Möbius and odd real trigonometric M-functions

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Abstract

We study two series of spaces of special real trigonometric polynomials of fixed degree having the maximal possible number of distinct critical values. Those are functions such that either $g(\varphi+\pi) \equiv -g(\varphi)$ or $g(-\varphi) \equiv -g(\varphi)$. For each of the spaces, we calculate the number of its connected components and identify, within the mirror arrangement of the Weyl group of series B, a convex polyhedral model for its closure.

A real trigonometric *M-polynomial* of degree n is one with the maximal number $2n$ of real critical points. In his recent paper [4], V. I. Arnold constructed a polyhedral model for the manifold of such polynomials and calculated the number of topologically different M-polynomials with all their critical values distinct. The model was provided by a convex cone in the space equipped with the mirror arrangement of the reflection group A_{2n-1}. The enumeration was done in terms of updown sequences (also called A-snakes) of [1, 2].

In the present note we establish a similar result for *Möbius* trigonometric M-polynomials, that is those with the property $g(\varphi+\pi) \equiv -g(\varphi)$. The polyhedral model is now a simplex within the mirror arrangement of the reflection group of series B. Analogous constructions are carried out for odd trigonometric functions.

Also we enumerate topological types of our M-functions with non-coinciding critical values. The enumeration is given by so-called β- and γ-snakes (see [2]), which have been lacking a direct singularity theory interpretation.

All our results describe properties of the real Lyashko-Looijenga mapping which associates to an M-function the ordered set of its critical values. They are parallel to the results on the sets of real M-functions in the other natural families (see [2, 6, 7] and papers cited there).

1 Möbius polynomials

1.1 Functions of degree 3

We start with an example that illustrates general properties of Möbius M-polynomials which we are going to establish.

Consider the 2-parameter family

$$f(\varphi, a, b) = \frac{1}{3}\cos 3\varphi + a\cos\varphi + b\sin\varphi$$

of all possible real Möbius trigonometric polynomials of degree 3 with the fixed leading term $\frac{1}{3}\cos 3\varphi$.

The *bifurcation diagram* Σ of the family is the set of all the values of the parameters (a, b) for which the function $f(\cdot, a, b)$ is non-generic. It consists of three curves:

- Σ_c, caustic, — functions $f(\cdot, a, b)$ have non-Morse critical points;
- Σ_m, Maxwell stratum, — functions $f(\cdot, a, b)$ have coinciding non-zero critical values;
- Σ_0, *special* Maxwell stratum, — functions $f(\cdot, a, b)$ have critical points on their zero-levels.

To find Σ_c one can proceed as follows. Rewrite the family as

$$F(\varphi, A, \delta) = \frac{1}{3}\cos 3\varphi + A\cos(\varphi - \delta), \qquad a = A\cos\delta, \quad b = A\sin\delta\,.$$

Let prime denote the derivation with respect to φ. Then the equation for Σ_c is

$$0 = -F'' - iF' = 2e^{3i\varphi} + e^{-3i\varphi} + Ae^{i(\varphi-\delta)}\ .$$

Thus, $Ae^{-i\delta} = -2e^{2i\varphi} - e^{-4i\varphi}$. This is a hypocycloid with 3 cusps, the bigger one of Fig.1.

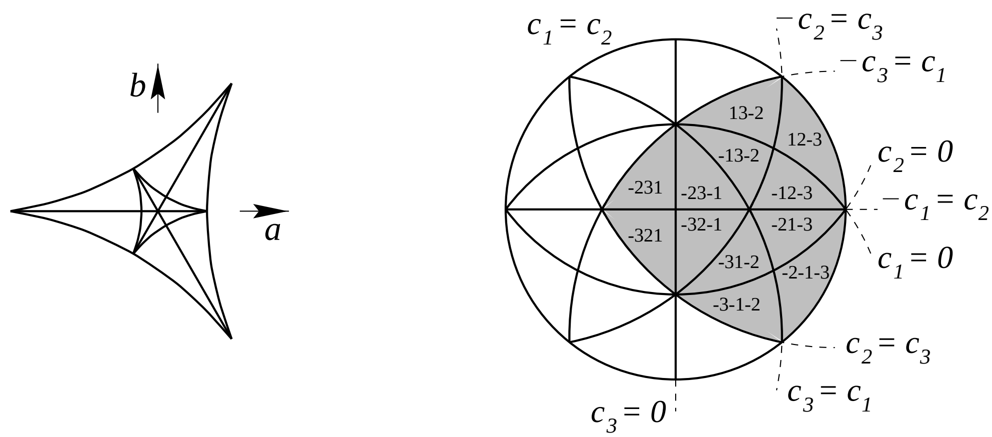

Figure 1: *Stratification of the space of Möbius functions of degree 3 with a fixed leading term and the mirror arrangement of the reflection group B_3. The spherical triangles are marked with the corresponding β_3-snakes (see Sect.1.2).*

The smaller hypocycloid of Fig.1 is Σ_0 defined by the equation $F - iF' = 0$, that is $Ae^{-i\delta} = -\frac{2}{3}e^{2i\varphi} + \frac{1}{3}e^{-4i\varphi}$.

The Maxwell stratum Σ_m consists of the three intervals. The calculations for it are, as usual, a bit more complicated. We do not show them here.

The set of M-functions $f(\cdot, a, b)$, with all 6 critical points real, is that inside Σ_c. The subdivision of its closure by the strata Σ_m and Σ_0 is seen to be homeomorphic to that of the spherical domain in the cone $c_1 \leq c_2 \geq c_3 \leq -c_1$ in the coordinate 3-space by the mirrors of the group B_3. The elementary triangles of the domain are in one-to-one correspondence with permutations described in the next section.

1.2 β-snakes

Consider a Möbius trigonometric M-polynomial of order n (n is odd). Walk counterclockwise along the source circle starting from some initial point so that the first critical point to meet is a local minimum. This provides us with a sequence $\{c_i\}$ of the critical values satisfying the relations

$$c_1 < c_2 > c_3 < \ldots > c_{2n+1} = c_1 \quad \text{and} \quad c_{i+n} = -c_i\,,\ i = 1, 2, \ldots, n\,.$$

The space of Möbius M-functions will be shown to be closely related to the space of such sequences. So let us consider the latter in some detail.

An arbitrary sequence of $2n$ real numbers subject to the above relations in which all the inequalities are allowed to be non-strict will be called a *Möbius snake of order* $2n$. A snake all of whose elements are distinct will be called *generic*.

Since a Möbius snake is completely defined by its initial half, the set S_n of all Möbius snakes of order $2n$ is the cone

$$c_1 \le c_2 \ge c_3 \le \ldots \ge c_n \le -c_1$$

in $\mathbf{R}^n$. The set S_n^{Σ} of non-generic snakes is its intersection with the set of mirrors

$$c_i = \pm c_j, \quad c_k = 0$$

of the reflection group B_n.

Theorem 1.1 *The number $\mathcal{M}_r$ of connected components of the set of generic Möbius snakes of order $2(2r+1)$ is equal to $(2r+1)\mathcal{S}_r$, where the numbers $\mathcal{S}_r$ are given by the exponential generating function*

$$\sum_{r=0}^{\infty} \mathcal{S}_r \frac{t^{2r}}{(2r)!} = \sec 2t\ .$$

Thus $\mathcal{M}_r = (2r+1)2^{2r}E_r$, where the sequence $\{E_r\}$ is that of the Euler numbers: 1, 1, 5, 61, 1385, … . Fig.1 is an illustration to $\mathcal{M}_1 = 12$.

Proof. Let $n = 2r+1$. Each of the connected components under consideration is contractible and contains one and only one n-sequence which is the initial half of a *normalised* Möbius snake, that is a Möbius snake which is a permutation of the numbers $\pm 1, \pm 2, \ldots, \pm n$. So we need to show that the number of the normalised snakes is that claimed in the theorem.

Let us delete the absolute maximum $c_\ell = n$ and minimum $c_{\ell+n} = -n$ of a normalised Möbius snake and consider the $(n-1)$-sequence $s_i = c_{\ell+i}$,

$i = 1, \dots, n-1$ (the indexation in the original snake is taken modulo $2n$). The obtained integer sequence satisfies the conditions:

$$s_1 < s_2 > s_3 < \dots s_{n-1} \qquad \text{and} \qquad \{|s_i|\} = \{1, 2, \dots, n-1\}\,.$$

Following [2], we call such a sequence a β_{n-1}-*snake* (we do not call it normalised since we will not need any others).

Denote by $\mathcal{B}_{n-1}$ the number of all possible β_{n-1}-snakes. The following lemma, allowing n to be of any parity (so that the last inequality in the above chain is ">" for even n), identifies the generating function $\mathcal{B}(t) = \sum_{k\geq 0} \mathcal{B}_k \frac{t^k}{k!}$.

Lemma 1.2 *(part of Theorem 24 of [2])*

$$\mathcal{B}(b) = \sec 2t + \tan 2t\,.$$

The lemma implies our theorem. Indeed, $\mathcal{S}_r = \mathcal{B}_{2r}$ and the maximal-minimal pair $(c_\ell, c_{\ell+n}) = (n, -n)$ of a normalised Möbius snake of order $2n = 2(2r+1)$ can stay in any of the $2r+1$ different positions. $\square$

1.3 Homeomorphism of the configurations

Consider the space $\mathbf{R}^{2r}$ of all Möbius trigonometric polynomials

$$\cos{(2r+1)\varphi} + \sum_{k=0}^{r-1} a_k \cos{(2k+1)\varphi} + b_k \sin{(2k+1)\varphi}$$

of degree $2r+1$ with fixed leading term. Let $M_r \subset \mathbf{R}^{2r}$ be the closure of the set of all M-functions, and $M_r^\Sigma \subset M_r$ its intersection with the bifurcation diagram of the family. The following theorem relates these two sets to the set $S_{2r+1} \subset \mathbf{R}^{2r+1}$ of all Möbius snakes of order $2(2r+1)$ and its subset S_{2r+1}^Σ of all non-generic snakes.

Theorem 1.3 *The pair (M_r, M_r^Σ) is homeomorphic to the intersection of the pair $(S_{2r+1}, S_{2r+1}^\Sigma)$ with a sphere in $\mathbf{R}^{2r+1}$ centered at the origin. The homeomorphism is a diffeomorphism at any of the internal points of M_r.*

Proof. Our statement will follow from the main result of [4] which asserts that the real analog of the Lyashko-Looijenga mapping [5, 3] provides a similar homeomorphism for trigonometric M-functions which are not necessarily Möbius. We recall the definiton of the mapping and the result.

Consider the space $\mathbf{R}^{2n-2}$ of *all* trigonometric polynomials of degree n with the fixed leading term $\cos(n\varphi)$ and no free term. Take one of M-functions in it and order its $2n$ critical points as in Section 1.2 starting from a local minimum. Since the closure $N_n \subset \mathbf{R}^{2n-2}$ of the set of all M-functions is contractible [4], this induces the ordering of the critical points of any function in the interior of N_n. Now to each of the functions in the interior we associate the ordered set $c_1 < c_2 > c_3 < \ldots < c_{2n} > c_1$ of its critical values. Shift all the values by their arithmetic mean. This gives a mapping into the hyperplane $c_1 + c_2 + \ldots + c_{2n} = 0$ in the $2n$-dimensional coordinate c-space equipped with the diagonals $c_i = c_j$. Its composition with the radial projection onto a sphere in it centred at the origin is a diffeomorphism between the interior of N_n and the spherical domain defined by the above chain of inequalities. The diffeomorphism maps the Maxwell stratum to the diagonals, and extends to a homeomorphism $\mathcal{L}_n$ of the closures [4].

Returning to the Möbius M-functions, consider M_r as a subset of N_{2r+1}. The composition of the restriction of $\mathcal{L}_{2r+1}$ to M_r with the further projection forgetting the last half of the coordinates in the c-space $\mathbf{R}^{4r+2}$ is the homeomorphism required in the claim of the theorem. □

In Fig.1, the ordering of the critical values is induced by that for the function $\frac{1}{3}\cos 3\varphi$ starting from its local minimum at $\varphi = \pi/3$.

1.4 Monodromy

Now consider trigonometric polynomials of order n with the leading term varying and no free term:

$$\sum_{k=1}^{n} a_k \cos k\varphi + b_k \sin k\varphi = \sum_{k=1}^{n} A_k \cos(k\varphi - \delta_k), \quad A_n \neq 0 .$$

The set of all such polynomials is $S^1 \times \mathbf{R}^{2n-1}$.

For any real τ, the shift

$$\varphi \mapsto \varphi + \tau, \qquad \delta_k \mapsto \delta_k + k\tau, \quad k = 1, \ldots, n,$$

does not change the value of a function. Thus, when δ_n changes from 0 to 2π, the bifurcation diagram of a subfamily with the fixed leading term maps onto itself with the twist by $2k\pi/n$ in the coordinate (a_k, b_k)-plane. For the diagram of Fig.1 this is the counterclockwise rotation by $2\pi/3$.

In terms of the critical values, the above monodromy cancels the difference between every n snakes obtained from each other by rotation of the source circle. Thus the number of different topological types of generic Möbius trigonometric M-functions is given by

Corollary 1.4 *In the set $S^1 \times \mathbf{R}^{2r+1}$ of all Möbius trigonometric polynomials of degree $2r+1$ with an arbitrary leading term, the subset of M-functions with all their critical values distinct has $2^{2r} E_r$ connected components each contractible onto a circle.*

2 Odd trigonometric polynomials

2.1 Degree 3 family

Again we start with an illustration to our other general statement.

Consider the family

$$\sin\varphi\,(\cos^2\varphi + a\cos\varphi + b)$$

of degree 3 odd trigonometric polynomials.

The set of non-generic functions in the family (Fig.2) consists of:

- Σ_c, caustic, — functions with non-Morse critical points not at multiples of π;
- Σ_b, *boundary* caustic, — functions with non-Morse critical points at multiples of π, thus on their zero-levels;
- Σ_m, Maxwell stratum, — functions with coinciding critical values on the open interval $(0,\pi)$;
- Σ_m^-, *anti*-Maxwell stratum, — functions with anti-coinciding critical values $c_i = -c_j$ on the open interval $(0,\pi)$;
- Σ_0, *special* Maxwell stratum, — functions with critical points on their zero-levels not at multiples of π.

In Fig.2, the asymptote for the anti-Maxwell strata is $b = -\frac{1}{2}$. The strata Σ_m^- and Σ_m intersect at $b = -\frac{1}{4}$, that is at $\frac{1}{4}\sin 3\varphi$. The strata Σ_m and Σ_0 meet at the origin. The intersections of Σ_b and Σ_m^- are at $(\pm 3(1-2^{-1/3}), 2 - 3\cdot 2^{-1/3})$.

The shaded region in the parameter space is that of M-functions. Its configuration is isomorphic to the spherical domain $0 < c_1 > c_2 < c_3 > 0$ in the 3-space containing the B_3 arrangement.

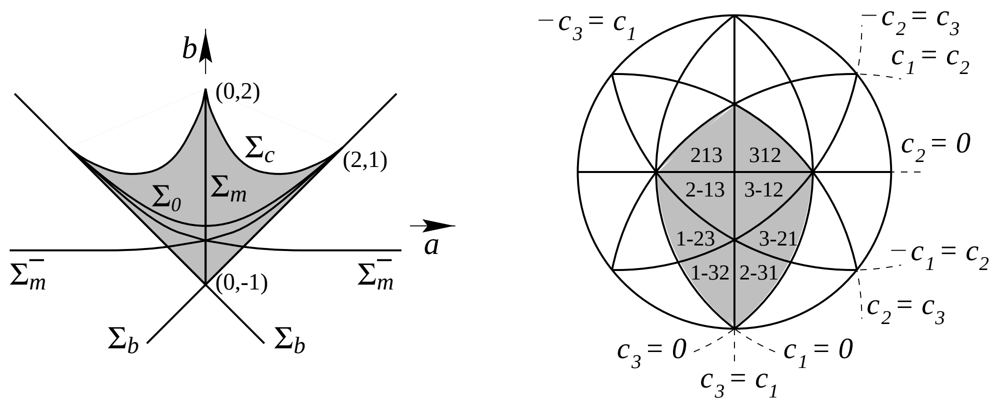

Figure 2: *Stratification of the space of odd functions of degree 3 compared with the B_3 mirror arrangement. The spherical triangles are marked with the β_3-snakes of critical values of the functions in the open interval* $(0, \pi)$.

2.2 The polyhedral model

Up to the choice of the sign of a function, we may assume that any real odd trigonometric polynomial of degree n enters the family

$$\Phi_n = \sin\varphi\left(\cos^{n-1}\varphi + a_1\cos^{n-2}\varphi + \ldots + a_{n-2}\cos\varphi + a_{n-1}\right).$$

Consider the restriction of the real Lyashko-Looijenga mapping $\mathcal{L}_n$ from the space of all trigonometric M-polynomials of degree n with the fixed leading term $\frac{1}{n}\sin n\varphi$ to M-functions of the family Φ_n. Let the ordering of the critical values for $\mathcal{L}_n$ be induced by that of the function $\frac{1}{n}\sin n\varphi$ with respect to the increase of the coordinate $\varphi \in (0, 2\pi)$ of its critical points. According to [2], our restriction of $\mathcal{L}_n$ maps the set of all M-functions of the family Φ_n diffeomorphically onto the spherical domain in $\mathbf{R}^{2n}$ lying in the cone

$$c_1 > c_2 < \ldots > c_{2n} < c_1\,, \qquad c_i = -c_{2n+1-i}\,, \quad i = 1, \ldots, n\,.$$

Thus, the set of all M-functions in Φ_n is connected. Denote by O_n its closure.

The subset of functions in O_n for which $c_1 = 0$ (and thus $c_{2n} = 0$) lies on the boundary of O_n. Indeed this special critical value in general corresponds to a cubical critical point at $\varphi = 0$.

So, for any M-function from Φ_n, the above ordering of the critical values is that given by the increase of the coordinate $\varphi \in (0, 2\pi)$ of its own critical points. In particular, for such a function the first critical point is a local maximum and $c_1 > 0$. Similarly, $(-1)^{n-1}c_n > 0$.

Now let $O_n^\Sigma \subset O_n$ be the subset of non-generic odd functions in the parameter space $\mathbf{R}^{n-1}$ of the family Φ_n. Denote by P_n the intersection of the cone

$$0 \le c_1 \ge c_2 \le c_3 \ge \ldots c_n\,, \qquad (-1)^{n-1}c_n \ge 0$$

with the sphere in the coordinate space $\mathbf{R}^n$ centred at the origin. Let P_n^Σ be the intersection of P_n with the mirror arrangement of the group B_n.

The above discussion proves

Theorem 2.1 *The pairs* (O_n, O_n^Σ) *and* (P_n, P_n^Σ) *are homeomorphic. The homemorphism between them is a diffeomorphism at the internal points.*

The homemorphism is provided by the composition of the restriction of the mapping $\mathcal{L}_n$ with the projection forgetting the critical values $c_{n+1}, \ldots, c_{2n}$.

2.3 Enumeration of the topological types

Theorem 2.2 *The numbers* $\mathcal{O}_n$ *of topologically distinct real odd trigonometric M-functions of degree* n *fit in the exponential generating function*

$$\sum_{n=0}^{\infty} \mathcal{O}_n \frac{t^n}{n!} = \sec 2t + \tan 2t\,.$$

For $n = 0$ this makes a reasonable sense: there is only one odd function, $g \equiv 0$, of degree 0 with the maximal possible number 1 of distinct critical values.

The topological equivalence of the theorem is that via odd diffeomorphisms of the source circle preserving its orientation and fixing $\varphi = 0$, and orientation-preserving odd diffeomorphisms of the target real axis. Thus the family Φ_n contains only half of all the types. For example, in Fig.2 we see $8 = \mathcal{O}_3/2$ of them.

Proof of the theorem. Consider for the moment only the topological types represented in Φ_n. From the previous section we see that their number is that of β_n-snakes (more precisely, of the negatives of them) which might be continued to the left and right by zeros so that the inequality chain stays alternating:

$$0 < p_1 > p_2 < p_3 > \dots p_n\,, \quad (-1)^{n-1}p_n > 0, \qquad \{|p_i|\} = \{1, 2, \dots, n\}\,.$$

An odd diffeomorphism of the real axis adding 1 to each natural number establishes a one-to-one correspondence between the above snakes and sequences

$$1 = \bar{p}_0 < \bar{p}_1 > \bar{p}_2 < \bar{p}_3 > \dots \bar{p}_n\,, \qquad \{|\bar{p}_i|\} = \{1, 2, \dots, n+1\}\,.$$

extendible by 0 to the right in the similar way. Giving up the restriction $\bar{p}_0 = 1$ we get exactly the definition of Arnold's γ_{n+1}-snakes [2].

Let $\mathcal{G}^k_{n+1}$ be the number of γ_{n+1}-snakes with the fixed beginning $\bar{p}_0 = k$, and $\mathcal{B}^k_{n+1}$ the number of β_{n+1}-snakes starting with $k = \bar{p}_0 < \bar{p}_1 > \dots$. We have to identify $\mathcal{G}^1_{n+1} = \mathcal{O}_n/2$.

Lemma 2.3 *(part of Theorem 15 of [2]) For $k > 0$,*

$$\mathcal{B}^{-k}_{n+1} = \mathcal{G}^{k-(n+2)}_{n+1} + \mathcal{G}^{(n+2)-k}_{n+1}\,.$$

Setting $k = n+1$ we get

Corollary 2.4 $$2\mathcal{G}^1_{n+1} = \mathcal{B}_n\,.$$

Indeed, according to the lemma, $\mathcal{B}^{-(n+1)}_{n+1} = \mathcal{G}^{-1}_{n+1} + \mathcal{G}^1_{n+1}$. On the other hand, $\mathcal{B}^{-(n+1)}_{n+1} = \mathcal{B}_n$ and $\mathcal{G}^{-1}_{n+1} = \mathcal{G}^1_{n+1}$.

According to Lemma 1.2, the exponential generating function for the numbers $\mathcal{B}_n$ is $\sec 2t + \tan 2t$. □

Remark 2.5 The exponential generating function $\sum_{n\geq 0} \mathcal{E}_{n-1}\frac{t^n}{n!}$ for the numbers $\mathcal{E}_n$ of topological types of real even trigonometric M-functions of degree n is easily seen to be $2(\sec t + \tan t)$, with a very formal setting $\mathcal{E}_{-1} = 2 = \mathcal{E}_0$.

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