px_2,\ x_1x_2+\lambda_{p+q-1}).$$

Note that the same map defines a miniversal deformation of the complete intersection $I_{p,q}$ [15, 16] (u is regarded as one of the deformation parameters). Therefore the discriminant of the projection $C_{p,q}$ coincides with the discriminant of the multiple point $I_{p,q}$. By Theorem 3.1, we obtain the basis of $\mathfrak{A}_\Delta$ for the first singularities in the series (the components of the field v_j multiplied by some integer are written out by (j + 1) rows of the matrix):

$$p=q=2:\ \begin{pmatrix} 2u & \lambda_1 & \lambda_2 & 2\lambda_3 \\ -6\lambda_2\lambda_3 & 4u-\lambda_1^2 & -8\lambda_3 & \lambda_1\lambda_3 \\ -6\lambda_1\lambda_3 & -8\lambda_3 & 4u-\lambda_2^2 & \lambda_2\lambda_3 \\ 4\lambda_3-2\lambda_1\lambda_2 & -3\lambda_2 & -3\lambda_1 & u \end{pmatrix},$$

$$p=3,\ q=2:$$

$$\begin{pmatrix} 6u & 4\lambda_1 & 2\lambda_2 & 3\lambda_3 & 5\lambda_4 \\ -12\lambda_3\lambda_4 & 9u-\lambda_1\lambda_2 & 6\lambda_1-2\lambda_2^2 & -15\lambda_4 & \lambda_2\lambda_4 \\ -3\lambda_2\lambda_3\lambda_4+45\lambda_4^2 & \begin{matrix}-6\lambda_1^2+2\lambda_1\lambda_2^2- \\ -36\lambda_3\lambda_4\end{matrix} & \begin{matrix}27u-15\lambda_1\lambda_2+ \\ +4\lambda_2^3\end{matrix} & -6\lambda_2\lambda_4 & 6\lambda_1\lambda_4-2\lambda_2^2\lambda_4 \\ -6\lambda_1\lambda_4 & -8\lambda_2\lambda_4 & -10\lambda_4 & 4u-\lambda_3^2 & \lambda_3\lambda_4 \\ -2\lambda_1\lambda_3+4\lambda_2\lambda_4 & -3\lambda_2\lambda_3+5\lambda_4 & -4\lambda_3 & -3\lambda_1 & u \end{pmatrix}.$$

Table 1 gives the matrix of components of the linearizations $(v_{ij}^0)^t$ of the basis fields of $\mathfrak{A}_\Delta(C_{p,q})$. It is easily constructed using the matrices of multiplication by x_1 and x_2 in the linear space

$$\mathcal{O}^2(2)/\{f_0^*(\mathfrak{m}(2))\,\mathcal{O}^2(2)+\mathcal{O}(2)\langle \partial f_0/\partial x_1,\ \partial f_0/\partial x_2\rangle\}\simeq \mathbf{C}\langle \partial F/\partial u|_{\lambda=0},\ \partial F/\partial\lambda_1|_{\lambda=0},\ \ldots,\ \partial F/\partial\lambda_{p+q-1}|_{\lambda=0}\rangle,$$

where $f_0=(x_1^p+x_2^q,\ x_1x_2)$. In what follows, knowledge of the linearizations of the generators of $\mathfrak{A}_\Delta$ will be needed in order to examine stability of the general function on $(\mathbf{C}^\tau, \Delta)$.

3.3. Stable Functions and Hypersurfaces. Stability of functions and hypersurfaces in the space of versal deformation of a simple function relative to the group of discriminant-preserving biholomorphisms was considered in [12]. Here we examine the same topic in the extended parameter space $\mathbf{C}^\tau$ of the R^+-miniversal deformation of a simple projection.

Definition. Δ-stability (Δ-equivalence) is stability (equivalence) relative to the biholomorphism group of the pair $(\mathbf{C}^\tau, \Delta)$.

The following proposition is proved by traditional methods (see, e.g., [3, 12]).

Proposition 3.2. The germs at $0 \in \mathbf{C}^\tau$ of the function h and the hypersurface h = 0 are Δ-stable if and only if resp.

$$\mathfrak{m}(\tau)=\mathcal{O}(\tau)\langle v_0h,\ldots,v_{\tau-1}h\rangle+\mathfrak{m}^2(\tau)$$

and

$$\mathfrak{m}(\tau)=\mathcal{O}(\tau)\langle v_0h,\ldots,v_{\tau-1}h,\ h\rangle+\mathfrak{m}^2(\tau).$$

COROLLARY 3.3. Let v (0) = 0 for any $v\in\mathfrak{A}_\Delta$ and let v_j^0 be the linearization of the basis field v_j. Then Δ-stability of the germ at $0 \in \mathbf{C}^\tau$ of the function h is equivalent to the condition

$$\mathbf{C}\langle u, \lambda_1,\ldots,\lambda_{\tau-1}\rangle \equiv \mathbf{C}\langle v_0^0h,\ldots,v_{\tau-1}^0h\rangle \bmod \mathfrak{m}^2(\tau),$$

and Δ-stability of the germ of the hypersurface h = 0 is equivalent to the condition

$$\mathbf{C}\langle u, \lambda_1,\ldots,\lambda_{\tau-1}\rangle \equiv \mathbf{C}\langle v_0^0h,\ldots,v_{\tau-1}^0h,\ h\rangle \bmod \mathfrak{m}^2(\tau).$$

COROLLARY 3.4. If v(0) = 0 for any $v\in\mathfrak{A}_\Delta$, then Δ-stable germs at $0 \in \mathbf{C}^\tau$ of a function and a hypersurface are Δ-equivalent to their linear parts.

It is also obvious that Δ-stability of the function h implies Δ-stability of the hypersurface h = 0.

3.4. Let us elucidate what simple projections have discriminants on which stable functions and hypersurfaces exist.

The R^+-classification of projections of smooth manifolds on the straight line coincides with the R-classification of smooth functions. The discriminants also coincide. Stable functions on discriminants of simple functions, as we have noted previously, were obtained in [12]: these are the lowest-weight parameters in the base of the quasihomogeneous miniversal deformation.

Discriminants of simple projections of plane nonsmooth curves (B_k, C_k, and F_4 in [5]) are cylinders over the discriminants of the series A (resp., A_1, A_{k-1}, A_2). The list of stable functions on such sets is also given in [12].

It remains to examine the projections $C_{p,q}$, $p \geq q \geq 2$, and F_p, $p \geq 7$, of spatial nonsmooth curves. It turns out that if stable objects exist, then their normal forms are entirely analogous to the normal forms from [12, 9].

Before stating the final result, we write out the R^+-miniversal deformations of projections of the series F:

The deformations are quasihomogeneous. The parameters λ_1 and λ_2 have minimal weights. The following theorem deals with extended parameter spaces C^τ of these particular deformations and of the deformations $C_{p,q}$ from 3.2.

THEOREM 3.5.

a) The germ at $0 \in C^{p+q}$, $p \geq q \geq 2$, of the hypersurface of general position is $\Delta(C_{p,q})$-stable only for $q = 2$. In this case, it is reducible to the normal form λ_{p-1} (the minimal weight parameter).

b) The germ at $0 \in C^{p+q}$, $p \geq q \geq 2$, of the hypersurface of general position is $\Delta(C_{p,q})$-stable only for $q = 2$ and $p > q = 3$. In the first case, it is Δ-equivalent to the germ $\lambda_{p-1} = 0$, in the second case to $\lambda_{p-1} + \lambda_{p+1} = 0$.

c) The germ at $0 \in C^p$, $p \geq 7$, of the function of general position is $\Delta(F_p)$-stable only for $p = 7$ and $p = 8$. For $p = 7$ it is Δ-equivalent to the germ $\lambda_1 + \lambda_2$ and for $p = 8$ to the germ λ_1.

d) The germ at $0 \in C^p$, $p \geq 7$, of the hypersurface of general position is $\Delta(F_p)$-stable only for $p = 7,8,9,10$. For $p = 7,9,10$ it is Δ-equivalent to the germ $\lambda_1 + \lambda_2$, and for $p = 8$ to $\lambda_1 = 0$.

Remarks. a) Since the maps defining R^+-miniversal deformations of the projections $C_{p,q}$ and F_p define miniversal deformations of the simple complete intersections $I_{p,q}$ and I_p [15, 16] (u is an additional parameter), Theorem 3.5 also classifies the stable objects on the discriminants of these multiple points.

b) Theorem 3.5 is consistent with part of the classifications of simple projections of manifolds from [7].

Quasihomogeneity of the miniversal deformations of projections plays a central role in the proof of the negative part of the theorem (on nonexistence of Δ-stable objects in many cases). Theorem 3.1 leads to

COROLLARY 3.6. If the R^+-miniversal deformation of a projection is quasihomogeneous, then the weight of the basis field $v_j \in \mathfrak{A}_\Delta$ is equal to the difference of weights of u and λ_j.

Proof of Theorem 3.5. We will only prove part a). The rest is proved similarly.

Thus, consider the singularity $C_{p,q}$. Assuming that the weight of x_1 in its quasihomogeneous deformation in 3.2 equals q, by Corollary 3.6 we obtain the Poincaré polynomial of the set of generators of $\mathfrak{A}_\Delta$

$$P_V(t) = (t^{pq}-1)(t^q-1)^{-1} + (t^{pq}-t^p)(t^p-1)^{-1} + t^{pq-p-q}.$$

The Poincaré polynomial of the set of linear homogeneous functions on $C^\tau = C^{p+q}$ (with 1 having the weight 0) is

$$P_L(t) = (t^{pq+q}-t^q)(t^q-1)^{-1} + (t^{pq}-t^p)(t^p-1)^{-1} + t^{p+q}.$$

The minimal weight of a parameter from C^{p+q} (λ_{p-1}) is q. The function of general position contains λ_{p-1} in its expansion. For $(p, q) \neq (2, 2)$, one of the generators of $\mathfrak{A}_\Delta$ is of weight 0, and the remaining generators are of positive weight; all the generators vanish at $0 \in C^{p+q}$. Applying Corollaries 3.3. and 3.4 and the spectral sequence in order to reduce the functions to normal form [3], we see that if the function is Δ-stable for $(p, q) \neq (2, 2)$, then it is Δ-equivalent to its linear term of weight q and $P_L(t) = t^q P_V(t)$ (in other words, the linear part of the function should be reducible to normal form by the zero differential of the spectral sequence).

This polynomial equality is possible only for $q = 2$. Thus, for $q > 2$, the functions have no $\Delta(C^{p,q})$-stable germs.

$\Delta(C_{p,2})$-stability of the germ at $0 \in C^{p+2}$ of the function λ_{p-1} now follows from Corollary 3.3 and the fact that the elements $v_j^0 \lambda_{p-1} = v_{p-1,j}^0$ of the matrix v_{ij}^0 introduced in 3.2 ($q = 2$) generate $\mathfrak{m}(p+2)$.

Note that any function h such that $(\partial h/\partial\lambda_1(0))^2 \neq (\partial h/\partial\lambda_2(0))^2$ for $p = q = 2$ or $\partial h/\partial\lambda_{p-1}(0) \neq 0$ for $p > q = 2$ has Δ-normal form λ_{p-1}.

4. BIFURCATION DIAGRAMS OF PROJECTIONS

4.1. Vector Fields. Consider the R^+-miniversal deformation $F \in \mathcal{O}^m(n+\tau)$ of the projection f. By versality, we have the expansions

$$u^j \partial F/\partial u = \sum_{i=0}^{\tau-1} w_{ij} \partial F/\partial\lambda_i + \sum_{s=1}^{n} h_{sj} \partial F/\partial x_s \bmod F^*(\mathfrak{m}(m))\,\mathcal{O}^m(n+\tau), \quad j = 1, \ldots, \tau-1.$$

Here $w_{ij}(\lambda)$ and $h_{sj}(x, u, \lambda)$ are germs of holomorphic functions; $\lambda_0 = u$, but $\lambda = (\lambda_1, \dots, \lambda_{\tau-1})$.

THEOREM 4.1 [8]. The algebra $\mathfrak{A}_\Sigma$ of the germs at $0 \in \mathbf{C}^{\tau-1}$ of the vector fields tangent to the bifurcation diagram of the projection f is generated by the fields $w_j = \sum_{i=1}^{\tau-1} w_{ij}\partial_{\lambda_i}$, $j = 1, \dots, \tau-1$ as a free $\mathcal{O}(\tau-1)$-module.

Remark. For $m = 1$ and $f(x, u) = f_0(x) + u$, Theorem 4.1 coincides with the theorem of Bruce [13] on vector fields that preserve the bifurcation diagram of the function f_0.

4.2. The Number of Linearly Independent Basis Fields. Let $F_\lambda = F|_{\lambda=\text{const}}$; let $u_1, \dots, u_r$ be all possible different critical values of the height function on the manifold $Y_\lambda = \{F_\lambda = 0\}$; let X_i be the set of x-components of the critical points of the height function on Y_λ on level $u = u_i$; let $T_{F_\lambda,(\bar{x},u_i)} \subset \mathcal{O}^m(n+1)_{(\bar{x},u_i)}$ be the tangent space to the R-equivalence class of the germ of the projection F_λ at the point $(\bar{x}, u_i)$.

The projection $p:(x, u) \mapsto u$ induces on $\bigoplus_{\bar{x}\in X_i} \mathcal{O}^m(n+1)_{(\bar{x},u_i)}$ a $\mathcal{O}(1)_{u_i}$-module structure. For example, the codimension of the RL-equivalence class of the germ of the projection f at the point $(0, 0) \in \mathbf{C}^{n+1}$ equals $\dim_{\mathbf{C}} \mathcal{O}^m(n+1)/(T_f + \mathcal{O}(1)\,\partial f/\partial u)$, and the corresponding codimension of the multigerm of the projection F_λ at the points $(\bar{x}, u_i)$, $\bar{x} \in X_i$, is

$$\nu_i = \dim_{\mathbf{C}} [(\bigoplus_{\bar{x}\in X_i} \mathcal{O}^m(n+1)_{(\bar{x},u_i)}/T_{F_\lambda,(\bar{x},u_i)})/\mathcal{O}(1)_{u_i}\partial F_\lambda/\partial u].$$

Proposition 4.2.

$$\operatorname{cork}(w_{ij}(\lambda))_{i,j=1}^{\tau-1} = \nu_1 + \dots + \nu_r.$$

We thus assert that at the point $\lambda \in \mathbf{C}^{\tau-1}$ the number of linearly independent vectors of basis fields from $\mathfrak{A}_\Sigma$ is equal to the dimension of the RL-equivalence class of the corresponding multisingularity (see [13, proposition 9]).

Proof. Consider the τ-dimensional linear space responsible for the R-codimension of the multigerm of the projection F_λ at all the critical points of the height function on Y_λ:

$$Q_{F_\lambda} = \bigoplus_{i=1}^{r} \Big[\bigoplus_{\bar{x}\in X_i} \mathcal{O}^m(n+1)_{(\bar{x},u_i)}/T_{F_\lambda,(\bar{x},u_i)}\Big].$$

Its basis is $\partial F_\lambda/\partial u,\ \partial F/\partial\lambda_1|_{\lambda=\text{const}}, \dots, \partial F/\partial\lambda_{\tau-1}|_{\lambda=\text{const}}$.

Consider in Q_{F_λ} the subspace W generated by the elements $u^\alpha\partial F_\lambda/\partial u$, $\alpha = 0, \dots, \tau-1$. From dimensional considerations we conclude that W also contains all the elements $u^\alpha\partial F_\lambda/\partial u$ with $\alpha \geq \tau$. Thus,

$$Q_{F_\lambda}/W = \bigoplus_{i=1}^{r} \Big[\Big(\bigoplus_{\bar{x}\in X_i} \mathcal{O}^m(n+1)_{(\bar{x},u_i)}/T_{F_\lambda,(\bar{x},u_i)}\Big)/\mathcal{O}(1)_{u_i}\partial F_\lambda/\partial u\Big].$$

Hence, $\operatorname{codim} W = \nu_1 + \dots + \nu_r$.

On the other hand, W is the image of the linear map of the space $\mathbf{C}\langle 1, u, \dots, u^{\tau-1}\rangle\,\partial F_\lambda/\partial u$ to the space Q_{F_λ} defined in the basis $\partial F_\lambda/\partial u,\ \partial F/\partial\lambda_1|_{\lambda=\text{const}}, \dots, \partial F/\partial\lambda_{\tau-1}|_{\lambda=\text{const}}$ by the matrix

$$\begin{pmatrix} 1 & w_{01} & \dots & w_{0,\tau-1} \\ 0 & w_{11} & \dots & w_{1,\tau-1} \\ \dots & \dots & \dots & \dots \\ 0 & w_{\tau-1,1} & \dots & w_{\tau-1,\tau-1} \end{pmatrix}.$$

All the functions w_{ij} are evaluated at the point λ. The codimension of W coincides with the dimension of the kernel of this map, i.e., with $\operatorname{cork}(w_{ij}(\lambda))_{i,j=1}^{\tau-1}$.

4.3. Fields Preserving the RL-Bifurcation Diagram. Let $F' \in \mathcal{O}^m(n+1+\nu)$ be the RL-miniversal deformation of the projection f, $\lambda' \in \mathbf{C}^\nu$ its parameter. If as before τ is the R-codimension of f, then there exist the expansions

$$u^j\partial F'/\partial u \equiv \sum_{i=0}^{\nu} w'_{ij}\partial F'/\partial\lambda'_i + \sum_{s=1}^{n} h'_{sj}\partial F'/\partial x_s \bmod F'^*(\mathfrak{m}(m))\,\mathcal{O}^m(n+1+\nu),\ j = \tau-\nu, \dots, \tau-1,$$

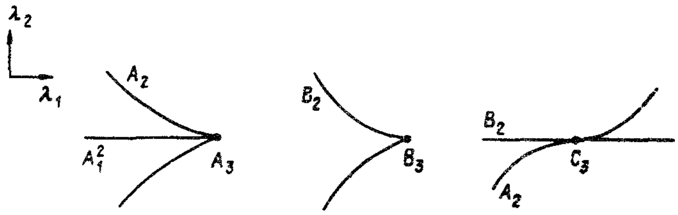

Fig. 1

where $w_{0j}(u, \lambda')$ are polynomials in u of degree not higher than $\tau - 1 - \nu$; $w'_{ij}(\lambda')$, $i > 0$ and $h'_{sj}(x, u, \lambda')$ are germs of holomorphic functions; $\lambda_0' = u$, $\lambda' = (\lambda_1', \ldots, \lambda_\nu')$.

Let $\Sigma' \subset C^\nu$ be the bifurcation diagram of the projection f (it is connected with Σ by $(C^\nu, \Sigma') \times C^{\tau-1-\nu} \simeq (C^{\tau-1}, \Sigma)$).

COROLLARY 4.3. The algebra $\mathfrak{A}_{\Sigma'}$ of the germs at $0 \in C^\tau$ of the vector fields tangent to Σ' is generated by the fields

$w'_j = \sum_{i=1}^{\nu} w'_{ij}\partial_{\lambda_i'}$, $j = \tau-\nu, \ldots, \tau-1$ as a free $\mathcal{O}$-module. Here $w_j'(0) = 0$ for all j.

The corollary follows from Theorem 4.1 and Corollary 4.2 if we note that

$$\min\{\alpha \mid u^\alpha \partial f/\partial u \in T_f\} = \tau - \nu.$$

4.4. Quasihomogeneous Mappings.

COROLLARY 4.4. Let f be a quasihomogeneous mapping. Then all the vector fields preserving its bifurcation diagram in the space of the R^+-miniversal deformation vanish at the origin.

The converse of this proposition would follow from a generalization of Saito's theorem on quasihomogeneous functions [24].

Conjecture 4.5 (criterion of quasihomogeneity of a complete intersection of positive dimension). Let the projection on the axis u of the complete intersection f(x, u) = 0 be of finite R-codimension, $f \in \mathcal{O}^m(n+1)$, $x \in C^n$, $u \in C$, $n \geq m$. The germ at zero of the set f = 0 is biholomorphically equivalent to the germ of a quasihomogeneous complete intersection if and only if

$$u\partial f/\partial u \in f^*(\mathfrak{m}(m))\mathcal{O}^m(n+1) + \mathcal{O}(n+1)\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle.$$

4.5. Terao Type Formula. In order to compute the generators of the module $\mathfrak{A}_\Sigma$ it is helpful to use the matrices $V = (v_{ij})_{i,j=0}^{\tau-1}$ of the components of the generators of the module $\mathfrak{A}_\Delta$ (see [30]). From the expansions in 3.1 it follows that V is the matrix of multiplication by u in the free $\mathcal{O}(\tau-1)$-module $\mathcal{O}^m(n+\tau)/T_f$ in the basis $\partial F/\partial u$, $\partial F/\partial \lambda_1$, ..., $\partial F/\partial \lambda_{\tau-1}$:

$$u\partial F/\partial \lambda_j = \sum_{i=0}^{\tau-1} v_{ij}\partial F/\partial \lambda_i.$$

Therefore, if $\pi'' : (u, \lambda) \mapsto \lambda$ and the vector field is represented as a column of its components, then Theorem 4.1 leads to

COROLLARY 4.6 [8]. The generators of the module $\mathfrak{A}_\Sigma$ are given by the formulas

$$w_j = \pi''_* [V^{j-1}\mathfrak{v}_0], \quad j = 1, \ldots, \tau-1,$$

where $\mathfrak{v}_0 = \sum_{i=0}^{\tau-1} v_{i0}\partial_{\lambda_i} = v_0 + u\partial_u$.

4.6. Functions of General Position and Hypersurfaces. Let us indicate the normal forms of the general germs at $0 \in C^{\tau-1}$ of functions (h) and hypersurfaces (Γ) relative to the group of biholomorphisms preserving Σ, for projections of R^+-codimensions 2 and 3. The normal forms were obtained using a spectral sequence [3] and the action of basis vector fields from $\mathfrak{A}_\Sigma$ on the functions. These fields differ from the fields of Theorem 4.1 in that they are multiplied by some integers, and we write them out only to the extent that it is needed for reduction of the object of general position to normal form.

Figures 1 and 2 show the bifurcation diagrams of the singularities. The notation of the projections follows [5, 7]. The bifurcation diagrams of the projections of smooth manifolds (A_k, D_k) are the bifurcation diagrams of the corresponding singularities of smooth functions (it is easy to see that the same also holds for more complicated projections of smooth manifolds). The bifurcation diagrams of the projections of series B coincide with the discriminants of functions of series A. In the normal forms below, $\alpha_i \in C$ are the modulus numbers.

A_3: $x^4+\lambda_1 x^2+\lambda_2 x+u=0$, $w_1=(2\lambda_1, 3\lambda_2)$,
Σ: $\lambda_2(8\lambda_1^3+27\lambda_2^2)=0$, $w_2=(9\lambda_2^2-2\lambda_1^3, -7\lambda_2\lambda_1^2)$,
h: $\lambda_1+\alpha\lambda_2$ Γ: $\lambda_1+\lambda_2=0$;

B_3: $u^3+\lambda_1 u+\lambda_2=0$, $w_1=(2\lambda_1, 3\lambda_2)$,
Σ: $4\lambda_1^3+27\lambda_2^2=0$, $w_2=(9\lambda_2, -2\lambda_1^2)$,
h: λ_1 Γ: $\lambda_1=0$ [12];

C_3: $x^3+\lambda_1 x^2+ux+\lambda_2=0$, $w_1=(\lambda_1, 3\lambda_2)$,
Σ: $\lambda_2(\lambda_1^3-27\lambda_2)=0$, $w_2=(9\lambda_2-\lambda_1^3, -4\lambda_1^2\lambda_2)$,
h: λ_1 Γ: $\lambda_1=0$;

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A_4: $x^5+\lambda_1 x^3+\lambda_2 x^2+\lambda_3 x+u=0$, $w_1=(2\lambda_1, 3\lambda_2, 4\lambda_3)$,
$w_2|_{\lambda_1=0}=(60\lambda_2\lambda_3, 40\lambda_3^2, -9\lambda_2^3)$, $w_3|_{\lambda_1=0}=(27\lambda_2^4-160\lambda_3^3, 117\lambda_2^3\lambda_3, 192\lambda_2^2\lambda_3^2)$,
h: $\lambda_1+\alpha_1\lambda_2+\alpha_2\lambda_2^2+\alpha_3\lambda_2^3+$
$+\alpha_4\lambda_2^4+\alpha_5\lambda_3+\alpha_6\lambda_3^2$,
Γ: $\lambda_1+\lambda_2+\alpha_5\lambda_3+\alpha_6\lambda_3^2=0$,

B_4: $u^4+\lambda_1 u^2+\lambda_2 u+\lambda_3=0$, $w_1=(2\lambda_1, 3\lambda_2, 4\lambda_3)$,
$w_2=(6\lambda_2, 8\lambda_3-2\lambda_1^2, -\lambda_1\lambda_2)$, $w_3=(16\lambda_3-8\lambda_1^2, -14\lambda_1\lambda_2, -8\lambda_1\lambda_3-3\lambda_2^2)$
h: λ_1 Γ: $\lambda_1=0$ [12];

C_4: $x^4+\lambda_1 x^3+\lambda_2 x^2+ux+\lambda_3=0$, $w_1=(\lambda_1, 2\lambda_2, 4\lambda_3)$,
$w_2|_{\lambda_1=0}=(\lambda_2^2+12\lambda_3, 0,0)$,
$w_3|_{\lambda_1=0}=(0, -\lambda_2^4+72\lambda_3^2+18\lambda_2^2\lambda_3,$
$\lambda_2^3\lambda_3+60\lambda_2\lambda_3^2)$,
h: $\lambda_1+\alpha_1\lambda_2+\alpha_2\lambda_2^2+\alpha_3\lambda_2^3$, $\alpha_1\neq 0$, Γ: $\lambda_1+\lambda_2=0$;

D_4: $x_1^2x_2+x_2^3+\lambda_1 x_1^2+\lambda_2 x_2+\lambda_3 x_1+u=0$,
(bifurcation diagram $D_4^{\pm}$: $\pm x_1^2x_2+x_2^3+\ldots=0$)
$w_1=(\lambda_1, 2\lambda_2, 2\lambda_3)$, $w_2|_{\lambda_1=0}=(\lambda_2^2-3\lambda_3^2, 0, 0)$,
$w_3|_{\lambda_1=0}=(0, -2\lambda_2^4-9\lambda_2^2\lambda_3^2+9\lambda_3^4, \lambda_2^3\lambda_3-15\lambda_2\lambda_3^3)$,
h: $\lambda_1+\alpha_1\lambda_2+\alpha_2\lambda_3+\alpha_3\lambda_2\lambda_3+\alpha_4\lambda_3^2+\alpha_5\lambda_2\lambda_3^2+\alpha_6\lambda_3^3+\alpha_7\lambda_2\lambda_3^3$,
$\alpha_1(27\alpha_1^2-\alpha_2^2)\neq 0$,
Γ: $\lambda_1+\lambda_2+\alpha\lambda_3$, $3\alpha^2\neq 1$;

F_4: $x^3+u^2+\lambda_1 x+\lambda_2 ux+\lambda_3=0$, $w_1=(4\lambda_1, \lambda_2, 6\lambda_3)$,
$w_2|_{\lambda_2=0}=(0, \lambda_1, 0)$ $w_3|_{\lambda_2=0}=(36\lambda_1\lambda_3, 0, 27\lambda_3^2-4\lambda_1^3)$,
h: $\lambda_2+\alpha\lambda_3$, $\alpha\neq 0$, Γ: $\lambda_2+\lambda_3=0$

$C_{2,2}$: $(x_1^2+x_2^2+\lambda_1 x_1+\lambda_2 x_2+u, x_1x_2+\lambda_3)=(0, 0)$
(bifurcation diagram $-C_{2,2}^{\pm}$: $(x_1^2\pm x_2^2+\ldots)=(0, 0)$)
$w_1=(\lambda_1, \lambda_2, 2\lambda_3)$ $w_2=(\lambda_1^3+32\lambda_2\lambda_3, \lambda_2^3+32\lambda_1\lambda_3, -\lambda_1^2\lambda_3-\lambda_2^2\lambda_3)$
$w_3=(\lambda_1^5-36\lambda_1^2\lambda_2\lambda_3+320\lambda_1\lambda_3^2-4\lambda_2^3\lambda_3, \lambda_2^5-36\lambda_1\lambda_2^2\lambda_3+$
$+320\lambda_2\lambda_3^2-4\lambda_1^3\lambda_3, 128\lambda_3^3-176\lambda_1\lambda_2\lambda_3^2-\lambda_1^4\lambda_3-\lambda_2^4\lambda_3)$
Σ: $\lambda_3(\lambda_1^2-\lambda_2^2)(4096\lambda_3^3+768\lambda_1\lambda_2\lambda_3^2+27\lambda_1^4\lambda_3-6\lambda_1^2\lambda_2^2\lambda_3+$
$+27\lambda_2^4\lambda_3+\lambda_1^3\lambda_2^3)=0$
Γ: $\lambda_1+\lambda_3+\alpha_1\lambda_2+\alpha_2\lambda_2^2=0$, $(\alpha_1-32\alpha_2)(\alpha_1^4-1)\alpha_1\neq 0$.

The picture of the bifurcation diagram $C_{2,2}^+$ in [18] is incorrect.

4.7. Series C and D. V. I. Arnol'd called our attention to the fact that the bifurcation diagram D_4 is a two-sheeted covering of the projection diagram C_4. The stratum B_2 is a bifurcation variety. A similar fact for caustics (diagrams with Maxwell strata A_1^2) has been previously noted by O. P. Shcherbak. This incidentally accounts for the name of the caustic C_4 — "pyrse" or "puramid" — are a combination of the names "purse" and "pyramid" of the caustics D_4^+ and D_4^-.

We will show that this is a universal phenomenon for the interrelationship of the diagrams of series C and D.

Proposition 4.7. The pair $(C^{k-1}, \Sigma(D_k))$ is a two-sheeted covering of the pair $(C^{k-1}, \Sigma(C_k))$. The bifurcation variety is the stratum B_2 to which one of the components of the Maxwell stratum is mapped.

Proof. The R^+-miniversal deformation of the projection D_k is

$$F(x, u, \lambda)=x_1^2x_2+g(x_2)+\lambda_{k-1}x_1+u=0,$$

where $g(x_2)=x_2^{k-1}+\lambda_1 x_2^{k-2}+\ldots+\lambda_{k-2}x_2$.

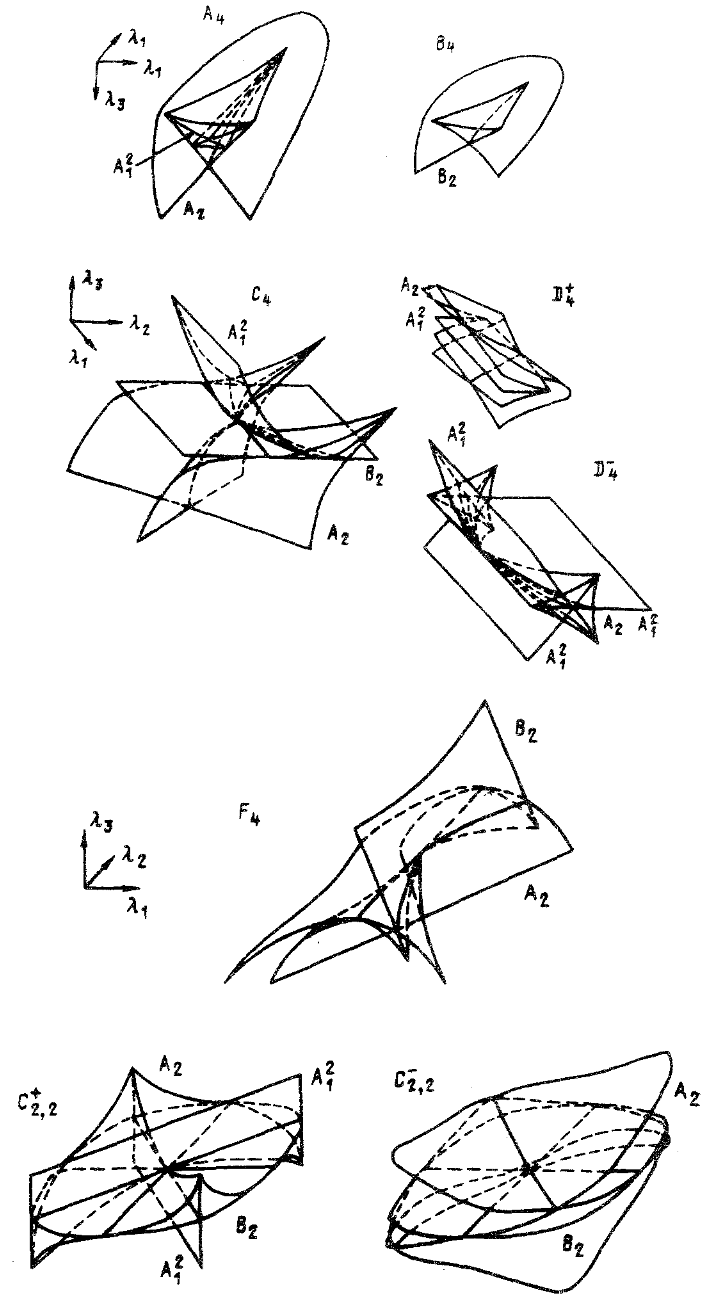

Fig. 2

The critical points of the height function on $F_\lambda = 0$ are defined by the equations

$$\partial F_\lambda/\partial x_1 = 2x_1x_2 + \lambda_{k-1} = 0, \quad \partial F_\lambda/\partial x_2 = x_1^2 + g' = 0.$$

$\lambda_{k-1} = 0$ is one of the components of the Maxwell stratum (for $\lambda_{k-1} = 0$, F_λ is even in x_1, and the height function has critical points outside the plane $x_1 = 0$ on $F_\lambda = 0$).

For $\lambda_{k-1} \neq 0$, the equation $\partial F_\lambda/\partial x_1 = 0$ can be solved for x_1. At the critical points of the height function we obtain

$$u = (\lambda_{k-1}^2 - 4x_2g)/4x_2.$$

The height function is non-Morse on $F_\lambda = 0$ if and only if the function $u(x_2) = (\lambda_{k-1}^2 - 4x_2g)/4x_2$, is non-Morse, i.e., the function u is non-Morse on the curve $4x_2g + 4ux_2 - \lambda_{k-1}^2 = 0$ in C^2 (for $\lambda_{k-1} \neq 0$ this curve is smooth). Replacing ${\lambda_{k-1}}^2$ with λ_{k-1} in the last equation, we obtain a R^+-miniversal deformation of the projection C_k.

4.8. Edge and Line Singularities. It is easy to show that analogs of the theorems of Zakalyukin [9, 10] and Bruce [13] hold for functions on a manifold with an edge [2, 3] and with isolated line singularities [26-29, 6, 22, 23]. Let us state this in the form of a proposition.

Let $(x_1, \dots, x_n) \in \mathbf{C}^n$ and let $F \in \mathcal{O}(n+q)$ be the R^+-miniversal deformation of a function on the manifold $\mathbf{C}^n$ with the edge $x_1 = 0$ or the miniversal deformation of a function on $\mathbf{C}^n$ with an isolated line singularity on the straight line $x_2 = \dots = x_n = 0$ [6]. Here $(\lambda_1, \dots, \lambda_q) \in \mathbf{C}^q$ are the deformation parameters.

Definition. The discriminant $\Delta \subset \mathbf{C}^{1+q}$ of a line singularity is the set of critical values of the projection $(x, \lambda_0, \lambda_1, \dots, \lambda_q) \mapsto (\lambda_0, \lambda_1, \dots, \lambda_q)$ restricted to $F + \lambda_0 = 0$.

For example, the plane $\lambda_0 = 0$ is one of the components of Δ.

Proposition 4.8. The free generators $v_j = \sum_{i=0}^{q} (v_{ij} - \delta_{ij}\lambda_0)\partial_{\lambda_i} \in \mathfrak{A}_\Delta$, $j = 0, \dots, q$, and $w_j = \sum_{i=1}^{q} w_{ij}\partial_{\lambda_i} \in \mathfrak{A}_\Sigma$, $j = 1, \dots, q$, are defined by the expansions

$$-F\partial(F+\lambda_0)/\partial\lambda_j \equiv \sum_{i=0}^{q} v_{ij}\partial(F+\lambda_0)/\partial\lambda_i, \quad v_{ij} \in \mathcal{O}(q),$$

and

$$F^j \equiv \sum_{i=0}^{q} w_{ij}\partial(F+\lambda_0)/\partial\lambda_i, \quad w_{ij} \in \mathcal{O}(q).$$

The congruences are respectively in the rings $\mathcal{O}(n+1+q)$ and $\mathcal{O}(n+q)$ modulo the ideals spanned by $x_1\partial F/\partial x_1$, $\partial F/\partial x_2, \dots, \partial F/\partial x_n$ for the edge singularity and by $\partial F/\partial x_1$, $x_r\partial F/\partial x_s$, $2 \leq r, s \leq n$, for the line singularity.

LITERATURE CITED

1. A. G. Aleksandrov, "Cohomology of quasihomogeneous complete intersection," Izv. Akad. Nauk SSSR, **49**, No. 3, 467-510 (1985).
2. V. I. Arnol'd, "Critical points on a manifold with an edge, the simple Lie groups B_k, C_k, and F_4, and singularities of evolutes," Usp. Mat. Nauk, **33**, No. 5, 91-105 (1978).
3. V. I. Arnol'd, V. N. Varchenko, and S. M. Gusein-Zade, Singularities of Differentiable Maps, Vol. I [in Russian], Nauka, Moscow (1982).
4. V. V. Goryunov, "The Poincaré polynomial of the space of residue forms on a quasihomogeneous complete intersection," Usp. Math. Nauk, **35**, No. 2, 205-206 (1980).
5. V. V. Goryunov, "The geometry of bifurcation diagrams of simple projections on the straight line," Funkts. Anal. Prilozhen., **15**, No. 2, 1-8 (1981).
6. V. V. Goryunov, "Bifurcation diagrams of some simple and quasihomogeneous singularities," Funkts. Anal. Prilozhen., **17**, No. 2, 23-37 (1983).
7. V. V. Goryunov, "Singularities of projections of complete intersections," Itogi Nauki i Tekhniki, Sovr. Probl. Mat., **22**, 167-206 (1983).
8. V. V. Goryunov, "Projections and vector fields tangent to the discriminant of a complete intersection," Funkts. Anal. Prilozhen., **21**, No. 2, 26-37 (1987).
9. V. M. Zakalyukin, "Evolution of wave fronts dependent on one parameter," Funkts. Anal. Prilozhen., **10**, No. 2, 69-70 (1976).
10. V. M. Zakalyukin, "Legendre maps in Hamiltonian systems," in: Some Problems of Mechanics [in Russian], MAI, Moscow (1977), pp. 11-16.
11. Le Dung Trang, "Computing the Milnor number of an isolated singularity of a complete intersection," Funkts. Anal. Prilozhen., **8**, No. 2, 45-49 (1974).
12. V. I. Arnold, "Wave front evolution and equivariant Morse lemma," Commun. Pure Appl. Math., **29**, No. 6, 557-582 (1976).
13. J. W. Bruce, "Vector fields on discriminants and bifurcation varieties," Bull. London Math. Soc., **17**, No. 3, 257-262 (1985).

14. M. Giusti, "Classification des singularites isolees d'intersections completes simples," C. R. Acad. Sci., **284**, No. 3, A167-A170 (1977).
15. M. Giusti, "Classification des singularites isolees simples d'intersections completes," Preprint, Ecole Polytechnique (1977).
16. M. Giusti, "Sur les singularites isolees d'intersections completes quasihomogenes," Ann. Inst. Fourier, **27**, No. 3, 163-192 (1977).
17. M. Giusti, "Classification des singularites isolees simples d'intersections completes," in: Singularities, Proc. Summer Inst., Arcata, Calif., 20 July—7 Aug, 1981, Part 1, Providence, R.I. (1983), pp. 457-494.
18. M. Golubitski and D. G. Schaeffer, Singularities and Groups in Bifurcation Theory, Vol. 1, Springer, New York (1985).
19. G.-M. Greuel, "Dualitat in der lokalen Kohomologie isolierter Singularitaten," Math. Ann., **250**, No. 2, 157-173 (1980).
20. H. A. Hamm, "Lokale topologische Eigenschaften komplexer Raume," Math. Ann., **191**, No. 3, 235-252 (1971).
21. E. Looijenga, Isolated Singular Points on Complete Intersections, London Math. Soc. Lect. Notes Ser., **77** (1984).
22. R. Pellikaan, "Residual Jacobi ideals of hypersurface singularities," Preprint 357, Utrecht Univ. Dept. of Math. (1984).
23. R. Pellikaan, "Hypersurface singularities and resolution of Jacobi modules," Proefschrift, Utrecht (1985).
24. K. Saito, "Quasihomogene isolierte Singularitaten von Hyperflachen," Invent. Math., **14**, No. 2, 123-142 (1971).
25. K. Saito, "Theory of logaritihmic differential forms and logarithmic vector fields," J. Fac. Sci. Univ. Tokyo, Sec. 1A, **27**, No. 2, 265-291 (1980).
26. D. Siersma, "Isolated line singularities," Preprint 217, Utrecht Univ. Dept of Math. (1981).
27. D. Siersma, "Isolated line singularities," in: Singularities, Proc. Summer Inst. Arcata, Calif., 20 July—7 Aug., 1981, Part 2, Providence, R.I. (1983), pp. 485-496.
28. D. Siersma, "Hypersurfaces with singular locus: a plane curve and transversal type A_1," Preprint 406, Utrecht Univ. Dept. of Math. (1986).
29. D. Siersma, "Singularities with critical locus: a 1-dimensional locus and transversal type A_1," Preprint 407, Utrecht Univ. Dept. of Math. (1986).
30. H. Terao, "The bifurcation set and logarithmic vector fields," Math. Ann., **263**, No. 3, 313-321 (1983).