SINGULARITIES OF PROJECTIONS OF FULL INTERSECTIONS

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The paper contains a survey of results in the theory of projections: a classification of simple projections of surfaces onto manifolds of low dimension is presented, assertions are formulated regarding the homotopy type of complements to bifurcation diagrams of projections onto a line as are theorems on rectification of vector fields in bundles in a neighborhood of manifolds with singularities, and the connection of the theory of projections with other areas of the theory of singularities (critical points of functions on a smooth manifold and on a manifold with boundary) is considered.

INTRODUCTION

In studying the dependence of various mathematical objects on parameters, we encounter the investigation of projections and their singularities. For example, in the theory of bifurcations of singular points of differential equations an essential role is played by the projection of the surface formed by the singular points in the product of the space of parameters and phase space onto the space of parameters. Singularities of this projection correspond to bifurcations of singular points.

An equivalence relation more rigid than in the general theory of singularities of mappings is the main distinguishing feature of the theory of projections. The projections of two submanifolds from the bundle spaces onto the bases are considered equivalent if there exists a diffeomorphism of the bundle spaces taking fibers of the first bundle into fibers of the second and the submanifolds being projected into one another. Such an equivalence is precisely an equivalence of deformations of full intersection cut out by the submanifolds being projected in distinguished fibers of the bundles. Therefore, for example, Golubitsky and Schaeffer [21] called the projections "imperfect bifurcations" (in contrast to "perfect"—versal bifurcations).

The beginning of a systematic study of singular projections was set by Arnol'd in [1] where, in investigating singularities in the problem of fastest passage about an obstacle, he obtained a list of 10 projections encountered in the projection of general smooth surfaces from three-dimensional Euclidean space onto the plane along any direction (in this situation for a common choice of the direction of projection only singularities of mappings of the plane onto the plane occur; these were indicated already in the classical work of Whitney [25]).

In the present paper we give a survey of results of the theory of projections: a classification is presented of simple projections of surfaces onto manifolds of low dimension, assertions are formulated regarding the homotopy type of complements to bifurcation diagrams of projections onto a line as are theorems on the rectification of vector fields and bundles in a neighborhood of certain manifolds with singularities, and the connection of the theory of projections with other areas of the theory of singularities (critical points of functions on a smooth manifold and on a manifold with boundary) is considered.

1. General Facts Concerning Projections of Surfaces

The projection of a submanifold V from a bundle space E onto the base B is a triple $V \to E \to B$ consisting of an imbedding and a projection.

An equivalence of projections $V_i \to E_i \to B_i$, $i = 1, 2$ is a commutative 3×2 diagram with vertical elements which are diffeomorphisms $h: E_1 \to E_2$ and $k: B_1 \to B_2$ whereby $hV_1 = V_2$.

Similar definitions hold for germs.

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Locally a bundle $E \to B$ is isomorphic to the trivial bundle $\mathbf{C}^n \times \mathbf{C}^p \to \mathbf{C}^p$ (to be specific we consider the complex case). We suppose that the manifold V projected is a full intersection: if the codimension of V in E is equal to m, then the germ of V is given by a system of m holomorphic equations

$$f_1 = \ldots = f_m = 0.$$

This system is determined up to multiplication by a germ on E of a nondegenerate holomorphic $m \times m$ matrix M. Therefore, equivalence of given systems $f = 0$ and $g = 0$ of germs of projections from $\mathbf{C}^n \times \mathbf{C}^p$ onto $\mathbf{C}^p$, $(x, u) \mapsto u$ implies the existence of a local diffeomorphism h of the form

$$h(x, u) = (X(x, u), U(u)),$$

for which $h^*g = Mf$.

The germs of the projection $(x, u) \mapsto u$ of the manifold given by the system of equations $f(x, u) = 0$ we call the projection of f.

Let $\mathscr{E}_{x,u}$ [or $\mathscr{E}(n+p)$] be the space of germs at zero of holomorphic functions of the variables x and u, $x \in \mathbf{C}^n$, $u \in \mathbf{C}^p$; let $\mathscr{E}^m_{x,u}$ be the space of germs of holomorphic mappings from $\mathbf{C}^{n+p}$ to $\mathbf{C}^m$; let $\mathfrak{m}(m) \subset \mathscr{E}(m)$ be a maximal ideal.

We introduce a quantity called the codimension of the projection of $f \in \mathscr{E}^m_{x,u}$:

$$\operatorname{codim} f = \dim_{\mathbf{C}} Q(f),$$

where

$$Q(f) = \mathscr{E}^m_{x,u} / \{ f^*(\mathfrak{m}(m)) \mathscr{E}^m_{x,u} + \mathscr{E}_{x,u} \langle \partial f / \partial x_1, \ldots, \partial f / \partial x_n \rangle + \mathscr{E}_u \langle \partial f / \partial u_1, \ldots, \partial f / \partial u_p \rangle \}.$$

This number is the codimension of the germ of the orbit of the projection of f relative to the equivalence group of projections which realizes transport of distinguished points (those at which the germs are considered). The first term in the denominator of Q(f) corresponds to infinitesimal replacements of f by mappings with the same preimage of zero, while the other two correspond to infinitesimal fibered changes of coordinates.

The following result is proved by means of the Mather–Wassermann technique [18, 24].

THEOREM 1 (Finite Determination). If the codimension of the projection of f is finite, then there exists a finite number k such that any other projection for which the k-jet at zero is the same as for the germ of f is equivalent to the projection of f.

It should be noted that Theorem 1, just as Theorems 2 and 3 below, is true not only in the holomorphic case but also in the analytic case as well as for C^∞ and formal projections.

An s'-parameter deformation of the projection of f is the germ of a mapping $\lambda \to f_\lambda$ of s'-dimensional complex space (the base of the deformation) into the space $\mathscr{E}^m(n+p)$ of all projections taking 0 into f. In the language of diagrams this means that we consider a germ of the diagram

$$(V', V) \to (E', E) \to (B', B) \to (\mathbf{C}^{s'}, 0),$$

where the first arrow is the imbedding of a full intersection and the two others are fiberings. This deformation is called versal if any other deformation

$$(V'', V) \to (E'', E) \to (B'', B) \to (\mathbf{C}^{s''}, 0)$$

of this same projection can be induced from it: there exists a commutative 4×2 diagram whose horizontal elements are the two diagrams indicated and whose vertical elements are germs of mappings of the spaces labeled with two primes into the spaces with one prime (here the mapping $E'' \to E'$ must be the identity on E).

Suppose that a mapping $F(x, u, \lambda)$, $\lambda \in \mathbf{C}^s$, gives a deformation of the projection $(x, u) \mapsto u$ of the surface $f(x, u) = 0$ (i.e., $f_\lambda = F|_{\lambda=\text{const}}$). We say that the deformation F is infinitesimally versal if the images of the vectors of the initial velocities of the deformation $(\partial F/\partial \lambda_i|_{\lambda=0})$ in the module Q(f) generate it as a space over $\mathbf{C}$.

THEOREM 2 [11]. A deformation of the projection of a full intersection is versal if and only if it is infinitesimally versal.

From this it follows, for example, that the dimension of the base of a miniversal (i.e., versal with the smallest possible number of parameters) deformation of a projection is equal to its codimension. For such a deformation it is possible to take $\lambda \mapsto f + \sum_1^{\nu} \lambda_i e_i$, $\lambda \in \mathbf{C}^{\nu}$, where $\nu = \operatorname{codim} f$, and $\{e_i\} \in \mathscr{E}^m_{x,u}$ are representers of a basis of Q(f).

For C^{∞} projections onto a line Theorems 1 and 2 were proved by Golubitsky and Schaeffer in [21]. There was hereby imposed on the projection the condition of finiteness of the number

$$\dim_{\mathbf{R}} \mathscr{E}^m(n+1)/\{f^*(\mathfrak{m}(m))\mathscr{E}^m(n+1)+\mathscr{E}(n+1)\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle\}.$$

This number is clearly no less than the codimension of the projection.

We now consider projections of zero codimension. It is easy to see that these are precisely versal deformations of full intersections V_0 [6] of a distinguished fiber of the bundle $E \to B$. For them there is a stability theorem [10] which in our case can be reformulated as follows.

THEOREM 3. The germ at $0 \in E$ of the projection $f^{-1}(0) \to E \to B$ of zero codimension is stable: for any mapping g sufficiently close to f there exist a representer $\tilde{g}$ of it and a point close to zero at which the germ of the projection $\tilde{g}^{-1}(0) \to E \to B$ is equivalent to the projection of the germ of the manifold $f^{-1}(0)$.

It should be noted that Theorems 1-3 are also valid in a more general situation — for cascades of projections, i.e., for diagrams of the form

$$V \to E_0 \to E_1 \to \ldots \to E_k,$$

where the first arrow is an imbedding of a full intersection and each of the following arrows is the projection of the space of the bundle E_i onto the base E_{i+1}, $\dim E_i = p_i + \ldots + p_k$ (the algebraic definition of the codimension of a projection can be generalized to this case in an obvious, but somewhat cumbersome, way).

CHAPTER I

COMPLEX PROJECTIONS ONTO A LINE

2. Projections onto a Line and Boundary Singularities

The most substantial results in the theory of projections are obtained for projections onto one-dimensional bases. The coordinate function of the base induces a function on the projected manifold. Therefore, as we shall see, projections onto a line have much in common with functions with isolated critical points on a smooth manifold.

1. We begin with a classification of simple projections onto $\mathbf{C}^1$.

We call the germ of a projection simple if a sufficiently small neighborhood of it intersects only a finite number of equivalence classes. More precisely, a germ is simple if there exists a finite collection of equivalence classes such that for any k any k-jet sufficiently close to the k-jet at zero of the collection of functions f giving the germ is the k-jet of a collection giving the germ of the projection of one of the classes indicated.

A superstructure over a projection $V \to E \to B$ is a projection $V \to E' \to B$ where V is imbedded in E' as a submanifold of the space of the subbundle $E \subset E'$. Stable equivalence of projections is equivalence of corresponding superstructures.

THEOREM 4 [2, 11, 12]. 1. Any simple projection of a full intersection onto the complex line is stably equivalent to the projection either of a hypersurface or a curve in three-dimensional space or a multiple point on the plane or in three-dimensional space.

2. Any simple projection of a hypersurface onto the line is equivalent to the germ at zero of the projection $(x, u) \mapsto u$ of the manifold $f(x, u) = 0$ where f is one of the functions of Table 1. In the table $q = x_r^2 + \ldots + x_n^2$ whereby $r = 3$ for D_μ and E_μ and $r = 2$ for the remaining singularities.

3. Simple germs of projections $(x, u) \mapsto u$ of curves $f = 0$ onto the line which do not reduce by stable equivalence to projections of plane curves form two series:

TABLE 1

Type	f	Type	f
$A_\mu,\ \mu\geqslant 0$	$u+x_1^{\mu+1}+q$	E_8	$u+x_1^3+x_2^5+q$
$D_\mu,\ \mu\geqslant 4$	$u+x_1^2x_2+x_2^{\mu-1}+q$	$B_\mu,\ \mu\geqslant 2$	$u^\mu+x_1^2+q$
E_6	$u+x_1^3+x_2^4+q$	$C_\mu,\ \mu\geqslant 3$	$ux_1+x_1^\mu+q$
E_7	$u+x_1^3+x_1x_2^3+q$	F_4	$u^2+x_1^3+q$

TABLE 2

n	Type	f	n	Type	f
1	$B_\mu,\ \mu\geqslant 1$	$x,\ u^\mu$	1	V_6	$x^2,\ u^2$
	$X_{k,l},\ 2\leqslant k\leqslant l$	$x^k+u,\ x^l$		V_7	$x^3+u^2,\ ux$
	$U_\mu,\ \mu\geqslant 5$	$x^2+u^{\mu-3},\ ux$	2	$\Gamma^{a,b}_{a+b+3}$	$x_1x_2,\ x_1^2+x_2^a,\ u+x_2^b$

$$C_{k,l},\ 2\leqslant k\leqslant l\ f=(x_1x_2,\ x_1^k+x_2^l+u);$$
$$F_{2k+1},\ \ k\geqslant 2\ f=(x_1^2+x_2^3,\ x_2^k+u);$$
$$F_{2k+4},\ k\geqslant 1\ f=(x_1^2+x_2^3,\ x_1x_2^k+u).$$

4. Simple projections of multiple points of f = 0 onto the line are listed in Table 2. Here n is the dimension of a fiber of the projection $(x, u) \mapsto u$; the series $\Gamma^{a,b}$ contains five members: b = 2, 3 for a = 2 and b = 2, 3, 4 for a = 3.

The codimensions of the projections $C_{k,l}$ and $X_{k,l}$ are equal to $k + l - 1$ and $k + l - 2$. The codimension of any other of the projections listed is one less than the lower index of the symbol denoting the projection.

The projections B_μ, $\mu \geqslant 1$ of part 2 (for n = 0) and of part 4 are stably equivalent (in part 2 $B_1 = A_1$).

2. A projection f abuts a projection g ($f \to g$) if by an arbitrarily small perturbation of f it is possible to obtain a projection equivalent to g.

Before enumerating abutments of simple projections onto the line, we indicate the coincidence of two different classifications of simple singularities.

Arnol'd pointed out that the manifolds in simple projections of hypersurfaces onto the line are precisely the zeros of simple function on a manifold with boundary [4]: the boundary is a distinguished fiber of the bundle of the enveloping space over the line. This fact can be obtained not by comparing two lists of simple singularities obtained independently but directly, by having only lists of simple and bordering boundary functions (a nonsimple singularity is called a bordering singularity if all singularities of smaller codimension to which it abuts are simple). The main role here is played by the fact that both simple and bordering boundary singularities are quasihomogeneous (see [4] or Sec. 7 of the present work).

A mapping $h(y_1,\ldots,y_k)$, $h = (h_1,\ldots,h_m)$ is called quasihomogeneous if there exist rational numbers $w_1,\ldots,w_k$ (weights of the variables y) and $d_1,\ldots,d_m$ (degrees of the coordinate functions of the mapping h) such that

$$h_i(e^{w_1t}y_1,\ldots,e^{w_kt}y_k)=e^{d_it}h_i(y_1,\ldots,y_k)$$

for any $t\in\mathbf{C}$. For a quasihomogeneous mapping there is the Euler relation

$$(d_1h_1,\ldots,d_mh_m)=\Sigma w_jy_j\partial h/\partial y_j.$$

Let T_1 be the tangent space to an orbit of a quasihomogeneous function f(x, u) in the maximal ideal $\mathfrak{m}(n+1)\subset\mathscr{E}_{x,u}$, relative to the equivalence group of germs of projections at zero; let T_2 be the tangent space to the orbit of f(x, u) relative to the group of germs at zero of diffeomorphisms of the space $\mathbf{C}^{n+1}$ which preserve the plane u = 0 (this is the group realizing

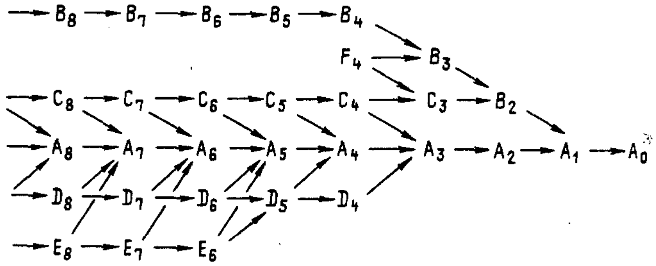

Diagram 1

$$\begin{array}{ll}
A_\mu \to A_{\mu-1},\ \mu>0 & B_\mu \to B_{\mu-1},\ \mu>1 \\
C_{k,l} \to C_{k-1,l},\ 1<k\leqslant l & F_\mu \to F_{\mu-1},\ \mu>3 \\
\downarrow \quad \searrow & \downarrow \\
A_{k+l-1} \quad C_{k,l-1},\ 1\leqslant k<l & C_{s,\mu-s-1},\ s=1,\ldots,\left[\frac{\mu-1}{2}\right]
\end{array}$$

Diagram 2

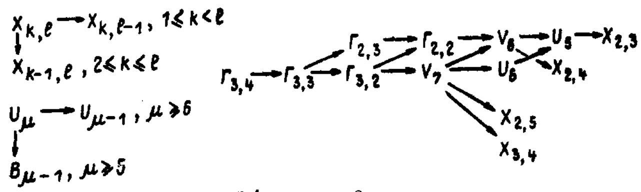

Diagram 3

equivalence of germs of functions on a manifold with boundary). Using the expression for a quasihomogeneous function in terms of the sum of its derivatives, we see that $T_1 = T_2$. For this reason a function is simple (bordering) on a manifold with boundary if and only if it gives a simple (bordering) projection of the hypersurface.

Because of the quasihomogeneity of simple and bordering projections onto the line (Theorem 4 and Secs. 7, 8) the lists of simple full intersections on a manifold with boundary and projections of surfaces of positive dimension onto the line also coincide (if a full intersection is a hypersurface, then we again obtain the list of simple boundary singularities of functions). As a whole the classification of boundary full intersections is rougher than the classification of projections onto the line: one distinguished hyperplane is preserved rather than the entire bundle on the hyperplane.

3. THEOREM 5 [11, 12]. All abutments of simple projections of hypersurfaces, curves, and multiple points onto the line can be obtained by composition on a finite number of the abutments indicated, respectively, in Diagrams 1, 2, and 3.

We here consider that $C_{1,l} = C_{1+l}$, $C_{1,1} = B_2$, $B_1 = A_1$, $F_3 = B_3$, $X_{1,l} = B_l$.

Projections of hypersurfaces abut one another if and only if the boundary functions of the same name abut one another.

Example. $F_\mu \to C_{\mu-1}$, $\mu \geqslant 4$:

$$(x_1^2+x_2^3-\varepsilon^2 x_2^2,\ x_1 p_\mu + x_2 q_\mu + u),$$

where p_μ and q_μ are defined recursively:

$$p_4=1, \quad q_4=\varepsilon;$$

$$p_\mu=(-1)^{\mu-1}\varepsilon p_{\mu-1}+q_{\mu-1}, \quad q_\mu=(x_2-\varepsilon^2)\,p_{\mu-1}+(-1)^\mu \varepsilon q_{\mu-1}.$$

3. Dynkin Diagrams

1. The coincidence of the lists of simple boundary functions and simple projections of hypersurfaces onto the line suggests a generalization to the case of projections of the definition of the intersection form of a boundary singularity (see [4]).

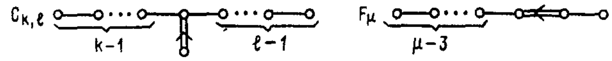

Diagram 4

We consider the projection $E \to \mathbf{C}$, $(x, u) \mapsto u$ of a manifold $V = f^{-1}(0)$ of positive dimension. In E we fix a ball of small radius r with center at zero. In this ball we consider a nonsingular fiber $V_\varepsilon = f^{-1}(\varepsilon)$, $|\varepsilon| \ll r$ of the mapping f. For a general choice of ε the intersection of V_ε with a fiber of the projection $(x, u) \mapsto u$ over zero is a smooth manifold V_ε^0 of complex codimension 1 in V_ε. Let $\tilde{V}_\varepsilon \to V_\varepsilon$ be a two-sheeted covering with ramification along V_ε^0, and let $\tilde{H}$ be the integer homologies of middle dimension of the manifold $\tilde{V}_\varepsilon$. On $\tilde{H}$ there acts the involution induced by permutation of the sheets of the covering. We denote by H^- the antiinvariant part of $\tilde{H}$. The intersection form on $\tilde{H}$ gives a bilinear form on H^-. It is called the intersection form of the projection [2].

The same mapping can be introduced for full intersections on a manifold with boundary [the boundary plays the role of the zero fiber of the projection $(x, u) \mapsto u$].

2. On the basis of the intersection form we can construct a Dynkin diagram. We hereby use the definitions of short and long cycles which are given in [4] for the symmetric case and in [2] for the skew-symmetric case. We consider projection as a full intersection on a manifold with boundary, and we shall construct for it a Dynkin diagram in the basis of the space H^- which is labeled in the sense of [2, 9, and 11].

For the singularities A_μ, B_μ, C_μ, D_μ, E_μ, and F_4 the diagrams in labeled bases coincide with the Dynkin diagrams of the corresponding Lie algebras [8].

THEOREM 6 [11]. There exist labeled bases in which the Dynkin diagrams of the intersection forms of curves $C_{k,l}$ and F_μ on the manifold $\mathbf{C}^3$ with boundary are as shown in Diagram 4.

The form of the diagram explains the notation chosen in denoting simple full intersections and projections.

For the curves $C_{k,l}$ and F_μ there are relations between the labeled cycles (one for the series C and two for F). In constructing a diagram from these cycles we must choose a basis and discard the rest. This feature contains an undesirable element of arbitrariness. A more invariant object in the present situation is the "complete" Dynkin diagram constructed on the basis of all labeled cycles regardless of their dependence.

3. We shall indicate a means of defining a vanishing basis of a projection onto the line which differs from that given in [11].

Let $\lambda \mapsto F_\lambda$ be a versal deformation of the projection, and let u be a coordinate function on the base of the projection. For a common value of λ the manifolds $V_\lambda = F_\lambda^{-1}(0)$ and $V_\lambda^0 = V_\lambda \cap \{u=0\}$ are smooth, and u is a Morse function on V_λ. If μ is the number of its critical points (and critical values), then the factor space V_λ/V_λ^0 is homotopically equivalent to a bouquet of μ spheres of middle dimension (μ is also the rank of the subgroup H^- of antiinvariant homologies of the covering $\tilde{V}_\lambda \to V_\lambda$ branched along V_λ^0). We note that if the projection in question is a quasihomogeneous projection of a hypersurface or is simple, then $\mu = \nu + 1$, where ν is the codimension of the projection (it is not hard to show that for the projection of an arbitrary hypersurface $\mu \geqslant \nu + 1$).

The motion of u along a complex line from the noncritical value 0 on V_λ to the μ critical values along μ nonintersecting paths defines a basis of μ vanishing semicycles in the relative homologies of the pair (V_λ, V_λ^0). To this basis there corresponds a basis of the antiinvariant homologies of the space $\tilde{V}_\lambda$ composed of short, labeled (in the new sense) cycles.

For projections onto the line of smooth hypersurfaces (including A_μ, D_μ, E_μ) this definition of a labeled basis gives the same Dynkin diagrams as the definition indicated above. For simple projections distinct from those indicated the new diagrams are strongly different from the classical diagrams. For example, in the symmetric case the diagram of the projection B_μ is μ isolated vertices, while in the skew-symmetric case it is a complete graph with two edges. For the projections C_μ and F_4 there exist labeled short bases in which the diagrams of their symmetric intersection forms look like the classical diagrams of D_μ and D_4.

THEOREM 7. There exist labeled bases in which the Dynkin diagrams of the projections $C_{k,l}$ and F_μ have the form presented in Diagram 5.

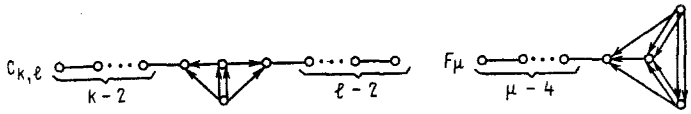

Diagram 5

CHAPTER II

THE GEOMETRY OF BIFURCATION DIAGRAMS

4. $k(\pi, 1)$-Theorems

We recall that the bifurcation diagram of zeros of a function on a smooth manifold is the set of those values of the parameter $\lambda \in \mathbf{C}^\mu$ of a miniversal deformation $\lambda \mapsto f_\lambda$ of it for which 0 is a critical value of f_λ. The set of those λ for which the function f_λ is not a Morse function is diffeomorphic to a cylinder over some singular hypersurface imbedded in $\mathbf{C}^{\mu-1}$. This hypersurface is called the bifurcation diagram of functions [3].

In analogy with these definitions we introduce the definitions for projections onto the line.

1. Let $F(x, u, \lambda)$ be a miniversal deformation of projection onto the line $(x, u) \mapsto u$ of a full intersection $f(x, u) = 0$. $F = 0$ is a smooth surface in $\mathbf{C}^{n+1+\nu}$.

The bifurcation diagram of zeros $W \subset \mathbf{C}^{1+\nu}$ of the projection f is the germ at zero of the set of critical values of the projection $(x, u, \lambda) \mapsto (u, \lambda)$ restricted to $F = 0$.

For example, if $f = 0$ is a multiple point, then W is the image of the restriction indicated.

The bifurcation diagram of zeros is a hypersurface in $C^{1+\nu}$ composed of those (u, λ) for which 0 is a critical value of the mapping

$$F_{u,\lambda} = F(\cdot, u, \lambda).$$

The bifurcation diagram $\Sigma \subset \mathbf{C}^\nu$ of the projection f is the union of the image of singular points of the diagram W under the projection $(u, \lambda) \mapsto \lambda$ and the set of critical values of this projection onto the regular part of W.

Σ is a germ of a hypersurface in $\mathbf{C}^\nu$ composed of those λ for which either the manifold $V_\lambda = F_\gamma^{-1}(0)$ is singular or the function u on this manifold is not a Morse function.

If $f = 0$ is a multiple point, then this corresponds to the coalescence of points into which $f = 0$ decomposes under deformation or to the fact that the coordinates u of different points coincide.

In the general case Σ consists of three components:

Σ_1 — the projection of the cusp edge of the manifold W (for points of Σ_1 the function u on V_λ has a degenerate critical point);

Σ_2 — the projection of the set of self-intersections of the diagram W (the function u assumes the same values at different critical points);

Σ_3 — the set of critical values of the projection $(u, \lambda) \mapsto \lambda$ onto the smooth part of W (the manifold V_λ is singular).

Examples. 1. A miniversal deformation of the projection C_3: $x^3 + \lambda_1 x^2 + ux + \lambda_2$. Here W is the set of those points (u, λ) for which this polynomial in x has a multiple root. It is diffeomorphic to a cylinder over a semicubic parabola. The component Σ_3 of the diagram Σ is the line $\lambda_2 = 0$, Σ_1 is the cubic parabola tangent to this line, and Σ_2 is empty (see Fig. 1).

2. If f is a simple projection of a smooth hypersurface $(f \in A_\mu, D_\mu, E_\mu)$, then the diagrams W and Σ coincide, respectively, with the bifurcation diagrams of zeros and functions of the analogous functions with isolated critical points. Here $\Sigma_3 = \emptyset$. Figure 2 shows the case $f \in A_3$.

2. For simple functions on a smooth manifold the Lyashko–Looijenga theorem holds.

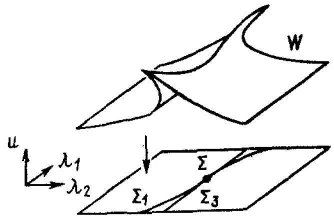

Fig. 1

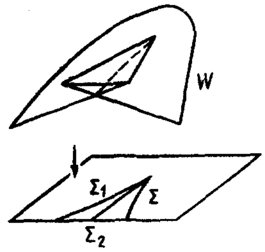

Fig. 2

THEOREM 7 [3, 23]. The germ of the complement of the base of a versal deformation to the bifurcation diagram of a simple holomorphic function with a μ-fold critical point is the space $k(\pi, 1)$ where π is a subgroup of finite index in the Artin braid group of μ strands.

We recall that the multiplicity of a critical point is the number of Morse points into which it decomposes under small perturbation. Lyashko later extended this theorem to functions on a manifold with boundary [17]. For projections there is an analogous assertion.

THEOREM 8 [11]. The germ of the complement of the base of a miniversal deformation to the bifurcation diagram of a simple projection of a full intersection of positive dimension onto $\mathbf{C}^1$ is the space $k(\pi, 1)$. Here π is a subgroup of finite index in the braid group of $\nu + 1$ strands (ν is the codimension of the projection, i.e., the dimension of the base of the miniversal deformation).

If in Theorem 8 we restrict ourselves only to projections of smooth manifolds, then we obtain precisely Theorem 7 (cf. Example 2, part 1). For the projections of B_μ, C_μ, F_4 the assertion of Theorem 8 differs from Lyashko's theorem for the analogous boundary functions: the bifurcation diagrams of the corresponding singularities and the fundamental groups π are different. Thus, the bifurcation diagram of the boundary function C_3 is a quadratic parabola with a tangent, and π has index 27 in the braid group of three strands. For the projection C_3 (Fig. 1) π has index 8 in this same group.

The proof of Theorem 8 follows the scheme of proof of Theorem 7.

In part 5, Sec. 3 it was noted that for a general value of the parameter $\lambda \in \mathbf{C}^\nu$ of a miniversal deformation F of a simple projection the function u has on the manifold $V_\lambda = F_\lambda^{-1}(0)$ precisely $(\nu + 1)$ critical values $u_1, \ldots, u_{\nu+1}$. It is not hard to see that the condition that λ be general is $\lambda \in \mathbf{C}^\nu \setminus \Sigma$, and u_j are the coordinates of points of intersection of the line $\lambda = \text{const}$ with the bifurcation diagram of zeros W in $\mathbf{C}^{1+\nu}$ (as λ approaches a regular point of any of the three components of the diagram Σ two points u_j coalesce; for example, for general degeneration of the manifold V_λ the value of the function u at its singular point is considered a twofold critical point).

Thus, to any $\lambda \in \mathbf{C}^\nu \setminus \Sigma$ we can assign a polynomial $z^{\nu+1} + \alpha_1 z^{\nu-1} + \ldots + \alpha_\nu$ with $(\nu + 1)$ distinct roots $u_j - (\Sigma u_i)/(\nu + 1)$. We obtain a mapping φ into the space $\mathbf{C}_\nu \setminus \Xi$ of polynomials of this form [this is the classifying space of the braid group of $(\nu + 1)$ strands]. It extends to a holomorphic mapping $\Phi: (\mathbf{C}^\nu, 0) \to (\mathbf{C}^\nu, 0)$.

The validity of Theorem 8 is equivalent to the fact that φ is a covering, and it follows from the fact that Φ is proper and φ is locally biholomorphic. The proof of these two facts is contained in [11], but one aspect of it — the lemma that the preimage of zero under the mapping Φ is zero — contains an error. We shall fill this gap.

3. LEMMA. $\Phi^{-1}(0) = 0$.

Proof. We must show that as soon as the function u has on V_λ a single critical value, then $\lambda = 0$.

For the projections A_μ, D_μ, E_μ the assertion follows from the connectedness of the Dynkin diagrams of the corresponding functions on a smooth manifold.

To prove the lemma in the remaining cases we note that all simple projections onto the line have quasihomogeneous versal deformations F (part 2 of Sec. 2) defined globally on $\mathbf{C}^{n+1+\nu}$ [11]. It suffices for us to consider the global mapping Φ constructed for such a deformation.

Each series must be investigated separately.

The case B_μ: $F = x^2 + u^\mu + \lambda_1 u^{\mu-2} + \ldots + \lambda_{\mu-1} = x^2 + p_\lambda(u)$.

A critical value of the function u on V_λ is defined by the condition $\partial F_\lambda/\partial x = 0$, i.e., $x = 0$. The critical values of u are the roots of the polynomial p_λ. For $\lambda \in \Phi^{-1}(0)$ the polynomial p_λ has a unique root: $p_\lambda = (x - a)^\mu$. Now the sum of the roots of p_λ is zero. This implies that $\Phi^{-1}(0) = 0$ for the series B.

The critical values of the function u on the curve $F_\lambda = 0$ are determined by the condition $J_\lambda = 0$ where $J_\lambda = \det(DF_\lambda/Dx)$. All projections to which the projections $C_{k,l}$ and F_μ abut are simple. Let r_0 be the number of Morse critical points into which a singular point of the function u decomposes under small perturbation of the simple projection $(x, u) \mapsto u$ of the germ at (x^0, u^0) of the curve $F_\lambda = 0$. It is equal to

$$\dim_{\mathbb{C}} \mathscr{E}_{x,u}^{x^0,u^0}/(F_\lambda, J_\lambda),$$

where the upper index of the space $\mathscr{E}$ indicates the point at which the holomorphic germs of functions of the variables x and u are considered. As noted, this number is 1 greater than the codimension of a simple germ of the projection.

Let $\{(x^i, u^0)\}$ be all critical points of the function u on V_λ at the level $u = u^0$. The condition $\lambda \in \Phi^{-1}(0)$ is equivalent to $\Sigma r_i = \nu + 1$.

The case $C_{k,\lambda}$, $1 \leqslant k \leqslant l$: $F = (x_1x_2 - \lambda_0,\ \ x_1^k + \lambda_1 x_1^{k-1} + \ldots + \lambda_{k-1}x_1 + u + \lambda_k x_2 + \ldots + \lambda_{k+l-2}x_2^{l-1} + x_2^l) = (x_1x_2 - \lambda_0,\ p_\lambda(x_1) + u + q_\lambda(x_2))$.

The equations of critical points on V_λ at the level $u = u^0$ are

$$x_1x_2 = \lambda_0, \quad p_\lambda + u^0 + q_\lambda = 0, \quad x_1 p'_\lambda - x_2 q'_\lambda = 0.$$

For $\lambda_0 \neq 0$, $x_2 = \lambda_0/x_1$, and we see that the point $(x_1^0, \lambda_0/x_1^0, u^0)$ is a critical point of the function u on V_λ if and only if x_1^0 is a nonzero multiple root of the polynomial

$$Q = x_1^l(p_\lambda(x_1) + u^0 + q_\lambda(\lambda_0/x_1)) = \prod (x_1 - a_i)^{s_i}.$$

Suppose $x_1^0 = a_0$. We compute the number r_0 of Morse singularities of the function u coalescing at the corresponding critical point. Noting that $a_0 \neq 0$, i.e., the function x_1 is invertible at the point of interest to us, we have

$$\begin{aligned} r_0 &= \dim_{\mathbb{C}} \mathscr{E}_{x_1,x_2,u}^{a_0,\lambda_0/a_0,u^0}/(x_1x_2 - \lambda_0,\ p_\lambda + u + q_\lambda,\ x_1p'_\lambda - x_2q'_\lambda) \\ &= \dim_{\mathbb{C}} \mathscr{E}_{x_1,x_2,u}^{a_0,\lambda_0/a_0,u^0}/(x_2 - \lambda_0/x_1,\ Q + (u - u^0)x_1^l, \\ &\partial[Q + (u-u^0)x_1^l]/\partial x_1) = \dim_{\mathbb{C}} \mathscr{E}_{x_1,u}^{0,0}/(x_1^{s_0}\prod_{i\neq 0}(x_1 - a_i + a_0)^{s_i} \\ &+ u(x_1 + a_0)^l,\ \ \partial[x_1^{s_0}\prod_{i\neq 0}(x_1 - a_i + a_0)^{s_i} + u(x_1+a_0)^l]/\partial x_1) = s_0 - 1, \end{aligned}$$

since $a_i \neq a_0 \neq 0$.

Hence, for $\lambda_0 \neq 0$ at the level $u = u^0$ there coalesce $r = \Sigma(s_j - 1)$ Morse critical points of the function u. Now $\Sigma s_j = k + l = \nu(C_{k,l}) + 1$. Therefore, $r \leqslant \nu(C_{k,l})$, and the preimage of zero under the mapping Φ contains no points with $\lambda_0 \neq 0$.

For $\lambda_0 = 0$ V_λ decomposes into two components:

$$V^1 = \{x_1 = 0,\ q_\lambda + u = 0\} \text{ and } V^2 = \{x_2 = 0,\ p_\lambda + u = 0\}.$$

The point $x = 0$, $u = 0$ is a singular point of the curve V_λ and hence is a critical point for the function u. Therefore, for $\lambda \in \Phi^{-1}(0)$ the only critical value of the function u must be zero.

The critical points of the function u on V^1 (or V^2) at the zero level are multiple roots of the polynomial q_λ (or p_λ).

Let

$$p_\lambda = x_1^{s_0}\prod (x_1 - a_i)^{s_i} \text{ and } q_\lambda = x_2^{t_0}\prod (x_2 - b_j)^{t_j},$$

where $a_i \neq a_{i'} \neq 0 \neq b_j \neq b_{j'}$.

Just as for $\lambda_0 \neq 0$, computation of the number r_i for a critical point $(a_i, 0, 0)$ [or $(0, b_i, 0)$] of the function u on V^2 (or V^1) different from the origin gives $r_i = s_i - 1$ (or $r_i = t_i - 1$).

For the origin

$$r_0 = \dim_{\mathbf{C}} \mathscr{E}_{x,u}/(x_1x_2,\ x_1^{s_0}\prod(x_1-a_i)^{s_i}+u+x_2^{t_0}\prod(x_2-b_j)^{t_j},$$
$$x_1[x_1^{s_0}\prod(x_1-a_i)^{s_i}]'-x_2[x_2^{t_0}\prod(x_2-b_j)^{t_j}]')=s_0+t_0.$$

Thus, at the level u = 0 there have coalesced at different places

$$r=t_0+s_0+\sum_{i\neq 0}(s_i-1)+\sum_{j\neq 0}(t_j-1)$$

Morse critical points of the function u. Since

$$s_0+\sum_{i\neq 0}s_i=k, \text{ and } \quad t_0+\sum_{j\neq 0}t_j=l,$$
$$\text{then} \quad r\leqslant l+k=\nu(C_{k,l})+1.$$

Equality is achieved only for $p_\lambda = x_1^k$, $q_\lambda = x_2^l$, i.e., for $\lambda = 0$.

Thus, $\Phi^{-1}(0) = 0$ for $f \in C_{k,l}$.

The case F_μ, $\mu \geqslant 4$: $F=(x_1^2+x_2^3+\lambda_1x_2+\lambda_2, p_\lambda(x_2)+u+x_1q_\lambda(x_2))$, where for $\mu = 2k + 1$

$$p_\lambda=x_2^k+\lambda_3x_2^{k-1}+\ldots+\lambda_{k+1}x_2 \quad\text{and}\quad q_\lambda=\lambda_{k+2}x_2^{k-2}+\ldots+\lambda_{2k},$$

for $\mu = 2k + 4$

$$p_\lambda=\lambda_3x_2^{k+1}+\ldots+\lambda_{k+3}x_2 \quad\text{and}\quad q_\lambda=x_2^k+\lambda_{k+4}x_2^{k-1}+\ldots+\lambda_{2k+3}.$$

It is not hard to see that the coordinates x_2 of critical points of the function u on the curve V_λ at the level $u = u^0$ are multiple roots of the polynomial.

$$Q=(p_\lambda+u^0)^2+(x_2^3+\lambda_1x_2+\lambda_2)q_\lambda^2$$

of degree $\mu - 1 = \nu(F_\mu)$.

After computations analogous to the case $C_{k,l}$ but more involved, noting that V_λ can have only one singularity, we obtain:

if V_λ is a smooth curve, then $r=\Sigma r_i\leqslant\nu(F_\mu)-1$;

if V_λ has a point of self-intersection, then $r\leqslant\nu(F_\mu)$;

if V_λ has a cusp point (here $\lambda_1 = \lambda_2 = 0$ and $u^0 = 0$), then $r\leqslant\nu(F_\mu)+1$.

In the last case equality is possible only for $Q = x_2^{\mu-1}$. Hence $\Phi^{-1}(0) = 0$ for the series F as well.

4. The assertion on the triviality of the highest homotopy groups is valid also for certain projections of multiple points onto the complex line (for the notation see part 2 of Sec. 2).

THEOREM 9 [12]. The germs of complements of bases of miniversal deformations to the bifurcation diagrams of the simple projections $X_{k,k}$, $k \geqslant 2$, $X_{2,l} \geqslant 3$ and U_μ, $\mu \geqslant 5$ are the spaces $k(\pi, 1)$.

For the projection $X_{2,l}$ π is a subgroup of index 2^{l-1} in the group of generalized braids BD_l; for U_μ it is a subgroup of index 2 in the group of generalized braids $BC_{\mu-1}$ [7].

Whether a similar theorem holds for the singularities $X_{k,l}$, $3 \leqslant k < l$; V_6; V_7 and $\Gamma^{a,b}$ is unknown.

We note that for the projection $U_5(x^2 + u^2, ux)$ one of the components of the bifurcation diagram (corresponding to coalescence of points of the set $F_\lambda = 0$) coincides with the bifurcation diagram of zeros $\Sigma'\subset\mathbf{C}^4$ of two quadrics in $\mathbf{C}^2$. Knörrer showed in [22] that the space $\mathbf{C}^4 \setminus \Sigma'$ has a trivial group π_2. He also proved that the complement to the bifurcation diagram of zeros of two quadrics of general position in $\mathbf{C}^3$ is the space $k(\pi, 1)$ [22].

TABLE 3

n	Notation	$f=(f_1, f_2)$
$\geqslant 2$	A_0	(x_n, x_1)
$\geqslant 1$	A_1, B_μ	$(x_n, f_2(x_1, \ldots, x_{n-1}, u))$, f_2— is the projection $(x_1, \ldots, x_{n-1}, u) \mapsto u$ of the hypersurface onto the line analogous to the boundary projection
$\geqslant 2$	$A_\mu,\ \mu \geqslant 2$ $C_\mu,\ F_4$	
$\geqslant 3$	D_μ, E_μ	
$\geqslant 1$	$B_\mu^*,\ \mu \geqslant 2$	$\left(x_1^\mu + x_2^2 + \ldots + x_n^2 + u,\ x_1\right)$
$\geqslant 2$	$C_\mu^*,\ \mu \geqslant 3$	$\left(x_1 x_2 + x_2^\mu + x_3^2 + \ldots + x_n^2 + u,\ x_1\right)$
	F_4^*	$(x_1^2 + x_2^3 + x_3^2 + \ldots + x_n^2 + u,\ x_1)$
1	$\tilde{X}_{k,l},\ k, l \geqslant 2$	$(x^k + u,\ x^l)$

5. We consider the problem of classification of projections of full intersections with boundary, i.e., of surfaces in $\mathbf{C}^{n+1}$ on which there is distinguished a submanifold of codimension 1.

The definitions of equivalence, stable equivalence, abutment, simplicity, codimension, and versal deformation are obvious.

THEOREM 10 [13]. Any simple germ of a projection of a full intersection with boundary onto $\mathbf{C}^1$ is stably equivalent to the germ at zero of the projection $(x, u) \mapsto u$ of the hypersurface $f_1 = 0$ with boundary $f_1 = f_2 = 0$ where $f = (f_1, f_2): \mathbf{C}^{n+1} \to \mathbf{C}^2$ is one of the mappings of Table 3.

The codimension ν of the projection $\tilde{X}_{k,l}$ is equal to $k + l - 2$ while for any other projection it is 1 less than the lower index of the symbol denoting it. All abutments of the projections of Theorem 10 are indicated in [13].

Let $\lambda \mapsto F_\lambda$, $\lambda \in \mathbf{C}^\nu$, be a miniversal deformation of projection onto the axis of u of the hypersurface $f_1 = 0$ with boundary $f_1 = f_2 = 0$. We set

$$V_\lambda = F_{1\lambda}^{-1}(0), \quad V'_\lambda = F_\lambda^{-1}(0).$$

The germ of the set of those λ for which at least one of the manifolds V_λ or V'_λ is singular or the function u is not a Morse function on at least one of them we call the bifurcation diagram $\Sigma \subset \mathbf{C}^\nu$ of the boundary projection f.

This definition in the cases A_μ–F_4 leads to bifurcation diagrams of projections of the same name, while in the cases B_μ^*–F_4^* it leads to bifurcation diagrams of corresponding functions on a manifold with boundary. The complements to these diagrams are Eilenberg–MacLane spaces.

THEOREM 11 [13]. Let Σ be the bifurcation diagram of the boundary projection $\tilde{X}_{k,l}$. Then the germ of the space $\mathbf{C}^\nu \setminus \Sigma$, $\nu = k + l - 2$, is the space $k(\pi, 1)$ where π is a subgroup of index $\binom{k+l-1}{l} k^{k+l-2}$ in the braid group of $(k + l - 1)$ strands.

5. Rectification of Vector Fields

In this section we consider still another property of projections which is analogous to the property of isolated singularities of functions. It turns out that in a neighborhood of the bifurcation diagrams of zeros of a number of projections it is possible to rectify vector fields while preserving the diagrams.

1. We call the tangent space to a manifold Δ at a singular point δ of it the limit of tangent spaces at regular points (if this limit exists).

Examples. 1. In the space $\mathbf{C}^3$ of coefficients of polynomials of degree 4

$$x^4 + ax^2 + bx + c$$

we consider the set corresponding to polynomials with multiple roots (a swallow tail). The tangent plane to it at zero is $c = 0$.

2. The Whitney umbrella is the surface in $\mathbf{C}^3$ given by the equation

$$x^2 = yz^2.$$

The limit of the tangent spaces to the umbrella on approach to zero does not exist.

For simple functions on a manifold with boundary there is the theorem of Lyashko.

THEOREM 12 [17]. Let v_1 and v_2 be vector fields defined in a neighborhood of the bifurcation diagram of zeros of a simple function on a manifold with boundary and transversal to the tangent plane to the diagram at zero. The germs of the fields v_1 and v_2 are equivalent relative to the group of germs at zero of diffeomorphisms of the enveloping space which preserve the diagram.

Examples. 1. Any field v transversal at zero to a swallow tail can be reduced to the local normal form $\partial/\partial c$ by a diffeomorphism of $\mathbf{C}^3$ preserving the swallow tail.

2. In three-dimensional space with coordinates (x, y, z) we consider the cylinder Δ

$$x^2 = y^3$$

over a semicubic parabola and a field v of general position. From Theorem 12 and its extensions to the projection C_3 (see Theorem 13 formulated below) we find: in a neighborhood of a general point of the cusp edge of the set Δ the field v is transversal to Δ and can be taken into the field $\partial/\partial x$ by a diffeomorphism preserving Δ while at exceptional points of the edge v is tangent to Δ and is locally equivalent to the field $\partial/\partial y + \partial z/\partial x$.

2. Let $F(x, u, \lambda)$ be a miniversal deformation of the projection $(x, u) \mapsto u$ of a full intersection $f(x, u) = 0$ onto the line; let ν be the codimension of this projection. The bifurcation diagram of zeros W of the projection f lies in $\mathbf{C}^{1+\nu}$.

We consider the germ at $0 \in \mathbf{C}^{1+\nu}$ of a smooth vector field v, $v(0) \neq 0$. Let u' be the time of motion along v from the germ at 0 of some smooth hypersurface transversal to v; let λ' be local coordinates on this hypersurface. We carry λ' by the field v to $\mathbf{C}^{1+\nu}$. $v = \partial/\partial u'$.

Let $G(x, u', \lambda')$ be the notation for the mapping F in the new coordinates:

$$G(x, u', \lambda') = F(x, u(u', \lambda'), \lambda(u', \lambda')).$$

G gives a deformation of the projection $(x, u') \mapsto u'$ of the surface $g(x, u') = 0$ where $g = G|_{\lambda'=0}$; $\lambda' \in \mathbf{C}^\nu$ is the parameter of this deformation. W is the bifurcation diagram of zeros of the projection g as well.

THEOREM 13 [11, 14]. Suppose that the projections onto lines of the surfaces

$$f(x, u) = 0 \text{ and } g(x, u') = 0$$

are equivalent and G is a miniversal deformation of g. If the projection f is simple, then there exists a germ of a diffeomorphism of the space $(\mathbf{C}^{1+\nu}, 0)$ preserving the diagram W and taking the germ of the field $\partial/\partial u$ into the germ of $\partial/\partial u'$.

Remarks. 1. The assertions of Theorems 12 and 13 for the singularities A_μ, D_μ, E_μ are equivalent.

2. In [14] Theorem 13 is formulated only for full intersections of positive dimension. However, it is easy to see that the scheme of proof presented below goes through also for simple projections of multiple points and for simple boundary projections.

Example. A versal deformation of the boundary projection $\tilde{X}_{2,2}$: $F = (x^2 + u, x^2 + \lambda_1 x + \lambda_2)$. The bifurcation diagram of zeros has two components (Fig. 3):

W_1: $u = 0$: the component consists of those (u, λ) for which the function $F_1(\cdot, u, \lambda)$ has the critical value 0;

W_2: $(u - \lambda_2)^2 + u\lambda_1^2 = 0$ (the Whitney umbrella): the component consists of those (u, λ) for which $0 \in \mathbf{C}^2$ lies in the image of the mapping $F(\cdot, u, \lambda)$.

By Theorem 13 for $\tilde{X}_{2,2}$ two fields transversal at zero to the planes $u = 0$ and $u = \lambda_2$ are equivalent relative to the group of diffeomorphisms preserving the umbrella and $u = 0$.

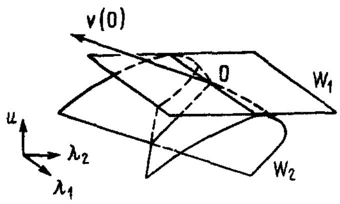

Fig. 3

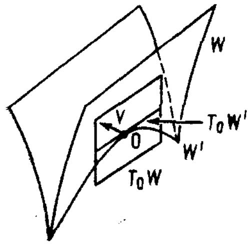

Fig. 4

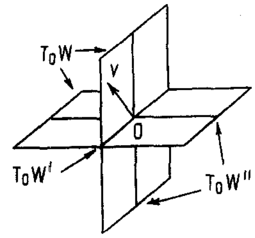

Fig. 5

We note that W_2 is the bifurcation diagram of zeros of the projection $X_{2,2}$ (not a boundary projection). In this case we find that fields transversal only to the plane $u = \lambda_2$ go over into one another by diffeomorphisms preserving only the umbrella.

3. Theorem 13 can be proved by following the proof of Theorem 12 in [17]. In this case the mapping Φ is used (see part 2, Sec. 4).

However, it is possible to proceed otherwise, namely, as in [13] in an analogous problem for quasihomogeneous projections which makes it possible to prove Theorem 13 not only in the holomorphic case but in the C^∞ and analytic cases as well. For this we note that the existence of the required diffeomorphism is equivalent to the existence of a germ at 0 of the change of coordinates

$$x=x(x', u', \lambda'),\ u=u'+\alpha(\lambda'),\ \lambda=\lambda(\lambda')$$

and a germ of a nondegenerate matrix $M(x', u', \lambda')$ such that

$$G(x', u', \lambda')=M(x', u', \lambda')F(x(x', u', \lambda'), u'+\alpha(\lambda'), \lambda(\lambda')).$$

We denote by $\mathcal{G}$ the group formed by the germs at 0 of the changes and matrices of the form indicated.

The result of the exposition we present for the special case $f\in C_\mu,\ \mu\geqslant 3$. Let F be a quasihomogeneous miniversal deformation of the projection C_μ:

$$F=x^\mu+\lambda_1 x^{\mu-1}+\ldots+\lambda_{\mu-2}x^2+ux+\lambda_{\mu-1}.$$

G is induced from F:

$$G=x^\mu+\lambda_1(u', \lambda')x^{\mu-1}+\ldots+\lambda_{\mu-2}(u', \lambda')x^2+u(u', \lambda')x+\lambda_{\mu-1}(u', \lambda'),\quad u(0, 0)=0,\quad \lambda(0, 0)=0.$$

$g = G|_{\lambda'=0}\in C_\mu$ and $(u', \lambda') \mapsto (u, \lambda)$ is a diffeomorphism. Therefore, up to changes of coordinates u' and λ' permitted by the group $\mathcal{G}$ it may be assumed that the principal part G (see [3]) has the form

$$G^0=x^\mu+\lambda_1' x^{\mu-1}+\ldots+\lambda_{\mu-2}' x^2+au'x+(\lambda_{\mu-1}'+bu'\lambda_1'),\quad a\neq 0.$$

A simple computation shows that the versality of G is equivalent to the inequality $a \neq \mu b$. We shall prove that in this case the mapping G is $\mathcal{G}$-equivalent to G^0.

G^0 is a versal deformation of the projection $g^0 = G^0|_{\lambda'=0}$ and is quasihomogeneous. The weight of u' is nonzero. Hence,

$$\begin{aligned}\mathscr{E}_{x,u'}&=\mathscr{E}_{x,u'}\langle g^0, \partial g^0/\partial x\rangle+\mathscr{E}_{u'}\partial g^0/\partial u'+\mathbf{C}\langle \partial G^0/\partial\lambda_i'|_{\lambda'=0}\rangle\\ &=\mathscr{E}_{x,u'}\langle g^0, \partial g^0/\partial x\rangle+\mathbf{C}\langle \partial g^0/\partial u', \partial G^0/\partial\lambda_i'|_{\lambda'=0}\rangle.\end{aligned}$$

Therefore,

$$\mathscr{E}_{x,u',\lambda'}=\mathscr{E}_{x,u',\lambda'}\langle G^0, \partial G^0/\partial x\rangle+\mathscr{E}_{\lambda'}\langle \partial G^0/\partial u', \partial G^0/\partial\lambda_i'\rangle+\mathfrak{m}_{\lambda'}\mathscr{E}_{x,u',\lambda'},$$

where $\mathfrak{m}_{\lambda'}\subset\mathscr{E}_{\lambda'}$ is a maximal ideal.

By Wassermann's lemma [24] the last term drops out. The equality hereby obtained means that G^0 has zero codimension relative to the group $\mathcal{G}$ to which parallel transports have been added. Using the Mather–Wassermann technique [18, 24], it is possible to prove that such a mapping is finitely $\mathcal{G}$-determined (cf. Theorem 1, Sec. 1). The element $G - G^0$ belongs to the tangent space to the $\mathcal{G}$-orbit of G^0. Its quasihomogeneous components have degrees greater than G^0. Therefore G is $\mathcal{G}$-equivalent to G^0.

It is now not hard to achieve the equivalence of G^0 and F by quasihomogeneous transformations of the group $\mathcal{G}$ (see [3]).

Analogous arguments prove Theorem 13 also for the other simple singularities.

4. We shall now indicate the geometric meaning of the conditions of Theorem 13 imposed on a rectifiable field v for nonboundary projections of full intersections of positive dimension. There are two conditions:

1. Projections onto lines of the surfaces f = 0 and g = 0 are equivalent.

2. G is a miniversal deformation of the projection g.

We denote by W' and W" the cusp edge and set of self-intersection of the diagram W; $T_0\Delta$ is the union of all tangent spaces at zero to different components of the set Δ; $\mathcal{L}$ is the factor algebra of the algebra L of vector fields on $\mathbf{C}^{1+\nu}$ tangent to W by its subalgebra L_0 preserving the origin. We recall that if $f\notin\mathbf{C}_{k,l}$, then $1 + \nu$ is the lower index of the symbol denoting the simple projection.

We consider conditions 1 and 2 for all cases of simple projections.

The cases A_μ, D_μ, E_μ, $C_{k,l}$, $2 \leqslant k \leqslant l$, $F_{\mu'}$, $\mu' \leqslant 7$.

1. v(0) is transversal to T_0W (for $C_{2,2}$ and F_7 T_0W consists of two hyperplanes and one hyperplane for the other singularities.

2. It follows from 1.

The cases C_μ, F_6.

1. $v(0)\in T_0W\setminus T_0W'\simeq\mathbf{C}^\nu\setminus\mathbf{C}^{\nu-1}$ (Fig. 4).

2. The diagrams of zeros in the cases considered are cylinders over the diagrams of zeros $A_{\mu-1}$ and $C_{2,3}$. Therefore, the factor algebra $\mathcal{L}$ is nontrivial and isomorphic to the line. Condition 2 is equivalent to the fact that $[v, l](0)\notin T_0W$ where l is any representer of a generator of $\mathcal{L}$. We note that for C_μ this coincides precisely with the condition $a \neq \mu b$ in the proof of Theorem 13.

The case F_5. Here the diagram of zeros is a cylinder over the diagram of zeros $C_{2,2}$. In suitable coordinates $\lambda_1,\ldots,\lambda_5\in\mathbf{C}^5$

$$T_0W=\{\lambda_1=0\}\cup\{\lambda_2=0\},\ T_0W'=\{\lambda_1=\lambda_2=0\},$$
$$T_0W''=\{\lambda_1=\lambda_3=0\}\cup\{\lambda_2=\lambda_4=0\}\ \text{(Fig. 5)}.$$

1. $v(0)\in T_0W\setminus(T_0W'\cup T_0W'')$.

2. $[v, l](0)\notin T_0W'\oplus\mathbf{C}v(0)$ where l is a representer of a generator of the algebra $\mathcal{L}=L/L_0\simeq\mathbf{C}$.

The case F_4. W is a cylinder over a semicubic parabola with a two-dimensional generator.

1. $v(0)\in T_0W'$, but $[v, v'](0)\notin T_0W$ where $v'\in L$, $v'(0)=v(0)$.

2. The mapping $\mathcal{L}\to T_0\mathbf{C}^4/T_0W'$, $l+L_0\mapsto[v, l](0)\bmod T_0W'$, is an isomorphism (it can be shown that this mapping is well defined).

The case B_μ. W is smooth and $\mathcal{L}\simeq C^{\mu-1}$.

1. At zero v has tangency of order $\mu - 1$ with W, i.e., for any $v'\in L$, $v'(0)=v(0)$, we have

$$v^0(0),\ldots, v^{\mu-2}(0)\in T_0W,\ v^{\mu-1}(0)\notin T_0W,$$

where v^k is the k-fold commutator of v with v',

$$v^k=[\ldots[[v, v'], v'],\ldots, v'].$$

2. The mapping of $\mathcal{L}$ into $\mathbf{C}^{\mu-1}$ whose k-th coordinate function is equal to $[v^{k-1}, l + L_0](0) \bmod T_0W$ is an isomorphism.

All the conditions enumerated here of the contact of the field v with the diagram W are satisfied for fields of general position on $\mathbf{C}^{1+\nu}$, i.e., the field obtained by a small perturbation of the field v has near zero the same singularity v has at zero.

5. Zakalyukin carried Theorem 12 over to quasihomogeneous (not necessarily simple) functions on a smooth manifold in the form of an assertion regarding the stability of a vector field transversal to the bifurcation diagram of zeros [16]. In this form Theorem 13 also carries over to quasihomogeneous projections onto the line.

Let $F(x, u, \lambda)$ be a quasihomogeneous miniversal deformation of a projection $(x, u) \mapsto u$ of a full intersection (possibly with boundary), and let ν be the codimension of this projection. The bifurcation diagram of zeros $W \subset \mathbf{C}^{1+\nu}$ is defined globally.

THEOREM 14 [13]. The vector field $\partial/\partial u$ is stable at $0 \in \mathbf{C}^{1+\nu}$: if v is a holomorphic field defined near zero sufficiently close to $\partial/\partial u$, then there exist a point $q \in \mathbf{C}^{1+\nu}$ near 0 and a germ of a diffeomorphism

$$(\mathbf{C}^{1+\nu}, W, 0) \to (\mathbf{C}^{1+\nu}, W, q),$$

taking the germ at zero of the field $\partial/\partial u$ into the germ of v at q.

A similar assertion holds in the real case as well.

An analogue of Zakalykin's theorem for functions on a manifold with boundary follows easily from Theorem 14 as a corollary. For this it is necessary to observe that if

$$H(x, \lambda) = h(x) + \lambda_1 + \sum_2^{\mu} \lambda_i \varphi_i(x)$$

is a quasihomogeneous, miniversal deformation of the function h on the manifold $\mathbf{C}^n$ with boundary $x_1 = 0$, then $(H(x, \lambda), x_1)$ is a miniversal deformation of the projection $(x, \lambda_1) \to \lambda_1$ of the hypersurface $h(x) + \lambda_1 = 0$ with boundary $h(x) + \lambda_1 = x_1 = 0$.

6. The stability we have considered relative to a group of diffeomorphisms preserving the diagram of zeros holds also for direction fields $\partial/\partial u$. No restrictions on the initial projection are hereby required. Stability follows directly from the stability of versal deformations of projections (Theorem 3, Sec. 1).

Somewhat more follows from Theorem 3: an analogous assertion regarding projections onto bases of arbitrary dimension.

Let $W \subset \mathbf{C}^{p+\nu}$ be the bifurcation diagram of zeros of the projection $(x, u) \mapsto u$ of some full intersection (the definition of W is obvious). Let $\lambda \in \mathbf{C}^{\nu}$ be the parameter of a versal deformation. Then the fibering $(u, \lambda) \mapsto \lambda$ and the fibering $\mathbf{C}^{p+\nu} \to \mathbf{C}^{\nu}$ sufficiently close to it are locally equivalent relative to the group of diffeomorphisms of the space $\mathbf{C}^{p+\nu}$ preserving W.

If a versal deformation $F(x, u, \lambda)$ of projection onto $\mathbf{C}^p$ is quasihomogeneous, then in the formal case infinitesimal equivalence of the fiberings $(u, \lambda) \mapsto \lambda$ and $\mathbf{C}^{p+\nu} \to \mathbf{C}^{\nu}$ can be realized by means of fields $v = (v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+\nu})$ tangent to W such that $\Sigma\alpha_i \partial v_i / \partial u_i = 0$. Here α_i are complex numbers for which there exist no relation $\Sigma\alpha_i w_i A_i = 0$, where $w_1, \ldots, w_p$ are the quasihomogeneous weights of the variables $u_1, \ldots, u_p$ and $A_1, \ldots, A_p$ are any positive integers. The proof follows the proof of Theorem 13 of part 3.

Example. Taking all $\alpha_i = 1$, we obtain in the real case an equivalence of fiberings which preserves not only W but also the volume element in each of the fibers λ = const.

CHAPTER III

CLASSIFICATION OF SIMPLE PROJECTIONS

In this chapter we present a classification of simple germs of projections

$$V \to E \to B,$$

for which the dimension of the manifold V projected is no less than the dimension of the base B of the projection. We call them projections "onto." As before, it is assumed that V is a full intersection.

6. Nonsimple Deformations of Full Intersection

Projection of V onto B defines a family of submanifolds in fibers of the bundle $E \to B$ — the family of intersections of V with the fibers. A germ of the projection $V \to E \to B$ gives a deformation of the submanifold V_0 of a labeled fiber. The codimension of V_0 in the fiber coincides with the codimension of V in E. The equivalence class of full intersection V_0 is an invariant of the equivalence of projections we consider. Hence, for simplicity of the projection $V \to E \to B$ simplicity of the full intersection V_0 is required.

We recall that the projections $V \to E' \to B$ and $V \to E'' \to B$ are called stably equivalent if E' and E'' can be imbedded as subbundles in some bundle $E \to B$ and the projections hereby induced are equivalent.

We denote by m the codimension of V in E and by n the dimension of a fiber of the bundle $E \to B$.

The next result follows from what has been said and the classification of simple full intersections (see [6]).

THEOREM 15. Any germ of a simple projection "onto" is stably equivalent to some projection for one of the following three values of the dimension parameters: 1) m = 1 (a hypersurface); 2) n = 3, m = 2; 3) n = 2, m = 2.

The classification of simple singularities in these three cases is contained in the respective Theorems 16, 18, and 19 presented below.

We shall present a number of considerations used in the classification.

As before, we write the mapping $E \to B$ in local coordinates in the form

$$(x, u) \mapsto u, \ x \in \mathbf{K}^n, \ u \in \mathbf{K}^p,$$

where **K** is the field of real or complex numbers. Equivalence of projections induces on the space of local equations $f(x, u) = 0$ of manifolds V the action of the group of pairs (h, M), where $h(x, u) = (X'(x, u), U(u))$ is the germ of a fibered diffeomorphism, and M is the germ of a nondegenerate $m \times m$ matrix:

$$(h, M)f = M(h^*f).$$

The labeled full intersection V_0 is given in the fiber enveloping it by zeros of the mapping $f_0 = f|_{u=0}$. Let $\Phi(x, \lambda)$, $\lambda \in \mathbf{K}^\mu$, be a versal deformation of the full intersection $f_0(x) = 0$. As any deformation of the surface $f_0 = 0$, the deformation f(x, u) is induced from Φ for some mapping of the bases

$$\Lambda : (\mathbf{K}^p, 0) \to (\mathbf{K}^\mu, 0):$$

there exist a germ on E of an $m \times m$ matrix N, N(0) = E, and a p-parameter deformation X of the identity mapping of $\mathbf{K}^n$ such that

$$N(x, u)f(x, u) = \Phi(X(x, u), \Lambda(u)).$$

The mapping Λ inducing the deformation f plays a basic role in the classification of projections.

We recall that we call the tangent space $T_\delta\Delta$ to the manifold Δ at a singular point δ of it the limit of the tangent planes at regular points (if it exists).

The base of the versal deformation Φ is decomposed into equivalence classes of full intersections. Let Z be one of them.

The mapping Λ is transversal to the class Z ($\Lambda \pitchfork Z$) if the image of its differential at zero is transversal to the space T_0Z in $T_0\mathbf{K}^\mu$ (in particular, the rank of the differential is not less than the codimension of Z in the base of Φ).

The concept of transversality introduced does not depend on the choice of Φ and Λ.

Theorem 1 on finite determination (Sec. 1) makes it possible in the classification of simple projections to restrict attention to the formal case.

It is not hard to see that the technique of Arnol'd for reducing functions to normal form by means of a spectral sequence [5, 7] carries over to the theory of projections.

We present several assertions which enable us to guarantee the absence of simple projections. We hereby speak of the projection $(x, u) \mapsto u$ of a surface $f = 0$ onto a p-dimensional base as a p-parameter deformation f of the full intersection $f_0 = 0$, $f_0 = f|_{u=0}$.

Proposition 1. We suppose there exists an abutment $f_0 \to g_0$ of full intersections. Suppose that the full intersection $g_0 = 0$ has no simple p-parameter deformation. Then $f_0 = 0$ also has none.

For simplicity of the projection f simplicity of the full intersection $f_0 = 0$ is required. Any simple full intersection has a quasihomogeneous, miniversal deformation $f_0(x) + \sum_1^{\mu} \lambda_i e_i(x)$ with positive weights of all variables. Here $e_1, \ldots, e_\mu$ is a monomial basis of the space

$$\mathscr{E}_x^m / \{f_0^* (\mathfrak{m}(m)) \mathscr{E}_x^m + \mathscr{E}_x \langle \partial f / dx_i \rangle \}.$$

We assume that the weights α_i of the variables λ_i are ordered: $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_\mu > 0$.

Suppose that the rank at zero of the mapping Λ is equal to $\mu - 1$. Then there is a number s, $1 \leqslant s \leqslant \mu$, such that the deformation f can be reduced to the form

$$f(x, u) = f_0(x) + \sum_{i=1,\, i \neq s}^{\mu} u_i e_i(x) + \lambda_s(u) e_s(x), \tag{1}$$

where $u = (u_1, \ldots, \hat{u}_s, \ldots, u_{p+1})$ are the parameters of the deformation, and the derivative at zero of the function λ_s with respect to all u_j, $j > s$, is zero.

Proposition 2 (on Reduction). Let $2\alpha_\mu \geqslant \alpha_s$. The absence for f_0 of simple deformations of the form (1) with $(\mu - 1)$ parameters implies their absence also with a large number of parameters $p \geqslant \mu$.

The classification theorems we formulate pertain to the real case. We point out the differences in the complex situation after the theorems.

7. Singular Projections of Hypersurfaces

1. Suppose that the germ of the manifold V projected is given by the single equation $f(x, u) = 0$. If the projection is simple, then the function $f_0 = f|_{u=0}$ is also simple and hence has one of the normal forms [3]

$$A_\mu,\ \mu \geqslant 0: x_1^{\mu+1} \pm x_2^2 \pm \ldots \pm x_n^2;$$
$$D_\mu,\ \mu \geqslant 4: x_1^2 x_2 \pm x_2^{\mu-1} \pm x_3^2 \pm \ldots \pm x_n^2;$$
$$E_6: x_1^3 + x_2^4 \pm x_3^2 \pm \ldots \pm x_n^2;$$
$$E_7: x_1^3 + x_1 x_2^3 \pm x_3^2 \pm \ldots \pm x_n^2;$$
$$E_8: x_1^3 + x_2^5 \pm x_3^2 \pm \ldots \pm x_n^2.$$

The abutments of these functions are the same as for the analogous boundary singularities (they are indicated in Diagram 1 in part 3, Sec. 2).

In Theorem 16 presented below $f(x, u) = f_0(x) + \Delta(x, u)$; X_μ (or Y_ν) is one of the simple singularities of functions of n (or $p - \mu + 1$) variables, i.e., X, Y = A, D, E; if the set of arguments of a function g contained in Table 4 is empty, then we assume that $g \in A_1$, $g = 0$; $e_1, \ldots, e_\mu$ is a monomial basis of the ring $\mathscr{E}_x / (\partial f_0 / \partial x_i)$, and e_μ is the Hessian of f_0 (the element of maximal quasihomogeneous degree); $q = \pm u_\mu^2 \pm \ldots \pm u_p^2$.

THEOREM 16 [15]. A germ of projection of a real hypersurface is simple if and only if it is equivalent to the germ at zero of the projection $(x, u) \mapsto u$ of the manifold $f = 0$ where f is one of the functions of Table 4.

The function $\varkappa(\mu)$:

μ	2	3	4	5	6	7	8	9	10	$\geqslant 11$
$\varkappa$	2	∞	3	5	4	6	5	7	6	7

All the projections indicated are pairwise inequivalent up to permutations of signs in the form q and the normal forms X_μ and Y_ν. For $p = 1$ $A_1^V = A_1$.

The abutments of simple projections for $p = 1$ are enumerated in Diagram 1, part 3, Sec. 2. For the remaining p any abutment is obtained by a finite composition of the abutments indicated in Diagram 6.

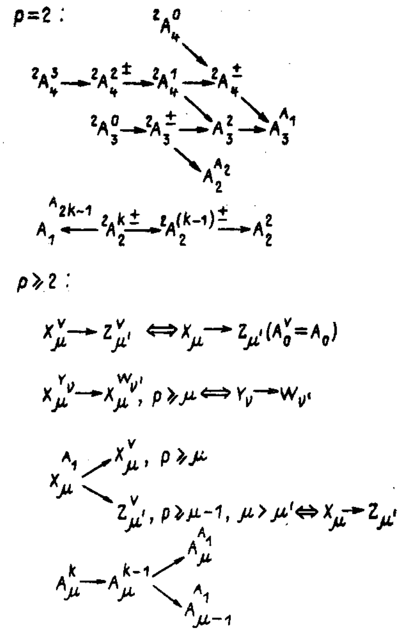

Diagram 6

Remarks. 1. The projections of smooth hypersurfaces with (n, p) = (1, 2) through codimension 1 constitute precisely the list of Arnol'd of all possible projections of general surfaces from three-dimensional space onto the plane [1]. All these projections are simple.

2. As a whole, the complex list of simple projections of hypersurfaces differs from the real list only in that in place of the sign ± it is necessary to insert the sign +, and it is also necessary to drop ${}^2A_2^k$. Over **C** this series abuts a projection containing a modulus (a continuous invariant) in normal form.

2. In Table 5 below the functions P_8, X_9, J_{10} are as follows:

$$P_8:\ x_1^3+x_2^3+x_3^3+\beta x_1x_2x_3\pm x_4^2\pm\ldots\pm x_n^2;$$
$$X_9:\ \pm x_1^4+\beta x_1^2x_2^2\pm x_2^4\pm x_3^2\pm\ldots\pm x_n^2;$$
$$J_{10}:\ x_1^3+\beta x_1x_2^4\pm x_2^6\pm x_3^2\pm\ldots\pm x_n^2.$$

Here $\beta\in\mathbf{C}$ is a modulus. We denote by $\mathrm{rk}\Lambda$ the rank of the mapping Λ at zero; the dots denote terms of higher degree of quasihomogeneity.

THEOREM 17. Any nonsimple germ of the projection of a hypersurface abuts at a germ at zero of a nonsimple projection $(x, u)\mapsto u$ of the manifold $f(x, u) = 0$ where f is one of the functions of Table 5 (here again $f = f_0 + \Delta$).

Remark on Table 5. Let $\mathcal{F}$ be the set of all functions of the form $f_0 + \Delta$ where Δ runs through all values indicated in the corresponding graph; let $\mathcal{O}$ be the germ of the orbit of $\mathcal{F}$ in $\mathcal{E}_{x,u}$ relative to the equivalence group of projections with transports. The codimension of $\mathcal{O}$ in $\mathcal{E}_{x,u}$ is the "codimension" indicated in the graph.

Some abutments of the projections of Table 5 to simple projections are indicated in Diagram 7.

3. Theorems 16 and 17 are proved by means of all the tools indicated in Sec. 6. The complete determinant of singularities in our problem is too broad (cf. [3]). Therefore, for illustration we present a small part of it consisting of five lemmas giving the classification of the projections of hypersurfaces for pairs (μ, p), $f_0\in\check{A}_\mu$, from the hatched region of Fig. 6 (the region is considered with boundary).

Fig. 6

TABLE 4

p	f_0	$\Delta(x, u)$	Codimension	Notation
		Smooth hypersurfaces $(\operatorname{grad} f\mid_{x=0, u=0} \neq 0)$		
$\geqslant 1$	A_0	0	0	A_0
1	$X_\mu,\ \mu > 0$	u	$\mu - 1$	X_μ
$\geqslant \mu$	$X_\mu,\ \mu > 0$	$\sum_1^\mu u_i e_i(x)$	0	X_μ^V
$\geqslant \mu - 1$	$X_\mu,\ \mu > 1$	$\sum_1^{\mu-1} u_i e_i(x) + g(u_\mu, \ldots, u_p)\, e_\mu(x)$, $g \in Y_\nu,\ \nu > 0$	ν	$X_\mu^{Y_\nu}$
	$A_\mu,\ \mu \geqslant 3$	$u_{\mu-1} x_1^{\mu-1} + \left(u_{\mu-1}^k + q\right) x_1^{\mu-2} + \sum_1^{\mu-2} \left(u_i x_1^{i-1}\right)$, $1 < k < \varkappa(\mu)$	k	A_μ^k
	$A_\mu,\ \mu \geqslant 4$	$u_{\mu-1} x^{\mu-1} + q x_1^{\mu-2} + \sum_1^{\mu-2} \left(u_i x_1^{i-1}\right)$	$\varkappa$	$A_\mu^\varkappa$
2	A_3	$u_1^2 x_1 \pm u_1^2 x_1^2 + u_2$	3	${}^2A_3^\pm$
		$u_1^2 x_1 + u_2$	4	${}^2A_3^0$
	A_4	$u_1 x_1 \pm u_1 x_1^3 + u_2$	2	${}^2A_4^\pm$
		$u_1 x_1 + u_2$	3	${}^2A_4^0$
		$u_1 x_1^2 + u_1 x_1^3 + u_2$	3	${}^2A_4^1$
		$u_1 x_1^2 \pm u_1^2 x_1^3 + u_2$	4	${}^2A_4^{2\pm}$
		$u_1 x_1^2 + u_2$	5	${}^2A_4^3$
		Singular hypersurfaces $(\operatorname{grad} f\mid_{x=0, u=0} = 0)$		
1	A_1	$\pm u^\mu$	$\mu - 1$	B_μ
> 1	A_1	$g(u_1, \ldots u_p),\quad g \in Y_\nu$	ν	$A_1^{Y_\nu}$
1	$A_\mu,\ \mu > 1$	$u x_1$	μ	$C_{\mu+1}$
	A_2	u^2	3	F_4
$\geqslant 2$		$u_1 x_1 + q$	2	A_2^2
2		$u_1^2 + u_2^2 \pm u_1^k x_1,\quad k \geqslant 2$	$2k$	${}^2A_2^{k\pm}$

Thus, let $f_0 \in A_\mu$ and $n = 1$.

LEMMAS.

1. Let $\mu \geqslant 0$ and $\operatorname{rk} \Lambda = \mu$. Then $f \in A_\mu^V$.
2. Let $\mu \geqslant 1$, $\operatorname{rk} \Lambda = \mu - 1$ and $\Lambda \pitchfork A_{\mu-1}$. Then $f \in A_\mu^{Y_\nu}$.
3. Let $\mu \geqslant 2$, $\operatorname{rk} \Lambda = \mu - 1$, $\Lambda \not\pitchfork A_{\mu-1}$ but $\Lambda \pitchfork A_{\mu-2}$. Then the projection f is equivalent to $x^{\mu+1} + u_{\mu-1} x^{\mu-1} + h(u_{\mu-1}, u_\mu, \ldots, u_p)\, x^{\mu-2} + \sum_{i=1}^{\mu-2} u_i x^{i-1}$, where h in $u_{\mu-1}$ has degree no higher than $\varkappa(\mu) - 1$.

TABLE 5

p	f_0	$\Delta(x, u)$	Codimension	Notation
1	A_2	$\beta u^2 x_1 + u^3$	4	X'_μ
	A_3	$\beta u x_1^2 + u^2$		
	D_4	$u x_1 + \beta u x_2$		
	$P_8,\ X_9,\ J_{10}$	u	$\mu-2$	X_μ
2	A_3	$\beta u_1^2 x_1^2 + u_1^3 x_1 + u_2 + \ldots$	4	${}^2A^*_\mu$
	A_4	$\beta u_1 x_1^3 + u_1^2 x_1 + u_2 + \ldots$		
	A_5	$u_1 x_1 + u_2 + \ldots$	3	${}^2A_5$
$\geqslant \max\{3, \mu-2\}$	$A_\mu,\ \mu \geqslant 2$	$\sum_1^\mu \lambda_i(u) x_1^{i-1}$, $\operatorname{rk}\Lambda = \mu-2$	$p-\mu+4$	$A_\mu(\mu-2)$
$\geqslant \mu-1$	$A_\mu,\ \mu \geqslant 3$	$\sum_1^\mu \lambda_i(u) x_1^{i-1}$, $\operatorname{rk}\Lambda = \mu-1,\ \Lambda \pitchfork A_{\mu-2}$	3	$A_\mu(\mu-1)$
$\geqslant \mu$	$A_\mu,\ \mu \geqslant 2$	$\sum_1^{\mu-2} u_i x_1^{i-1} + \left(u_{\mu-1}u_\mu + \beta u_\mu^3 + \sum_{\mu+1}^{p} \pm u_j^2 + \ldots\right) x_1^{\mu-2} + u_{\mu-1} x_1^{\mu-1}$	3	A^*_μ
$\geqslant \mu-1$	$D_\mu,\ E_\mu$	$\sum_1^\mu \lambda_i(u) e_i(x),\ \Lambda \pitchfork D_{\mu-1}$	2	X^*_μ
$\mu-2$	$D_\mu,\ E_\mu$	$\sum_1^\mu \lambda_i(u) e_i(x)$, $\operatorname{rk}\Lambda = \mu-2$	2	$X_\mu(\mu-2)$
$\geqslant \mu-3$	$P_8,\ X_9,\ J_{10}$	$\sum_1^\mu \lambda_i(u) e_i(x)$	$\max\{0, \mu-p-1\}$	X^d_μ
$\geqslant \mu+1$	$A_\mu,\ D_\mu,\ E_\mu$	$\sum_1^{\mu-1} u_i e_i(x) + g(u_\mu, \ldots, u_p) e_\mu(x)$, $g(0)=0,\ g \in Y_\nu = P_8,\ X_9,\ J_{10}$	$\nu-1$	$X^{Y_\nu}_\mu$

If the quadratic part of the restriction $h|_{u_{\mu-1}=0}$ is hereby nondegenerate, then $f \in A'_\mu$. Otherwise f nonsimply abuts A^*_μ.

4. Let $\mu \geqslant 3$, $\operatorname{rk}\Lambda = \mu - 1$ but $\Lambda \pitchfork A_{\mu-2}$. Then f is nonsimple and belongs to the class $A_\mu(\mu - 1)$.

5. Let $\mu \geqslant 2$, $p \geqslant 3$ and $\operatorname{rk}\Lambda = \mu - 2$. Then f is nonsimple and belongs to the class $A_\mu(\mu - 2)$.

We note that Lemma 4 implies the absence of simple projections of hypersurfaces with singularities for p = 2 and $f \in X_\mu$, where $\mu \geqslant 3$.

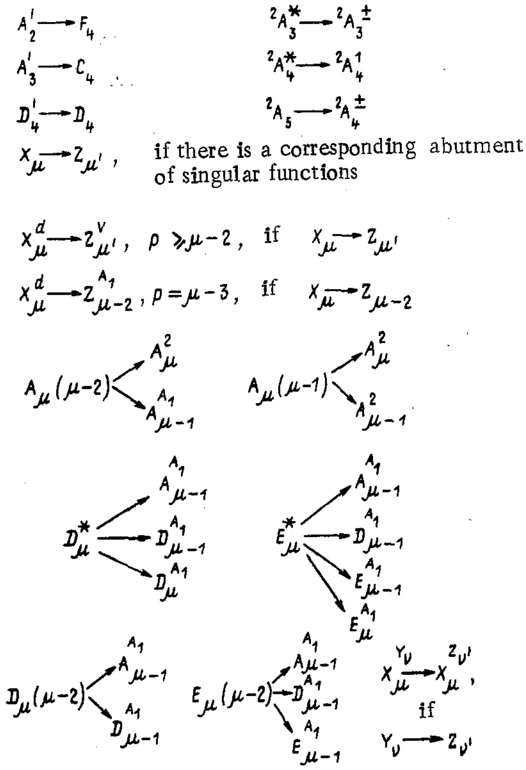

Diagram 7

8. Projections of Manifolds of Codimension 2

1. THEOREM 18. A simple projection of a manifold onto a space of dimension 1 less is simple if and only if it is stably equivalent either to a simple projection of the hypersurface or to a versal deformation of a simple germ of a curve in three-dimensional space.

The list of simple curves $\mathbf{C}^3$ not equivalent to plane curves was obtained by Giusti [19]. Any such curve can be given by equations $f_0 = 0$, where f_0 is one of the mappings of Table 6.

The analogous real list differs from Giusti's list by the placement of the signs + or − of (S_μ, T_8) and addition of the singularity $\tilde{T}_7(x_1^2 + x_3^3,\ x_2^2 + x_3^2)$, $\mathbf{C}$-equivalent to T_7.

All abutments of the projections of Theorem 18 can be obtained from the list in Sec. 7 of abutments of simple projections of hypersurfaces by the addition of abutments of versal deformations $X_\mu^V \to Z_{\mu'}^V$.

We note that all abutments of simple curves are so far not known. For example, in [11]

$$W_8 \to D_6,\quad Z_9 \to D_7 \text{ and } Z_{10} \to D_8$$

were added to the diagram of abutments compiled by Giusti. Indeed, the first two of these abutments pass, respectively, through the curves E_7 and U_8:

$$W_8 \to E_7\colon\ (x_1^2 + 64x_2^3 - 46\varepsilon^2 x_1 x_2 + 429\varepsilon^4 x_2^2 + 20\varepsilon^3 x_2 x_3 - \varepsilon^2 x_3^2,$$

TABLE 6

Type	f_0	Type	f_0
S_μ, $\mu \geqslant 5$	$x_1^2 + x_2^2 + x_3^{\mu-3},\quad x_2x_3$	U_8	$x_1^2 + x_2x_3 + x_3^3,\quad x_1x_2$
T_7	$x_1^2 + x_2^3 + x_3^3,\quad x_2x_3$	U_9	$x_1^2 + x_2x_3,\quad x_1x_2 + x_3^4$
T_8	$x_1^2 + x_2^3 + x_3^4,\quad x_2x_3$	W_8	$x_1^2 + x_3^3,\quad x_2^2 + x_1x_3$
T_9	$x_1^2 + x_2^3 + x_3^5,\quad x_2x_3$	W_9	$x_1^2 + x_2x_3^2,\quad x_2^2 + x_1x_3$
U_7	$x_1^2 + x_2x_3,\quad x_1x_2 + x_3^3$	Z_9	$x_1^2 + x_3^3,\quad x_2^2 + x_3^3$
		Z_{10}	$x_1^2 + x_2x_3^2,\quad x_2^2 + x_3^3$

TABLE 7

Type	f_0	Type	f_0
$C^{\pm}_{k,l},\ 2\leqslant k\leqslant l$	$x_1x_2,\ x_1^k\pm x_2^l$	$F_{2m+1},\ m\geqslant 3$	$x_1^2+x_2^3,\ x_2^m$
$\tilde{C}_{2k},\ k\geqslant 3$	$x_1^2+x_2^2,\ x_2^k$	$F_{2m+4},\ m\geqslant 2$	$x_1^2+x_2^3,\ x_1x_2^m$
$H^{\pm}_{m+5},\ m\geqslant 4$	$x_1^2\pm x_2^m,\ x_1x_2^2$	G_{10}	$x_1^2,\ x_2^4$

$$4x_1x_2+x_3^2-16\varepsilon^4x_1+208\varepsilon^6x_2+16\varepsilon^5x_3);$$
$$U_8\to D_7:\ (x_1^2+x_2x_3+2\varepsilon^3x_1+\varepsilon^4x_3,$$
$$x_1x_2+x_1x_3^2-3\varepsilon x_1^2-\frac{3}{2}\varepsilon^2x_1x_3+\frac{3}{8}\varepsilon^3x_3^2+\frac{1}{2}\varepsilon x_2x_3\Big).$$

2. Before presenting the classification of simple projections of manifolds onto bases of the same dimension we recall the classification of simple plane 0-dimensional full intersections (by Theorem 15 our projections in the present case are deformations of multiple points of the plane).

The simple multiple points in $\mathbf{C}^2$ were classified by Giusti [20, 6]. The corresponding real list is given in Table 7 (our notation differs from that of Giusti).

Depending on the parity of the indices in the real case and always in the complex case, singularities differing only in sign are equivalent. Over $\mathbf{C}$, $\tilde{C}_{2k}\sim C_{k,k}$.

All abutments of real multiple points are obtained by composition of a finite number of abutments of Diagram 8.

In Table 8 presented below we retain the notation of Theorem 16 with the sole difference that now $e_1,\ldots,e_\mu$ is a monomial basis ordered with respect to weights of the space

$$\mathscr{E}_x^2/\{f_0^*(\mathfrak{m}(2))\mathscr{E}_x^2+\mathscr{E}_x\langle\partial f_0/\partial x_i\rangle\},\quad x=(x_1,x_2);$$

X_μ is the class of f_0. Sometimes, with special mention, we write the normal form $C^{+}_{2,2}$ as $(x_1^2,\ x_2^2)$. The singularities $\tilde{C}_4(x_1^2+x_2^2,\ x_2^2)$ are equivalent to $C^{+}_{2,2}$.

THEOREM 19. The germ of a projection of a real surface onto a manifold of the same dimension is simple if and only if it is stably equivalent either to a simple projection of a hypersurface or to the germ at zero of the projection $(x,\ u)\mapsto u$ of one of the manifolds $f=0$ of Table 8.

The value of $\varkappa$ (in the series F^r_{10}) is not known exactly (it is possible that $\varkappa=\infty$).

In the last row of the table X_μ is any of the singularities $C^{\pm}_{k,l}$–G_{10}.

Over the field $\mathbf{C}$ projections differing only in the placement of the signs + and – are equivalent; the index deformations $\tilde{C}_{2k}$ and $C_{k,k}$ and also the singularities ${}^2C^{2r}_{2,2}$ and ${}^2C^{r,r}_{2,2}$ are the same.

Diagram 9 presents some abutments of the projections of Theorem 19. Among them are all abutments of real singularities for p = 1 and complex singularities for p = 1, 2.

THEOREM 20. Any nonsimple projection of a manifold onto a space of the same dimension abuts either one of the nonsimple projections of Tables 5 and 9 or one of the projections $X_\mu^{Y_\nu}(X_\mu\in C^{\pm}_{2,l},\ C_{3,l},\ F_7\div F_{10},\ G_{10};\ Y_\nu\in P_8,\ X_9,\ J_{10})$, or some deformation of a bordered multiple point in $\mathbf{R}^n$.

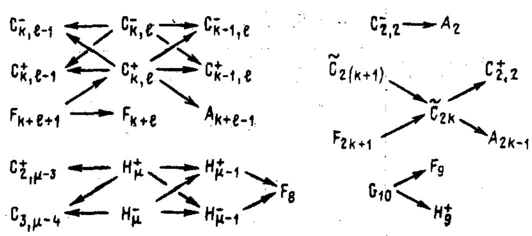

Diagram 8

TABLE 8

p	f_0	$\Delta(x, u)$	Codimension	Notation
1	$C^{\pm}_{k,l}$, $2\leqslant k\leqslant l$, $\widetilde{C}_{2r}$, $r\geqslant 2$, F_{μ}, $\mu\geqslant 7$	$(0, u)$	$\mu-1$	X_{μ}
	$C^{+}_{2,2}$, (x_1^2, x_2^2)	(x_2^3, u)	4	F_5
	$C_{2,3}$	$(u, 0)$	5	F_6
2	$C^{-}_{2,2}$	$(u_1, u_2+u_1^k x_1)$, $k\geqslant 1$	$2k$	${}^2C^{2k}_{2,2}$
	$C^{+}_{2,2}$, (x_1^2, x_2^2)	$(u_1+u_2^r x_2, u_2+u_1^s x_1)$, $1\leqslant r\leqslant s$	$r+s$	${}^2C^{r,s}_{2,2}$
$\geqslant 3$		$(u_1+u_3x_2, u_2+hx_1)$ $h=u_3+g(u_4, \ldots, u_p)$	ν	$C^{+,Y_{\nu}}_{2,2}$
		$h=u_1^k\pm u_3^2+q$, $k\geqslant 1$	$k+1$	$C^{2k}_{2,2}$
		$h=u_1+u_3^3+q$	3	$C^{3}_{2,2}$
$\geqslant 4$		$h=u_1^k+u_3u_4\pm u_4^l+$ $+\sum_5^p\pm u_j^2$, $k\geqslant 1$, $l\geqslant 3$	$k+l-1$	$C^{l,k}_{2,2}$
		$h=u_1\pm u_3^2+u_4^3+\sum_5^p\pm u_j^2$	4	$C^{3,1,2}_{2,2}$
$\geqslant\mu-1$	$C^{-}_{2,2}$, F_8, $C^{\pm}_{2,l}$, $l>2$	$\sum_1^{\mu-1}u_ie_i(x)+$ $+g(u_{\mu}, \ldots, u_p)e_{\mu}(x)$	ν	$X^{Y_{\nu}}_{\mu}$
	$C_{2,3}$	$(u_2, u_1+u_3x_2+u_4x_2^2+$ $+(u_4^r+q)x_1)$, $r>1$	r	$C^{r}_{2,3}$
	$C_{3,l}$, $l>3$	$\left(u_l, \sum_1^{l-1}u_ix_2^{i-1}+u_{l+1}x_1+\right.$ $\left.+u_{l+2}x_1^2+hx_2^{l-1}\right)$ $h=\pm u_{l+2}+g(u_{\mu}, \ldots, u_p)$	ν	$C^{Y_{\nu}}_{3,l}$
$\geqslant\mu-1$	$C_{3,l}$, $l>3$	$h=\pm u^r_{l+2}+q$, $r>1$	r	$C^{r}_{3,l}$
	F_7, F_9, F_{10}	$\sum_1^{\mu-1}u_ie_i(x)+(u_{\mu-1}+$ $+g(u_{\mu}, \ldots, u_p))e_{\mu}(x)$	ν	$F^{Y_{\nu}}_{\mu}$
	F_{10}	$\sum_1^{\mu-1}u_ie_i(x)+(u_9^r+q)e_{10}(x)$, $1<r<\varkappa>4$	r	F^{r}_{10}
	G_{10}	$\sum_1^{\mu-1}u_ie_i(x)+$ $+(u_9+g(u_{10}, \ldots, u_p))e_{10}(x)$, $e_9=(x_2^3, 0)$, $e_{10}=(0, x_1x_2^2)$	ν	$G^{Y_{\nu}}_{10}$
$\geqslant\mu$	X_{μ}	$\sum_1^{\mu}u_ie_i(x)$	0	X^{V}_{μ}

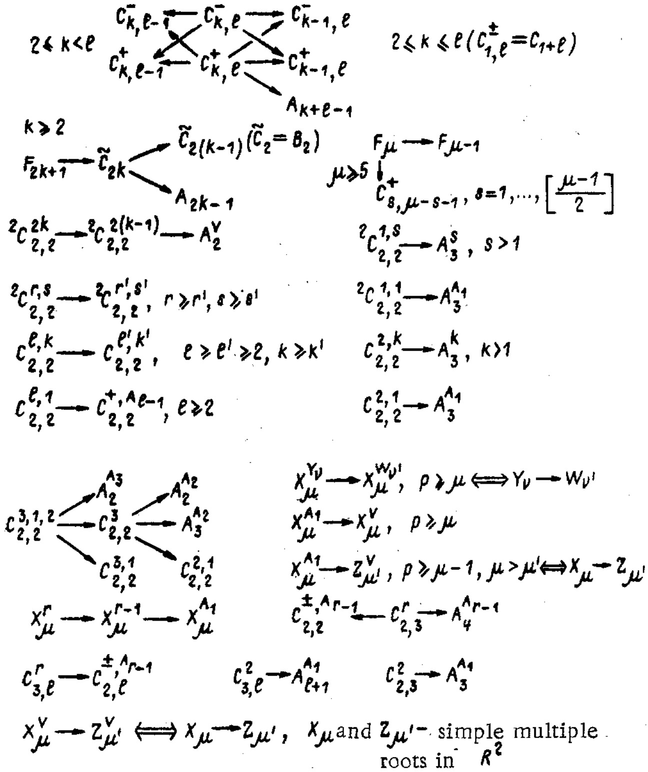

Diagram 9

Diagram 10 shows the position of the projections of Table 9 in the hierarchy of projections relative to simple singularities.

3. We present the codimensions C and C_0 of sets of nonsimple projections "onto," respectively, of nonsingular and not necessarily nonsingular surfaces depending on the values of the dimensional triples (n, m, p):

a) n = m = 1,
$$\frac{p \mid 1 \mid 2 \mid \geqslant 3}{C \mid \infty \mid 3 \mid 2};$$

n = m = 2,
$$\frac{p \mid 1 \mid 2 \mid 3,4 \mid 5-9 \mid \geqslant 10}{C \mid \infty \mid 3 \mid 2 \mid 1 \mid 0};$$

n = m $\geqslant$ 3: the same as for n = m = 2 except for the case p = 9: C(9) = 0;

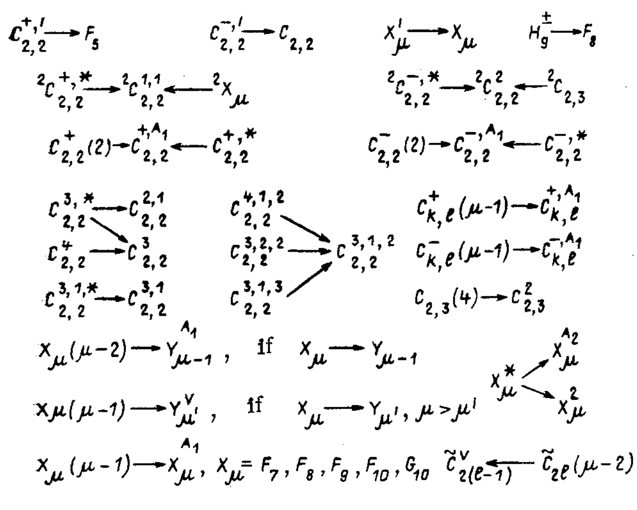

Diagram 10

TABLE 9

p	f_0	$\Delta(x, u)$	Codimension	Notation
1	$C^{\pm}_{2,2}$	$(u^2,\ ux_1+\beta ux_2+\gamma u^2)$	5	$C^{\pm,'}_{2,2}$
	$C^{\pm}_{2,4}$	$(u,\ \beta ux_2)$	6	X'_μ
	$C_{3,3},\ \tilde{C}_6$	$(u,\ \beta ux_1+\gamma ux_2)$	6	
	$H^{\pm}_9$	$(u,\ \beta u)$	8	$H^{\pm}_9$
2	$C^{\pm}_{2,2}$	$\operatorname{rk}(Df/D(x,u))\vert_0=1$	3	${}^2C^{\pm,*}_{2,2}$
	$C_{2,3},\ \tilde{C}_6$	$\operatorname{rk}(Df/D(x,u))\vert_0=2$	$\mu-2$	${}^2X_\mu$
$\geqslant 3$	$C^{\pm}_{2,2}$	$\operatorname{rk}(Df/D(x,u))\vert_0= =\operatorname{rk}\Lambda=2$	p	$C^{\pm}_{2,2}(2)$
		$\operatorname{rk}(Df/D(x,u))\vert_0=1,\ \operatorname{rk}\Lambda=3$	3	$C^{\pm,*}_{2,2}$
	$C^{+}_{2,2}$ $(x_1^2,\ x_2^2)$	$(u_1+u_3x_2,\ u_2+hx_1)$ $h=u_3^3\pm u_1u_3+\beta u_1^2+ +\sum_4^p \pm u_i^2+\dots$	4	$C^{3,*}_{2,2}$
		$h=u_1+\beta u_3^4+ +\sum_4^p \pm u_i^2+\dots$	4	$C^{4}_{2,2}$
$\geqslant 4$		$h=u_3^2+\beta u_3u_4^2\pm u_4^4+ +\sum_5^p \pm u_i^2+\dots$	5	$C^{4,1,2}_{2,2}$
		$h=u_3^3+\beta u_3^2u_4+u_4^3+ +\sum_5^p \pm u_i^2+\dots$	5	$C^{3,1,3}_{2,2}$
		$h=u_3^2+u_4^3\pm u_1u_4+ +\beta u_1^2+\sum_5^p \pm u_i^2+\dots$	5	$C^{3,2,2}_{2,2}$

p	f_0	$\Delta(x, u)$	Codimension	Notation
$\geqslant 5$	$C^{+}_{2,2}$ $(x_1^2,\ x_2^2)$	$h=u_4^3+u_5^3+u_3u_4+ +\beta u_3u_5+\sum_6^p \pm u_i^2+\dots$	4	$C^{3,1*}_{2,2}$
$\mu-2$	$C^{\pm}_{k,l},\ \tilde{C}_{2r},\ F_8,\ H^{\pm}_9,$ $(k,\ l)>(2,\ 3),\ \ r\geqslant 3$	$\operatorname{rk}\Lambda=\mu-2$	2	$X_\mu(\mu-2)$
$\geqslant \mu-2$	$C_{2,3}$		$p-1$	
$\geqslant \mu-1$	$C_{3,3},\ C^{\pm}_{k,l},\ \tilde{C}_{2r},$ $F_{\mu'},\ H^{\pm}_{\mu''},\ 4\leqslant k\leqslant l,$ $r\geqslant 3,\ \mu'\geqslant 11,\ \mu''\geqslant 9$	$\operatorname{rk}\Lambda=\mu-1$	1	$X_\mu(\mu-1)$
	$C^{\pm}_{k,l},\ k=2,\ 3,\ l>3$	$\operatorname{rk}\Lambda=\mu-1,\ \ \Lambda\pitchfork C^{\pm}_{k,l-1}$	2	
	$F_7,\ F_8,\ F_9\ F_{10},\ G_{10}$	$\operatorname{rk}\Lambda=\mu-1,\ \ \Lambda\pitchfork F_{\mu-1}$ $(F_6=C^{+}_{2,4})$	2	
	$C_{2,3}$	$(u_2,\ u_1+u_3x_1+u_4x_2^2+ +h(u)\,x_2),$ $\partial h/\partial u_l\vert_{u=0}=0,\ \ l>2$	3	
$\geqslant \mu$	$C_{3,l},\ l>3$	$\Lambda\pitchfork C^{\pm}_{2,l},\ \ \Lambda\in A_2$	3	X^*_μ
	$C_{2,3}$	$\Lambda\pitchfork C^{\pm}_{2,2},\ \ \Lambda\in A_2$		
	F_{10}	$\sum_1^9 u_ie_i(x)+h(u)\,e_{10}(x),$ $h(0)=\frac{\partial h}{\partial u_j}(0)=0,$ $j>8,\ \ \Lambda\in A_2$		

b) n = m + 1 = 2, $\frac{p \mid 1 \mid 2\text{–}6 \mid 7 \mid \geqslant 8}{C \mid 7 \mid 2 \mid 1 \mid 0}$;

n = m + 1 ⩾ 3 the same if p ≠ 4, 5, 6; otherwise C = 1;

c) n = m + 2 = 3 or n > m + 2 ⩾ 4

$$\frac{p \mid 1 \mid 2\text{–}5 \mid 6 \mid \geqslant 7}{C \mid 6 \mid 2 \mid 1 \mid 0};$$

n = m + 2 ⩾ 4 the same if p ≠ 5, 6; C(5) = 1, C(6) = 0;

d) for any n and m, n ⩾ m,

$$C_0(1)=4 \qquad C_0(p)=C(p) \text{ for } p\geqslant 2.$$

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NEWTON POLYHEDRA (RESOLUTION OF SINGULARITIES)

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Some results are presented on the resolution of singularities and compactification of an algebraic manifold determined by a system of algebraic equations with fixed Newton polyhedra and rather general coefficients. Resolution and compactification are carried out by means of smooth toric manifolds which are described in the first half of the survey.

The Newton polyhedron of a polynomial depending on several variables is the convex hull of the exponents of the monomials contained in the polynomial with nonzero coefficients. The Newton polyhedron generalizes the concept of degree and plays an analogous role. Discrete characteristics of a joint level line of several polynomials in multidimensional complex space are the same for almost all values of the coefficients and are computed in terms of Newton polyhedra. Among the discrete characteristics computed are the number of solutions of a system of n equations in n unknowns, the Euler characteristic, the arithmetic and geometric genus of full intersections, and the Hodge numbers of a mixed Hodge structure on the cohomology of full intersections.

A Newton polyhedron is defined not only for polynomials but also for germs of analytic functions. For germs of analytic functions of general position with given Newton polyhedra the multiplicity of a joint solution of a system of analytic equations, the Milnor number and zeta function of the monodromy operator, the asymptotics of oscillating integrals, and the Hodge numbers of a mixed Hodge structure on vanishing cohomology are computed; in the two-dimensional case and the multidimensional quasihomogeneous case the modality of a germ of a function is computed.

In the answers quantities characterizing both the sizes of the polygons (the volume and the number of integer points contained inside the polygon) and their combinatorics (the number of faces of different dimensions and the numerical characteristics of their abutments) are encountered. These and other results connected with Newton polyhedra can be found in the works [1-9, 11-16, 18-24, 26-28].

A large part of the computations with Newton polyhedra is carried out by means of toric manifolds. "Elementary" computations in which it is possible to get by without their help are are most often exceptional. The basic step in applying toric manifolds consists in the explicit construction of a resolution of singularities and subsequent nonsingular compactification of the joint level line of several polynomials having sufficient general coefficients and fixed Newton polyhedra. The present paper is devoted to toric manifolds from the point of view of their applications to the resolution of singularities and compactification.

In the first half of the paper we present a detailed construction of smooth toric manifolds. Usually the description of these manifolds is presented in terms of spectra of rings which are common in algebraic geometry but are little suited for specialists in mathematical analysis. In our exposition the entire algebraic apparatus is reduced to linear algebra and to the simplest properties of integral lattices.

The second half of the paper is devoted to theorems on compactification and resolution of singularities. In the first of these (part 2.4) a nonsingular compactification of a joint

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