Géométrie/*Geometry*

Local invariants of mappings of oriented surfaces into 3-space

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Abstract — The numbers of pinch and triple points are obvious order 1 (in the sense of Vassiliev [7]) invariants of mappings of surfaces into 3-space. We show that, besides them, there is exactly one integer invariant of such mappings that depends only on local perestroikas of the image. Our study provides new restrictions on the numbers of different perestriokas during sphere eversions.

Invariants locals des applications des surfaces orientées dans espace de dimension 3

Résumé — Les nombres des points de type parapluie et des points triples sont évidemment des invariants d'ordre 1 (au sens de Vassiliev [7]) des applications des surfaces dans l'espace de dimension 3. Nous démontrons qu'il n'y a en plus qu'un invariant entier de telles applications, qui ne dépend que des perestroikas locales de l'image. Notre étude fournit des restrictions nouvelles sur les nombres des perestroikas différentes pendant eversions des sphères.

Version française abrégée — Localement, l'image d'une application générique d'une surface fixée fermée M dans $\mathbf{R}^3$ est soit une feuille lisse, soit une intersection transverse de deux ou trois feuilles lisses, soit un parapluie de Whitney. Les applications avec des images plus compliquées forment une hypersurface discriminante Δ dans l'espace Ω de dimension infinie des toutes les applications C^∞ de M dans R^3.

Le discriminant coupe Ω en composantes connexes. Un invariant numerique est une fonction localement constante sur $\Omega \setminus \Delta$. Le long d'un chemin générique de Ω, nous regardons les sauts d'un invariant au passage du discriminant. Nous disons que notre invariant est local si chaque saut est complètement défini par le type de difféomorphism d'une perestroika locale de l'image au moment du passage du discriminant.

Soit Imf l'image d'une application générique $f : M \to \mathbf{R}^3$ d'une surface orientée. Prenons un point u dans $\mathbf{R}^3 \setminus \mathrm{Im}f$. Considérons une 2-sphère petite, avec l'orientation extérieure, centrée en u. La contraction radiale de l'image sur la sphère définit une application composée de M sur la sphère. Soit $deg(u)$ le degré de cette application.

$\mathbf{R}^3 \setminus \mathrm{Im}f$ a un nombre fini de composantes connexes D. $deg(u)$ est constant dans chacune d'entre elles. Nous noterons la valeur correspondante par $deg(D)$.

Nous introduisons l'integral de la fonction deg par rapport à la caractéristique de Euler χ:

$$\int_{\mathbf{R}^3\setminus\mathrm{Im}f} deg(u)d\chi(u) = \sum_D deg(D)\chi(D),$$

où la somme est prise sur toutes les composantes connexes de $\mathbf{R}^3 \setminus \mathrm{Im}f$ (cf. [8, 9]).

Il y a 8 (resp. 3) composantes connexes locales du complémentaire de l'image au voisinage du point triple t (resp. un point de type parapluie p). Nous définissons les degrés $deg(t)$ et $deg(p)$ comme les moyennes arithmétiques des 8 ou 3 degrés correspondants.

On pose (cf. [9])

$$I_{\int}(f) = \int_{\mathbf{R}^3\setminus\mathrm{Im}f} deg(u)d\chi(u) - \sum_t deg(t) - \frac{1}{2}\sum_p deg(p),$$

où la somme est prise sur tous les points triple t et parapluie p de l'image.

Théorème. *L'espace des invariants locals entiers des applications lisses d'une surface orientée dans $\mathbf{R}^3$ est de dimension trois. Soit g le genre de la surface source. Alors, les invariants élémentaires sont:*

1. *le nombre i_t des points triples de l'image,*
2. *le nombre i_p des paires des points parapluies,*
3. $(I_{\int} + i_t + i_p + g - 1)/2$.

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1. Generic degenerations. Locally, the image of a generic mapping of a fixed closed surface M to $\mathbf{R}^3$ is either a smooth sheet, or transversal intersection of either 2 or 3 smooth sheets, or a Whitney umbrella. Mappings with more complicated images form a *discriminant* hypersurface Δ in the infinite-dimensional space Ω of all C^∞ maps $M \to \mathbf{R}^3$.

The discriminant subdivides Ω into connected components. A numerical *invariant* is a way to assign numbers to each of these components.

Moving along a generic path in Ω, we watch the jumps, as we pass the discriminant, of the values of an invariant. We say that our *invariant is local* if every jump is completely determined by the diffeomorphism type of the local perestroika of the image at the instant of crossing of the discriminant.

1.1. The top strata. There are 7 types of local events on the image that take place at generic points of Δ, i.e. occur along generic curves in Ω [4]:

(E) elliptic tangency of two smooth sheets;

(H) hyperbolic tangency of two smooth sheets;

(T) (for 'triple') tangency of the line of intersection of two smooth sheets to a third smooth sheet (all the three sheets are pairwise transversal);

(Q) (for 'quadruple') four smooth sheets intersecting at the same point;

(C) (for 'cup') a smooth sheet passes through a pinch point;

(B) birth of a bubble with two pinch points joined by an interval of selfintersection;

(K) (for 'cones' in Russian) the hyperbolic version of B.

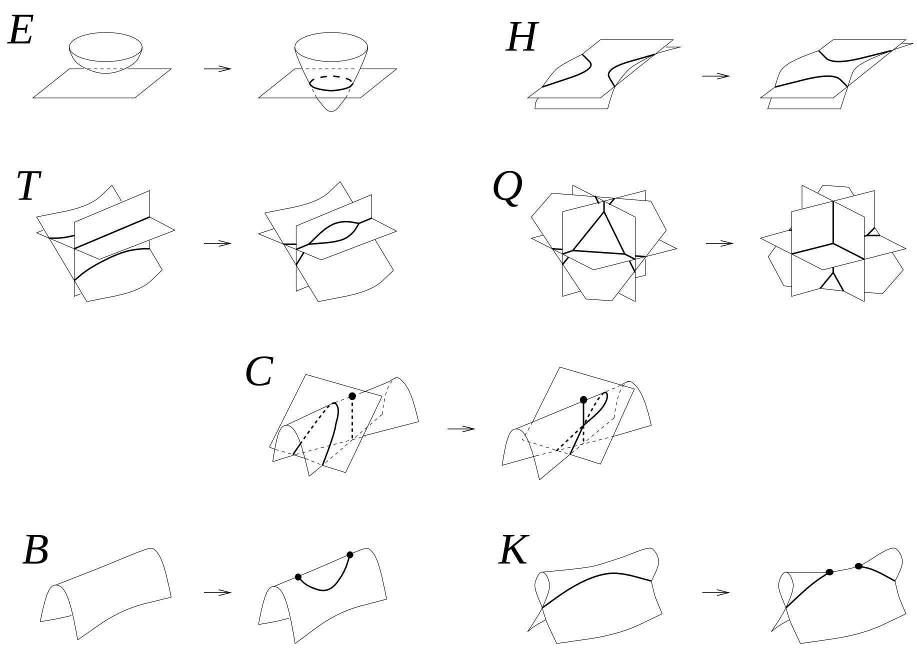
A local coordinate form for B and K is $(x, y) \mapsto (x, y^2, y(x^2 \pm y^2 - \lambda))$.

The seven perestroikas define seven top-dimensional (of codimension 1 in Ω) strata of the discriminant.

1.2. Coorientation of the strata. We say that *passings* through the strata E, T, C, B and K in the direction specified by the illustration are *positive.* In the first four cases amongst those five, this is equivalent to the appearence of a new 2-cycle on a generic image.

To coorient H and Q (at least partially), we assume from now on that the source surface M is oriented. Fix the orientation of $\mathbf{R}^3$ as well. We get a canonical coorientation of the image at immersive points requiring that the frame $\langle$coorienting normal, the image of a positive frame of $M\rangle$ is a positive frame in $\mathbf{R}^3$.

We now add the induced coorientations of the sheets in all the 7 perestroikas of the illustration. This splits them in 20 subcases listed below.



There, considering a 2-sphere that consists of pieces of several smooth sheets (with their own coorientations), we call a piece *positive* if it has the outer coorientation. The outer coorientation is also called *positive*, the inner one is called *negative*.

We split the strata:

- E, T, Q in $E^j, j = 0,1,2$, $T^j, j = 0,1,2,3$, $Q^j, j = 2,3,4$. Here j is the number of positive pieces of the new-born sphere (there is the vanishing tetrahedron in the Q-cases which has $4 - j$ positive pieces);

- H in H^+ and H^-: the sheets, at the tangency point, have coinciding, respectively opposite, coorientations;

- C in $C^{\alpha,\beta}, \alpha, \beta = +, -$: for the appearing cup-shaped 2-sphere, α and β are the signs of the coorientations of the lateral (having the pinch point) and bottom pieces respectively;

- B in B^+ and B^-: the superscript is the coorientation of the new-born sphere;

- K in K^+ and K^-: the index is the coorientation of the local 'tubular' part before the perestroika.

We continue with the coorientation of the discriminant.

Q^4 and Q^3. We coorient those strata in the direction of increase of the number of positive faces of the local tetrahedron.

H^-. Consider the two points of M at which we get tangency for the degenerate mapping. We say that the crossing of the stratum is positive if the relative motion of the images of the two points occurs in the direction opposite to the coorientations of the sheets.

There is no way to coorient Q^2 and H^+ by only local means, and we leave them without coorientation. This allows them to participate only in mod 2 invariants.

2. Lists of local invariants. We express a local invariant as a linear combination of the strata of the discriminant in which each coefficient is the increment of the invariant for the positive crossing of the stratum.

Theorem 1. *The space of integer local invariants of smooth mappings of an oriented surface into $\mathbf{R}^3$ is 3-dimensional. The following are basic invariants:*

1. $I_t = 2T + C$;

2. $I_p = B + K$;

3.$I_3 = E^2 - E^0 + H^- + T + C^{++} + C^{+-} + B^+ + K^+$.

Here T, C, B, K are the sums of all the corresponding 4 or 2 substrata.

The proof of the theorem is a rather routine study of generic 2-parameter families of mappings (cf. [1]).

Theorem 1 identifies the invariants up to a choice of constants of integration. Set I_t and I_p to vanish on an embedding. Then those invariants are respectively the number of triple points and the number of pairs of pinch points of the image of a mapping.

Theorem 2. *The space of mod 2 local invariants of smooth mappings of an oriented surface into $\mathbf{R}^3$ is 4-dimensional. Basic invariants are those of Theorem 1 and*

4. $I_4 = E^1 + H^+ + C^{+-} + C^{-+}$.

We say that a self-tangency of two smooth sheets of the image is *direct* (resp. *inverse*) if their coorientations coincide (resp. are opposite).

The invariants I_3 and I_4 measure modified numbers of inverse and direct selftangencies in a generic homotopy between two mappings.

3. Immersions. We now restrict our attention to the subset of immersions in Ω.

3.1. Local invariants.

Theorem 3. *The space of integer local invariants of immersions of an oriented surface to* $\mathbf{R}^3$ *is 2-dimensional. Basic invariants are:*

$$I_{t/2} = T \qquad \text{and} \qquad I_3 = E^2 - E^0 + H^-.$$

With the normalization $I_{t/2}(embedding) = 0$, $I_{t/2}(f)$ is the number of pairs of triple points of the image of f.

Theorem 4. *The space of mod 2 local invariants of immersions of a 2-sphere to 3-space is 4-dimensional. In addition to the basic invariants of Theorem 3 there are two more:*

$$I_4 = E^1 + H^+ \qquad \textit{and} \qquad I_q = Q.$$

3.2. Sphere eversions. Those are turnings of a positive sphere inside out in $\mathbf{R}^3$ by a generic regular (i.e. with no pinch points) homotopy [6].

Let $N(S)$ be the number of crossings of the stratum S (the signs of the crossings of an oriented stratum are respected) during a sphere eversion.

Corollary 5. *The numbers of perestroikas during a generic sphere eversion are subject to the following relations:*

$$N(T) = 0, \quad N(E^2) - N(E^0) + N(H^-) = -1, \quad N(E^1) + N(H^+) = 0 \bmod 2$$

This follows from consideration of a generic path in the space of mappings $S^2 \to \mathbf{R}^3$ along which a positive sphere becomes a negative one via the birth of a negative bubble and death of a positive one. The right-hand sides of the relations are the increments of the local invariants of Theorems 1 and 2 along this path.

Remark. One more restiction, $N(Q) = 1 \bmod 2$, was proved by Max and Banchoff [3]. This implies that, during an eversion, there are at least two positive crossings of T.

4. Intergal invariant. We introduce now an invariant that is very similar to the integral in Rokhlin's complex orientation formula for real algebraic plane curves [5, 8].

4.1. Degrees. Let $\mathrm{Im}f$ be the image of a generic mapping of a surface $f : M \to \mathbf{R}^3$. Take a point u in $\mathbf{R}^3$ not on the image. Consider a small 2-sphere, with the outer coorientation, centered at u. The radial contraction of the image onto the sphere defines a through mapping from M to the sphere. We denote by $deg(u)$ the degree of this mapping.

$\mathrm{Im}f$ subdivides the ambient 3-space into a finite number of connected components D. $deg(u)$ is constant on each of them. We denote the corresponding value by $deg(D)$.

We define an integral of function deg against Euler charactestics χ:

$$\int_{\mathbf{R}^3 \setminus \mathrm{Im}f} deg(u) d\chi(u) = \sum_D deg(D)\chi(D),$$

where D runs through all the connected components of $\mathbf{R}^3 \setminus \mathrm{Im}f$ (cf. [8, 9]).

There are 8 (resp. 3) local connected components of the complement to the image around a triple point t (resp. a pinch point p). We set the degrees $deg(t)$ and $deg(p)$ to be the arithmetical means of the corresponding 8 or 3 degrees.

4.2. The invariant. We set (cf. [9])

$$I_{\int}(f) = \int_{\mathbf{R}^3 \setminus \mathrm{Im}f} deg(u) d\chi(u) - \sum_t deg(t) - \frac{1}{2}\sum_p deg(p),$$

where t and p run through all the triple and pinch points of the image.

Theorem 5. *$I_{\int}$ is a local invariant. Up to an additive constant,*

$$I_{\int} = 2I_3 - I_t - I_p.$$

Thus all the three local integer invariants of Theorem 1 may be defined only in terms of geometry of the image of the mapping f. There is no need to choose any homotopy in Ω between the distinguished mapping and f.

Remark. $I_{\int}(f)$ has also an integral expression in terms of the smoothed image of f (cf. [9]).

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