To Vitya, with my great respect

Logarithmic vector fields for the discriminants of composite functions

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Abstract

The $\mathcal{K}_f$-equivalence is a natural equivalence between map-germs $\varphi : \mathbf{C}^m \to \mathbf{C}^n$ which ensures that their compositions $f \circ \varphi$ with a fixed function-germ f on $\mathbf{C}^n$ are the same up to biholomorphisms of $\mathbf{C}^m$. We show that the discriminant Σ in the base of a $\mathcal{K}_f$-versal deformation of a germ φ is Saito's free divisor provided the critical locus of f is Cohen-Macaulay of codimension $m+1$ and all the transversal types of f are A_k singularities. We give an algorithm to construct basic vector fields tangent to Σ. This is a generalisation of classical Zakalyukin's algorithm [14, 15] to write out basic fields tangent to the discriminant of an isolated function singularity. The case of symmetric matrix families in two variables is done in detail. For simple singularities, it is directly related to Arnold's convolution of invariants of Weyl groups [1].

A *logarithmic* vector field for a hypersurface $V \subset \mathbf{C}^k$ is a derivation that multiplies an equation of V by a holomorphic function. This is a generalisation of tangency to a smooth hypersurface. Such fields are dual to 1-forms logarithmic along V, hence the name. We denote the set of all logarithmic fields for V by $Der(-log\, V)$.

A hypersurface V is a *free divisor* in the sense of Saito if $Der(-log\, V)$ is a free module over functions on $\mathbf{C}^k$. In this case the module $Der(-log\, V)$ is generated by k vector fields and the determinant of the components of these fields in any coordinates gives an equation of V.

The first examples of what is now called free divisors was discovered by Arnold and Zakalyukin [1, 14, 15] in their study of evolution of wave fronts. These were discriminants of finite reflection groups and of isolated function singularities. Later on it was shown that some other natural equivalences of holomorphic maps and functions yield bifurcation diagrams which are also free divisors (see, for example, [2]). In this paper we demonstrate that discriminants of composite functions, under the dimensional assumptions that provide nice deformational properties, are free divisors too.

The paper is structured as follows.

Section 1 recalls the definition of the $\mathcal{K}_f$-equivalence which is a natural equivalence of compositions of maps $\varphi : \mathbf{C}^m \to \mathbf{C}^n$ with a fixed function f on $\mathbf{C}^n$.

In Section 2, we prove our main result that the discriminant Σ in the base of a $\mathcal{K}_f$-versal deformation is a free divisor provided the critical locus of f is Cohen-Macaulay of codimension $m+1$ in $\mathbf{C}^n$ and all the transversal singularity types of f are A_k. Here we also give an algorithm to construct basic vector fields tangent to Σ in a way that generalises Zakalyukin's algorithm for discriminants of isolated function singularities.

In Section 3, we analyse what our constructive algorithm becomes in the case of symmetric-matrix-valued functions of two variables. For simple matrix singularities in this situation we consider an alternative algorithm similar to Arnold's convolution of invariants of Weyl groups [1]. In Section 3.2, we prove a version of the Splitting Lemma for composite functions. This, for example, explains why certain parts of classifications of matrix singularities in [3] and [4] in a sense coincide.

Finally, in Section 4, we formulate a Conjecture on multiplication of vector fields on the base of a $\mathcal{K}_f$-miniversal deformation induced from the multiplication on the base of an $\mathcal{R}$-miniversal deformation of the isolated function singularity.

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1 Equivalence of compositions

Consider a fixed function-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and various map-germs $\varphi : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ into its domain. We would like to consider maps φ up to an equivalence which preserves the singularity type of the composition $f \circ \varphi$. First of all, this means that we allow changes of coordinates on the source $\mathbf{C}^m$. On the other hand, we should allow the image $\varphi(x)$ of a point $x \in \mathbf{C}^m$ move within the same level set of the function f. The most universal way to do this is to consider families $\{h_x\}$ of diffeomorphisms of $\mathbf{C}^n$ preserving all the levels of f and numbered by the parameter $x \in \mathbf{C}^m$, with the action on φ by the rule $\varphi(x) \mapsto h_x(\varphi(x))$. The equivalence of maps φ obtained by joining these two parts together is called the *$\mathcal{K}_f$-equivalence* (see [8]).

The (extended) tangent space to the $\mathcal{K}_f$-orbit of φ is

$$T\mathcal{K}_f\varphi = t\varphi(\theta_m) + \varphi^*(Der(-log\, f))\,.$$

Here θ_m is the space of all holomorphic vector field-germs on $(\mathbf{C}^m, 0)$ and $Der(-log\, f) \subset \theta_n$ are the fields annihilating f. The pullback of the latter by φ^* involves taking the module over functions on $\mathbf{C}^m$.

The codimension of $T\mathcal{K}_f\varphi$ in the space $\mathcal{O}_m^n$ of all variations of φ is denoted $\tau_{\mathcal{K}_f}(\varphi)$ and called the *Tjurina number of* φ. Since $\mathcal{K}_f$ is Damon's geometric subgroup of the contact equivalence group, all the standard theorems of singularity theory apply [6]: finite determinacy, infinitesimal criterion for versality, etc. For example, $\mathcal{O}_m^n/T\mathcal{K}_f$ can be considered as the base of a $\mathcal{K}_f$-miniversal deformation of a germ φ.

2 Vector fields tangent to the discriminant of a composition

2.1 The $\mathcal{K}_f$ discriminant

For the rest of the paper, except for Section 3.2, we shall assume that the critical locus of f is Cohen-Macaulay (that is the quotient $\mathcal{O}_n/J_f$ by the jacobian ideal is such) and has codimension $m+1$ in $\mathbf{C}^n$. According to [8], this guarantees nice properties for the maps φ. In particular, the Tjurina number of φ coincides with the Milnor number of the composition:

Theorem 2.1 [8] $\tau_{\mathcal{K}_f}(\varphi) = \mu(f \circ \varphi)$.

Let $\Phi = \Phi(x, \lambda)$ be a $\mathcal{K}_f$-miniversal deformation of a germ φ, $\lambda \in \mathbf{C}^\tau$. The base of this deformation contains the *discriminant* hypersurface $\Sigma = \Sigma_1 \cup \Sigma_2 \subset \mathbf{C}^\tau$: component Σ_1 corresponds to the non-transversality of the perturbations $\Phi(-, \lambda)$ to the regular part of $f = 0$, and Σ_2 to the images of the perturbations meeting the critical locus of f. These are the only two ways to get the levels $f \circ \Phi(-, \lambda) = 0$ singular.

Theorem 2.2 *Assume all transversal types of f are A_k singularities. Then the discriminant $\Sigma \subset \mathbf{C}^\tau$ of a map-germ φ is Saito's free divisor.*

Proof. Let $F = F(X, \Lambda)$ be an $\mathcal{R}$-miniversal deformation of the function $(f \circ \varphi)(X)$, $\Lambda \in \mathbf{C}^{\mu=\tau}$. Denote by $\Delta \in \mathbf{C}^\mu$ the ordinary discriminant of $f \circ \varphi$.

The deformation $f \circ \Phi$ is induced from F, that is

$$f(\Phi(x, \lambda)) = F(X(x, \lambda), \Lambda(\lambda))$$

for suitable parameter substitution $\Lambda = \Lambda(\lambda)$ and family $X = X(x, \lambda)$ of diffeomorphisms of $\mathbf{C}^m$ depending on λ, $X(x, 0) = x$. The inverse image of Δ under the base map $\Lambda : \mathbf{C}^\tau \to \mathbf{C}^\mu$ is the discriminant Σ.

Notice that in order to distinguish between the bases of the two different deformations we shall be writing $\mathbf{C}^\tau$ and $\mathbf{C}^\mu$ in spite of the dimensions being equal.

Theorem 2.3 [7] *The critical locus of the inducing map $\lambda \mapsto \Lambda(\lambda)$ is Σ_2.*

This theorem is valid without any constraints on the transversal types of the function f.

The set Σ_2 splits into hypersurfaces corresponding to the images of the perturbations $\Phi(-, \lambda)$ meeting various components of the critical locus of f.

Lemma 2.4 *Assume the transversal type of f along certain component of its critical locus is A_k. Then the branching order of the map Λ along the corresponding component of Σ_2 is $k + 1$.*

Proof. Take a generic point on that component of the critical locus of f. In local coordinates about this point, $f = y_0^{k+1} + y_1^2 + \ldots + y_m^2$. A generic non-transversality of a map $\mathbf{C}^m \to \mathbf{C}^n$ to the component has normal form $(y_0, y_1, \ldots, y_m, y_{m+1}, \ldots, y_{n-1}) = (0, x_1, \ldots, x_m, 0, \ldots, 0)$. A $\mathcal{K}_f$-miniversal deformation Φ of this map replaces 0 by $\lambda \in \mathbf{C}$ in the y_0-component. Hence $f \circ \Phi = \lambda^{k+1} + x_1^2 + \ldots + x_m^2$. This is induced from the $\mathcal{R}$-miniversal deformation of the A_1 singularity by the map $\Lambda = \lambda^{k+1}$. □

From now on we shall assume that all the transversal types of f, along all components of its critical locus, are A_k's (an individual k for an individual component).

Lemma 2.5 *A vector field on $\mathbf{C}^\mu$ is liftable to $\mathbf{C}^\tau$ if and only if it is tangent to Δ. The result of the lifting is tangent to $\Sigma \subset \mathbf{C}^\tau$.*

Proof. A mapping of branching order $k + 1$, near its generic critical point, is a cylinder over the map $z = w^{k+1}$ which lifts $z\partial_z$ to $\frac{1}{k+1} w\partial_w$. Application of Hartog's theorem finishes the proof. □

We can now finish the proof of Theorem 2.2 too. Namely, we know that the divisor $\Delta \subset \mathbf{C}^\mu$ is free. Take a basis of logarithmic vector fields for Δ. Lift them against the map Λ to $\mathbf{C}^\tau$. From the proof of the last Lemma, the determinant of the components (in any chosen coordinates on $\mathbf{C}^\tau$) of the lifted fields has an order 1 zero at a generic point of Σ. □

Corollary 2.6 *The map Λ lifts an $\mathcal{O}_\mu$-basis of $Der(-log\,\Delta)$ to an $\mathcal{O}_\tau$-basis of $Der(-log\,\Sigma)$.*

Example. This is not true for other transversality types of f. For example, take $f(y) = y_1(y_1^2 - y_2^2) \in D_4$ and $\Phi(x) = (x, \alpha x + \beta)$. Then $\Delta = \Delta(A_2)$ and $\Sigma = \Sigma_2 = \{\beta = 0\}$. Generators of $Der(-log\,\Delta)$ lift to $\beta\partial_\alpha$ and $\beta\partial_\beta$. The divisor of the determinant of the components of these fields is 2Σ. The desired multiplicity 1 of Σ got increased by the modality of the singularity which has α as a modulus.

2.2 Basic fields

We would now like to derive an algorithm to construct a basis of $Der(-log\,\Sigma)$.

A vector field $V(\Lambda) = \sum_j V_j(\Lambda)\partial_{\Lambda_j}$ on $\mathbf{C}^\mu$ whose components in certain coordinates $\Lambda = (\Lambda_1, \ldots, \Lambda_\mu)$ are read from the decomposition

$$\psi(X)F(X,\Lambda) = \sum_{k=1}^{m} F_{X_k}(X,\Lambda)U_k(X,\Lambda) + \sum_{j=1}^{\mu} F_{\Lambda_j}(X,\Lambda)V_j(\Lambda) \qquad (1)$$

is tangent to the discriminant Δ. Here ψ is a holomorphic function-germ on $\mathbf{C}^m$, $X = (X_1, \ldots, X_m)$, $F_{X_k} = \partial F/\partial X_k$, and the U_k and V_j are holomorphic functions.

According to the classical result by Zakalyukin [14, 15], we obtain an $\mathcal{O}_\mu$-basis of $Der(-log\,\Delta)$ by running ψ in (1) through a linear basis of the local ring $\mathcal{O}_m/J_{f\circ\varphi}$. By Corollary 2.6, the basis of $Der(-log\,\Delta)$ lifts to that of $Der(-log\,\Sigma)$. However, in concrete examples, the map $\Lambda : \mathbf{C}^\tau \to \mathbf{C}^\mu$ between the deformation bases may be rather complicated. So, let us try to avoid it and find out what decomposition (1) corresponds to in terms of the deformation Φ itself.

To shorten our formulae, we rewrite (1) identifying the fields V and $U = \sum_k U_k(X)\partial_{X_k}$ with the columns of their components:

$$\psi(X)F(X,\Lambda) = F_X(X,\Lambda)U(X,\Lambda) + F_\Lambda(X,\Lambda)V(\Lambda)\,.$$

Pull this back to the variables (x,λ) using the inducing substitution $(X,\Lambda) = (X(x,\lambda), \Lambda(\lambda))$:

$$\psi(X(x,\lambda))F(X(x,\lambda),\Lambda(\lambda)) = F_x\, X_x^{-1}\, U(X(x,\lambda),\Lambda(\lambda)) + (F_\lambda - F_x\, X_x^{-1}\, X_\lambda)\, \Lambda_\lambda^{-1}V(\Lambda(\lambda)) =$$

$$= F_x\, X_x^{-1}\left(U - X_\lambda\, \Lambda_\lambda^{-1}V(\Lambda(\lambda))\right) + F_\lambda\, \Lambda_\lambda^{-1}V(\Lambda(\lambda))\,. \qquad (2)$$

Here we kept the arguments only where it was necessary to emphasise them. The liftability of V means that the column $\Lambda_\lambda^{-1}V(\Lambda(\lambda))$ is holomorphic. We denote this column (or actually vector field on $\mathbf{C}^\tau$) by $v(\lambda)$. Since $X(-,\lambda)$ is a diffeomorphism for any fixed λ, we see that the column $X_x^{-1}\left(U - X_\lambda\, \Lambda_\lambda^{-1}V(\Lambda(\lambda))\right)$ is also holomorphic. We denote it by $u(x,\lambda)$.

Let y be coordinates on $\mathbf{C}^n$. Since F pulled back to the (x,λ)-variables is $f\circ\Phi$, we have $F_x = f_y\Phi_x$ and $F_\lambda = f_y\Phi_\lambda$. Putting these into (2), we get:

$$\psi(X(x,\lambda))f(\Phi(x,\lambda)) = f_y\Phi_x u(x,\lambda) + f_y\Phi_\lambda v(\lambda)\,. \qquad (3)$$

Now assume f is quasihomogeneous: $f = f_yE$ for certain vector field-column $E = E(y)$ on $\mathbf{C}^n$. Then

$$\psi(X(x,\lambda))f_yE(\Phi(x,\lambda)) = f_y\left(\Phi_x u(x,\lambda) + \Phi_\lambda v(\lambda)\right)\,. \qquad (4)$$

Removing f_y we obtain:

$$\psi(X(x,\lambda))E(\Phi(x,\lambda)) = \Phi_x u(x,\lambda) + \Phi_\lambda v(\lambda) + R\,. \qquad (5)$$

Here $R \in \Phi^*(Der(-log\, f))$. The latter follows from the fact that a free resolution of the local algebra $\mathcal{O}_n/J_f$ of f pulled back by Φ has only zero-dimensional homology while the d_1-image in it is exactly $\Phi^*(Der(-log\, f))$ (see the proof of Theorem 1.2 in [8]).

Assume ψ in (5) runs through a basis $\psi_1, \dots, \psi_\mu \in \mathcal{O}_m$ of the local algebra of $f \circ \varphi$. The resulting $\tau = \mu$ decompositions give us a basis of $Der(-log\, \Sigma)$. However, since $X(x,0) = x$, according to Nakayama's lemma, we can actually slightly simplify the decompositions taking in them $\psi_i(x)$ instead of $\psi_i(X(x,\lambda))$.

Hence we have obtained

Theorem 2.7 *Assume a function-germ $f : \mathbf{C}^n \to \mathbf{C}$ is quasihomogeneous and $E \in \theta_n$ is an Euler vector field preserving f. Let $\Phi = \Phi(x,\lambda)$ be a $\mathcal{K}_f$-miniversal deformation of a map-germ $\varphi : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$, $x \in C^m$, $\lambda \in \mathbf{C}^\tau$. Choose a linear basis $\psi_1(x), \dots, \psi_\tau(x)$ of the local ring of the composition $f \circ \varphi$ and fix some coordinates $(x,\lambda) = (x_1, \dots, x_m, \lambda_1, \dots, \lambda_\tau)$. Then there exist decompositions*

$$\psi_i \Phi^*(E) \equiv \sum_{k=1}^{m} \Phi_{x_k} u_{ik} + \sum_{j=1}^{\tau} \Phi_{\lambda_j} v_{ij} \mod \Phi^*(Der(-log\, f))$$

in which the $u_{ik} = u_{ik}(x,\lambda)$ and $v_{ij} = v_{ij}(\lambda)$ are holomorphic functions. The holomorphic vector fields $v_i = \sum_{j=1}^{\tau} v_{ij}\partial_{\lambda_j}$, form an $\mathcal{O}_\tau$-basis of $Der(-log\, \Sigma)$.

In fact the existence of the decompositions follows already from the $\mathcal{K}_f$-versality of Φ.

Notice that we are back to Zakalyukin's algorithm to obtain basic vector fields tangent to the discriminant of an isolated function singularity g if, for example, we take $n = m+1$, $f = y_{m+1}$ and an embedding of the graph $y_{m+1} = g(y_1, \dots, y_m)$ as φ.

Remark. If there is another field $E' \in \theta_n$ preserving f then using it in the decompositions of the Theorem instead of E will not affect the fields v_j at all since $E - E' \in Der(-log\, f)$.

Example. Take $f(y_1, y_2) = y_1^3 + y_2^2$ and $\varphi : \mathbf{C} \to \mathbf{C}^2$, $x \mapsto (y_1, y_2) = (x, 0)$. Then $\Phi = (x, \alpha x + \beta)$, and the discriminant components are $\Sigma_1 = \{4\alpha^3 = 27\beta\}$ and $\Sigma_2 = \{\beta = 0\}$. To avoid fractions, it is better to take the field $E = 2y_1\partial_{y_1} + 3y_2\partial_{y_2}$ which multiplies f by 6. The $\mathcal{O}_2$-module $Der(-log\, f)$ has just one generator $2y_2\partial_{y_1} - 3y_1^2\partial_{y_2}$. The decompositions of the Theorem are

$$\psi_i(x) \begin{pmatrix} 2x \\ 3(\alpha x + \beta) \end{pmatrix} =$$

$$= u_i(x,\alpha,\beta) \begin{pmatrix} 1 \\ \alpha \end{pmatrix} + v_{i1}(\alpha,\beta) \begin{pmatrix} 0 \\ x \end{pmatrix} + v_{i2}(\alpha,\beta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + w_i(\alpha,\beta) \begin{pmatrix} 2(\alpha x + \beta) \\ -3x^2 \end{pmatrix} .$$

For a basis of the local ring of $f \circ \varphi = x^3$, this gives:

$$\psi_1 = 1 : \quad v_1 = \alpha\partial_\alpha + 3\beta\partial_\beta\,, \quad u_1 = 2x\,, \quad w_1 = 0\,;$$

$$\psi_2 = 3x : \quad v_2 = (9\beta - 2\alpha^3)\partial_\alpha - 2\alpha^2\beta\partial_\beta\,, \quad u_2 = 6x^2 + 2\alpha(\alpha x + \beta)\,, \quad w_2 = -\alpha\,.$$

Of course, the first field is just the Euler field on the parameter space.

3 Symmetric matrices

The main motivation for the results of the previous section was to understand the situation with matrix singularities [3, 4, 5, 9, 12]. Let us see what Theorem 2.7 gives, for example, for families of symmetric matrices.

So, we take for $\mathbf{C}^n$ the space $Sym_r \simeq \mathbf{C}^{r(r+1)/2}$ of symmetric matrices and for f the determinantal function det on it. The critical locus of this function has codimension 3. Therefore, we are considering for φ 2-parameter symmetric matrix families $S_0 : (\mathbf{C}^2, 0) \to Sym_r$. The transversal type of the function det is A_1.

The $\mathcal{K}_{det}$*-equivalence* of such families is that up to diffeomorphisms of the domain and up to transformations $S_0 \sim M^t S_0 M$, where $M : (\mathbf{C}^2, 0) \to SL_r$.

Infinitesimally this gives:

$$T\mathcal{K}_{det} S_0 = \{A^t S_0 + S_0 A + \sum_k S_{0,x_k} h_k\}$$

where $A = A(x)$ is an arbitrary family of traceless matrices, the S_{0,x_k} are the partial derivatives, $h_k = h_k(x)$ and $x = (x_1, x_2)$.

Let $S = S(x, \lambda)$ be a $\mathcal{K}_{det}$-miniversal deformation of S_0, $\lambda \in \mathbf{C}^\tau$, $\tau = \mu(det\, S_0)$. In the matrix discriminant $\Sigma = \Sigma_1 \cup \Sigma_2 \subset \mathbf{C}^\tau$, the component Σ_i corresponds to non-transversality of the perturbations of S_0 to the set of all matrices of corank i.

The results of the previous section yield

Corollary 3.1 *The matrix discriminant $\Sigma \subset \mathbf{C}^\tau$ is a free divisor. The components of the basic vector fields $v_i = \sum_j v_{ij}(\lambda)\partial_{\lambda_j}$ can be found from the decompositions*

$$\psi_i S = A_i^t S + S A_i + \sum_{k=1,2} S_{x_k} u_{ik} + \sum_{j=1}^{\tau} S_{\lambda_j} v_{ij}\,, \qquad (6)$$

where the $\psi_i = \psi_i(x)$, $i = 1, \ldots, \mu$, form a linear basis of $\mathcal{O}_2/J_{\det S_0}$, the $A_i = A_i(x, \lambda)$ are families of traceless matrices, and $u_{ik} = u_{ik}(x, \lambda)$.

If the corank of the matrix $S_0(0)$ is 1, the family S_0 is $\mathcal{K}_{det}$-equivalent to a diagonal family $diag\{g(x), 1, \ldots, 1\}$ and we are again back to Zakalyukin's algorithm for basic fields tangent to the functional discriminants.

Proof of the Corollary. For $f = det$, there is an obvious choice $E = \frac{1}{r}\sum_{p\le q} \mathcal{S}_{pq}\partial_{\mathcal{S}_{pq}}$ of the Euler field of Theorem 2.7 when we assign the same weight $1/r$ to each of the entries $\mathcal{S}_{pq}$ of a generic symmetric matrix $\mathcal{S}$. The S^*-pullback of this field is $\frac{1}{r}S$. □

Example. Assume S quasihomogeneous, that is its entries S_{pq} are quasi-homogeneous functions of degrees d_{pq}, such that $d_{pq} + d_{st} = d_{pt} + d_{sq}$ for all p, q, s, t. Hence $d_{pq} = c_p + c_q$, with $c_p = d_{pp}/2$. Let C be the matrix $diag\{c_1, \ldots, c_r\}$, $c = tr(C) = deg(\det S)/2$, and w_{x_k} and w_{λ_j} the weights of the variables.

Clearly, the Euler field $e = \sum_j w_{\lambda_j}\lambda_j\partial_{\lambda_j}$ is tangent to the discriminant $\Sigma \subset \mathbf{C}^\tau$. Let us check which decomposition of type (6) this field is coming from.

The derivative of S in the direction of the Euler lift $\tilde{e} = \sum w_{x_k} x_k \partial_{x_k} + \sum_j w_{\lambda_j}\lambda_j\partial_{\lambda_j}$ of e to $\mathbf{C}^{2+\tau}$ satisfies the decomposition

$$\tilde{e}(S) = \sum_k S_{x_k} w_{x_k} x_k + \sum_j S_{\lambda_j} w_{\lambda_j}\lambda_j = (d_{pq} S_{pq}) = CS + SC = (C - \frac{c}{r} I_r)S + S(C - \frac{c}{r} I_r) + \frac{2c}{r} S\,.$$

Hence the decomposition of the Corollary that gives the Euler field e is

$$-\frac{2c}{r}S = (C - \frac{c}{r}I_r)S + S(C - \frac{c}{r}I_r) + \sum_k S_{x_k} w_{x_k} x_k + \sum_j S_{\lambda_j} w_{\lambda_j} \lambda_j \,.$$

3.1 Simple symmetric matrices in two variables

In the case of simple function singularities, in addition to Zakalyukin's method to find basic vector fields tangent to the discriminant there is also Arnold's method [1]. This method is based on the identification of the base of an $\mathcal{R}$-miniversal deformation of a simple function with the orbit space of the corresponding Weyl group. The identification sends the discriminant to the set of irregular orbits. Choosing coordinates $\sigma_1, \ldots, \sigma_\mu$ on the orbit space that is, a set of basic invariants of the group, one can take

$$\sum_{j=1}^{\mu} (\sigma_{i,z}, \sigma_{j,z}) \partial_{\sigma_j} \,, \qquad i = 1, \ldots, \mu,$$

for a basis of the $\mathcal{O}_\mu$-module of the logarithmic fields. Here $\sigma_{i,z}$ is the gradient of σ_i with respect to some coordinates z on the configuration space of the group and the brackets denote the standard dot product.

For simple symmetric matrices in two variables, there is an opportunity to obtain a similar description of basic logarithmic fields since there also exists an interpretation of the base of a $\mathcal{K}_{det}$-miniversal deformation as a quotient space of a reflection group. Namely, according to [9] simple singularities $S_0 : (\mathbf{C}^2, 0) \to Sym_r$, $\operatorname{cork} S_0(0) > 1$, are classified by pairs (G, H) formed by a Weyl group $G = A_\mu, D_\mu, E_\mu$ and its reflection subgroup Y satisfying certain lattice conditions. The function $det\, S_0$ has a simple isolated singularity traditionally denoted G as well. Denoting by $W \subset \mathbf{C}^\mu$ the mirror set of group G in its complex configuration space, we have factorisation isomorphisms

$$(\mathbf{C}^\mu, W)/H \simeq (\mathbf{C}^\mu, \Sigma) \qquad \text{and} \qquad (\mathbf{C}^\mu, W)/G \simeq (\mathbf{C}^\mu, \Delta) \,.$$

The inducing map Λ we used in Section 2 is the natural map from the intermediate quotient $(\mathbf{C}^\mu, W)/H$ to the final quotient $(\mathbf{C}^\mu, W)/G$.

We keep the notation σ_i for the coordinates on $(\mathbf{C}^\mu, \Delta)$ and introduce coordinates $\rho_1, \ldots,$ ρ_μ on $(\mathbf{C}^\mu, \Sigma)$. From Arnold's result and the factorisation relation between the orbit spaces we immediately obtain

Theorem 3.2 *The vector fields*

$$\sum_{j=1}^{\mu} (\sigma_{i,z}, \rho_{j,z}) \partial_{\rho_j} \,, \qquad i = 1, \ldots, \mu,$$

form an $\mathcal{O}_\rho$-basis of $Der(-log\, \Sigma)$.

Indeed, these are the images of the G-invariant fields $\sigma_{i,z}$ under the factorisation map $\mathbf{C}^\mu_z \xrightarrow{/H} \mathbf{C}^\mu_\rho$.

3.2 Stabilisation of composite functions

Once we have touched classification of simple symmetric matrix singularities, it should also be explained why some of them in a sense appear as simple singularities of square matrices. Namely, the list of all 3-parameter simple singularities of arbitrary square matrices [4] can be obtained from the complete classification of simple 2×2 symmetric matrices in 2 variables [3] by the move

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \longrightarrow \begin{pmatrix} a & b+x_3 \\ b-x_3 & c \end{pmatrix}, \qquad a,b,c \in \mathcal{O}_2\,,$$

which has already been used in [9]. The reason turns out to be that passing from the determinantal function on the space $\mathbf{C}^3$ of 2×2 symmetric matrix to that on the space $\mathbf{C}^4$ of arbitrary order 2 square matrices we are adding the square of a new variable while a map to $\mathbf{C}^3$ is being extended to the map to $\mathbf{C}^4$ as a trivial one-parameter unfolding.

Here is a general approach. In it, we are not imposing any constraints on the critical locus of f and on the dimensions m and n.

Splitting Lemma for composite functions *Consider two germs, $f : (\mathbf{C}^n,0) \to (\mathbf{C},0)$ and $\varphi : (\mathbf{C}^m,0) \to (\mathbf{C}^n,0)$. Assume that f is critical at the origin and that the rank of the 2-jet of the composition $f \circ \varphi$ at $0 \in \mathbf{C}^m$ is r. Then one can choose coordinates $(x_1,\ldots,x_m) \in \mathbf{C}^m$ and $(y_1,\ldots,y_n) \in \mathbf{C}^n$ in which*

$$f(y) = y_1^2 + \ldots + y_r^2 + \widehat{f}(y_{r+1},\ldots,y_n)$$

and φ is $\mathcal{K}_f$-equivalent to the mapping

$$y_1 = x_1, \ \ldots\ , \ y_r = x_r, \quad (y_{r+1},\ldots,y_n) = \widehat{\varphi}(x_{r+1},\ldots,x_m)$$

such that the composition $\widehat{f} \circ \widehat{\varphi}$ has the trivial 2-jet at the origin.

The Lemma reduces the $\mathcal{K}_f$-classification of the maps φ satisfying its condition to the $\mathcal{K}_{\widehat{f}}$-classification of the maps $\widehat{\varphi}$. It also allows to have for compositions an analog of the notion of stabilisation of functions which is so useful in the $\mathcal{R}$-equivalence: we can call φ a *stabilisation* of $\widehat{\varphi}$, even without requesting $j_0^2(\widehat{f} \circ \widehat{\varphi}) = 0$.

Proof. Let $\mathcal{T} \subset \mathbf{C}^m$ be a germ of an r-dimensional submanifold transversal to the kernel of $j_0^2(f \circ \varphi)$. The map φ embeds $\mathcal{T}$ into $\mathbf{C}^n$ so that the restriction of f to $\varphi(\mathcal{T})$ is a Morse function. Choose coordinates $y_1,\ldots,y_r$ on $\varphi(\mathcal{T})$ in which this restriction is $y_1^2 + \ldots + y_r^2$. According to the standard Morse Lemma with parameters, we can extend them to a coordinate system $y_1,\ldots,y_n$ in $\mathbf{C}^n$ so that $f(y) = y_1^2 + \ldots + y_r^2 + \widehat{f}(y_{r+1},\ldots,y_n)$ and the equations of $\varphi(\mathcal{T})$ are $y_{>r} = 0$.

Now introduce the coordinates $x_i = \varphi^*(y_i)$, $i = 1,\ldots,r$, on $\mathcal{T}$. Consider the map φ as an $(m-r)$-parameter deformation of its restriction to $\mathcal{T}$, that is of the map $\varphi_0 : (\mathbf{C}^r,0) \to (\mathbf{C}^n,0)$ given by the formulas $y_i = x_i$ if $i \le r$ and $y_i = 0$ if $i > r$.

Lemma 3.3 *The $(n-r)$-parameter deformation*

$$y_i = x_i \ \text{ if } \ i \le r \quad \text{and} \quad y_i = \lambda_{i-r} \ \text{ if } \ i > r$$

of φ_0 is $\mathcal{K}_f$-versal.

Proof. Any variation of the first r coordinate functions of φ_0 can be obtained by the change of the coordinates on the source $\mathbf{C}^r$. Any variation of the last $n-r$ coordinate functions of φ_0 which does not move the image of the origin is contained in the $\mathcal{O}_r$-module generated by the elements $\varphi_0^*(\widehat{f}_{y_\ell}\partial_{y_k} - 2y_k\partial_{y_\ell}) = -2x_k\partial_{y_\ell}$, $k \le r$, $\ell > r$. □

The deformation in the Lemma is not necessarily miniversal. However, any other deformation of φ_0 is still induced from it. This finishes the proof of our Splitting Lemma: the map $\widehat{\varphi}$ plays the rôle of the inducing map between the bases of the deformations. □

4 Multiplication on the tangent sheaf of the base of a $\mathcal{K}_f$-miniversal deformation

We return to our main case when the fixed function f has a Cohen-Macaulay critical locus of codimension $m+1$.

According to [10], one can define a multiplication on the $\mathcal{O}_\tau$-module θ_τ of vector fields on the base of a $\mathcal{K}_f$-miniversal deformation $\Phi = \Phi(x,\lambda)$ of a map-germ φ. Namely, let $\mathcal{C}$ be the union of all critical points of all the functions $f(\Phi(-,\lambda))$:

$$\mathcal{C} = \{\partial(f(\Phi(x,\lambda))/\partial x_k = 0\,,\ k=1,\dots,m\} \subset \mathbf{C}^m \times \mathbf{C}^\tau\,.$$

Associate to the coordinate field ∂_{λ_i} the partial derivative $p_i = \partial(f\circ\Phi)/\partial\lambda_i$ restricted to $\mathcal{C}$. As it has been shown in [10], product of any two such derivatives belongs to the $\mathcal{O}_\tau$-module spanned by these derivatives:

$$p_i p_j = \sum_{k=1}^{\tau} c_{ij}^k(\lambda) p_k\,. \tag{7}$$

This allows to set $\partial_{\lambda_i}\partial_{\lambda_j} = \sum_{k=1}^{\tau} c_{ij}^k(\lambda)\partial_{\lambda_k}$.

In fact, this multiplication is induced by the mapping $\Lambda : \mathbf{C}^\tau \to \mathbf{C}^\mu$ of Section 2 from the multiplication of vector fields on the base $\mathbf{C}^\mu$ of an $\mathcal{R}$-miniversal deformation of $f\circ\varphi$, that is from the multiplication which appears in the Frobenius structure on the base of a miniversal deformation of an isolated function singularity (see [13]). Namely, in order to obtain the value at some point $\lambda_0 \in \mathbf{C}^\tau$ of the product of two elements $u, v \in \theta_\tau$, we send the vectors $u(\lambda_0)$ and $v(\lambda_0)$ to $T_{\Lambda(\lambda_0)}\mathbf{C}^\mu$, multiply them in this tangent space and lift the result back to $T_{\lambda_0}\mathbf{C}^\tau$. This works well off the critical locus Σ_2 of Λ.

Via exactly this inducing approach, the F-manifold structure has been introduced on $\mathbf{C}^\tau \setminus \Sigma_2$ in [7]. However, the existence of the product decompositions (7) demonstrates that the multiplication holomorphically extends to the entire base $\mathbf{C}^\tau$. The integrability of the multiplication survives this extension since the underlying Lagrangian manifold is stable (see [10]). But there is no unity for the extended multiplication. This is readily demonstrated by the simplest codimension 1 composition used in the Proof of Lemma 2.4. For it, $p = (k+1)\lambda^k$ and hence $p^2 = (k+1)\lambda^k p$. Therefore, in general, all the elements of θ^2 have order k tangency with the components of Σ_2 corresponding to the components of the critical locus of f along which the transversal type is A_k.

Thus, for at least the case of functions f with A_k transversal types only,

$$\theta_\tau^2 \subset Der(-log\,\Sigma_2)\,.$$

The inclusion is very likely to be true without any constraints on the transversal types.

Experiments with simple symmetric matrix singularities suggest a stronger

Conjecture 4.1 (see also [11]) *Assume that the only transversal type of f is A_1. Then*

$$\theta_\tau^2 = Der(-log\,\Sigma_2)\,.$$

Perhaps a more general equality that allows various A_k transversal types and takes into account the higher order tangencies with the relevant components of Σ_2 (as noticed above) is also true.

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