# Cyclically equivariant function singularities and unitary reflection groups $\mathbf{G(2m,2,n)}$, $\mathbf{G_9}$, $\mathbf{G_{31}}$

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**Abstract**

We study singularities of functions equivariant with respect to an action of $\mathbf{Z}_m \subset SU(2)$. We list all the functions with a finite monodromy group. We show that such a monodromy group is one of the Shephard-Todd groups $G(3,1,n)$, $G(2m,2,n)$, $G_9$ and $G_{31}$.

Discoveries of close interrelations between apparently distant subjects are among the most inspiring in mathematics. One of the best demonstrations of this was by Arnold who found that the classification of simple function singularities follows that of Weyl groups [1, 2]. Soon after doing this he proposed to search for singularity theory realisations of finite groups generated by complex reflections. The first examples of such realisations were given in [7]. In the present paper we continue with the program and relate some other Shephard-Todd groups to function singularities. The groups come out as the only possible finite monodromy groups of functions equivariant with respect to an action of the group $\mathbf{Z}_m \subset SU(2)$.

The idea to study such functions is naturally suggested by inspection of the sets of the degrees of basic invariants of Shephard-Todd groups. The degree set $D(G)$ of a finite reflection group $G$ serves as its passport in the problems of singularity theory. For example, the existence of the inclusion

$D(H_4) \subset D(E_8)$ helped O. Scherbak to relate the group $H_4$ to function singularities [12].

The group $E_8$ plays a special rôle for us as well. One of the remarkable inclusions in the list of unitary reflection groups is $D(G_{31}) \subset D(E_8)$ ($G_{31}$ is the group number 31 in the Shephard-Todd list [13]). Translation of this to the language of singularity theory gives a subfamily in the versal deformation of the function $E_8$ which consists of all the odd degree functions (see Figure 1), that is, of the functions with the $\mathbf{Z}_2$-symmetry:

$$f(-x,-y) = -f(x,y)\,.$$

Calculation of the monodromy group of the subfamily (in fact, of its one-variable stabilisation), which is another traditional object of singularity theory, shows that this group is exactly $G_{31}$.

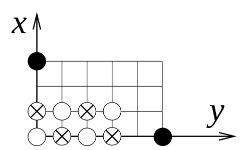


Figure 1: *Deforming function $E_8$ : $x^3 + y^5$ (black monomials) by arbitrary linear combinations of the eight distinguished monomials. The monomials corresponding to the degrees of the group $G_{31}$ are crossed.*

The simplest generalisation of the symmetry group provides Shephard-Todd groups $G(3,1,n)$, $G(2m,2,n)$ and $G_9$ (see Section 1) as monodromy groups of the equivariant function singularities.

The paper is organised in the following way.

Section 1 reviews some generalities about the unitary reflection groups we are dealing with.

Section 2 introduces the equivariant functions of two variables, the equivalence of these functions, vanishing homology and corresponding Picard-Lefschetz operators. In Section 3 we describe stabilisations of all these objects. We need the stabilisations as we are looking for the intersection forms of the right parity on the homology of the Milnor fibres.

In Section 4 we list all the equivalence classes of the equivariant functions with finite monodromy of the stabilisation. Their monodromy groups are exactly the unitary reflection groups under consideration. On the other hand, the monodromy of the corresponding functions on $\mathbf{C}^2$ are what one can naturally take for a skew-Hermitian analogue of these groups. We list, in the form of diagrams, their generating skew-Hermitian reflections.

We finish Section 4 with the study of the discriminants of our singularities and unitary groups.

# 1 The unitary reflection groups

A *complex reflection* in $\mathbf{C}^n$ is a unitary transformation identical on a hyperplane, which is called the *mirror* of the reflection. The complete list of finite irreducible groups generated by complex reflections was obtained by Shephard and Todd [13]. It contains Coxeter groups as its proper subset.

The Shephard-Todd list consists of three infinite series (Weyl groups $A_n$, cyclic $\mathbf{Z}_m$ and three-index $G(p,q,n)$) and 34 exceptional groups. In the present paper we are dealing with two subseries in $G(p,q,n)$ and two exceptional groups, $G_9$ and $G_{31}$ (the lower index is the number of a group in the table in [13]). We briefly recall the description of these groups. In our considerations a mirror is identified by its normal, which we call a *root*.

## 1.1 Groups $\mathbf{G(p,q,n)}$

The group $G(p,q,n)$ (all the parameters are natural numbers, $q$ divides $p$, and $n \geq 2$) is a subgroup in $U(n)$. It is generated by the rotation by the angle of $2\pi q/p$ corresponding to the root $u_1$ (the $u_i$ are the unit coordinate vectors) and by $n$ reflections of order 2 defined by the roots

$$u_2 - u_1\,,\ u_3 - u_2\,,\ \ldots, u_n - u_{n-1} \text{ and } u_2 - e^{2\pi i/p} u_1\,. \qquad (1)$$

For example, in two cases, when either $q = p$ or $q = 1$, just $n$ reflections are sufficient to generate the group.

The series contains Coxeter groups: $G(2,2,n) = D_n$, $G(2,1,n) = B_n$ and $G(p,p,2) = I_2(p)$.

Information about generating reflections of a group can be represented by a graph analogous to a Dynkin diagram of a Coxeter group (cf. [5, 11]). Our conventions are as follows:

- a vertex of a graph represents a root;
- a vertex is white if the root is *long*, of length $\sqrt{2}$, and black if the root is *short*, of length 1;
- the order of the reflection is written at the vertex (order 2 is omitted);
- the Hermitian product $(v_1, v_2)$ of the roots is written on an oriented edge $v_1 \to v_2$;
- there is no edge between two orthogonal roots;
- the product $-1$ is not written;
- orientation of an edge equipped with a real number is omitted.

Figure 2 shows graphs of the groups $G(3,1,n)$ and $G(2m,2,n)$ which we relate to function singularities in this paper. Groups $G(p,1,n)$ have already appeared in a singularity theory context in [7].

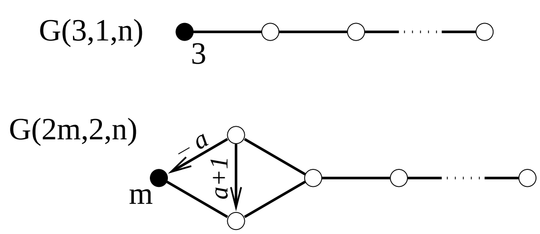


Figure 2: *Graphs of the groups* $G(3,1,n)$ *(n vertices) and* $G(2m,2,n)$ *(*$n+1$ *vertices). Notation:* $a = e^{\pi i/m}$.

**Remark 1.1** It is convenient to include groups $G(1,1,n)$ into the series. According to the previous description, group $G(1,1,n)$ is the permutation group of the coordinates in $\mathbf{C}^n$. Hence $\mathbf{C}^n$ splits into the standard representation of $A_{n-1}$ on the hyperplane $x_1 + \ldots + x_n = 0$ and the one-dimensional trivial representation.

It is also natural to set $G(m,1,1) = G(mq,q,1) = \mathbf{Z}_m$.

The orbit space of any Shephard-Todd group is smooth. Basic invariants of the group $G(p,q,n)$ have degrees $p, 2p, \ldots, (n-1)p, np/q$ (hence the order of the group is $p^n n!/q$). For such invariants one can take the first $n-1$ elementary symmetric functions of $x_1^p, \ldots, x_n^p$ and $(x_1 \cdot \ldots \cdot x_n)^{p/q}$. Thus the orbit space $\mathbf{C}^n/G(p,q,n) \simeq \mathbf{C}^n$ can be identified with the space $\mathbf{C}^n_{\alpha_1,\ldots,\alpha_n}$ of polynomials in $y$:

$$y^n + \alpha_1 y^{n-1} + \ldots + \alpha_{n-1} y + \alpha_n^q \,.$$

The *discriminant* $\Sigma$ of the group $G(q,q,n)$, that is, the space of its irregular orbits, is the set of all such polynomials with multiple roots. The discriminant of any other group of the series is the set of the polynomials with either multiple or zero roots.

**Remark 1.2** The latter means that $\mathbf{C}^n \setminus \Sigma(G(p,q,n))$, $p \neq q$, is an unramified $q$-fold covering of the complement to the discriminant of the Weyl group $B_n$. Hence $\mathbf{C}^n \setminus \Sigma(G(p,q,n))$ is a $k(\pi,1)$-space. Similar assertion is true for the groups $G(q,q,n)$ as well [9].

## 1.2 Groups $G_9$ and $G_{31}$

Group $G_9 \subset U(2)$ is generated by two reflections, one of order 4 and the other of order 2. The degrees of its basic invariants are 8 and 24. The discriminant coincides with that of the Weyl group $G_2$ [10] which consists of a cubical parabola and its tangent. As an abstract group, $G_9$ is a quotient-group of Brieskorn's braid group associated with the Weyl group $G_2$ [3] (that is, generated by two elements subject to the relation $(ab)^3 = (ba)^3$) by the relations $a^4 = 1$ and $b^2 = 1$.

Group $G_{31}$ is generated by five order 2 reflections in $\mathbf{C}^4$. We formulate a conjecture on the shape of the discriminant of $G_{31}$ in Section 4.2. The defining relations for the group, as well as for all the other Shephard-Todd groups, can be found in [4].

Figure 3 shows graphs of generating reflections for the two groups. The sum of the two upper roots in the diagram of $G_{31}$ is equal to the lower centre root. The $G_9$ diagram shows in particular that $G_9$ contains the Coxeter group $I_2(8)$.

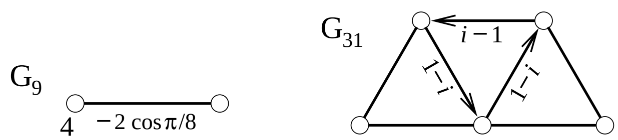


Figure 3: *Graphs of the groups $G_9$ and $G_{31}$.*

Now we leave reflection groups for a while and switch to the study of function singularities.

# 2 Equivariant functions on the plane

## 2.1 The equivalence

The standard $SU(2)$ representation of the group $\mathbf{Z}_m$ is generated by the transformation

$$g:\ x, y\ \mapsto\ x\cdot\varepsilon, y\cdot\varepsilon^{-1}\,,\quad \text{where } \varepsilon = e^{2\pi i/m}\,.$$

**Definition 2.1** A function $f$ on $\mathbf{C}^2$ is $\mathbf{Z}_m$*-equivariant* if $f\circ g = \varepsilon f$.

**Remark 2.2** One could consider more general $\mathbf{Z}_m$-equivariant functions, such that $f\circ g = \varepsilon^r f$ for some integer exponent $r$. But in what follows we prefer generic equivariant functions to have smooth zero-levels which allows to study intersection forms on these levels without any complications. This requirement is satisfied only in three cases, $r = 0, \pm 1$. For any other choice of the exponent, the level $f = 0$ is singular at the origin. The case of $\mathbf{Z}_m$-invariant functions, $r = 0$, was studied in [8] and led to the first appearance of the reflection group $H_3$ in singularity theory. The case $r = -1$ reduces to $r = 1$ in an obvious way.

The ring $\mathcal{I}$ of $\mathbf{Z}_m$-invariant holomorphic function-germs on $\mathbf{C}^2$ at the origin is generated by 3 elements: $x^m$, $y^m$ and $xy$. The space $\mathcal{E}$ of germs at the origin of our $\mathbf{Z}_m$-equivariant functions is an $\mathcal{I}$-module generated by $x$ and $y^{m-1}$:

$$\mathcal{E} = \mathcal{I}\langle x, y^{m-1}\rangle .$$

The group $\mathcal{R}_{\mathbf{Z}_m}$ of biholomorphism-germs of $(\mathbf{C}^2, 0)$ commuting with the $\mathbf{Z}_m$-action acts on $\mathcal{E}$ providing equivalence of equivariant functions. This equivalence satisfies all the axioms of Damon's good geometric equivalence [6]. Hence the standard theorems of finite determinacy and on the form of versal deformations are valid for the $\mathcal{R}_{\mathbf{Z}_m}$-equivalence. For example, the Lie algebra of the group $\mathcal{R}_{\mathbf{Z}_m}$ is an $\mathcal{I}$-module generated by four $\mathbf{Z}_m$-invariant vector fields

$$x\partial_x\,, \quad y^{m-1}\partial_x\,, \quad y\partial_y\,, \quad x^{m-1}\partial_y\,.$$

Therefore the tangent space $Tf$ to the equivalence class of an equivariant function $f \in \mathcal{E}$ is

$$Tf = \mathcal{I}\langle xf_x\,,\ y^{m-1}f_x\,,\ yf_y\,,\ x^{m-1}f_y\rangle \subset \mathcal{E}$$

and an $\mathcal{R}_{\mathbf{Z}_m}$-miniversal deformation of $f$ can be taken in the form

$$f + \lambda_1\varphi_1 + \ldots + \lambda_\tau\varphi_\tau\,, \tag{2}$$

where the $\varphi_j \in \mathcal{E}$ represent a basis of the linear space $\mathcal{E}/Tf$ and the $\lambda_j \in \mathbf{C}$ are parameters. We assume here the *equivariant Tjurina number* $\tau$ of $f$ to be finite.

The base $\mathbf{C}^\tau$ of an $\mathcal{R}_{\mathbf{Z}_m}$-miniversal deformation of an equivariant function $f$ contains the *equivariant discriminant* $\Delta$ of $f$, that is, the set of those values of the deformation parameters for which the zero-level set of the corresponding function is not smooth. The discriminant is a hypersurface which has two top-dimensional strata if $m \geq 3$ and only one if $m = 2$. The strata correspond to two possible degenerations of zero-levels in generic one-parameter families of $\mathbf{Z}_m$-equivariant functions:

- $m$ Morse singularities $A_1$ occupying one complete $\mathbf{Z}_m$-orbit in $\mathbf{C}^2$, all out of the origin;
- singularity $D_m$ at the origin, $m \geq 3$ (we set $D_3 = A_3$).

Indeed, a $\mathbf{Z}_m$-equivariant series

$$f(x,y) = x \cdot \psi_1(x^m, y^m, xy) + y^{m-1} \cdot \psi_2(x^m, y^m, xy)\,, \quad m \geq 3\,,$$

has a singularity at the origin if and only if the monomial $x$ is absent. In this case the principal weighted-homogeneous part of $f$ is, in general,

$$\text{either}\ \ D_m:\ ax^2y + by^{m-1}, \quad m \geq 4\,,$$

$$\text{or}\ \ A_3:\ ax^4 + bx^2y + cy^2\,, \quad m = 3\,, \qquad a, b, c \in \mathbf{C}\,.$$

The $mA_1$ stratum of the discriminant may be reducible. The $D_m$ stratum is obviously always smooth.

**Remark 2.3** An $\mathcal{R}_{\mathbf{Z}_m}$-miniversal deformation of an equivariant function $f$ can be extended to an $\mathcal{R}$-miniversal deformation of $f$. Therefore the equivariant discriminant $\Delta(f)$ is the section of the ordinary, non-equivariant bifurcation diagram of zeros of $f$ lying in the base of the larger deformation.

## 2.2 Equivariant homology

Let $V$ be the zero-level of a generic member of an $\mathcal{R}_{\mathbf{Z}_m}$-miniversal deformation of an equivariant function $f$ (in fact $V$ is a Milnor fibre of $f$). The action of $\mathbf{Z}_m$ on $V$ induces the action on the homology of $V$ which splits $H_1(V, \mathbf{C})$ into a direct sum of the character subspaces:

$$H_1(V) = \oplus_{\chi^m=1} H_\chi\,.$$

The generator $g_*$ acts on $H_\chi$ by multiplication by $\chi$.

Let $V'$ be the orbit space $V/\mathbf{Z}_m$, $pr: V \to V'$ the projection, and $\mu$ the first Betti number of $V'$.

**Proposition 2.4** $\quad \operatorname{rk} H_\chi = \mu\,.$

*Proof.* The $\mathbf{Z}_m$-action on $V$ has one fixed point, the origin $0 \in V \subset \mathbf{C}^2$. The action is free on $V \setminus \{0\}$.

Smooth curve $V'$ retracts to a bouquet of $\mu$ circles wedged at $pr(0)$. The curve $V$ retracts to the inverse image of this bouquet which consists of $m\mu$ circles wedged at 0. The group $\mathbf{Z}_m$ cyclically permutes each $m$ of them making the $pr$-preimage of one of the circles in $V'$. Each such orbit contributes 1 to the rank of each $H_\chi$. $\square$

**Definition 2.5** The number $\mu = \text{rk}\, H_\chi$ is called the *equivariant Milnor number* of $V$.

**Proposition 2.6** $\quad \mu = \tau\,.$

*Proof.* Consider the local algebra $Q_f$ of $f$, that is, the quotient of the ring of all holomorphic function-germs on $(\mathbf{C}^2, 0)$ by the ideal generated by the first partial derivatives of $f$. We have

$$\dim_{\mathbf{C}} Q_f = m\mu\,,$$

as the latter is the ordinary Milnor number of $f$.

Slightly adjusting the arguments of [14] to the case of cyclically equivariant functions, one can establish the relation between the natural representations of $\mathbf{Z}_m$ on $Q_f$ and on $H_1(V, \mathbf{C})$. The relation implies that the dimensions of all the character subspaces in $Q_f$ are the same, equal to $\mu$. The character $\varepsilon$ subspace of $Q_f$ is exactly the quotient $\mathcal{E}/Tf$. $\square$

Standard elements in each of the $H_\chi$, analogous to ordinary vanishing cycles, can be obtained in the following way.

Let $*$ be a point in $\mathbf{C}^\tau \setminus \Delta$ which corresponds to the function whose zero-level set is $V$. Take a generic straight line $\ell \simeq \mathbf{C}$ in $\mathbf{C}^\tau$ through $*$. It transversally meets the discriminant at certain points, exactly one of which in the case $m \geq 3$ belongs to the $D_m$ stratum and all the others are in the $mA_1$ stratum.

Moving along a non-self-intersecting path from $*$ to a point of $\ell \cap (mA_1)$, we define on $V$ the $m$ vanishing 1-cycles $\sigma_0, \ldots, \sigma_{m-1}$. They do not intersect, and we orient and order them so that the generator $g$ of $\mathbf{Z}_m$ permutes them cyclically: $g(\sigma_j) = \sigma_{(j+1) \bmod m}$. For each $\chi$ we introduce the cycle

$$\sum_{s=0}^{m-1} \chi^{-s} \sigma_s \in H_\chi\,.$$

We call it a *long vanishing $\chi$-cycle.*

To choose a similar equivariant cycle vanishing at a point of the $D_m$ stratum, we consider a miniversal equivariant deformation of the singularity $D_m$ and analyse the family of zero-levels:

$$V_\alpha = \{x^2 y + y^{m-1} + 2\alpha x = 0\} \subset \mathbf{C}^2\,. \tag{3}$$

In Figure 4 we show a way to glue $V_{\alpha\neq 0}$ from two copies of the coordinate $y$-axis which are cut along rays from the $m$ branching points $y = \alpha^{2/m}$. The left sheet has a puncture at the origin. In the same figure we show the system of 1-cycles on which the generator $g$ acts by the clockwise shift by one element. Taking the cycles with the coefficients shown in the figure we obtain an element of $H_\chi(V_\alpha)$ which we call a *short vanishing $\chi$-cycle*. It vanishes along the straight path from $\alpha$ to 0.

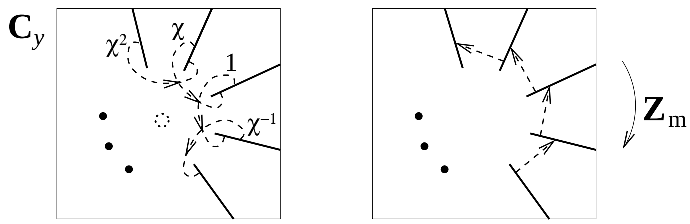


Figure 4: *The $D_m$ curve $x^2y + y^{m-1} + 2\alpha x = 0$ as a branched double covering of the $y$-axis and a $\mathbf{Z}_m$-symmetric system of vanishing 1-cycles on it.*

Now a system of paths on $\ell$ from the point $*$ to all the points of $\ell \cap \Delta$, without mutual- and self-intersections, defines a system of long and short vanishing $\chi$-cycles in each of the $H_\chi(V)$. We call it a *distinguished* system of vanishing $\chi$-cycles.

We have a traditional

**Proposition 2.7** *A distinguished system of vanishing $\chi$-cycles generates the character subspace $H_\chi$.*

A distinguished system may not form a basis.

**Remarks 2.8** Both long and short $\chi$-cycles are defined up to multiplication by $-1$ and by powers of $\chi$.

The $m$ cycles of Figure 4 form a distinguished (in the ordinary, non-symmetric sense) basis in vanishing homology of the $D_m$ function singularity. It corresponds to the family of curves $x^2y + y^{m-1} + 2\alpha x + \beta = 0$, in which $\alpha \neq 0$ is fixed and $\beta$ varies. Its Dynkin diagram is a regular $m$-gon (see figures in Section 4.1).

## 2.3 Monodromy group

### 2.3.1 Long vanishing $\chi$-cycles

The self-intersection of a long vanishing $\chi$-cycle $e$ is 0.

Similar to [7], the Picard-Lefschetz operator of the $\mathbf{Z}_m$-equivariant monodromy on $H_1(V, \mathbf{C})$, corresponding to the path in the line $\ell$ along which $e$ vanishes (as well as the other $m-1$ cycles in the other character subspaces), splits into a direct sum of operators $h_\chi : H_\chi \to H_\chi$,

$$h_\chi : \ a \ \mapsto \ a - (a, e)e/m \,,$$

where $(\cdot, \cdot)$ is the skew-Hermitian form on $H_1(V, \mathbf{C})$ defined by the intersection number of cycles.

### 2.3.2 Short vanishing $\chi$-cycles

This case is a bit more complicated.

Figure 4 shows that the self-intersection number of a short $\chi$-cycle $e$ is $m(\chi^{-1} - \chi)$.

To understand the Picard-Lefschetz operator, we consider first its action on the $\chi$-cycle $e$ itself. This is described by the monodromy of the family (3). We see, by the weighted-homogeneous argument, that the change of the parameter $\alpha \mapsto \alpha \cdot e^{2\pi i}$ induces the mapping

$$(x,\, y) \ \mapsto \ (x \cdot e^{2\pi i \cdot (m-2)/m},\, y \cdot e^{2\pi i \cdot 2/m}) = (x \cdot \varepsilon^{-2},\, y \cdot \varepsilon^{2}) = g^{-2}(x,\, y)$$

of $V_\alpha$ into itself. Therefore $h_\chi(e) = \chi^{-2} e$.

This implies that, for an arbitrary element $a \in H_\chi$, we have

$$h_\chi : \ a \ \mapsto \ a + (\chi^{-2} - 1)\frac{(a, e)}{(e, e)} e = a + \chi^{-1}(a, e)e/m \,.$$

The final result is easily checked to be true even for $\chi = \pm 1$.

# 3 Stabilisation

An attempt to stabilise a $\mathbf{Z}_m$-equivariant function $f = f(x, y)$, by adding a square of a new variable, gives rise to a $\mathbf{Z}_{2m}$-equivariant function. Indeed, the function $\widetilde{f}(x,y,z) = f(x,y) + z^2$ must still be sent by the generator $g$ of $\mathbf{Z}_m$ to $\varepsilon \widetilde{f}$ which means that $g$ has to multiply $z$ by $\varepsilon^{1/2}$.

In what follows we denote the objects related to the one-variable stabilisations of $\mathbf{Z}_m$-equivariant functions by putting tilde over the notations of the corresponding objects we had in the two-variable case. For example, the generator $\widetilde{g}$ of $\mathbf{Z}_{2m}$ is the transformation

$$\widetilde{g}: \ (x,y,z) \ \mapsto \ (\widetilde{\varepsilon}^2 x, \widetilde{\varepsilon}^{-2} y, \widetilde{\varepsilon} z)\,, \quad \text{where } \widetilde{\varepsilon} = e^{\pi i/m}\,,$$

and the functions $\widetilde{f}(x,y,z) = f(x,y) + z^2$ we are considering now are such that

$$\widetilde{f} \circ \widetilde{g} = \widetilde{\varepsilon}^2 \widetilde{f}\,. \tag{4}$$

One can straightforwardly develop the $\widetilde{\mathcal{R}}_{\mathbf{Z}_{2m}}$-equivalence theory for function-germs on $(\mathbf{C}^3, 0)$ satisfying the equivariant condition (4), without requiring that the germs be stabilisations of functions on the plane. Within this theory, a miniversal deformation of a stabilisation $\widetilde{f} = f + z^2$ is obtained by the stabilisation of an $\mathcal{R}_{\mathbf{Z}_m}$-miniversal deformation of $f$ (just add $z^2$ to the family (2)).

The symmetry group $\mathbf{Z}_{2m}$ acts on the zero-level $\widetilde{V}$ of a generic member of an $\widetilde{\mathcal{R}}_{\mathbf{Z}_{2m}}$-miniversal deformation of a function $\widetilde{f} = f + z^2$. The corresponding action on $H_2(\widetilde{V}, \mathbf{C})$ is generated by the operator $\widetilde{g}_*$ such that $\widetilde{g}_*^m = -id$ (indeed, the transformation $\widetilde{g}^m(x,y,z) = (x,y,-z)$ changes orientation of all suspended cycles). Therefore we have a splitting

$$H_2(\widetilde{V}) = \oplus_{\widetilde{\chi}^m = -1} H_{\widetilde{\chi}}\,.$$

The operator $\widetilde{g}_*$ acts on a rank $\mu$ subspace $H_{\widetilde{\chi}}$ by multiplication by $\widetilde{\chi}$.

As in the planar case, distinguished vanishing generators for the $H_{\widetilde{\chi}}$ come from the $mA_1$ and $D_m$ bifurcations. The following minor modifications are needed.

Consider the $m$ 2-cycles on $\widetilde{V}$ vanishing at a point of the $mA_1$ stratum. We orient them so that

$$\widetilde{g}_* : \ \widetilde{\sigma}_0 \mapsto \widetilde{\sigma}_1 \mapsto \ldots \mapsto \widetilde{\sigma}_{m-2} \mapsto \widetilde{\sigma}_{m-1} \mapsto -\widetilde{\sigma}_0\,. \tag{5}$$

This gives a long vanishing $\widetilde{\chi}$-cycle

$$\widetilde{e} = \sum_{s=0}^{m-1} \widetilde{\chi}^{-s} \widetilde{\sigma}_s \in H_{\widetilde{\chi}}\,. \tag{6}$$

The self-intersection number of this cycle is $-2m$, and the corresponding Picard-Lefschetz operator on $H_{\widetilde{\chi}}$ is a Hermitian reflection of order 2:

$$h_{\widetilde{\chi}}: \ a \mapsto a - 2\frac{(a,\widetilde{e})}{(\widetilde{e},\widetilde{e})}\widetilde{e} = a + (a,\widetilde{e})\widetilde{e}/m\,.$$

To obtain a short vanishing $\widetilde{\chi}$-cycle, we suspend the 1-cycles of Figure 4 and number and orient them so that the action of $\widetilde{g}_*$ is again given by (5). Then the short $\widetilde{e} \in H_{\widetilde{\chi}}$ is again defined by (6). Now $(\widetilde{e},\widetilde{e}) = m(-2+\widetilde{\chi}+\widetilde{\chi}^{-1})$.

Similar to Section 2.3.2, consider the quasi-homogeneous mapping, induced in the family

$$\widetilde{V}_\alpha = \{x^2y + y^{m-1} + z^2 + 2\alpha x = 0\}$$

by the rotation $\alpha \mapsto \alpha \cdot e^{2\pi i}$. The consideration shows that the monodromy operator, corresponding to a short $\widetilde{\chi}$-cycle $\widetilde{e}$, multiplies $\widetilde{e}$ by $\widetilde{\chi}^{-2}$ and hence defines on $H_{\widetilde{\chi}}$ a Hermitian reflection

$$h_\chi: \ a \mapsto a + (\widetilde{\chi}^{-2} - 1)\frac{(a,e)}{(\widetilde{e},\widetilde{e})}\widetilde{e} = a + \frac{1+\widetilde{\chi}^{-1}}{1-\widetilde{\chi}}(a,\widetilde{e})\widetilde{e}/m$$

of the order equal to the order of $\widetilde{\chi}^{-2}$.

# 4 Elliptic singularities

**Definition 4.1** A $\mathbf{Z}_m$-equivariant function-germ $f$ on $(\mathbf{C}^2, 0)$ is called *elliptic* if the $\mathbf{Z}_{2m}$-equivariant monodromy group of its stabilisation $\widetilde{f} = f + z^2$ is finite.

**Theorem 4.2** *The complete list of $\mathcal{R}_{\mathbf{Z}_m}$-equivalence classes of elliptic $\mathbf{Z}_m$-equivariant function-germs is as follows (the notation includes the symmetry* *group):*

| notation | normal form | $\tau$ | restrictions | unitary group |
|---|---|---|---|---|
| $A_0/\mathbf{Z}_m$ | $x$ | 0 | $m \geq 2$ | – |
| $A_{3n}/\mathbf{Z}_3$ | $y^2 + x^{3n+1}$ | $n$ | $n \geq 1$ | $G(3,1,n)$ |
| $D_{mn}/\mathbf{Z}_m$ | $x^2y + y^{mn-1}$ | $n$ | $m \geq 2,\ n \geq 1,\ mn \geq 4$ | $G(2m,2,n)$ |
| $E_8/\mathbf{Z}_2$ | $x^3 + y^5$ | 4 | – | $G_{31}$ |
| $E_8/\mathbf{Z}_4$ | $x^3 + y^5$ | 2 | – | $G_9$ |

*If $\widetilde{\chi}$ is a primitive root of unity of order $2m$, the monodromy group acting on the character vanishing homology subspace $H_{\widetilde{\chi}}$ of the corresponding function-germ on $\mathbf{C}^3$ is the unitary reflection group of the last column of the table.*

The proof that there are no other elliptic singularities is very similar to the proof of the analogous classification theorem in [7] and we are not giving it here. In Section 4.1 we show that the monodromy groups are as claimed.

**Remarks 4.3** (a) We use the settings of Remark 1.1: $G(3,1,1) = \mathbf{Z}_3$ and $G(2m,2,1) = \mathbf{Z}_m$.

(b) In the case $A_{3n}/\mathbf{Z}_3$, the monodromy acts on $H_{\widetilde{\chi}=-1}$ as $G(1,1,n)$ (see Remark 1.1).

(c) In the case $D_{mn}/\mathbf{Z}_m$, for a non-primitive root $\widetilde{\chi}$, $\widetilde{\chi}^m = -1$, the monodromy group on the character subspace $H_{\widetilde{\chi}}$ is $G(ord(\widetilde{\chi}),2,n)$.

(d) All possible adjacencies of the elliptic singularities come from the adjacencies of the $\mathcal{R}$-simple functions.

## 4.1 The intersection diagrams

The proof of the claim that the monodromy groups of the stabilised elliptic equivariant singularities are exactly the unitary reflection groups follows from the comparison of the Picard-Lefshetz operators, corresponding to a distinguished set of vanishing $\widetilde{\chi}$-cycles, with the generators given by the group graphs in Section 1. In this section we draw similar graphs, *Dynkin diagrams*, for the singularities in which the roots are the vanishing $\chi$- or $\widetilde{\chi}$-cycles and the (skew-)Hermitian product of the roots is the intersection index of the cycles (in the three-variable case, the product is negative-definite as all the participating non-symmetric functions are $\mathcal{R}$-simple). The Dynkin diagrams obtained show that, for the functions on $\mathbf{C}^3$ and $\widetilde{\chi} = e^{\pm\pi i/m}$, only very simple rescaling of the cycles is required to make their intersection numbers equal to the negatives of the Hermitian products of the corresponding roots of the groups. For the other primitive characters $\widetilde{\chi}$, an additional minor change in the set of generating reflections of the group also gives the distinguished set of the Picard-Lefschetz operators.

The Dynkin diagrams for the plane curves introduce skew-Hermitian versions of the unitary reflection groups under consideration.

We obtain the intersection diagrams for the character subspaces from symmetric Dynkin diagrams of the corresponding $\mathcal{R}$-simple functions by the operation of $m$-folding which is very similar to the folding used in [12] and [7]. This is analogous to the folding by which one gets the canonical Dynkin diagrams of the Weyl groups $B_k$, $C_k$, $F_4$ from those of $A_{2k-1}$, $D_{k+1}$, $E_6$. In fact the $m$-folding has already been described in Sections 2 and 3 where we obtained one vanishing $\chi$- or $\widetilde{\chi}$-cycle from the $m$ ordinary.

Our conventions for drawing the Dynkin diagram of a distinguished system of vanishing $\chi$-cycles in $H_\chi$ of the equivariant function singularity on $\mathbf{C}^2$ are as follows (cf. Section 2.2):

- a long vanishing $\chi$-cycle, of square 0, is represented by a white vertex;
- a short vanishing $\chi$-cycle, of square $m(\chi^{-1} - \chi)$, is represented by a black vertex (in fact there is no difference between short and long cycles when $m = 2$);
- we number the paths in the line $\ell$ counter-clockwise as they leave the base point $*$ (see Section 2.2) and give the same numbers (written in bold) to the corresponding vanishing $\chi$-cycles;
- there is no edge between two skew-orthogonal $\chi$-cycles;
- an oriented weighted edge, $v_1 \xrightarrow{a} v_2$, stands for $(v_1, v_2) = ma$;
- the weight $a = 1$ is not written.

Drawing cyclically symmetric Dynkin diagrams of ordinary functions, we follow the same agreements taking formally $m = 1$ and making all the vertices white.

The conventions for the diagrams of the stabilised singularities, in three variables, are almost the same as above except for that we do not need to orient an edge with a real weight and the weight $a = -1$ is represented by a dashed edge. Of course, the squares of short and long $\widetilde{\chi}$-cycles are now $m(-2 + \widetilde{\chi} + \widetilde{\chi}^{-1})$ and $-2m$, and the orders of the related Picard-Lefschetz operators are the order of $\widetilde{\chi}^2$ and 2 (see Section 3).

Traditional calculations imply

**Proposition 4.4** *There exist distinguished bases in the vanishing homology of simple singularities $A_{3n}$, $D_{mn}$ and $E_8$, in two and three variables, with the intersections given by the symmetric Dynkin diagrams of Figures 5 and 6. They define, in the way described in Sections 2.2 and 3, the distinguished systems of vanishing $\chi$- and $\widetilde{\chi}$-cycles in the character homology subspaces of the equivariant functions with the Dynkin diagrams represented in the same Figures.*

Below are some comments on the construction of the Dynkin diagrams of the equivariant functions and on the coincidence of the monodromy and reflection groups.

$\mathbf{A_{3n}/Z_3}$.

All the vanishing $\chi$- and $\widetilde{\chi}$-cycles in the distinguished bases are in one-to-one correspondence with the triples of the cycles in the $A_{3n}$ diagrams lying on the concentric circles. The fact that the monodromy group on $H_{\widetilde{\chi}}$, $\widetilde{\chi} = e^{\pm\pi i/3}$, is $G(3,1,n)$ is obvious.

$\mathbf{D_{mn}/Z_n}$.

The diagrams for the singularities $D_{5n}/\mathbf{Z}_5$ given in Figure 5 provide a clear pattern for the entire series. The numbering of the vertices in the tails of the Dynkin diagrams of the character spaces of this series is $5, 4, 7, 6, 9, 8, \ldots$, ending with $\ldots, n, n-1, n+1$ if $n$ is odd $n$ and $\ldots, n-2, n+1, n$ if $n$ is even. There is one relation between the cycles:

$$e_2 = e_1 - e_3 \text{ in } H_\chi \text{ and } \widetilde{e}_2 = \widetilde{e}_1 + \widetilde{e}_3 \text{ in } H_{\widetilde{\chi}}. \qquad (7)$$

Geometrically, the second cycle is the image of the first under the square root of the monodromy operator corresponding to the third (short) cycle. All the

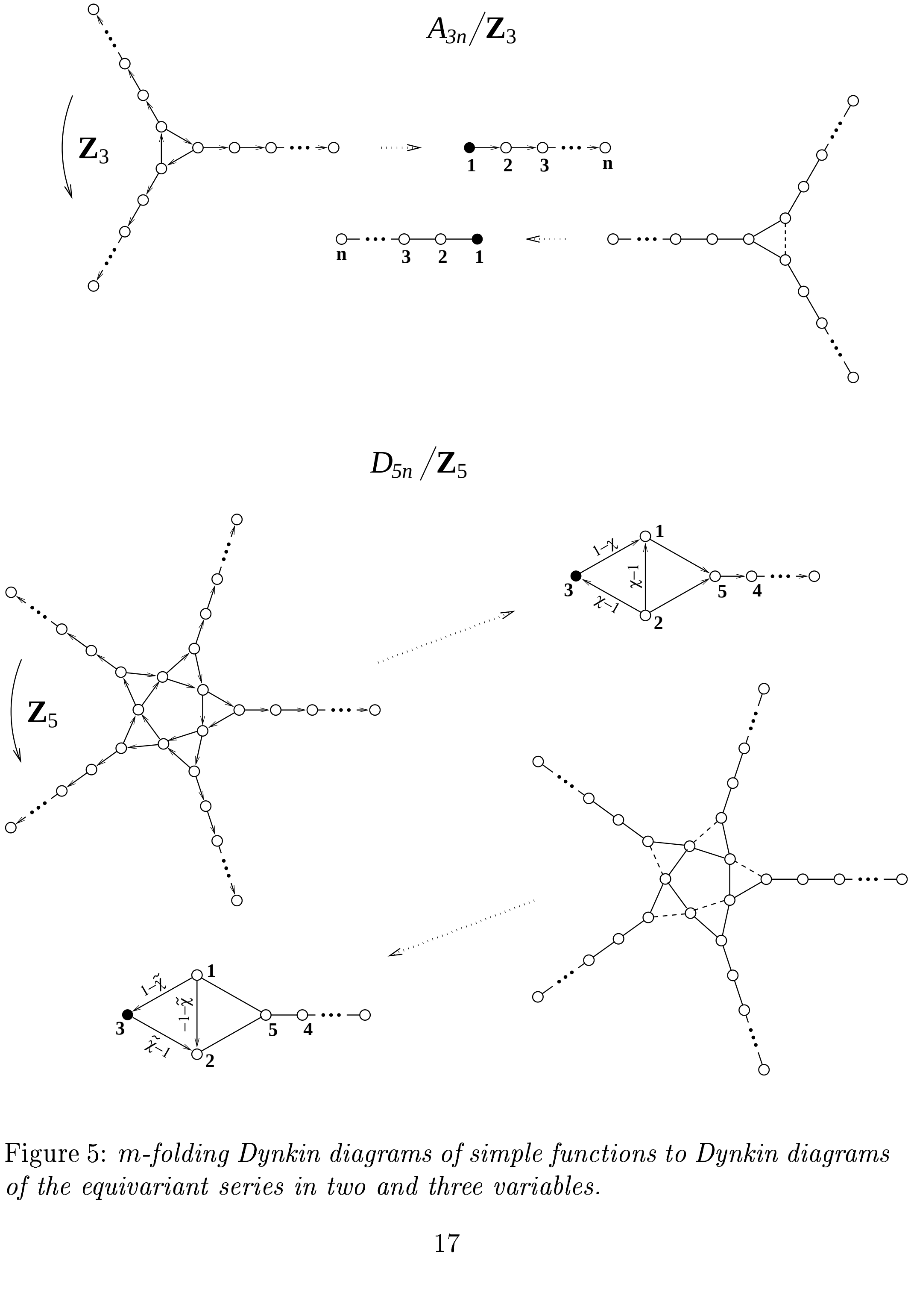


Figure 5: *m-folding Dynkin diagrams of simple functions to Dynkin diagrams of the equivariant series in two and three variables.*

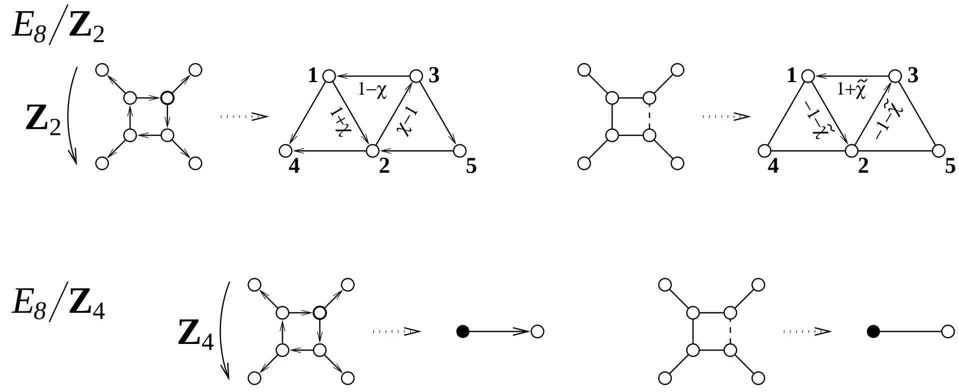


Figure 6: *$m$-folding Dynkin diagrams of function $E_8$ to Dynkin diagrams of the exceptional equivariant singularities in two and three variables.*

character cycles, except the second, are again in one-to-one correspondence with the concentric $m$-tuples of the cycles in the $D_{mn}$ diagram.

The division of all the long $\widetilde{\chi}$-cycles by $\sqrt{m}$ and of the short one by $(\widetilde{\chi}-1)\sqrt{m}$, along with the change of the sign of the intersection form, provide the intersection diagram of the group $G(2m, 2, n)$ (Figure 2) in which $a = \widetilde{\chi}$. For a primitive $\widetilde{\chi}$, reflecting the rescaled $\widetilde{e}_1$ an appropriate number of times (if any) in $\widetilde{e}_3$, we make $a = e^{\pi i/m}$. Therefore $G(2m, 2, n)$ is indeed the monodromy group of $H_{\widetilde{\chi}}$.

**$\mathbf{E_8/Z_2}$.**

The Dynkin diagrams of the equivariant singularities are constructed in the obvious way from those of $D_6/\mathbf{Z}_2$ by adding one more character cycle corresponding to two opposite peripherical cycles in the $E_8$ diagrams. The linear relation on the vanishing character cycles in this case is (7) again. When $\widetilde{\chi} = -i$, the diagram of the three-variable singularity is, up to the sign, exactly that of the group $G_{31}$ given in Figure 3. For $\widetilde{\chi} = i$, reorientation of the triangle 123 and reflection of the Dynkin diagram about the vertical axis also provide us with the diagram of Figure 3.

**$\mathbf{E_8/Z_4}$.**

For $\widetilde{\chi} = e^{\pm\pi i/4}$, the sign change of the form and rescaling transform the Dynkin diagram of the three-variable singularity into the standard diagram of $G_9$ (Figure 3). For $\widetilde{\chi} = e^{\pm 3\pi i/4}$, the same procedure gives the graph similar to that of $G_9$, but with the edge of weight $-2\cos 3\pi/8$. As $G_9 \supset I_2(8)$, this graph aslo defines $G_9$.

## 4.2 Discriminants of the singularities and of the groups

The Tjurina number of an equivariant elliptic singularity coincides with the dimension of the standard representation of the related reflection group.

In all the cases when the group discriminants are known, it is easy to check that they coincide with the discriminants of the function singularities.

**Example 4.5** The discriminants of the complex singularities $D_{2m}/\mathbf{Z}_m$ and $D_{3m}/\mathbf{Z}_m$ are shown in Figure 7. There and in what follows, we omit the group from the notation of the equivariant singularities in the figures. If $m = 2$, the $D_2$ stratum is just another component of the $2A_1$ stratum (clearly $D_2 = 2A_1$ as groups).

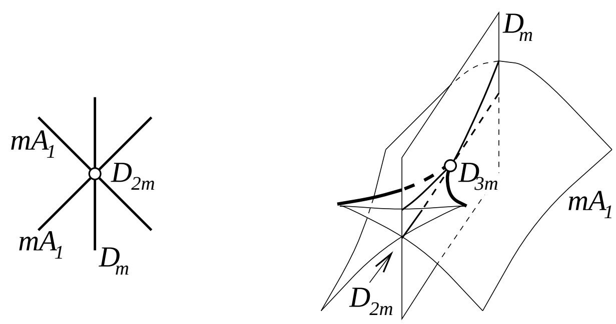


Figure 7: *Discriminants of the complex singularities $D_{2m}/\mathbf{Z}_m$ and $D_{3m}/\mathbf{Z}_m$.*

**Example 4.6** It makes sense to consider real versions of $\mathbf{Z}_2$-equivariant singularities, along with the corresponding discriminants. The discriminants of the initial members of the real series

$$D^{\pm}_{2m}/\mathbf{Z}_2 : \ \pm x^2y + y^{2m-1}$$

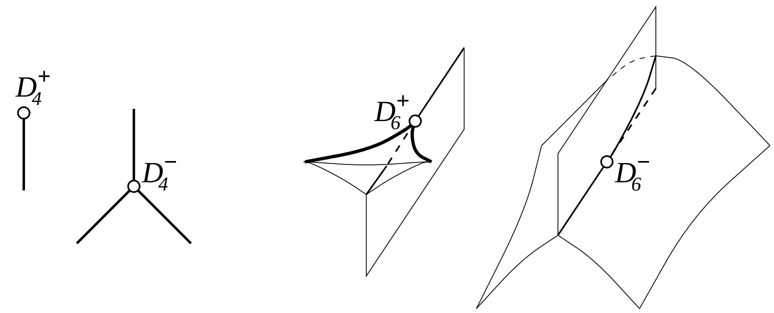


Figure 8: *Discriminants of the real singularities $D_4^\pm/\mathbf{Z}_2$ and $D_6^\pm/\mathbf{Z}_2$.*

are shown in Figure 8. They are just appropriate fragments of the discriminants of the previous figure.

The only unknown group discriminant is that of $G_{31}$. Hence we have

**Conjecture 4.7** *The discriminant of the group $G_{31}$ is isomorphic to the discriminant of the singularity $E_8/\mathbf{Z}_2$.*

A nice description of the latter is as follows.

Consider the family of planar curves defined by the $\mathcal{R}_{\mathbf{Z}_2}$-miniversal deformation of $E_8/\mathbf{Z}_2$:

$$y^5 + 2x^3 + 2\alpha xy^2 + 2\beta y^3 - \gamma x - \delta y = 0\,. \tag{8}$$

We set $x = uy$ in (8), then divide the result by $y$ and introduce $v = y^2$. This provides a family of hyperelliptic curves which after the substitution $w = v + u^3 + \alpha u + \beta$ takes the form

$$w^2 = (u^3 + \alpha u + \beta)^2 + \gamma u + \delta\,. \tag{9}$$

It is easy to see that our transformations did not change the discriminant. Hence the discriminant of the function $E_8/\mathbf{Z}_2$ is a section of the standard discriminant $A_5$ by a smooth hypersurface. The discriminant is represented in Figure 9 by its three spatial sections. Halving the surfaces along the $D_4$ stratum, one obtains the sections of the discriminant of the real $\mathbf{Z}_2$-equivariant singularity.

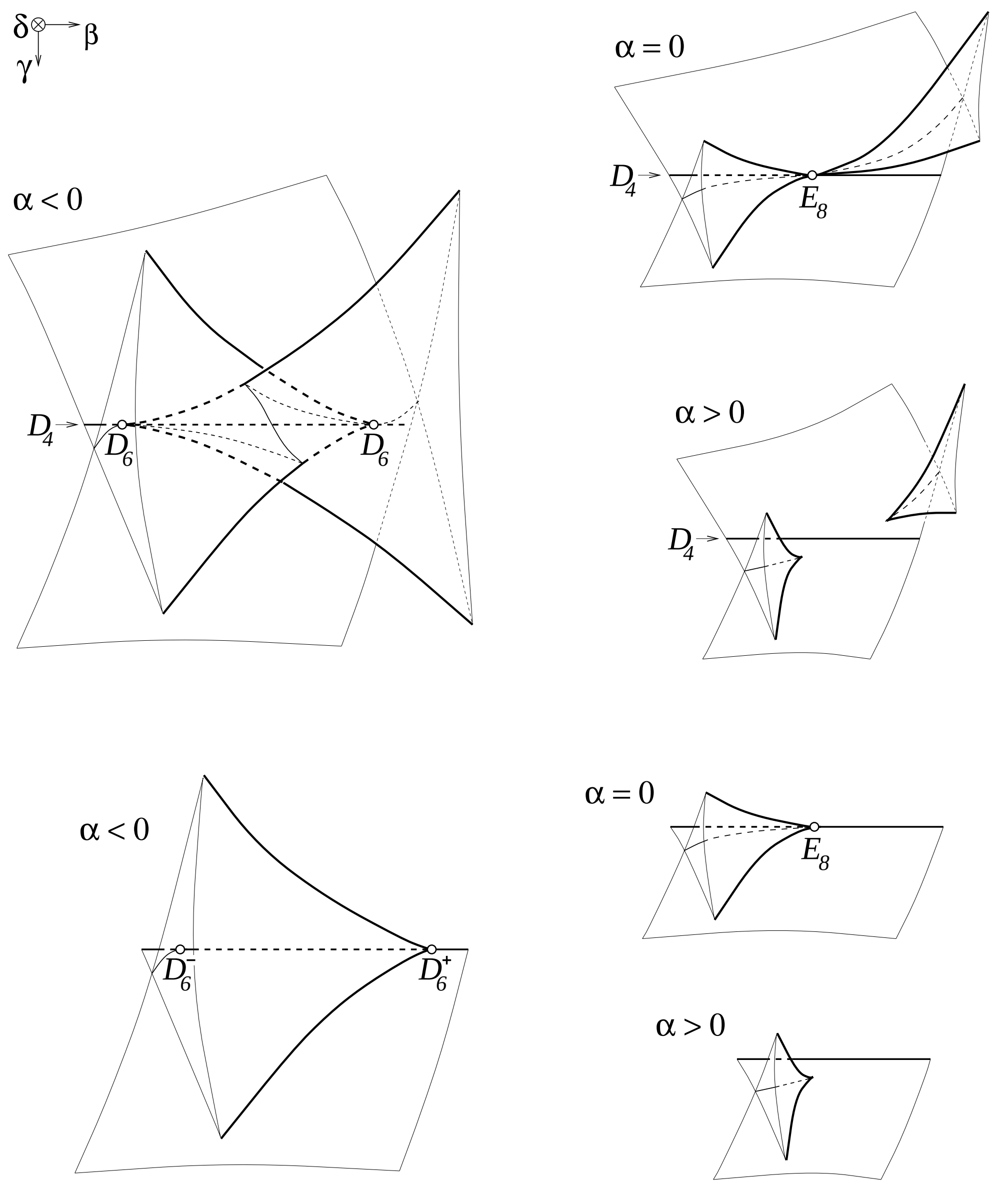


Figure 9: *Three-dimensional sections of the discriminant of the singularity* $E_8/\mathbf{Z}_2$ *and of its real version.*

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