Lagrangian and Legendrian varieties and stability of their projections

V.V.Goryunov and V.M.Zakalyukin

The study of singular Lagrangian and Legendrian varieties was initiated about twenty-five years ago by Arnold when he was investigating singularities in the variational problem of obstacle bypassing [1]. The first examples of such varieties, open swallowtails, were related to the discriminants of the non-crystallographic Coxeter groups [8, 14]. Incorporating these examples into a general context, Givental [8] introduced the notion of stability of Lagrangian and Legendrian varieties with respect to perturbations of symplectic structure and Lagrangian or, respectively, Legendrian projection only, keeping the diffeomorphic type of the variety fixed.

Later, in [13], it was shown that this stability notion has an explicit geometrical meaning in terms of generating families, versal deformations of function singularities and inducing mappings.

The interest in theory of singular Lagrangian and Legendrian varieties has been growing recently due to its possible applications to Frobenius structures, D-modules and in other areas.

The first half of these notes contains generalities about Lagrangian and Legendrian singularities. The second half is devoted to stable Lagrangian projections playing the central rôle in the geometry of Hamiltonian systems and, in particular, in the theory of F-manifolds. There we extend the results of [13] to a natural modification of Givental's stability notion and show that a wide class of Lagrangian and Legendrian varieties associated to matrix singularities (see [5, 6, 11]) and singularities of composed mappings [10] are stable.

The lectures are based on the books [2, 4, 3] and paper [12].

1 Symplectic and contact geometry

1.1 Symplectic geometry

A symplectic form ω on a manifold M is a closed 2-form, non-degenerate as a skew-symmetric bilinear form on the tangent space at each point. So $d\omega = 0$ and ω^n is a volume form, $\dim M = 2n$.

Manifold M equipped with a symplectic form is called symplectic. It is necessarily even-dimensional.

If the form is exact, $\omega = d\lambda$, the *symplectic area* of a 2-chain S is $\int_{\partial S} \lambda$. When λ exists and is fixed M is called *exact symplectic*.

Examples.
1. Let $K = M = \mathbf{R}^{2n} = \{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ be a vector space, and

$$\lambda = pdq = \sum_{i=1}^{n} p_i dq_i\,, \qquad \omega = d\lambda = dp \wedge dq\,.$$

In these coordinates the form ω is constant. The corresponding bilinear form on the tangent space at a point is given by the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

NOTICE: for any non-degenerate skew-symmetric bilinear form on a linear space, there exists a basis (called Darboux basis) in which the form has this matrix.

2. $M = T^*N$. Take for λ the *Liouville form* defined in an invariant (coordinate-free) way as

$$\lambda(\alpha) = \pi(\alpha)\Big(\rho_*(\alpha)\Big)\,,$$

where

$$\alpha \in T(T^*N)\,, \quad \pi : T(T^*N) \to T^*N \quad \text{and} \quad \rho : T^*N \to N\,.$$

This is an exact symplectic manifold. If $q_1, \ldots, q_n$ are local coordinates on the base N, the dual coordinates $p_1, \ldots, p_n$ are the coefficients of the decomposition of a covector into linear combination of the differentials dq_i:

$$\lambda = \sum_{i=1}^{n} p_i dq_i\,.$$

A diffeomorphism $\varphi : M_1 \to M_2$ which sends the symplectic structure ω_2 on M_2 to the symplectic structure ω_1 on M_1,

$$\varphi^* \omega_2 = \omega_1\,,$$

is called a symplectomorphism between (M_1, ω_1) and (M_2, ω_2). When the (M_i, ω_i) are the same, a symplectomorphism preserves the symplectic structure. In particular, it preserves the volume form ω^n.

Symplectic group.
For $K = (\mathbf{R}^{2n}, dp \wedge dq)$ of our first example, the group $Sp(2n)$ of linear symplectomorphisms is isomorphic to the group of matrices S such that

$$S^{-1} = -JS^tJ\,.$$

Here t is for transpose.

The dimension k of a linear subspace $L^k \subset K$ and the rank r of the restriction of the bilinear form ω on it are the complete set of $Sp(2n)$-invariants of L.

Define the skew-orthogonal complement $L^\angle$ of L as

$$L^\angle = \{v \in K | \omega(v,u) = 0 \quad \forall u \in L\}\,.$$

So $\dim L^\angle = 2n-k$. The kernel subspace of the restriction of ω to L is $L \bigcap L^\angle$. Its dimension is $k - r$.

A subspace is called isotropic if $L \subset L^\angle$ (hence $\dim L \leq n$).
Any line is isotropic.

A subspace is called co-isotropic if $L^\angle \subset L$ (hence $\dim L \geq n$).
Any hyperplane H is co-isotropic. The line $H^\angle$ is called the characteristic direction on H.

A subspace is called Lagrangian if $L^\angle = L$ (hence $\dim L = n$).

Lemma 1. *Each Lagrangian subspace $L \subset K$ has a regular projection to at least one of the 2^n coordinate Lagrangian planes (p_I, q_J), along the complementary Lagrangian plane (p_J, q_I). Here $I \bigcup J = \{1, \ldots, n\}$ and $I \bigcap J = \emptyset$.*

A Lagrangian subspace L which projects regularly onto the q-plane is the graph of a self-adjoint operator S from the q-space to the p-space with its matrix symmetric in the Darboux basis.

Even in a non-linear setting symplectic structure has no local invariants (unlike Riemannian structure) according to

Darboux Theorem. *Any two symplectic manifolds of the same dimension are locally symplectomorphic.*

Weinstein's Theorem. *A submanifold of a symplectic manifold is defined, up to a symplectomorphism of its neighbourhood, by the restriction of the symplectic form to the tangent vectors to the ambient manifold at the points of the submanifold.*

In a similar local setting, the inner geometry of a submanifold defines its outer geometry:

Givental's Theorem. *A germ of a submanifold in a symplectic manifold is defined, up to a symplectomorphism, by the restriction of the symplectic structure to the tangent bundle of the submanifold.*

Proof of Givental's Theorem. It is sufficient to prove that if the restrictions of two symplectic forms, ω_0 and ω_1, to the tangent bundle of a submanifold $G \subset M$ at point A coincide, then there exits a local diffeomorphism of M fixing G point-wise and sending one form to the other. We may assume that the forms coincide on $T_A M$.

We again use the homotopy method, aiming to find a family of diffeomorphism-germs g_t, $t \in [0,1]$, such that

$$g_t|_G = id_G\,, \quad g_0 = id_M\,, \qquad g_t^*(\omega_t) = \omega_0 \quad (*) \qquad \text{where} \quad \omega_t = \omega_0 + (\omega_1 - \omega_0)t\,.$$

Differentiating $(*)$ by t, we again get

$$\mathrm{Lie}_{v_t}(\omega_t) = d(i_{v_t}\omega_t) = \omega_0 - \omega_1$$

where v_t is the vector field of the flow g_t. Using the "relative Poincaré lemma", it is possible to find a 1-form α so that $d\alpha = \omega_0 - \omega_1$ and α vanishes on G. Then the required vector field v_t exists since ω_t is non-degenerate. □

Darboux theorem is a particular case of Givental's theorem: take a point as a submanifold.

If at each point x of a submanifold L of a symplectic manifold M the subspace T_xL is Lagrangian in the symplectic space T_xM, then L is called Lagrangian.

Examples.
1. In T^*N, the following are Lagrangian submanifolds: the zero section of the bundle, fibres of the bundle, graph of the differential of a function on N.

2. The graph of a symplectomorphism is a Lagrangian submanifold of the product space (it has regular projections onto the factors). An arbitrary Lagrangian submanifold of the product space defines a so-called Lagrangian relation.

3. Weinstein's theorem implies that a tubular neighbourhood of a Lagrangian submanifold L in any symplectic space is symplectomorphic to a tubular neighbourhood of the zero section in T^*N.

A fibration with Lagrangian fibres is called Lagrangian.

Locally all Lagrangian fibrations are symplectomorphic (the proof is similar to that of Darboux theorem).

A cotangent bundle is a Lagrangian fibration.

Let $\psi : L \to T^*N$ be a Lagrangian embedding and $\rho : T^*N \to N$ the fibration. The product $\rho \circ \psi : L \to N$ is called a Lagrangian mapping. It critical values

$$\Sigma_L = \{q \in N | \exists p : (p,q) \in L,\ \mathrm{rank}\, d(\rho \circ \psi) < n\}$$

form the caustic of the Lagrangian mapping. The equivalence of Lagrangian mappings is that up to fibre-preserving symplectomorphisms of the ambient symplectic space. Caustics of equivalent Lagrangian mappings are diffeomorphic.

Hamiltonian vector fields.
Given a real function $h : M \to \mathbf{R}$ on a symplectic manifold, define a Hamiltonian vector field v_h on M by the formula

$$\omega(\cdot, v_h) = dh\,.$$

This field is tangent to the level hypersurfaces $H_c = h^{-1}(c)$:

$$\forall a \in H_c \quad dh(T_a H_c) = 0 \qquad \Longrightarrow \qquad T_a H_c = v_h^{\angle}\,, \quad \text{but} \quad v_h \in v_h^{\angle}\,.$$

The directions of v_h on the level hypersurfaces H_c of h are the characteristic directions of the tangent spaces of the hypersurfaces.

Associating v_h to h, we obtain a Lie algebra structure on the space of functions:

$$[v_h, v_f] = v_{\{h,f\}} \qquad \text{where} \quad \{h, f\} = v_h(f)\,,$$

the latter being the Poisson bracket of the Hamiltonians h and f.

A Hamiltonian flow (even if h depends on time) consists of symplectomorphisms. Locally (or in $\mathbf{R}^{2n}$), any time-dependent family of symplectomorphisms that starts from the identity is a phase flow of a time-dependent Hamiltonian. However, for example, on a torus $\mathbf{R}^2/(\mathbf{Z}^2)$ (which is the quotient of the plane by an integer lattice) the family of constant velocity displacements are symplectomorphisms but they cannot be Hamiltonian since a Hamiltonian function on a torus must have critical points.

Given a time-dependent Hamiltonian $\widetilde{h} = \widetilde{h}(t, p, q)$, consider the extended space $M \times T^*\mathbf{R}$ with auxiliary coordinates (s, t) and the form $pdq - sdt$. An auxiliary (extended) Hamiltonian $\widehat{h} = -s + \widetilde{h}$ determines a flow in the extended space generated by the vector field

$$\dot{p} = -\frac{\partial \widehat{h}}{\partial q} \qquad\qquad \dot{q} = -\frac{\partial \widehat{h}}{\partial p}$$

$$\dot{t} = -\frac{\partial \widehat{h}}{\partial s} = 1 \qquad\qquad \dot{s} = \frac{\partial \widehat{h}}{\partial t}$$

The restrictions of this flow to the $t = const$ sections are essentially the flow mappings of $\widetilde{h}$.

The integral of the extended form over a closed chain in $M \times \{t_o\}$ is preserved by the $\hat{h}$-Hamiltonian flow. Hypersurfaces $-s + \widetilde{h} = const$ are invariant. When $\widetilde{h}$ is autonomous, the form pdq is also a relative integral invariant.

A (transversal) intersection of a Lagrangian submanifold $L \subset M$ with a Hamiltonian level set $H_c = h^{-1}(c)$ is an isotropic submanifold L_c. All Hamiltonian trajectories emanating from L_c form a Lagrangian submanifold $exp_H(L_c) \subset M$. The space Ξ_{H_c} of the Hamiltonian trajectories on H_c inherits, at least locally, an induced symplectic structure. The image of the projection of $exp_H(L_c)$ to Ξ_{H_c} is a Lagrangian submanifold there. This is a particular case of a symplectic reduction which will be discussed later.

Example. The set of all oriented straight lines in $\mathbf{R}^n_q$ is T^*S^{n-1} as a space of characteristics of the Hamiltonian $h = p^2$ on its level $p^2 = 1$ in $K = \mathbf{R}^{2n}$.

1.2 Contact geometry

An odd-dimensional manifold M^{2n+1} equipped with a maximally non-integrable distribution of hyperplanes (contact elements) in the tangent spaces of its points is called a contact manifold.

The maximal non-integrability means that if locally the distribution is determined by zeros of a 1-form α on M then $\alpha \wedge (d\alpha)^n \neq 0$ (cf. the Frobenius condition of complete integrability being $\alpha \wedge d\alpha = 0$.)

Examples.
1. A projectivised cotangent bundle PT^*N^{n+1} with the projectivisation of the Liouville form $\alpha = pdq$. This is also called a space of contact elements on N. The spherisation of PT^*N^{n+1} is a 2-fold covering of PT^*N^{n+1} and its points are co-oriented contact elements.

2. The space J^1N of 1-jets of functions on N^n. (Two functions have the same m-jet at a point x if their Taylor polynomials of degree k at x coincide). The space of all 1-jets at all points of N has local co-ordinates $q \in N$, $p = df(q)$ which are the partial derivatives of a function at q, and $z = f(q)$. The contact form is $pdq - dz$.

Contactomorphisms are diffeomorphisms preserving the distribution of contact elements.

Contact Darboux theorem. *All equidimensional contact manifolds are locally contactomorphic.*

An analog of Givental's theorem also holds.

Symplectisation.
Let $\widetilde{M}^{2n+2}$ be the space of all linear forms vanishing on contact elements of M. The space $\widetilde{M}^{2n+2}$ is a "line" bundle over M (fibres do not contain the zero forms). Let

$$\widetilde{\pi} : \widetilde{M} \to M$$

be the projection. On $\widetilde{M}$, the symplectic structure (which is homogeneous of degree 1 with respect to fibres) is the differential of the canonical 1-form $\widetilde{\alpha}$ on $\widetilde{M}$ defined as

$$\widetilde{\alpha}(\xi) = p(\widetilde{\pi}_* \xi), \qquad \xi \in T_p\widetilde{M}.$$

A contactomorphism F of M lifts to a symplectomorphism of $\widetilde{M}$:

$$\widetilde{F}(p) := (F^*_{F(x)})^{-1} p \, .$$

This commutes with the multiplication by constants in the fibres and preserves $\widetilde{\alpha}$. The symplectisation of contact vector fields (= infinitesimal contactomorphisms) yields Hamiltonian vector fields with homogeneous (of degree 1) Hamiltonian functions $h(rx) = rh(x)$.

Assume the contact structure on M is defined by zeros of a fixed 1-form β. Then M has a natural embedding $x \mapsto \beta_x$ into $\widetilde{M}$.

Using the local model $J^1\mathbf{R}^n$, $\beta = pdq - dz$, of a contact space we get the following formulas for components of the contact vector field with a homogeneous Hamiltonian function $K(x) = h(x_\beta)$ (notice that $K = \beta(X)$ where X is the corresponding contact vector field):

$$\dot{z} = pK_p - K, \quad \dot{p} = -K_q - pK_z, \quad \dot{q} = K_p.$$

where the subscripts mean the partial derivations.

Various homogeneous analogs of symplectic properties hold in contact geometry (the analogy is similar to that between affine and projective geometries).

In particular, a hypersurface (transversal to the contact distribution) in a contact space inherits a field of characteristics.

Contactisation.
To an exact symplectic space M^{2n} associate $\widehat{M} = \mathbf{R} \times M$ with an extra coordinate z and take the 1-form $\alpha = \lambda - dz$. This gives a contact space.

Here the vector field $\chi = -\frac{\partial}{\partial z}$ is such that $i_\chi \alpha = 1$ and $i_\chi d\alpha = 0$. Such a field is called a Reeb vector field. Its direction is uniquely defined by a contact structure. It is transversal to the contact distribution. Locally, projection along χ produces a symplectic manifold.

A Legendrian submanifold Λ of M^{2n+1} is an n-dimensional integral submanifold of the contact distribution. This dimension is maximal possible for integral submanifolds.

Examples.
1. To a Lagrangian $L \subset T^*M$ associate $\Lambda \subset J^1M$:

$$\Lambda = \{(z, p, q) \mid z = \int pdq, \ (p, q) \in L\} \, .$$

Here the integral is taken along a path on L joining a distinguished point on L with the point (p, q). Such an Λ is Legendrian.

2. The set of all covectors annihilating tangent spaces to a given submanifold (or variety) $W_0 \subset N$ form a Legendrian submanifold (variety) in PT^*N.

3. If the intersection I of a Legendrian submanifold Λ with a hypersurface Γ in a contact space is transversal, then I is transversal to the characteristic vector field on Γ. The set of characteristics emanating from I form a Legendrian submanifold.

A Legendrian fibration of a contact space is a fibration with Legendrian fibres. For example, $PT^*N \to N$ and $J^1N \to J^0N$ are Legendrian. Any two Legendrian fibrations of the same dimension are locally contactomorphic.

The projection of an embedded Legendrian submanifold Λ to the base of a Legendrian fibration is called a Legendrian mapping. Its image is called the wave front of Λ.

Examples.
1. Embed a Legendrian submanifold Λ into J^1N. Its projection to J^0N, wave front $W(\Lambda)$, is a graph of a multivalued action function $\int pdq + c$ (again we integrate along paths on the Lagrangian submanifold $L = \pi_1(\Lambda)$, where $\pi_1 : J^1N \to T^*N$ is the projection dropping the z co-ordinate). If $q \in N$ is not in the caustic Σ_L of L, then over q the wave front $W(\Lambda)$ is a collection of smooth sheets.

If at two distinct points $(p', q), (p'', q) \in L$ with a non-caustical value q, the values z of the action function are equal, then at (z, q) the wave front is a transversal intersection of graphs of two regular functions on N.

The images under the projection $(z, q) \mapsto q$ of the singular and transversal self-intersection loci of $W(\Lambda)$ are respectively the caustic Σ_L and so-called Maxwell (conflict) set.

2. To a function $f = f(q)$, $q \in \mathbf{R}^n$, associate its Legendrian lifting $\Lambda = j^1(f)$ (also called the 1-jet extension of f) to $J^1\mathbf{R}^n$. Project Λ along the fibres parallel to the q-space of another Legendrian fibration

$$\pi_1^\wedge(z, p, q) \mapsto (z - pq, p)$$

of the same contact structure $pdq - dz = -qdp - d(z - pq)$. The image $\pi_1^\wedge(\Lambda)$ is called the Legendre transform of the function f. It has singularities if f is not convex.

This is an affine version of the projective duality (which is also related to Legendrian mappings). The space PT^*P^n (P^n is the projective space) is isomorphic to the projectivised cotangent bundle $PT^*P^{n\wedge}$ of the dual space $P^{n\wedge}$. Elements of both are pairs consisting of a point and a hyperplane, containing the point. The natural contact structures coincide. The set of all hyperplanes in P^n tangent to a submanifold $S \subset P^n$ is the front of the dual projection of the Legendrian lifting of S.

2 Generating families

2.1 Lagrangian case

Consider a co-isotropic submanifold $C^{n+k} \subset M^{2n}$. The skew-orthogonal complements $T_c^{\angle}C$, $c \in C$, of tangent spaces to C define an integrable distribution on C. Indeed, take two regular functions whose common zero level set contains C. At each point $c \in C$, the vectors of their Hamiltonian fields belong to $T_c^{\angle}C$. So the corresponding flows commute. Trajectories of all such fields emanating from $c \in C$ form a smooth submanifold I_c integral for the distribution.

By Givental's theorem, any co-isotropic submanifold is locally symplectomorphic to a coordinate subspace $p_I = 0$, $I = \{1, \dots, n-k\}$, in $K = \mathbf{R}^{2n}$. The fibres are the sets $q_J = const$.

Proposition 2. *Let L^n and C^{n+k} be respectively Lagrangian and co-isotropic submanifolds of a symplectic manifold M^{2n}. Assume L meets C transversally at a point a. Then the intersection $X_0 = L \bigcap C$ is transversal to the isotropic fibres I_c near a.*

The proof is immediate. If T_aX_0 contains a vector $v \in T_aI_c$, then v is skew-orthogonal to T_aL and also to T_aC, that is to any vector in T_aM. Hence $v = 0$.

Isotropic fibres define the fibration $\xi : C \to B$ over a certain manifold B of dimension $2k$ (defined at least locally). We can say that B is the manifold of isotropic fibres.

It has a well-defined induced symplectic structure ω_B. Given any two vectors u, v tangent to B at a point b take their liftings, that is vectors $\widetilde{u}, \widetilde{v}$ tangent to C at some point of $\xi^{-1}(b)$ such that their projections to B are u and v. The value $\omega(\widetilde{u}, \widetilde{v})$ depends only on the vectors u, v. For any other choice of liftings the result will be the same. This value is taken for the value of the two-form ω_B on B.

Thus, the base B gets a symplectic structure which is called a symplectic reduction of the co-isotropic submanifold C.

Example. Consider a Lagrangian section L of the (trivial) Lagrangian fibration $T^*(\mathbf{R}^k \times \mathbf{R}^n)$. The submanifold L is the graph of the differential of a function $f = f(x, q)$, $x \in \mathbf{R}^k$, $q \in \mathbf{R}^n$. The dual coordinates y, p are given on L by $y = \frac{\partial f}{\partial x}$, $p = \frac{\partial f}{\partial q}$. Therefore, the intersection $\widetilde{L}$ of L with the co-isotropic subspace $y = 0$ is given by the equations $\frac{\partial f}{\partial x} = 0$. The intersection is transversal iff the rank of the matrix of the derivatives of these equations, with respect to x and q, is k. If so, the symplectic reduction of $\widetilde{L}$ is a Lagrangian submanifold L_r in $T^*\mathbf{R}^n$ (it may be not a section of $T^*\mathbf{R}^n \to \mathbf{R}^n$).

This example leads to the following definition of a generating function (the idea is due to Hörmander).

Definition. A generating family of the Lagrangian mapping of a submanifold $L \subset T^*N$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ \quad (p, q) \quad \Big| \quad \exists x : \frac{\partial F(x, q)}{\partial x} = 0, \quad p = \frac{\partial F(x, q)}{\partial q} \quad \right\}.$$

Here $q \in N$, and x is in the fibre over q. We also assume that the following Morse condition is satisfied:

$$0 \text{ is a regular value of the mapping } \quad (x,q) \mapsto \frac{\partial F}{\partial x}\,.$$

The latter guarantees L being a smooth manifold.

Remark. The points of the intersection of L with the zero section of T^*N are in one-to-one correspondence with the critical points of the function F.

Existence.
Any germ L of a Lagrangian submanifold in $T^*\mathbf{R}^n$ has a regular projection to some (p_J, q_I) co-ordinate space. In this case there exists a function $f = f(p_J, q_I)$ (defined up to a constant) such that

$$L = \left\{ \quad (p,q) \quad | \quad q_J = -\frac{\partial f}{\partial p_J}\,, \quad p_I = \frac{\partial f}{\partial q_I} \quad \right\}.$$

Then the family $F_J = xq_J + f(x, q_I)$, $x \in \mathbf{R}^{|J|}$, is generating for L. If $|J|$ is minimal possible, then $\mathrm{Hess}_{xx} F_J = \mathrm{Hess}_{p_J p_J} f$ vanishes at the distinguished point.

Uniqueness.
Two family-germs $F_i(x,q)$, $x \in \mathbf{R}^k$, $q \in \mathbf{R}^n$, $i = 1, 2$, at the origin are called <u>$\mathcal{R}_0$-equivalent</u> if there exists a diffeomorphism $\mathcal{T} : (x,q) \mapsto (X(x,q), q)$ (i.e. preserving the fibration $\mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$) such that $F_2 = F_1 \circ \mathcal{T}$.

The family $\Phi(x,y,q) = F(x,q) \pm y_1^2 \pm \ldots, \pm y_m^2$ is called a <u>stabilisation</u> of F.

Two family-germs are called <u>stably $\mathcal{R}_0$-equivalent</u> if they are $\mathcal{R}_0$-equivalent to appropriate stabilisations of the same family (in a lower number of variables).

Lemma 3. *Up to addition of a constant, any two generating families of the same germ L of a Lagrangian submanifold are stably $\mathcal{R}_0$-equivalent.*

2.2 Legendrian case

Definition. A <u>generating family</u> of the Legendrian mapping $\pi|_L$ of a Legendrian submanifold $L \subset PT^*(N)$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ \quad (p,q) \quad | \quad \exists x : \quad F(x,q) = 0\,, \quad \frac{\partial F(x,q)}{\partial x} = 0\,, \quad p = \frac{\partial F(x,q)}{\partial q} \quad \right\},$$

where $q \in N$ and x is in the fibre over q, provided that the following Morse condition is satisfied:

$$0 \text{ is a regular value of the mapping } (x,q) \mapsto \{F, \frac{\partial F}{\partial x}\}\,.$$

Definition. Two function family-germs $F_i(x, q)$, $i = 1, 2$, are called V-equivalent if there exists a fibre-preserving diffeomorphism $\Theta : (x, q) \mapsto (X(x, q), q)$ and a function $\Psi(x, q)$ not vanishing at the distinguished point such that $F_2 \circ \Theta = \Psi F_1$.

Two function families are called stably V-equivalent if they are stabilisations of a pair of V-equivalent functions (may be in a lower number of variables x).

Theorem 4. *Any germ $\pi|_L$ of a Legendrian mapping has a generating family. All generating families of a fixed germ are stably V-equivalent.*

3 Stability of projections of Lagrangian varieties

3.1 0-stability

We shall slightly modify the standard notions introduced earlier.

3.1.1 The Lagrangian setup

A singular Lagrangian (sub)variety L of a symplectic space M^{2n} is an n-dimensional analytic subset of M which is Lagrangian in the ordinary sense at all its regular points. A Lagrangian projection π is a projection $\pi : M \to B^n$ defining a fibre bundle whose fibres are Lagrangian.

Fibres of any Lagrangian fibration posses a well-defined affine structure. Indeed, local coordinates on the base rise to regular functions on the total space, which are pairwise in involution. Hence their Hamiltonian vector fields do not vanish, commute and are tangent to the fibres.

The restriction $\pi|_L$ of the Lagrangian projection π to a Lagrangian subvariety $L \subset M$ is called a Lagrangian mapping.

Two Lagrangian mappings, of Lagrangian subvarieties L' and L'', are called equivalent if there exists a symplectomorphism of the ambient symplectic spaces sending L' to L'' and fibres of one Legendrian projection to fibres of the other. In particular, L' and L'' are symplectomorphic.

The germ of a Lagrangian map $\pi|_L$ of a variety L at its point m is called stable if the germ of any Lagrangian map $\widetilde{\pi}|_L$ close to $\pi|_L$ at any point $\widetilde{m}$ close to m is equivalent to the germ of $\pi|_L$ at a point near m. Notice that only the fibration π is allowed to vary in this context while the subvariety L is fixed.

According to Givental [8], the stability introduced is essentially equivalent to the following versality of the map-germ $\pi|_L$.

Let $\mathcal{O}_L$ be the algebra of regular functions on L and $\mathbf{m}_{B,m}$ the maximal ideal in the algebra $\mathcal{O}_{B,m}$ of function-germs on the base B at the point $\pi(m)$. We define the local algebra of the germ of $\pi|_L$ at m as

$$Q_m = \mathcal{O}_L/(\pi|_L)^*(\mathbf{m}_{B,m})\mathcal{O}_L\,.$$

The algebra Q_m is the algebra of restrictions of functions on L to the intersection of L with the fiber $F_{\pi(m)} = \pi^{-1}(\pi(m))$.

Denote by A_m the subspace of affine (with respect to the corresponding affine structure) functions on the fibre $F_{\pi(m)}$ and by $r : A_m \to Q_m$ the restriction homomorphism sending a function on the fibre to its restriction to $L \cap F_{\pi(m)}$.

The germ of $\pi|_L$ at $m \in L$ is called versal if r is surjective, and miniversal if r is an isomorphism.

Let p, q be local Darboux coordinates on M about m: $p(m) = p_0$, $q(m) = 0$ and $\pi(p, q) = q$. The Weierstrass preparation theorem implies that the versality of $\pi|_L$ at m is equivalent to the existence of a representation of any analytic function-germ φ on M at $m = (p_0, 0)$ in the form

$$\varphi(p, q) = \psi(p, q) + \sum_{j=1}^{n} a_j(q)p_j + a_0(q)\,, \tag{1}$$

where the a_j, $\ j \geq 0$ are analytic function-germs on the base B, and the function-germ ψ vanishes on L.

Remark. The decomposition means that any function-germ on M at m is a sum of a function vanishing on L and a function affine on the individual fibres. Therefore, any Hamiltonian vector field near m is a sum of a Hamiltonian vector field tangent to L and a Hamiltonian vector field preserving the fibration π. Hence the homotopy method implies that any symplectomorphism-germ of M at m close to the identity is a composition of a symplectomorhism preserving L and a symplectomorphism preserving the standard projection π. Since any perturbation of the germ of π in the class of Lagrangian projections is a composition of π with an appropriate symplectomorphism, the versality implies stability. See [8] for more details.

We now turn to a restricted version of the above setup. Namely, we take $M = T^*B$ to be the cotangent bundle of a manifold B^n, and distinguish the zero section T_0 of T^*B. Let $Sym_0(M)$ be the subgroup of symplectomorphisms of M preserving T_0.

Two Lagrangian mappings of Lagrangian subvarieties of a cotangent bundle are called 0-equivalent if they are equivalent via a symplectomorphism from $Sym_0(M)$.

Replacement of the equivalence by the 0-equivalence in the stability definition yields a definition of the 0-stability of Lagrangian map-germs.

The zero section T_0 determines a linear structure on fibres of a cotangent bundle. Replacing the space A_m of affine functions on F_m by its well-defined subspace A^0_m of linear functions, we obtain the definition of the 0-versality which is equivalent to the existence of the representation of any function-germ φ on M at m such that $\varphi|_{T_0} = 0$ in the form

$$\varphi(p, q) = \psi(p, q) + \sum_{j=1}^{n} a_j(q)p_j\,, \tag{2}$$

where the germs a_j and ψ are similar to those in (1) and we assume that the Darboux coordinates p vanish on the zero section T_0.

Like before, the 0-versality implies the 0-stability.

For the benefit of the exposition, we continue now with the complex case only. Everything below transfers absolutely straightforwardly to the real situation.

Lemma 5. *The projection $\pi : T^*\mathbf{C}^n \to \mathbf{C}^n$, $(p,q) \mapsto q$, of a Lagrangian germ L at the origin is 0-stable if and only if the germs of the products $p_i p_j$, $i,j = 1,\ldots,n$, have decompositions*

$$p_i p_j = \varphi_{ij}(p,q) + \sum_{k=1}^{n} p_k c_{ij}^k(q) \tag{3}$$

in which the function-germs φ_{ij} and c_{ij}^k are holomorphic, and the φ_{ij} vanish on L.

Proof. The "only if" part is obvious. To prove the "if" part, we notice that the ideal I generated by all the quadratic polynomials $P_{ij}(p) = p_i p_j - \sum c_{ij}^k(0) p_k$, $i,j = 1,\ldots,n$, in the space of all holomorphic function-germs on the fibre F_0 is of finite codimension. Modulo I, any function-germ on F_0 is an affine function in p. After the projection to the local algebra Q_0, that is after a further reduction modulo the functions vanishing on L (more precisely on $L \cap F_0$), such a function is still affine in p. Hence, the 0-versality condition holds. □

For the stability (rather than 0-stability) version of the lemma see [8].

The *suspension of a Lagrangian fibration* $\pi : M \to B$ is its direct product

$$\widehat{\pi} = (\pi, \pi_0) : \widehat{M} = M \times T^*\mathbf{C} \to B \times \mathbf{C}$$

with the canonical projection $\pi_0 : T^*\mathbf{C} \to \mathbf{C}$.

A *suspension of a Lagrangian variety* $L \subset M^{2n}$ is an $(n+1)$-dimensional Lagrangian variety $\widehat{L} \subset M \times T^*\mathbf{C}$ which is the product of L with the line $\ell = \{p_{n+1} = const \neq 0\}$ in $T^*\mathbf{C}$ endowed with the standard Darboux coordinates p_{n+1}, q_{n+1}.

The propositions below follow immediately from the definitions.

Proposition 6. *A map-germ $\pi|_L$ at $m \in M$ is (mini)versal if and only if its suspension $\widehat{\pi}|_{\widehat{L}}$ is 0-(mini)versal at a point of the line $m \times \ell$ (hence at all the points of this line) in $\widehat{M}$.*

Example. A germ of the standard projection π of a Lagrangian submanifold $L \subset T^*\mathbf{C}^n = \{p,q\}$ determined by a generating family $f = f(x,q)$ with parameters $q \in \mathbf{C}^n$ and variables $x \in \mathbf{C}^k$,

$$L = \{(p,q) | \exists x : \partial f/\partial x = 0, p = \partial f/\partial q\},$$

is stable if and only if the family-germ $f(\cdot,\cdot)$ is an $\mathcal{R}^+$-versal deformation of the function-germ $f(\cdot,0)$. The projection is 0-stable if $f(\cdot,\cdot)$ is an $\mathcal{R}$-versal deformation of the function germ $f(\cdot,0)$.

Proposition 7. *Consider a germ of a Lagrangian subvariety L in $\widehat{M} = T^*\mathbf{C}^n \times T^*\mathbf{C}$ at a point not in the zero section. Assume $\widehat{\pi}|_L$ is 0-versal and L belongs to a regular hypersurface in $\widehat{M}$ transversal to the $\partial_{p_{n+1}}$-direction. Then $\widehat{\pi}|_L$ is 0-equivalent to a suspension of a versal map-germ $\pi|_{L'}$ of a Lagrangian subvariety $L' \subset M = T^*\mathbf{C}^n$.*

3.1.2 The Legendrian case

A singular n-dimensional subvariety of a contact space is called Legendrian if at all its regular points it is Legendrian in the ordinary sense. For standard (and equivalent) local models of contact $(2n+1)$-spaces we use the projectivised cotangent bundle $PT^*\mathbf{C}^{n+1}$ and the space $J^1(\mathbf{C}^n, \mathbf{C}) = \{p, q, z\}$ of one-jets of functions on $\mathbf{C}^n$ endowed with the contact form $\alpha = dz - pdq$. The definitions of Legendrian mappings, stability and others are analogous to the Lagrangian case (see also [8]).

Symplectisation and contactisation functors relate Lagrangian and Legendrian germs as follows.

A. The projection $\rho : (p, q, z) \mapsto (p, q)$ maps a Legendrian variety $\Lambda \subset J^1(\mathbf{C}^n, \mathbf{C})$ to the Lagrangian variety $\rho(\Lambda) \subset T^*\mathbf{C}^n$.

B. Local Lagrangian fibration and its zero section determine uniquely the Liouville primitive form $\alpha = pdq$ of the symplectic form $\omega = d\alpha$. Given a Lagrangian germ $L \subset T^*\mathbf{C}^n$ at a point m, denote by $L_{0,m}$ the subset of points $s \in L$ such that the integral of α along some path γ on L joining m and s vanishes.

For simplicity we assume the values of the integral do not depend on the local path γ, that is the cohomology class of α vanishes on L (see [8] for examples of the opposite).

If $L_{0,m}$ does not meet the zero section of $T^*\mathbf{C}^n$, then its projectivisation is a Legendrian (or isotropic) variety in $PT^*\mathbf{C}^n$. Its projection $W_0(L, m) = \pi(L_{0,m}) \subset \mathbf{C}^n$ is called the 0-wave front of L.

C. For a Lagrangian germ $L \subset T^*\mathbf{C}^n$ at a point m, the set $\Lambda_{L,m} \subset J^1(\mathbf{C}^n, \mathbf{C})$ of points (p, q, z) such that $s = (p, q) \in L$ and the integral of α along a path in L joining m and s equals z is a Legendrian variety in $J^1(\mathbf{C}^n, \mathbf{C})$.

A germ of a symplectomorphism $\theta \in Sym_0(T^*\mathbf{C}^n)$ preserving π preserves α. Hence if $\theta(L') = L''$ then $\theta(L'_{0,m}) = L''_{0,\theta(m)}$. In Darboux coordinates, θ has the form:

$$\theta : (p, q) \mapsto (P, \check{\theta}(q)),$$

where $\check{\theta}$ is the underlying diffeomorphism of the base and $P = (\check{\theta}^{-1})^* p$ is the value at p of the linear operator on the fibres dual to the inverse of the derivative of θ. In particular, $\check{\theta}(W_0(L', m)) = W_0(L'', \theta(m))$.

The proof of the following statement is straightforward.

Proposition 8. *Consider a Legendrian germ $\Lambda \subset J^1(\mathbf{C}^n, \mathbf{C})$. Assume the variety $\widehat{\rho(\Lambda)}$ does not meet the zero section and that its standard Lagrangian projection is 0-stable. Then the projection of Λ to $J^0(\mathbf{C}^n, \mathbf{C})$ is Legendrian stable.*

Conversely, if the projection of Λ to $J^0(\mathbf{C}^n, \mathbf{C})$ is Legendrian stable and Λ is quasihomogeneous with positive weights then $\widehat{\rho(\Lambda)}$ is 0-stable.

3.2 Stability of induced mappings

3.2.1 The critical-value theorem

The images under a Lagrangian mapping $\pi|_L$ of singular points of the Lagrangian variety L along with the images of critical points of the restriction of $\pi|_L$ to the regular part of L form the caustic Σ_L of the Lagrangian mapping.

The caustic of a Lagrangian germ L at a point m of finite multiplicity μ is a proper analytic subset of the base B of codimension at least 1.

For $q \notin \Sigma_L$ close to the distinguished point $\pi(m)$, the inverse image $\pi^{-1}(q) \bigcap L$ consists of μ distinct points m_i close to m. We can assume that locally π is the standard fibration $T^*B \to B$. This allows us to introduce the Maxwell set $M_L \subset B$ as the closure of the set of the points $q \notin \Sigma_L$ for which the μ values of z on $\Lambda_{L,m} \bigcap (\rho \circ \pi)^{-1}(q)$ are not all distinct. If μ is finite, the Maxwell set is a germ of a proper analytic subset of the base. The union of the caustic and Maxwell set is called the bifurcation diagram $\mathrm{Bif}(\pi, L)$ of the Lagrangian projection.

Consider the Lagrangian projection $\pi : T^*\mathbf{C}^n \to \mathbf{C}^n$ of a Lagrangian variety-germ L. Let $g : \mathbf{C}^k \to \mathbf{C}^n$ be a germ of a smooth mapping. If the choice of the base points of the germs is consistent, we define the induced Lagrangian mapping $g^*(\pi|_L)$ as the projection of $g^*(L) \subset T^*\mathbf{C}^k$ to $\mathbf{C}^k$.

Theorem 9. *Assume the germs $\pi|_L$ at $m, m \notin T_0$ and $g^*(\pi|_L)$ are 0-miniversal and 0-stable respectively. Then the critical value set Ξ_g of the mapping g belongs to the union $W_0(L, m) \bigcup \mathrm{Bif}(\pi, L)$.*

Here we consider a source point of a mapping as critical if the derivative at the point is not surjective. In particular, all the source is critical if its dimension is less than that of the target, in which case the theorem implies that g maps $\mathbf{C}^k$ into $W_0(L, m) \bigcup \mathrm{Bif}(\pi, L)$.

The stability analog of the theorem was proved in [13].

Proof. Take a point $q_0 \in \mathbf{C}^n \setminus \Sigma_L$ close to the base point. Its $\pi|_L$-inverse image consists of n distinct points $m_1, \ldots, m_n \in F_{q_0}$, all different from the origin. The multi-germ of $\pi|_L$ at the finite set $\{m_1, \ldots, m_n\}$ is 0-versal (the decomposition (2) holds for multi-germs). This is equivalent to the m_i being linearly independent in the fibre F_{q_0}: the restriction of any function from the fibre to this set coincides with the restriction of a linear function.

Consider now $\lambda_0 \in g^{-1}(q_0)$. Let $I \subset T_{q_0}\mathbf{C}^n$ be the image of the derivative $g_* : T_{\lambda_0}\mathbf{C}^k \to T_{q_0}\mathbf{C}^n$. The pullback mapping $g^* : F^n_{q_0} \to F^k_{\lambda_0}$ between the fibres of the cotangent bundles is a composition of the factorisation pr of F_{q_0} by the subspace $I^\vee$ of covectors annihilating I and an embedding. Assume the dimension r of $I^\vee$ is positive, that is λ_0 is a critical point of g. The 0-stability of $g^*(\pi|_L)$ implies that the pr-images of the linearly independent points $m_1, \ldots, m_n \in F_{q_0}$ form a linearly independent set in the $(n-r)$-dimensional space $F_{q_0}/I^\vee$ (the image points counted without the multiplicities). As a result, the vertex set $\{m_0 = 0, m_1, \ldots, m_n\}$ of the n-simplex $\sigma \subset F_{q_0}$ is mapped to the vertex set $\{m'_0 = 0, m'_1, \ldots, m'_{n-r}\}$ of an $(n-r)$-simplex in $F_{q_0}/I^\vee$. In particular, the rank r subspace $I^\vee$ is spanned by all the differences $m_i - m_j$ such that $pr(m_i) = pr(m_j)$, that is by the vectors in all the faces of σ

contracted by pr to points (the sum of the dimensions of such faces is r).

Near each of the m_i, $i = 1, \dots, n$, the Lagrangian variety L is locally the graph of the differential of a function $z = \psi_i(q)$, $\psi(q_0) = 0$. The linearly independent points $m_i \in F_{q_0}$ are the differentials of the ψ_i at q_0.

For any pair $i \neq j$, denote by $\Delta_{ij} \subset T_{q_0}\mathbf{C}^n$ the hyperplane tangent to the hypersurface $\psi_i(q) - \psi_j(q) = 0$.

For any ℓ, let $\Delta_\ell \subset T_{q_0}\mathbf{C}^n$ be the hyperplane tangent to the hypersurface $\psi_\ell(q) = 0$. The hyperplanes Δ_ℓ and Δ_{ij} are dual to the directional lines of the 1-dimensional faces of the simplex $\sigma \subset F_{q_0}$.

The condition for the multi-germ $g^*(\pi|_L)$ to be 0-stable at the points of F_{λ_0} is equivalent to the subspace I being the intersection of all the Δ_ℓ such that $pr(m_\ell) = 0$ and all the Δ_{ij} such that $pr(m_i) = pr(m_j) \neq 0$. Hence I is the intersection of the subspaces in $T_{q_0}\mathbf{C}^n$ dual to certain faces of the simplex σ. Since I belongs to the tangent cone at q_0 of the critical value set Ξ_g, the regular strata of Ξ_g near q_0 coincide with the integral manifolds of the distributions defined similarly to I in the spaces $T_q\mathbf{C}^n$ by subsets of the faces of the relevant n-simplices in the fibres F_q.

According to [4] (items 7.1 and 7.2), among such integral manifolds, those having the highest dimension and containing $\pi(m)$ in their closures are the regular strata of the caustic, Maxwell set and, as it is easy to see, wavefront $W_0(L, m)$. Hence $\Xi_g \subset W_0(L, m) \bigcup \mathrm{Bif}(\pi, L)$. □

Theorem 10. *If g is the germ of a proper mapping between spaces of the same dimension, then the 0-stability of $g^*(\pi|_L)$ is equivalent to g being a ramified covering with the ramification locus contained in $W_0(L, m) \bigcup \mathrm{Bif}(\pi, L)$.*

Proof. In this case the regular strata of Ξ_g are $(n-1)$-dimensional. By Theorem 9, the 0-stability implies the ramification property. To prove the converse it is sufficient to notice that outside the ramification locus the induced map $g^*(\pi|_L)$ is 0-miniversal. Also it is versal at points of the regular strata of the ramification set, as it can be seen from the pull-back mapping g^* action on the corresponding simplex in the fibre. Hence any holomorphic function-germ $\varphi(p, q)$ possesses a decomposition (2) with the coefficients $a_j(q)$ uniquely determined on the complement of the analytic subset of codimension at least 2. Now Hartog's theorem extends the decomposition to an entire neighbourhood of the base point. □

Remark. Assume the Lagrangian variety-germ L at $m \in T^*\mathbf{C}^n$ is a suspension of a Lagrangian germ L' at a point not contained in the zero section of $T^*\mathbf{C}^{n-1}$. The base $\mathbf{C}^n$ of the suspended Lagrangian fibration contains a distinguished coordinate function, let it be q_n, corresponding to the second factor of the decomposition $L \simeq L' \times \mathbf{C}$. The caustic and Maxwell set for L are also isomorphic to the products of the caustic and Maxwell set for L' with a line, the q_n-axis. On the contrary, the hyperplane tangent to the wavefront $W_0(L, m)$ at m is $dq_n = 0$.

If, under the conditions of Theorem 10, the ramification locus Ξ_g contains an $(n-1)$-dimensional component of the caustic or of the Maxwell stratum then the direction ∂_{q_n} belongs to the image I of the differential of g at points arbitrary close to m. Hence the composition $q_n \circ g$ is not singular at the base point. On the other hand, if the ramification locus contains an $(n-1)$-dimensional component of the wavefront $W_0(L,m)$, then the composition $q_n \circ g$ must be singular at the base point. Otherwise, the hyperplanes tangent to Ξ_g near the base point are not close to the hyperplane $dq_n = 0$.

3.2.2 Composite functions

An interesting class of 0-stable Lagrangian projections is provided by versal deformations of composite mappings [6].

Given a function-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ consider the group $\mathcal{K}_f$ (see [6]) which consists of diffeomorphism-germs Θ of the product space $(\mathbf{C}^m \times \mathbf{C}^n, (0,0))$ fibred over the projection to the first factor $\Theta : (x,y) \mapsto (X(x), Y(x,y))$, $x \in \mathbf{C}^m$, $y \in \mathbf{C}^n$, and such that $f(Y(x,y)) = f(y)$ for any (x,y).

The group $\mathcal{K}_f$ acts naturally on the space of map-germs $\varphi : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ sending the graph of one map to the graph of the other.

Assume a map-germ φ at the origin has a finite Tjurina number τ with respect to the group $\mathcal{K}_f$. Let $\Phi(x,\lambda) = \varphi(x) + \sum \lambda_s \varphi_s(x)$, $\lambda \in \mathbf{C}^\tau$, be a $\mathcal{K}_f$-miniversal deformation of φ. Introduce the composition $F = f \circ \Phi$.

Theorem 11. *The Lagrangian projection defined by the generating family-germ $F(x,\lambda)$ is* 0*-stable.*

Proof. Let $t \in (\mathbf{C}, 0)$ be an additional parameter. Consider the deformation

$$F_{ij} = f \circ (\Phi + t\frac{\partial F}{\partial \lambda_j}\varphi_i)$$

of the composite function $f \circ \varphi$. Since $F_{ij}|_{t=0} = F$ and Φ is K_f-versal, there exists a family of K_f-equivalencies depending on t and inducing F_{ij} from F:

$$F_{ij}(x,\lambda,t) = f \circ \left(\varphi(X(x,\lambda,t)) + \sum_{s=1}^{\tau} \Lambda_s(\lambda,t)\varphi_s(X(x,\lambda,t))\right).$$

Moreover, we choose the family so that for $t = 0$ the mapping $(x,\lambda) \mapsto (X, \Lambda)$ is the identity mapping.

Differentiating this equality with respect to t at $t = 0$ we obtain

$$\frac{\partial F}{\partial \lambda_i}\frac{\partial F}{\partial \lambda_j} = \sum \frac{\partial F}{\partial x_r}\frac{\partial X_r}{\partial t} + \sum \frac{\partial F}{\partial \lambda_k}\frac{\partial \Lambda_k}{\partial t}.$$

Since $\partial F/\partial \lambda_i = p_i$ and $\partial F/\partial x_r = 0$ on the Lagrangian variety defined by the generating family F, this means that the 0-stability criterium of Lemma 5 holds for it. $\square$

Assume the germ at the origin of a composed function $h = f \circ \varphi$ has a finite multiplicity μ. The deformation $F = f \circ \Phi$ of h is induced from an $\mathcal{R}$-miniversal deformation H of h by a map-germ $g : (\mathbf{C}^\tau, 0) \to (\mathbf{C}^\mu, 0)$ between the deformation bases.

Corollary 12. *If $\tau = \mu$ and the inducing mapping g is proper, then g is a covering ramified over the* 0*-wavefront of the* 0*-stable Lagrangian manifold defined by the generating family H.*

Proof. The claim is trivial when the function-germ f is regular (if so, the mapping g is a diffeomorphism). So we may assume that f has critical point at the origin. In this case the composition of g with the projection $\mathbf{C}^\mu \to \mathbf{C}$ along the hyperplane tangent to the discriminant of h at the origin is singular at $0 \in \mathbf{C}^\tau$. Now Theorems 11, 10 and the Remark after Theorem 10 imply the result. □

Remark. Under the conditions of Corollary 12, the $\mathcal{K}_f$-discriminant of φ is a free divisor.

Example. The covering mapping inducing the determinantal function of a versal matrix deformation of a simple matrix singularity from a versal deformation of the determinantal function of the unperturbed matrix (see [11]) is a particular case of Corollary 12. At this point, one should consider either symmetric matrices in 2 variables or arbitrary square matrices in 3 variables. Skew-symmetric matrices in 5 variables will also do.

The matrix setting of the Example has been generalised in [10] to compositions $f \circ \varphi$ with functions f not necessarily determinantal. One of the main results of [10] states that $\mu = \tau$ provided the critical locus of f is Cohen-Macaulay and has codimension $m+1$ in $\mathbf{C}^n$. In this case the critical locus $\mathcal{C}$ of the inducing map g turns out to be the set of all those points in $\mathbf{C}^\tau$ which correspond to perturbations of φ whose images meet the critical locus of f [7]. Clearly, g maps this set to the discriminant of the function $f \circ \varphi$ which agrees with the theorems of section 3.2.1.

Now, the space of linear functions on a fibre F_q of the cotangent bundle $T^*B \to B$ is the tangent space T_qB. So the functions c_{ij}^k defined in (3) for a 0-versal Lagrangian map-germ determine a point-wise associative multiplication on the germs of vector fields on the base. When B is the base of a $\mathcal{K}_f$-miniversal deformation this is exactly the multiplication considered in [7]. The only difference is that in [7] certain hypersurfaces were removed from B to guarantee the multiplication has a unity. However, degeneracy of the multiplication is an interesting question on its own. For example, experiments suggest the following

Conjecture 13. *Let $\mathcal{T}$ be the space of vector fields on the base of a $\mathcal{K}_f$-miniversal deformation of a map-germ $\varphi : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$. Assume the critical locus of f is Cohen-Macaulay and has codimension $m+1$ in $\mathbf{C}^n$. Assume also that the transversal type of f is A_1. Then*

$$\mathcal{T}^2 = Der(-log\,\mathcal{C})$$

where $\mathcal{C}$ is the critical locus of the inducing map g.

The inclusion $\mathcal{T}^2 \subset Der(-log\,\mathcal{C})$ follows immediately from the results of [9]. Perhaps this inclusion should not depend on the transversality type of f at all. The results of [9] also indicate that the Conjecture can be generalised to higher A_k transversality types if we increase to k the order of tangency of the fields in $\mathcal{T}^2$ to the relevant components of $\mathcal{C}$.

References

[1] V. I. Arnold, *Critical points of functions on a manifold with boundary, the simple Lie groups B_k, C_k, F_4 and singularities of evolutes*, Russian Math. Surveys **33** (1978), no. 5, 99–116.

[2] V. I. Arnold, *Singularities of caustics and wave fronts*, Kluwer Academic Publ., Dordrecht-Boston-London, 1990.

[3] V. I. Arnold, V. V. Goryunov, O. V. Lyashko and V. A. Vassiliev, *Singularities II. Classification and Applications*, Encyclopaedia of Mathematical Sciences, vol.39, Dynamical Systems VIII, Springer Verlag, Berlin a.o., 1993.

[4] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable maps. Vol. I*, Monographs in Mathematics **82**, Birkhäuser, Boston, 1985.

[5] J. W. Bruce, *On families of symmetric matrices*, Moscow Math. J. **3** (2003), no. 2, 335–360.

[6] J. W. Bruce, V. V. Goryunov and V. M.Zakalyukin, *Sectional singularities and geometry of families of planar quadratic forms*, in: Trends in singularities, 83–97, Trends Math., Birkhäuser, Basel, 2002.

[7] I. de Gregorio and D. Mond, *F-manifolds from composed maps*, to appear in: Saõ Carlos-Sur-Mer, Proceedings of VIIIth Workshop on Real and Complex Singlarities, Luminy 2004.

[8] A. B. Givental, *Singular Lagrangian manifolds and their Lagrangian mappings*, J. Soviet Math. **52** (1990), no. 4, 3246–3278.

[9] V. V. Goryunov, *Logarithmic vector fields for the discriminants of composite functions*, to appear in Moscow Math. Journal.

[10] V. V. Goryunov and D. Mond, *Tjurina and Milnor numbers of matrix singularities*, J. London Math. Soc. **72** (2005), 205–224.

[11] V. V. Goryunov and V. M. Zakalyukin, *Simple symmetric matrix singularities and the subgroups of Weyl groups A_μ, D_μ, E_μ*, Moscow Math. Journal **3** (2003), no. 2, 507–530.

[12] V. V. Goryunov and V. M. Zakalyukin, *On stability of projections of Lagrangian varieties*, Funct. Anal. Appl. **38** (2004), no. 4, 249–255.

[13] R. M. Roberts and V. M. Zakalyukin, *Stability of Lagrangian manifolds with singularities*, Funct. Anal. Appl. **26** (1992), no. 3, 174–178.

[14] O. P. Shcherbak, *Wave fronts and reflection groups*, Russian Math. Surveys **43** (1988), no. 3, 149–194.