A Bennequin number estimate for transverse knots

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Abstract

We show that the Bennequin number of a transverse knot in the standard contact 3-space or solid torus is bounded by the negative of the lowest degree of the framing variable in its HOMFLY polynomial. For $\mathbf{R}^3$, this fact was established earlier by Fuchs and Tabachnikov [7] by comparison of the results of [1] and [5, 11]. We develop a different, direct approach based on the lowering of the polynomial to transversally framed regular planar curves and the results of [4]. We show, by providing explicit examples, that for knots in $\mathbf{R}^3$ with at most 8 double points in their diagrams the estimate is exact.

It is very well known (see, for example, [8]) that any topological knot type in a contact 3-manifold has a Legendrian representative. If the contact structure is coorientable a Legendrian knot gets a natural framing. The question of Legendrian representability of an arbitrary framed knot type has in general a negative answer: according to the classical result by Bennequin [1], the self-linking numbers (called in this case also *Bennequin numbers*) of all canonically framed Legendrian representatives of a fixed unframed knot type in $\mathbf{R}^3$ are bounded from one side.

Paper [1] gives an estimate for the Bennequin number in the standard contact 3-space in terms of the genus of a knot. This has a disadvantage of being insensitive to passing from a knot to its mirror image. One feels this immediately, considering the two trefoils: the Bennequin estimate is exact for one of them and is far from being such for the other.

In recent papers [7, 4, 2, 3] there was obtained a series of estimates for the Bennequin number of a Legendrian knot in the standard 3-space (and its analog for the solid torus introduced by Tabachnikov in [13]) which do respect the change of the orientation of the ambient space and are more efficient than that original by Bennequin. According to them, the self-linking number of a Legendrian knot K is at least the negative of the lowest power of the framing variable in (the unframed versions of) the HOMFLY and Kauffman polynomials of K (all the signs here follow our further choice of the orientations). The estimate coming from the Kauffman polynomial is usually better: for example, it copes with the Legendrian trefoils immediately, while the HOMFLY information falls a little short of being sufficient.

Another type of knots which are natural to consider in a contact 3-manifold are *transverse knots*, that is those everywhere transverse to the contact distribution. Their theory is parallel to the Legendrian one: any knot type has a transverse representative, they possess the canonical framing if the contact structure is parallelisable, the self-linking (Bennequin) number of a canonically framed transverse knot in the standard contact 3-space is at least the Euler characteristic of any Seifert surface of the knot [1]. Similar to the Legendrian case, the Bennequin number of a transverse knot in $\mathbf{R}^3$ cannot be less than the lowest degree of the framing variable in its HOMFLY polynomial [7] (it is not very reasonable to consider the Kauffman polynomial in the transverse setting, since the set of transverse knots is not closed under the main skein relation of that polynomial). In fact, it did not take too much effort to obtain the latter estimate in [7]: it came immediately from the comparison of some intermediate results of [1] with the results of [5, 11]. As in the Legendrian case, this easier estimate has proved to be more effective than that by Bennequin (once again, the example of the trefoils is the simplest to demonstrate this).

The main goal of the present note is to obtain an estimate on the Bennequin number of a transverse knot in the standard solid torus $ST^*\mathbf{R}^2$ analogous to that of [7] for the 3-space. We do this by direct methods, rather different from those of [7]. Namely, we follow the approach of [4, 2, 3] to the study of Legendrian knot invariants in the 3-space and solid torus via invariants of their planar fronts, and study transverse knots in $ST^*\mathbf{R}^2$ via their projections to the plane. The latter are regular curves equipped with a natural transverse framing. We lower the framed version of Turaev's HOMFLY polynomial for the solid torus [14] to our planar curves. After this, a straightforward application of the results of [4] immediately implies the desired estimate. Passing to the universal cover of our solid torus, we get another proof of the Fuchs-Tabachnikov estimate.

We also give experimental results which demonstrate rather surprising exactness of the Fuchs-Tabachnikov estimate for transverse representations of all the knots in $\mathbf{R}^3$ with at most 8 double points in their diagram.

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1 Links in the solid torus and 3-space, and framed planar curves

1.1 The two standard contact spaces

A *cooriented contact element* at a point of $\mathbf{R}^2$ is a line in the tangent space at this point along with a choice of one of the two half-planes into which it divides the tangent plane. Any such element can be represented by the unit normal n to the line pointing into the chosen half-plane (Fig.1). Thus

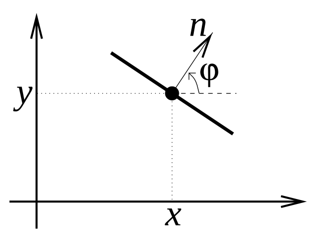


Figure 1: *A cooriented contact element on the plane and its coordinates in the solid torus $ST^*\mathbf{R}^2$.*

the variety of all cooriented contact elements of the plane is the spherisation $ST^*\mathbf{R}^2$ of its cotangent bundle. This solid torus possesses the standard contact structure, that is maximally non-integrable field of tangent planes: at each point of $ST^*\mathbf{R}^2$ one takes the tangent plane which is mapped by the canonical projection $p : ST^*\mathbf{R}^2 \to \mathbf{R}^2$ onto the contact element represented by that point. In the coordinates of Fig.1 this is the field of kernels of 1-form $\alpha = (\cos\varphi)dx + (\sin\varphi)dy$. The standard contact structure is naturally cooriented: the coorientation of a contact element lifts against p to the coorientation of the tangent plane.

In what follows we will need orientation on $ST^*\mathbf{R}^2$. We fix it to be the orientation of $\mathbf{R}^2$ followed by the direction of positive (counter-clockwise) rotation in the plane. This is opposite to the orientation $\alpha \wedge d\alpha$ traditionally taken in contact geometry.

Along with $ST^*\mathbf{R}^2$ we will be considering its universal cover $\mathbf{R}^3$ with its standard contact structure (induced via the covering mapping). The orientation of the 3-space will be that inherited from the solid torus.

1.2 Links as framed planar curves

Any link L, that is an embedded collection of circles, in $ST^*\mathbf{R}^2$ can be put by an arbitrary small perturbation in a generic position with respect to the projection p. Then its p-image F_L in the plane is an immersed collection of circles whose only singularities are transverse double points. At every point $p(a) \in F_L$ consider the normal n_a coorienting the contact element $a \in L$. We equip F_L with the unit framing e, $e(p(a)) \in a$, such that the pair $\{e(p(a)), n_a\}$ orients the plane positively (see Fig.2). For our further considerations, framing e is more visual than n_a itself.

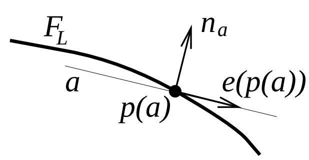


Figure 2: *Canonical framing e of the projection of a generic curve from $ST^*\mathbf{R}^2$ to the plane.*

For a Legendrian link, for example, thus defined framing is everywhere tangent to the planar curve. But this is an infinitely-degenerate case. In general position, e is transverse to F_L except for a finite number of isolated points of their tangency. The latter are projections of the points of tangency of the link L to the contact structure.

For a generic link L, the two framing vectors at any double point of F_L are distinct. On the other hand, making an elementary change of the link topology via a generic homotopy in $ST^*\mathbf{R}^2$ involving a double point, we instantaneously observe a regular planar curve with a double point at which the two vectors coincide (Fig.3). Such a double point will be called *an elementary framing degeneracy point.*

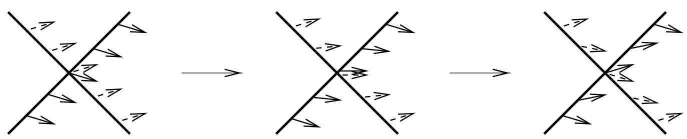


Figure 3: *An elementary framing degeneracy at a double point of a framed planar curve.*

Reversing our construction we can start with a generic framed planar curve F. Then the direction of the framing lifts F to a link L_F in $ST^*\mathbf{R}^2$.

To represent a link in $\mathbf{R}^3$ in the similar way, we need each of the components of a framed immersed planar collection F of circles to have the rotation (or Whitney winding) number zero. If F has more than one component, to make its lifting to the universal cover of $ST^*\mathbf{R}^2$ well-defined we must choose a point on each of the components and specify one of countably many possibilities to assign a phase φ to the framing at that point. The elementary framing degeneracy at a double point now requires coincidence of the corresponding real phases (not reduced modulo 2π).

1.3 Transverse links and transverse invariants

Definition 1.1 A link in a contact space is called *transverse* if it is everywhere transverse to the contact structure.

Definition 1.2 A transverse link L in $ST^*\mathbf{R}^2$ is *positive* if it is oriented and the framing e, followed by the orientation of F_L, orients the plane positively.

For a generic transverse link $L \subset ST^*\mathbf{R}^2$, the framing e is everywhere transverse to F_L. We can consider e up to homotopy which does not change the topology of L. This means that we may assume e to be normal to F_L except for small neighbourhoods of some double points. Such double points will be called *abnormal* and marked by a dot aside in the figures (see Fig.4). Such specification of the double points, along with the coorientation of F_L by e, is obviously sufficient to restore the link type of $L \subset ST^*\mathbf{R}^2$. All the other information about the framing may be suppressed.

For a one-component or positive transverse link L, even the coorientation of F_L can be omitted. Thus any such link in the solid torus can be identified with an oriented planar curve with some double points marked. The $\mathbf{R}^3$-version of this approach is obvious.

Example 1.3 The move of Fig.3 is a homotopy from a normal to abnormal double point.

Figure 4: *The convention to depict normal and abnormal double points of transversally framed planar curves.*

We will be searching for invariants of transverse links in terms of transversally framed regular planar curves. This means that we will be interested only in such invariants of those curves which change only in generic 1-parameter families that involve curves with elementary framing degeneracy double points. Such invariants will be called *transverse* invariants. Thus we are ignoring generic degenerations of transversally framed regular planar curves which have no relation to the framing, that is self-tangencies and triple points. Only families of immersed planar curves will be considered. Their framing is not allowed to be tangent at any time. At a double point, any interaction between a framing vector and the tangent to the other branch is ignored.

Remark 1.4 The above shows that, in fact, we will be working within the space of immersions of a collection of finite number of circles into $ST^*\mathbf{R}^2$

which are transversal to the contact structure. Let us call such immersions *transverse curves.*

The description of the set of connected components of the space of 1-component transverse curves is as follows.

A generic oriented transverse knot K in $ST^*\mathbf{R}^2$ has two obvious invariants read from its projection F_K to the plane: the rotation number $w(F_K) \in \mathbf{Z}$, and the orientation $\varepsilon \in \mathbf{Z}_2$ of the frame made up by the framing e and the velocity of F_K.

The Whitney-Graustein theorem [15] implies

Proposition 1.5 *Connected components of the space of 1-component oriented transverse curves in the standard solid torus are enumerated by the numbers* $(w, \varepsilon) \in \mathbf{Z} \times \mathbf{Z}_2$.

1.4 Self-linking of transverse links

Consider any oriented framed link L in $ST^*\mathbf{R}^2$. Following Tabachnikov [13], we define its self-linking number as the index of intersection of a small shift of L along the framing with a film that realises homology between L and the multiple of the fibre of $p : ST^*\mathbf{R}^2 \to \mathbf{R}^2$ over a sufficiently distant point of the plane.

This is a natural generalisation of the traditional definition for $\mathbf{R}^3$.

For a generic transversally framed oriented planar curve, consider a counter-clockwise rotation of its framing by a small angle. Lifting this to $ST^*\mathbf{R}^2$ or, if possible, to its universal cover we obtain a canonical framing of the corresponding transverse link.

Definition 1.6 The self-linking number β of a canonically framed transverse oriented link in the solid torus or 3-space is called the *Bennequin number* of the link.

In the solid torus, the Bennequin number of a link L is read from the link diagram as usual, that is as half the sum of the signs of crossings of L and its shift along the framing (see Fig.5).

To calculate β for a transverse link L in $\mathbf{R}^3$, given as an oriented and cooriented regular curve $F_L \subset \mathbf{R}^2$ with marked double points (see section 1.3), one may resolve all the double points of F_L according to the phases

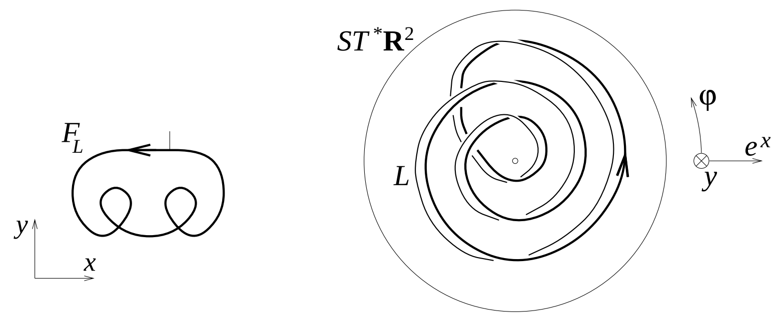


Figure 5: *Lifting of a planar curve to a positive transverse knot in $ST^*\mathbf{R}^2$ with the canonical framing. In absence of abnormal double points, φ is the direction of the velocity. The Bennequin number is 5.*

$\varphi \in \mathbf{R}$ and thus obtain a link diagram of L. Then the canonical framing is blackboard with respect to the projection back to $\mathbf{R}^2$. Thus, $\beta(L)$ is the sum of the signs of all the double points.

1.5 Transverse representation of topological knot types

Proposition 1.7 *Any oriented topological link type in $\mathbf{R}^3$ and $ST^*\mathbf{R}^2$ has a positive transverse representative.*

This follows, for example, from the validity of the same statement for Legendrian representatives whose fronts are regular curves (the constructive proof of this fact for knots given in [4] can be easily generalised for links). Indeed, a shift of such a Legendrian representative by a small non-zero constant along the fibres of the fibration $p : ST^*\mathbf{R}^2 \to \mathbf{R}^2$ provides a transverse link.

According to [1], the Bennequin numbers of canonically framed transverse knots of a fixed topological type are bounded from below (for our way to orient the solid torus). This was proved by Bennequin for knots in $\mathbf{R}^3$, but implies the result for the solid torus as well via a contact embedding of the solid torus into the 3-space. Thus, the analog of Proposition 1.7 for framed knots and links is not true.

In the next section we derive a new restriction on the Bennequin number of a transverse knot in the standard solid torus. This is the main result of our paper.

2 The HOMFLY polynomials

From now on we are considering only positive transverse links and corresponding transversally framed planar curves.

2.1 The solid torus

Following Turaev [14], we introduce the framed version of the HOMFLY polynomial of an oriented link in a solid torus [4].

Theorem 2.1 *For any framed link L in a solid torus, there exists an element $P(L) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \xi_{\pm 1}, \xi_{\pm 2}, \ldots]$ which is uniquely defined by the skein relations and initial data of Fig.6.*

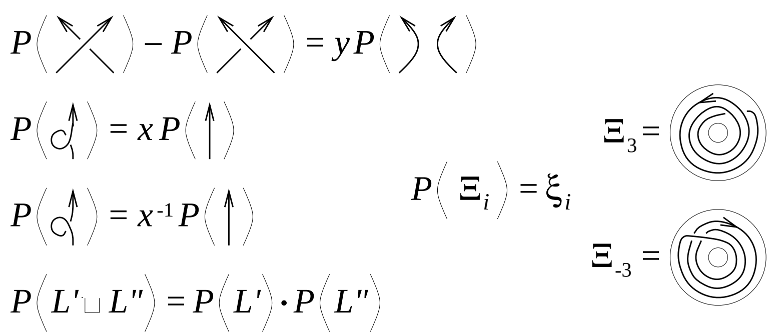


Figure 6: *Definition of the framed version of the HOMFLY polynomial for oriented links with the blackboard framing in a solid torus. In the last line, the links L' and L'' are mutually unlinked.*

Example 2.2 *For an unknot with trivial framing* $P = \frac{x-x^{-1}}{y}$.

Now we assume a link $L \subset ST^*\mathbf{R}^2$ to be transverse. We set $P(F_L) = P(L)$ and would like to calculate $P(L)$ entirely in terms of the planar curves. To do this we use the following obvious observation concerning the representation of section 1.3 (see Fig.3).

Lemma 2.3 *A generic homotopy of positive transverse curves in $ST^*\mathbf{R}^2$ passing from a negative crossing to a positive one is seen, in terms of the underlying framed planar curves, as a homotopy from a normal to abnormal double point via an elementary framing degeneracy.*

Now, for positive transverse links in their represention as oriented regular planar curves with some of the double points marked (as in section 1.3), the rules of Fig.6 imply the set of rules shown in Fig.7. The collections F' and F'' of its last line are lying in disjoint half-planes. According to the second rule of Fig.6, the relation between the transverse generators z_i we are using now and the blackboard generators ξ_i is $z_i = x^{|i|}\xi_i$: it is easy to show that the lift of Z_i has the unframed knot type of Ξ_i, and its canonical framing differs from the blackboard one of Ξ_i by $2|i|$ positive half-twists (see Fig.5, where $i = 3$).

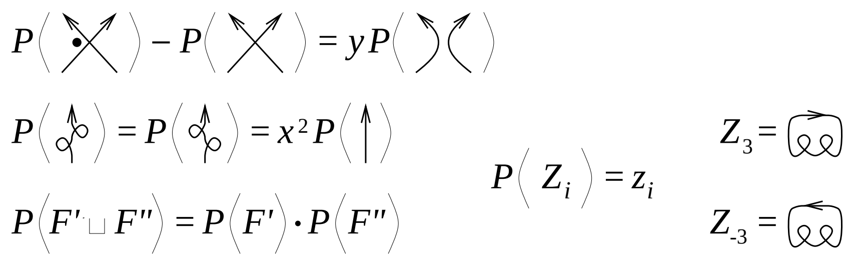


Figure 7: *Lowering of the definition of Fig.6 to generic transversally framed planar curves.*

Theorem 2.4 *The rules of Fig.7 uniquely define a transverse invariant $P(F) \in \mathbf{Z}[x, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$ for any generic transversally framed planar curve F.*

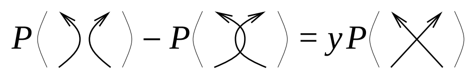
The major observation here is the absence of negative powers of the framing variable x. That is what our forthcoming estimate will be based on.

Corollary 2.5 *The polynomial $P(L)$ of any positive transverse link L in $ST^*\mathbf{R}^2$ is a genuine polynomial in x, not a Laurent one.*

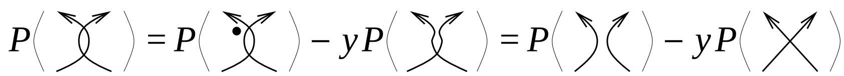
Remark 2.6 Allowing components with both positive and negative orientations we would not be able to stay within the set of transverse links after application of the main skein relation.

To prove Theorem 2.4 we need the following

Lemma 2.7



Proof.



□

Proof of Theorem 2.4. If a transversally framed planar curve has no abnormal double points, we calculate its polynomial P using the relation of the above lemma instead of the main skein relation of Fig.7. The renewed set of rules is exactly that which uniquely defined the framed version of the HOMFLY polynomial of a regular plane curve in [4]. According to [4], the element so defined is unique and lies in $\mathbf{Z}[x, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$.

If a planar curve has some abnormal double points, we start with applying the main skein relation of Fig.7 to reduce their number. Now the assertion of the theorem follows by induction. □

2.2 The standard contact 3-space

The set of rules which uniquely defines the framed version of the HOMFLY polynomial $P_0(L) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ for a link L with the blackboard framing in $\mathbf{R}^3$ is that of Fig.6 with all the information about the variables ξ_i omitted. According to what was said about the representation of transverse links in the universal cover of $ST^*\mathbf{R}^2$ in section 1.2, the restriction of P_0 to positive transverse links in the standard $\mathbf{R}^3$, in terms of transversally framed planar curves, is that defined by Fig.7 without mentioning the z_i. The main skein relation is now applicable only when the homotopy between the normal and abnormal framings in its left-hand side passes through a double point at which the phases $\varphi \in \mathbf{R}$ of the planar framing coincide. In all the other cases the change normal-abnormal does not affect the polynomial (along with the link type in the 3-space). We refer to thus obtained rules as *modified.*

As in the previous subsection, we have

Theorem 2.8 *For any transverse link L_0 in the standard 3-space, the modified rules of Fig.7 uniquely define an element $P_0(L_0) \in \mathbf{Z}[x, y^{\pm 1}]$.*

2.3 The estimate

Let ℓ be the self-linking number of an oriented framed knot L or L_0 in either the solid torus or 3-space respectively. Then the polynomials

$$P^u(L) = x^{\ell(L)} P(L) \qquad \text{and} \qquad P_0^u(L_0) = x^{\ell(L_0)} P(L_0)$$

are invariants of unframed knot types [14, 6].

Corollary 2.5 and Theorem 2.8 imply

Theorem 2.9 *Let k be the lowest power of the framing variable x in the unframed version of the HOMFLY polynomial of an oriented knot K in the solid torus or 3-space. Equip the ambient space with the standard contact structure. Then the Bennequin number β of any transverse knot, whose topological type is that of K, is at least $-k$.*

Remark 2.10 The Bennequin number β of a transverse knot is odd. Indeed, according to Proposition 1.5, one can pass from any transverse knot to the lift of either a basic curve of Fig.7 or of the curve ∞ by a series of ordinary change-crossings which change β by ± 2. Now the lift of any basic curve, as well as that of the curve ∞, has odd β.

Example 2.11 Due to Example 2.2, for an unknot $\beta \geq 1$ [1]. The lift of the curve ∞ is minimal, with $\beta = 1$.

Example 2.12 The transverse knots represented by the curves Z_i of Fig.7 have minimal possible Bennequin numbers $2|i|-1$ allowed for their topological type. On the other hand, the original Bennequin's estimate provides, via a contact embedding of the solid torus into the 3-space which unknots all such knots, a weaker lower bound 1 independent of i.

Example 2.13 For $(2,q)$-torus knots in $\mathbf{R}^3$, our lower bounds are $2-q$ and $2+q$ for the left- and right-handed cases respectively. Those bounds are exact as Fig.8 shows. For better illustration the double points there are resolved according to the phase φ.

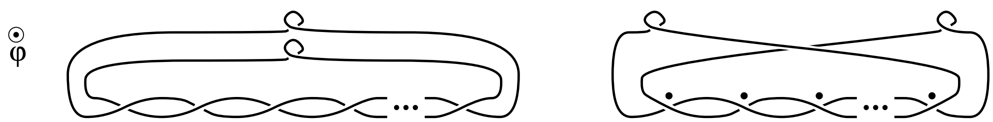


Figure 8: *Transverse representatives of left- and right-handed* $(2,q)$*-torus knots in the standard 3-space with minimal Bennequin numbers* $2-q$ *and* $2+q$*.*

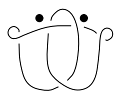
3 Minimal representatives of knots in the 3-space with a few double points in a knot diagram

Proposition 3.1 *The estimate of Theorem 2.9 is exact for all knots in the standard contact 3-space with at most 8 double points in a knot diagram.*

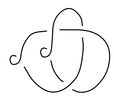
This rather surprising fact follows from the explicit examples given in three pages below. There we show the minimal realisations of the table knots [12, 9] and, in case of difference in the topology, of their mirror images. The lowest degrees of the HOMFLY polynomials P_0^u were taken from [10] (the entry for the knot 8_{13} there required correction).

Remark 3.2 We do not know any examples of more complicated knots for which the estimate is not exact.

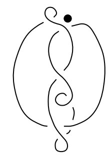
Remark 3.3 Similar to the approach used in this note, it looks very convenient to study links in the variety of all cooriented contact elements of any Riemannian surface in terms of framed curves on the surface.



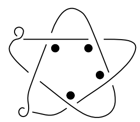
3_1 $\beta = 5$



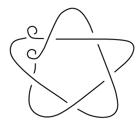
3_1 $\beta = -1$



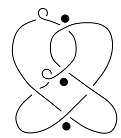
4_1 $\beta = 3$



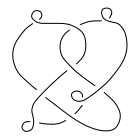
5_1 $\beta = 7$



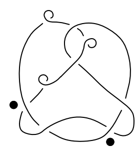
5_1 $\beta = -3$



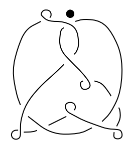
5_2 $\beta = 7$



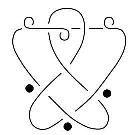
5_2 $\beta = -1$



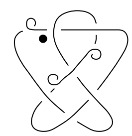
6_1 $\beta = 5$



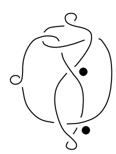
6_1 $\beta = 3$



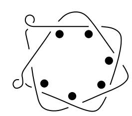
6_2 $\beta = 5$



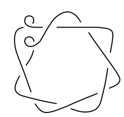
6_2 $\beta = 1$



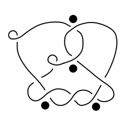
6_3 $\beta = 3$



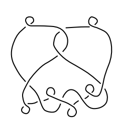
7_1 $\beta = 9$



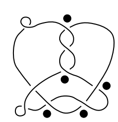
7_1 $\beta = -5$



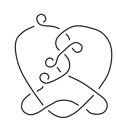
7_2 $\beta = 9$



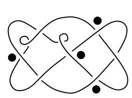
7_2 $\beta = -1$



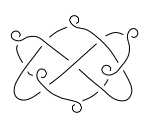
7_3 $\beta = 9$



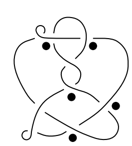
7_3 $\beta = -3$



7_4 $\beta = 9$



7_4 $\beta = -1$



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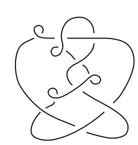

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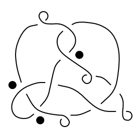
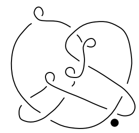
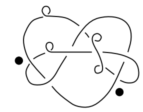


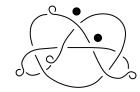
7_6 β = 7



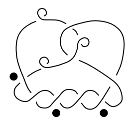
7_6 β = 1



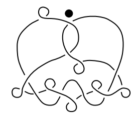
7_7 β = 5



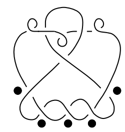
7_7 β = 3



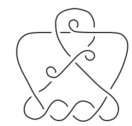
8_1 β = 7



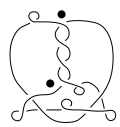
8_1 β = 3



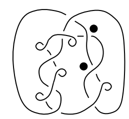
8_2 β = 7



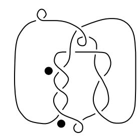
8_2 β = -1



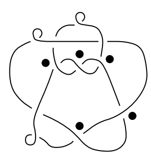
8_3 β = 5



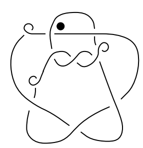
8_4 β = 5



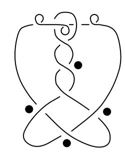
8_4 β = 3



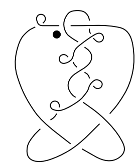
8_5 β = 7



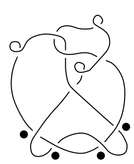
8_5 β = -1



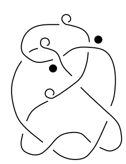
8_6 β = 7



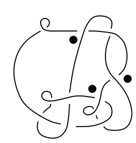
8_6 β = 1



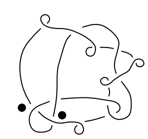
8_7 β = 5



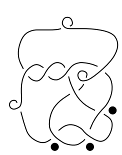
8_7 β = 1



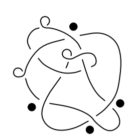
8_8 β = 5



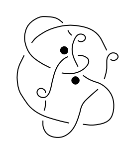
8_8 β = 3



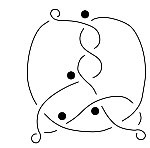
8_9 β = 3



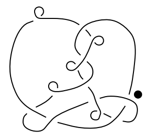
8_{10} β = 5



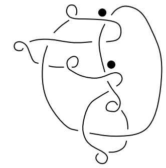
8_{10} β = 1



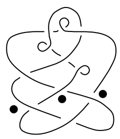
8_{11} β = 7



8_{11} β = 1

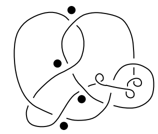


8_{12} β = 5

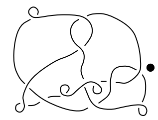


8_{13} β = 5

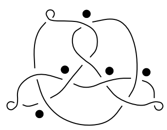
8_{13} β = 3



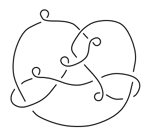
8_{14} β = 7



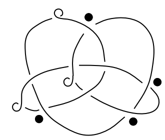
8_{14} β =1



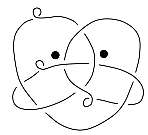
8_{15} β = 11



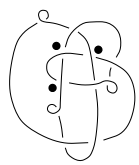
8_{15} β = -3



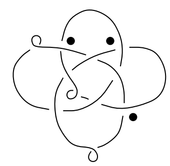
8_{16} β = 5



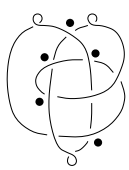
8_{16} β = 1



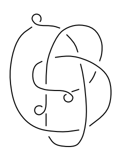
8_{17} β = 3



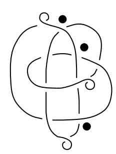
8_{18} β = 3



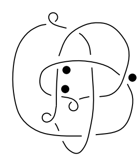
8_{19} β = 11



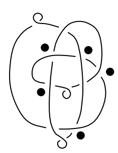
8_{19} β = -5



8_{20} β = 5



8_{20} β = 1



8_{21} β = 7

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