Local invariants of mappings of surfaces into three-space

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Abstract

Following Arnold's and Viro's approach to order 1 invariants of curves on surfaces [1, 2, 3, 20], we study invariants of mappings of oriented surfaces into Euclidean 3-space. We show that, besides the numbers of pinch and triple points, there is exactly one integer invariant of such mappings that depends only on local bifurcations of the image. We express this invariant as an integral similar to the integral in Rokhlin's complex orientation formula for real algebraic curves. As for Arnold's J^+ invariant [1, 2, 3], this invariant also appears in the linking number of two legendrian lifts of the image. We discuss a generalization of this linking number to higher dimensions.

Our study of local invariants provides new restrictions on the numbers of different bifurcations during sphere eversions.

In [17, 18] Vassiliev introduced the notion of a finite order invariant of knots. Finite order invariants can be defined for mappings in more general settings so long as the discriminant has codimension 1 in the function space. In [1] Arnold defined three invariants of order 1 for plane curves. These invariants have a local nature: they do not distinguish different components of the same top strata of the discriminant. Recently Viro generalized Arnold's invariants to the case of curves on surfaces [19]. Viro's generalization is based on Rokhlin's complex orientation formula for real algebraic curves [12, 13].

In this paper we consider mappings of oriented surfaces to Euclidean 3-space. We study special order 1 invariants which we call local. The notion

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of locality is slightly more rigid than the one used by Arnold: we require that the values of an invariant get the same increment once the images of the mappings experience the same local bifurcation.

We give a complete list of integer and mod 2 local invariants of generic mappings and of generic immersions (Sections 2 and 3). For example, we show that, besides the numbers of pinch and triple points, there is exactly one more integer local invariant, which we call I_3, of generic mappings of surfaces into $\mathbf{R}^3$. Our invariants provide new restrictions on the numbers of bifurcations during sphere eversions (subsection 3.2).

In Section 4 we relate I_3 to a local invariant $I_\int$ which is very similar to the integral form of Rokhlin's complex orientation formula [14, 19, 20].

Arnold found a formula for his direct selftangency plane curve invariant J^+ using linking numbers of corresponding legendrian curves [2, 3]. In Section 5 $I_\int$ is expressed in terms of linking numbers in $ST^*\mathbf{R}^3$ of two varieties homeomorphic to the source surface. One of the varieties is a modified legendrian lift of the image of a generic mapping, the second one is the 'negative' of the first.

In Section 6, following [2, 3] and Section 5, we introduce two linking numbers for a generic immersed hypersurface in $\mathbf{R}^n$. These numbers are local invariants dual to the direct and inverse selftangencies of a hypersurface.

Section 7 contains proofs of all theorems of the previous sections.

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1 Generic degenerations

Locally, the image of a generic mapping of a fixed closed surface M to $\mathbf{R}^3$ is either a smooth sheet, or transversal intersection of either 2 or 3 smooth sheets, or a Whitney umbrella (the image of $(x, y) \mapsto (x, y^2, xy)$) (see Fig.1). Mappings with more complicated images form a *discriminant* hypersurface Δ in the infinite-dimensional space Ω of all C^∞ maps $M \to \mathbf{R}^3$.

The discriminant subdivides Ω into connected components. A numerical *invariant* is a way to assign numbers to each of these components.

Moving along a generic path in Ω, we watch the jumps, as we pass the

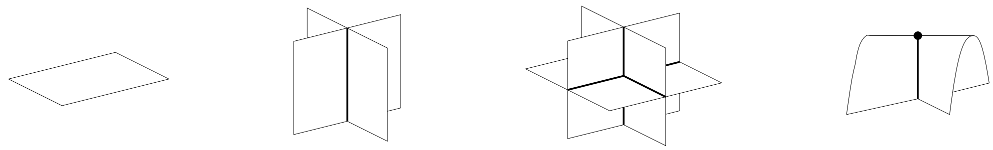

Figure 1: *Local singularities of images of generic maps of surfaces into 3-space*

discriminant, of the values of an invariant. We say that our *invariant is local* if every jump is completely determined by the diffeomorphism type of the local bifurcation of the image at the instant of crossing of the discriminant.

Remark. Arnold's plane curve invariant St [1, 2, 3] is not local in our sense since the coorientation of the corresponding stratum of the discriminant involves global information about the image.

1.1 The top strata

There are 7 types of local events on the image that take place at generic points of Δ, i.e. occur along generic curves in Ω [11, 10, 8] (Figs. 2 and 3):

(E) elliptic tangency of two smooth sheets;

(H) hyperbolic tangency of two smooth sheets;

(T) (for 'triple'), tangency of the line of intersection of two smooth sheets to a third smooth sheet (all the three sheets are pairwise transversal);

(Q) (for 'quadruple'), four smooth sheets intersecting at the same point;

(C) (for 'cup'), a smooth sheet passes through a pinch point;

(B) birth of a bubble with two pinch points joined by an interval of selfintersection;

(K) (for 'cones' in Russian), the hyperbolic version of B.

A local coordinate form for B and K is $(x, y) \mapsto (x, y^2, y(x^2 \pm y^2 - \lambda))$ (the real parameter λ increases from the left to the right in Fig.3 being zero in the middle).

The seven bifurcations define seven top (i.e. top-dimensional) strata of the discriminant.

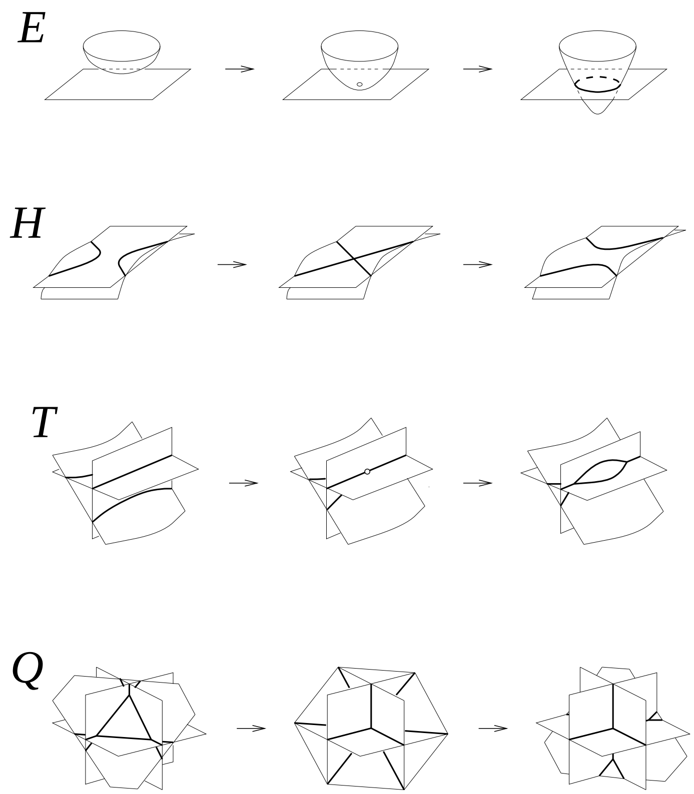

Figure 2: *Local bifurcations of images in generic 1-parameter families of mappings involving only smooth sheets*

1.2 Coorientation of the strata

In order to assign a jump of an invariant to a stratum, we will need to coorient the stratum (unless we work with a $\mathbf{Z}_2$-invariant).

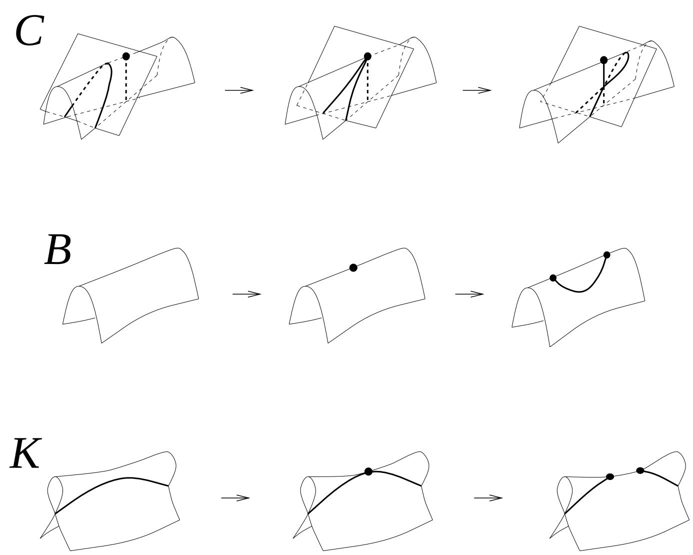

Figure 3: *Local bifurcations of images in generic 1-parameter families of mappings involving pinch points*

There is a natural way to coorient 5 of our 7 strata. This corresponds to moving from the left to the right in Figs. 2 and 3. Namely, we say that a *passing* through the stratum *is positive* if the passing gives birth to something new on a generic image:

- E, a new circle of selfintersection;
- T, two new triple points;
- C, one new triple point;
- B, K, two new pinch points.

In all these cases, except K, this is equivalent to the appearence of a new 2-cycle on a generic image.

In the two remaining cases, H and Q, the similar procedure fails: the initial and final pictures of Fig.2 in each of the two series are locally diffeomorphic.

1.3 Refinement for oriented surfaces

From now on we assume the source surface M oriented. This allows us to refine the stratification of the discriminant.

Let us fix orientations of M and $\mathbf{R}^3$. We obtain a canonical coorientation of the image at immersive points requiring that the frame $\langle$normal of the coorientation, the image of a positive frame of $M\rangle$ is a positive frame in $\mathbf{R}^3$.

We now add the induced coorientations of the sheets in all the 7 bifurcations of the previous subsection. This splits them in 20 subcases listed below (see Fig.4 for 15 of them). Considering a 2-sphere that consists of pieces of several smooth sheets (with their own coorientations), we call a piece *positive* if it has the outer coorientation. The outer coorientation is also called *positive*, the inner one is called *negative*.

We split the strata:

- E, T, Q in $E^j, j = 0, 1, 2$, $T^j, j = 0, 1, 2, 3$, $Q^j, j = 2, 3, 4$. Here j is the number of positive pieces of the appearing sphere (there is the vanishing tetrahedron in the Q-cases which has $4 - j$ positive pieces);

- H in H^+ and H^-: the sheets, at the tangency point, have coinciding, respectively opposite, coorientations;

- C in $C^{\alpha,\beta}, \alpha, \beta = +, -$: for the appearing cup-shaped 2-sphere, α and β are the signs of the coorientations of the lateral (having the pinch point) and bottom pieces respectively;

- B in B^+ and B^-: the superscript is the coorientation of the appearing sphere;

- K in K^+ and K^-: the index is the coorientation of the local 'tubular' part before the bifurcation.

We continue with the coorientation of the discriminant.

Q. We coorient the Q^4 and Q^3 strata in the direction of increase of the number of positive faces of the local tetrahedron (Fig.5). But there is still no local (in the sense introduced above) way to distinguish between the two sides of the Q^2 stratum.

H. To coorient H^- we consider the two points of M at which we get tangency for the degenerate mapping. We say that the crossing of the stratum is

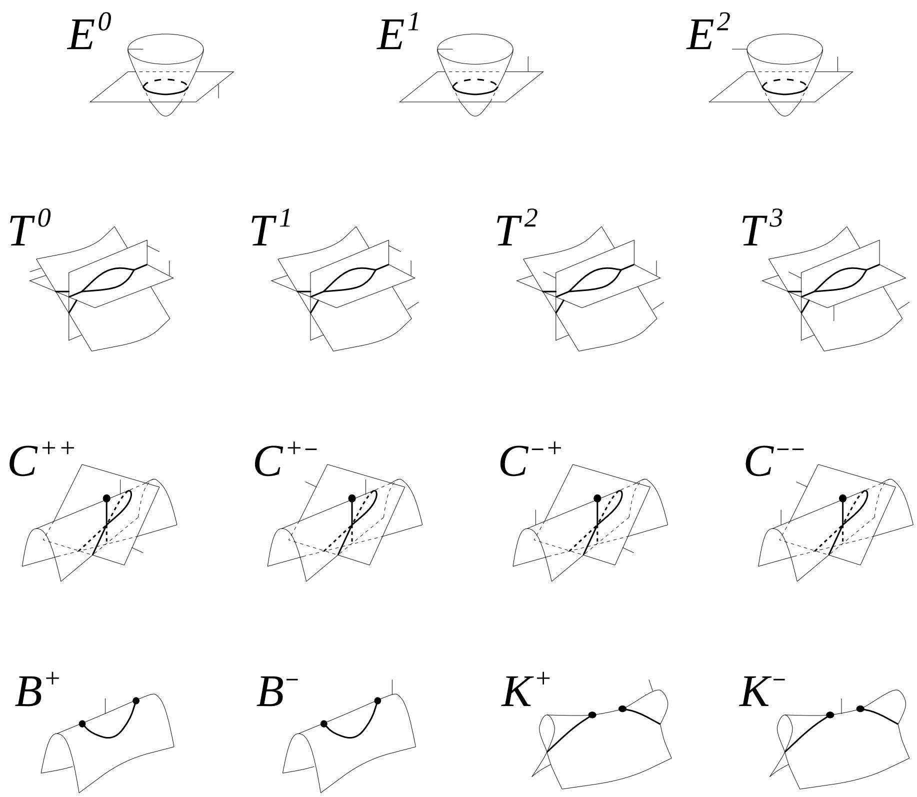

Figure 4: *Local images of oriented surfaces to the positive sides of the top strata of the discriminant*

positive if the relative motion of the images of the two points occurs opposite to the coorientations of the sheets (Fig.5).

Once again, there is no way to coorient H^+ by only local means.

In what follows the top strata of the discriminant are always taken with the coorientations introduced in this section. Q^2 and H^+ are not cooriented.

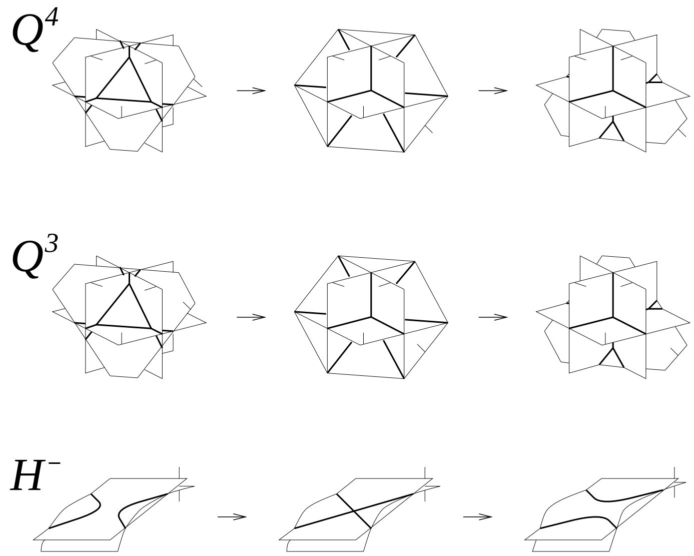

Figure 5: *Positive directions of bifurcations* Q^4, Q^3 *and* H^-

2 Lists of local invariants

2.1 Local invariants in terms of jumps along the top strata

Given an invariant I of generic mappings, we can enlarge its domain $\Omega \setminus \Delta$ to include generic points f_s of a cooriented top stratum $S \subset \Delta \subset \Omega$ [17, 18]. For this we take two points, f_+ and f_-, close to f_s in Ω, to the positive and negative sides of S respectively. We set $I(f_s) = I(f_+) - I(f_-)$. If I is a local invariant, the value obtained does not depend on a particular choice of generic $f_s \in S$.

Definition 2.1 $s(I) = I(f_s)$ is the *jump* of the local invariant I along the stratum S.

To assign a jump to a local $\mathbf{Z}_2$-invariant we do not need any coorientation of S.

On the other hand, a local invariant I on $\Omega \setminus \Delta$ is defined, up to an additive constant, by its jumps along the 20 strata of the discriminant. To calculate the value $I(f)$ in this situation, we need to:

(1) choose the value of I on a distinguished generic mapping f_0;

(2) join f_0 and f by a generic path γ in Ω;

(3) calculate the indices of intersection $\gamma \cap S$ of γ with each of the 20 strata S;

(4) set $I(f) = \sum_S (\gamma \cap S) s(I) + I(f_0)$.

Of course, this must be independent of the choice of γ. Since Ω is contractible, the independence is equivalent to vanishing of the increment of I (counted in the above way) along any small loop in Ω around the set of nongeneric points of Δ. This set has codimension 2 in Ω.

To shorten the notation, we will write a local invariant as a linear combination of the strata with the coefficients equal to the jumps. For the moment the invariants are considered up to the additive constants.

Checking of the independence condition (which is a rather routine study of generic 2-parameter families of mappings) provides:

Theorem 2.2 *The space of integer local invariants of smooth mappings of an oriented surface into* $\mathbf{R}^3$ *is 3-dimensional. The following are basic invariants:*

(1) $I_t = 2T + C$*;*

(2) $I_p = B + K$*;*

(3) $I_3 = E^2 - E^0 + H^- + T + C^{++} + C^{+-} + B^+ + K^+$*.*

Here T, C, B, K *are the sums of all the corresponding 4 or 2 substrata.*

Of course, the best choice of the additive constants for the first two invariants is the vanishing on an embedding of the surface. Then I_t is the number of triple points and I_p is the number of pairs of pinch points of the image of a mapping. We fix this normalizations of these two invariants for the rest of the paper. We will make comments on the constant for I_3 in subsection 4.4.

Definition 2.3 Tangency of two smooth sheets of the image that have coinciding (resp. opposite) coorientations is called *direct* (resp. *inverse*) *selftangency*.

Difference of values of I_3 (more precisely, of $2I_3 - I_t$) on two mappings is a modification of the number of opposite selftangencies in a generic homotopy between these mappings.

Theorem 2.4 *The space of mod 2 local invariants of smooth mappings of an oriented surface into* $\mathbf{R}^3$ *is 4-dimensional. Basic invariants are the ones of Theorem 2.2 and*

(4) $I_4 = E^1 + H^+ + C^{+-} + C^{-+}$.

In a way similar to I_3, I_4 measures a modified mod 2 number of direct selftangencies in a generic homotopy between two mappings.

The proof of both theorems is given in subsection 7.1.

2.2 mod 2 winding numbers

An oriented circle immersed into a plane has a winding number. By one of many equivalent definitions, this is the number of rotations of the tangent vector when its point of application runs along the circle once. A circle immersed into a 2-sphere has a well-defined mod 2 winding number: to obtain a plane, we puncture the sphere at any point not on the circle and take the parity of the plane winding number. This parity does not depend on the choice of the puncture and, for a generic curve, is opposite to the parity of the number of its double points.

Now take a generic mapping of a surface into $\mathbf{R}^3$. Consider the inverse image $\mathcal{D}$ of the set of singular points of the image. This is a collection of circles immersed into the source surface.

Proposition 2.5 *For generic mappings of a sphere, the mod 2 invariant* $I_t + I_p$ *is the parity* $w_2(\mathcal{D})$ *of the total winding number of* $\mathcal{D}$.

Proof. $w_2(\mathcal{D})$ is the mod 2 sum of the number of connected components and the number of double points of $\mathcal{D}$. The parity of the number of double points of $\mathcal{D}$ is the same as the parity of the number of triple points of the mapping.

The latter changes only across the stratum C. The number of connected components mod 2 of $\mathcal{D}$ changes only across B and K. Since both $w_2(\mathcal{D})$ and $I_t + I_p$ vanish on embeddings, the claim follows.

3 Immersions

3.1 Local invariants of immersions

Let $\Omega_i \subset \Omega$ be the space of C^∞ immersions of an oriented surface to $\mathbf{R}^3$. Restriction of local invariants from Ω to Ω_i defines local invariants of immersions. Almost always this exhausts all local invariants of immersions (as before, the invariants are considered modulo additive constants):

Theorem 3.1 *The space of integer local invariants of immersions of an oriented surface to $\mathbf{R}^3$ is two-dimensional. Basic invariants are:*

$$I_{t/2} = T \qquad \textit{and} \qquad I_3 = E^2 - E^0 + H^-.$$

With the normalization $I_{t/2}(\textit{embedding}) = 0$, $I_{t/2}(f)$ is the number of pairs of triple points of the image of f.

There is something extra over $\mathbf{Z}_2$:

Theorem 3.2 *The space of mod 2 local invariants of immersions of a 2-sphere to 3-space is four-dimensional. In addition to the basic invariants of Theorem 3.1 there are two more:*

$$I_4 = E^1 + H^+ \qquad \textit{and} \qquad I_q = Q.$$

Here I_4 is the restriction of the invariant of Theorem 2.4 and will appear for any oriented surface, not only for the sphere. The reason why we formulate the theorem for spheres only is as follows.

Considering bifurcations in all possible generic 2-parameter families of immersions of a surface (as in proofs of the first three theorems) it is easy to show that a small loop in Ω_i around the set of nongeneric points of the discriminant intersects the stratum Q an even number of times. But Ω_i has nontrivial fundamental group. A well-defined invariant should have zero increments along all elements of $\pi_1(\Omega_i)$. I do not know generators of this fundamental group in the case of an arbitrary surface. But for the 2-sphere, π_1 of the space of *marked* immersions (i.e. the ones that send a distinguished ordered 2-frame of the tangent space to S^2 at a distinguished point to a distinguished ordered 2-frame at a distinguished point of $\mathbf{R}^3$; for our needs it is enough to consider only marked immersions) is freely generated by a loop [15] described in [9]. This loop intersects Q twice.

It would be very interesting to work out whether I_q survives the $\pi_1(\Omega_i)$-test for surfaces of positive genus.

Of course, there are no such difficulties with π_1 for the restrictions of the local invariants from Ω, which is contractible.

Proofs of Theorems 3.1 and 3.2 are given in subsection 7.2.

3.2 Sphere eversions

Definition 3.3 An *eversion* of a sphere is a turning of a positive sphere inside out in $\mathbf{R}^3$ by a generic regular (i.e. with no pinch points) homotopy.

Eversions of surfaces are possible due to the classical result of Smale [15, 16].

Let $N(S)$ be the number of crossings of the stratum S (the signs of the crossings of a cooriented stratum are respected) during a sphere eversion.

Corollary 3.4 *The numbers of bifurcations during a generic sphere eversion are subject to the following relations:*

$$N(T) = 0, \qquad N(E^2) - N(E^0) + N(H^-) = -1, \qquad N(E^1) + N(H^+) = 0 \bmod 2$$

Proof. Consider a generic path in the space of mappings of S^2 into $\mathbf{R}^3$ along which a positive sphere becomes a negative one via the birth of a negative bubble and death of a positive one (Fig.6). The right-hand sides of the first three relations are the increments of the local invariants of Theorems 2.2 and 2.4 along this path.

Another way to prove the Corollary, not using any particular path, follows from the description of an integral invariant of the next section.

Remarks. a) There is no suprise in the restriction $N(T) = 0$: having generated triple points, we must kill them to complete an eversion.

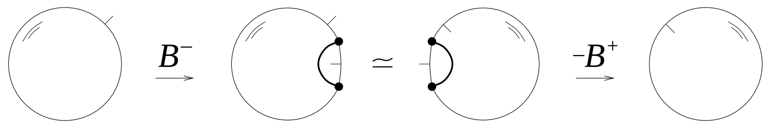

Figure 6: *Nonregular homotopy turning a sphere inside out*

b) One more restriction, $N(Q) = 1 \bmod 2$, was proved by Max and Banchoff by explicit presentation of an eversion with one quadruple point [9]. This implies, by the way, that, during an eversion, there are at least two positive crossings of T to create the vertices of the tetrahedron vanishing on Q. The basic loop in Ω_i, mentioned in the comments after Theorem 3.2, is a composition of the eversion of [9] with its 'mirror image'.

4 Integral invariant

We introduce now an invariant of generic mappings from oriented surfaces to 3-space that is very similar to the integral in Rokhlin's complex orientation formula for real algebraic plane curves [12, 13, 14, 19]. This invariant turns out to be local, very closely related to I_3.

4.1 Degrees

Let $\mathrm{Im} f$ be the image of a generic mapping of a surface $f : M \to \mathbf{R}^3$. Take a point u in $\mathbf{R}^3$ not on the image. Consider a small 2-sphere, with the outer coorientation, centered at u. The radial contraction of the image onto the sphere defines a through mapping from M to the sphere. We denote by $deg(u)$ the degree of this mapping.

$\mathrm{Im} f$ subdivides the ambient 3-space into a finite number of connected components D. $deg(u)$ is constant on each of them. We denote the corresponding value by $deg(D)$.

We define an integral of function deg against Euler charactestics χ setting [19, 20]:

$$\int_{\mathbf{R}^3 \setminus \mathrm{Im} f} deg(u) d\chi(u) = \sum_D deg(D)\chi(D),$$

where D runs through all the connected components of $\mathbf{R}^3 \setminus \mathrm{Im} f$.

There are 8 (resp. 3) local connected components of the complement to the image around a triple point t (resp. a pinch point p). We set the degrees $deg(t)$ and $deg(p)$ to be the arithmetical means of the corresponding 8 or 3 degrees. $deg(t)$ is a semi-integer that also coincides with the arithmetical mean of degrees of any of four pairs of the 'opposite' local components. $deg(p)$ is the degree of the 'largest' of the three components around p (Fig.7).

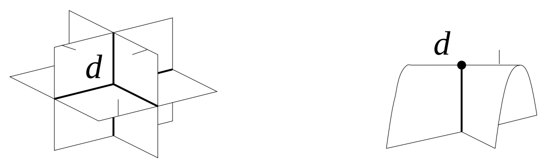

Figure 7: *A triple point of degree $d - \frac{3}{2}$ and a pinch point of degree d. In both cases d is the degree of the distinguished component of $\mathbf{R}^3 \setminus$ Image*

4.2 The invariant

We set

$$I_{\int}(f) = \int_{\mathbf{R}^3 \setminus \mathrm{Im} f} deg(u) d\chi(u) - \sum_t deg(t) - \frac{1}{2} \sum_p deg(p),$$

where t and p run through all the triple and pinch points of the image.

Example 4.1 The value of $I_{\int}$ on a standard immersion of a surface of genus g (Fig.8), with the outer coorientation, is $1 - g$.

Theorem 4.2 *$I_{\int}$ is a local invariant. Up to an additive constant,*

$$I_{\int} = 2I_3 - I_t - I_p.$$

Thus all the three local integer invariants of Theorem 2.2 may be defined only in terms of geometry of the image of the mapping f. There is no need to choose any homotopy in Ω between the distinguished mapping and f.

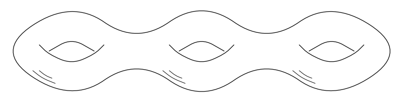

Figure 8: *Standard immersion of genus 3 surface*

The assertion of Theorem 4.2 is parallel to the assertion of Viro's Corollary 3.2.B in [20] for the invariant J^- of plane curves.

The proof of Theorem 4.2 consists in the comparison of the jumps of $I_\int$ and the above combination of the local invariants. It will be given in subsection 7.3.

Remark. The definition of $I_\int$ is similar to the formula for calculating the winding number w of a generic plane curve [20]:

$$w = \sum deg(D) - \sum deg(d),$$

with D and d running through all the connected components of the complement to the curve and all the double points, $deg(D)$ defined in the obvious way, and $deg(d)$ being the arithmetical mean of the four numbers.

4.3 Smoothings of images

The intergal invariant $I_\int$ has another description, in terms of a smoothed image.

Lemma 4.3 *There is a canonical way to smooth singularities of the image of a generic mapping of an oriented surface to 3-space. This smoothing is given by local pictures of Fig.9.*

Let $\widetilde{\mathrm{Im}} f$ be the smoothed image of a generic map f. As in subsection 4.1, for each point u of $\mathbf{R}^3 \setminus \widetilde{\mathrm{Im}} f$, we define $\widetilde{deg}(u)$.

Theorem 4.4

$$I_\int(f) = \int_{\mathbf{R}^3 \setminus \widetilde{\mathrm{Im}} f} \widetilde{deg}(u)\chi(u) - \sum_p deg(p).$$

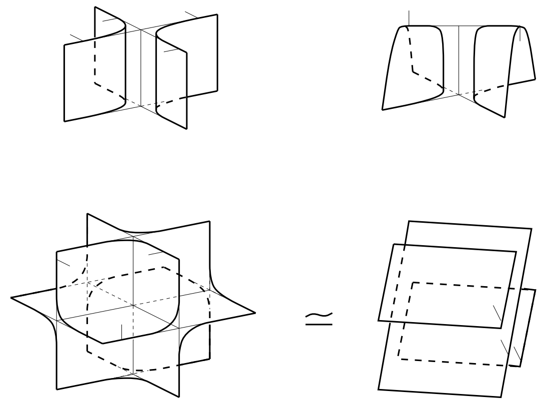

Figure 9: *Local smoothings of the image of a generic map*

Here p runs through all the pinch points of Imf.

The assertion of Theorem 4.4 is parallel to the assertion of Viro's Theorem 3.1.A in [20] for the invariant J^- of plane curves. The proof of Theorem 4.4 is given in subsection 7.4.

4.4 Connected summations

Let us choose a system of constants $a(g)$ for the system of local invariants $I_\int + a(g)$ on the spaces of mappings of surfaces of genus g to $\mathbf{R}^3$, in order to get a good invariant on the union of all these spaces. As in [1], for the notion of "goodness" let us take additivity with respect to the connected summation. It turns out that we have to distinguish two types of such summation. The sets of the constants $a(g)$ for the two types are distinct.

A *connected sum* of two surfaces $\mathcal{M}_1$ and $\mathcal{M}_2$, separated by a plane, in $\mathbf{R}^3$ is defined as shown in Fig.10: we cut a small disc out from the 'exterior' part of each of the surfaces and join the surfaces by a thin cylinder circle$\times$interval embedded in $\mathbf{R}^3 \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)$. If we want to save the orientations of the summands, there is an obvious restriction on $\mathcal{M}_1$ and $\mathcal{M}_2$. Coherently with the coorientations of the summands, the connecting cylinder must have either the outer or inner coorientation. We call these two options the *positive* and *negative summations* respectively.

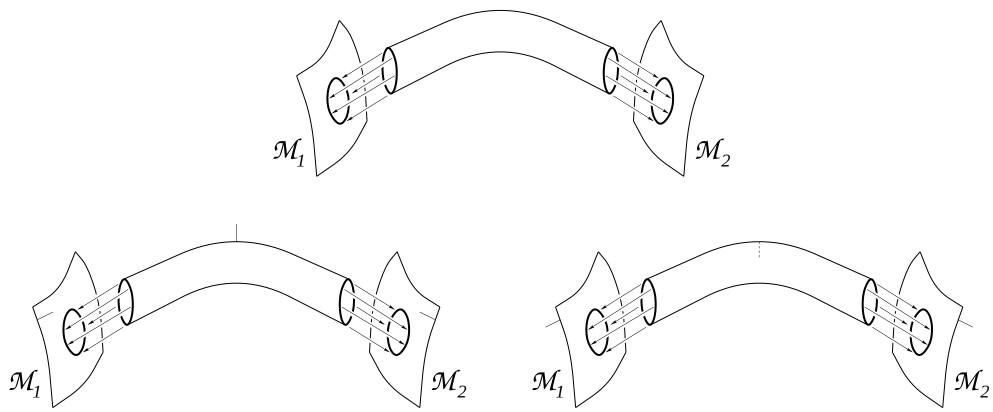

Figure 10: *Connected summation of surfaces and its two types: positive and negative*

Theorem 4.5 *The invariant* $I_\int + \alpha g + g - 1$*, where* α *is an arbitrary number, is additive with respect to the positive connected summation. The invariant* $I_\int + \beta g - g + 1$*, where* β *is an arbitrary number, is additive with respect to the negative connected summation.*

The Theorem easily follows from the summations of the standard surfaces (Fig.8). α and β are values of the invariants on the standard tori, with the outer and inner coorientations respectively.

Remark. The invariants I_t and I_p normalized to vanish on embeddings are obviously additive under both connected summations. For I_3 to be positively (resp. negatively) additive we set its value on the standard surface with the outer (resp. inner) coorientation to be $(\alpha g + g - 1)/2$ (resp. $(\beta g - g + 1)/2$).

The best behaviour of $I_\int$ is under *twisted* summation, when we use a tube with two pinch points as a connecting bridge (Fig.11). $I_\int$ is additive with respect to such a summation. But now we need to shift I_p: to make it twisted-additive we set $I'_p = I_p + 2$.

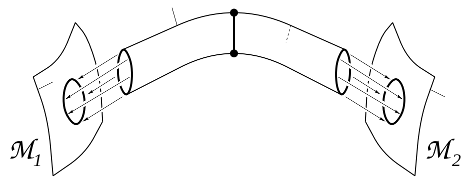

Figure 11: *Twisted connected summation*

5 Linking invariant

In [2, 3] Arnold gave an interpretation of his plane curve invariant J^+ in terms of linking numbers of corresponding legendrian curves in different covers of $ST^*\mathbf{R}^2$. Below we introduce a similar presentation for $I_\int$. We need there some extra work since a generic mapping of a surface into 3-space has critical points, while a generic plane curve does not.

5.1 Legendrian lift of a generic image

Let f be a generic mapping of an oriented surface M into oriented $\mathbf{R}^3$. We are going to associate to f a subvariety L_f^+ of the contact variety $ST^*\mathbf{R}^3$ of cooriented tangent 2-planes of $\mathbf{R}^3$. L_f^+ will be homeomorphic to M, and the image of L_f^+ under the projection $\pi : ST^*\mathbf{R}^3 \to \mathbf{R}^3$ will be the image $\mathrm{Im} f$ of f.

The part of L_f^+ over nonpinch points of $\mathrm{Im} f$, is the usual legendrian lift against π: it associates to a point a the cooriented (as in subsection 1.3) 2-plane tangent to a sheet of $\mathrm{Im} f$ passing through a. This divorces transversal selfintersections of $\mathrm{Im} f$.

Over a pinch point of $\mathrm{Im} f$ the legendrian lift gets a hole. Namely, easy computation shows:

Lemma 5.1 *Consider a Whitney umbrella W cooriented on the complement to its pinch point p. Consider the closure of the legendrian lift of $W \setminus p$ to $ST^*\mathbf{R}^3$. Then the boundary of the closure is an equator of the 2-sphere S_p^2. The equator consists of all covectors vanishing on the image of the differential of a parametrization of the umbrella by plane.*

Here S^2_p is the fibre of the bundle π over a point p.

To patch the hole in a canonical way, we need to recall the orientation of the fibre induced by an orientation of $\mathbf{R}^3$ [4]. The induced orientation of the fibre S^2 comes from the orientation of the boundary of a small 3-disc in $\mathbf{R}^3$. The set of outer normals of the boundary becomes a fibre when the radius of the ball tends to zero (vectors are identified with covectors by means of a metric on $\mathbf{R}^3$).

Let $\nu : \mathbf{R}^2 \to W$ be a parametrization of the umbrella by an oriented plane. We patch the hole on the legendrian lift of the punctured umbrella with a half of the fibre S^2_p. We chose the half so that the orientation induced from the plane by the lift extends to the patched surface as the canonical orientation of the fibre.

Lemma 5.2 *Choose a vector v in the image of the differential of ν at the pinch point so that v is directed to the half of the umbrella with the outer coorientation (Fig.12). Then the patching hemi-sphere is the set of covectors nonpositive on v.*

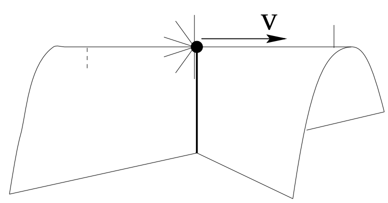

Figure 12: *Covectors added at a pinch point to patch the legendrian lift*

Doing the patching for all pinch points of the image of a generic mapping f of M, we get the subvariety $L^+_f \subset ST^*\mathbf{R}^3$.

5.2 Intersection with the 3-chain

There is another way to lift Imf to $ST^*\mathbf{R}^3$: we send a point to the *negative* of the coorientation normal (cf. [2, 3]). Let us lift a punctured neighbourhood of a pinch point is this negative way. The closure of the lift has the equator of Lemma 5.1 as the boundary. The orientation of the boundary equator is easily seen to be the same as for the lift in the previous subsection. Thus, the patches compatible with the orientations are also the same. We denote the new lift of the image (with the old patches used) by L_f^- (it is homeomorphic to M as well).

Now we define the linking number of L_f^+ and L_f^-.

Let us choose a direction in $\mathbf{R}^3$ and shift L_f^- by all possible vectors of this direction. We get in $ST^*\mathbf{R}^3$ a 3-chain V_f^- with the boundary L_f^-. We orient it as the image of $\mathbf{R}_+ \times M$, where $\mathbf{R}_+$ is the nonnegative axis oriented from 0 to ∞.

In order to bring L_f^+ in more or less general position with respect to V_f^-, we slightly shift every point of L_f^+ in the direction of the corresponding normal vector. Namely, a point $(a, n) \in ST^*\mathbf{R}^3$ moves to $(a + \varepsilon n, n)$, for small constant $\varepsilon > 0$ (a is a point in 3-space, n is a (co)vector applied at a). Let $L_{f,\varepsilon}^+$ be the shifted variety.

We take the canonical orientation of $ST^*\mathbf{R}^3$ by the frame ⟨positive frame of the base, positive frame of the fibre⟩ [4].

Definition 5.3 *The linking invariant* $\ell^-(f)$ is the intersection number of $L_{f,\varepsilon}^+$ and V_f^-.

Since the cohomology of $\mathbf{R}^3$ is trivial, the definition is independent of the chosen direction.

Example 5.4 The value of ℓ^- on the standard genus g surface (Fig.8) with the outer coorientation is $g - 1$. For the inner coorientation it is $1 - g$.

It is easily seen that in regular homotopies $\ell^-(f)$ can change only at inverse selftangencies. In general the following holds:

Theorem 5.5 $\qquad \ell^- = -I_{\int} + 2I_p.$

A wrong choice of the patch for the lift of an umbrella would imply appearance of an extra term, like the sum of degrees of pinch points, in the right hand side of this formula. Such a sum is not a local invariant.

A sketch of the proof of Theorem 5.5 is given in subsection 7.5.

6 Higher dimensions

The constructions of linking invariants in [2, 3] and the previous section suggest introduction of the following two linking numbers for generic immersed oriented closed hypersurface $\mathcal{M}$ in Euclidean space of arbitrary dimension n. These numbers count direct and inverse selftangencies in generic regular homotopies.

As above, just changing 3 for n, $\mathcal{M}$ defines two legendrian subvarieties, L^+ and L^-, in $ST^*\mathbf{R}^n$. There are no patches now. Choosing a direction in $\mathbf{R}^n$, we get n-chains V^+ and V^-. As in subsection 5.2 we perturb and get L^+_ε.

Definition 6.1 $\ell^+(\mathcal{M})$ and $\ell^-(\mathcal{M})$ are indices of intersection of L^+_ε with V^+ and V^- respectively.

There is a discriminant in the space of C^∞ immersions of a fixed hypersurface into $\mathbf{R}^n$. The top strata correspond to tangency of a smooth sheet to the transversal intersection of $r \leq n$ other smooth sheets (any r of these $r+1$ sheets are in general position; for $r = n$ 'tangency' means 'passing through') [11].

Lemma 6.2 *The invariant ℓ^+ (resp ℓ^-) can change only across the strata of direct (resp. inverse) selftangency of two smooth sheets.*

The claim is obvious from the definition.

We can locally represent the selftangency bifurcation as moving a hyperplane $x = t$, $t \in \mathbf{R}$, through the graph $x = -y_1^2 - \ldots - y_u^2 + z_1^2 + \ldots + z_v^2$, $u+v = n-1$. Assume the frame $(x, y_1, \ldots, y_u, z_1, \ldots, z_v)$ gives the orientation of the ambient space and x is the direction to produce the n-chains V^+ and V^-. Changing the parameter t from negative to positive, with coorientations of the sheets as in Fig.13, we see:

(1) in inverse selftangency bifurcation ℓ^- has a jump ± 2;

(2) in direct selftangency bifurcation in even-dimensional ambient space ℓ^+ has a jump ± 2;

(3) in direct selftangency bifurcation in odd-dimensional ambient space ℓ^+ does not change.

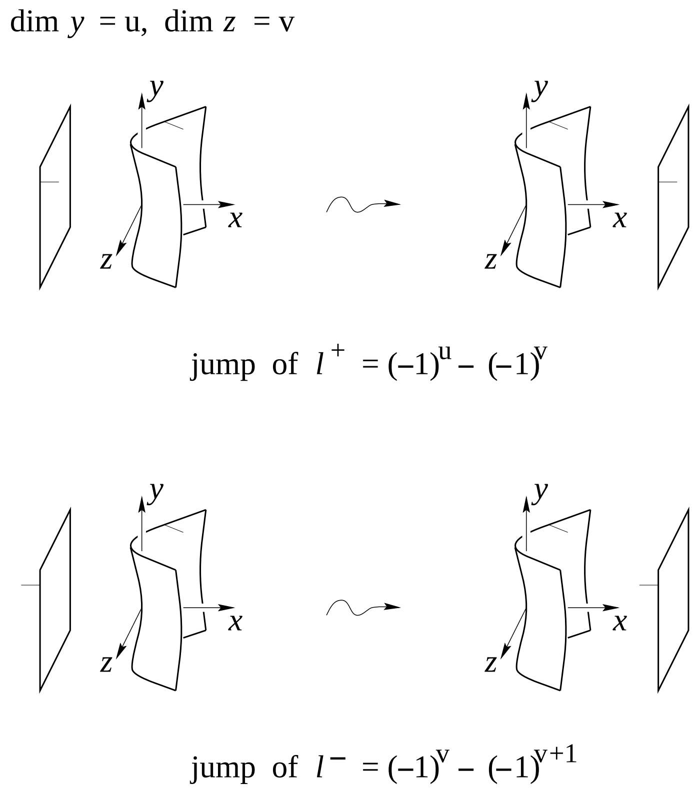

Figure 13: *Jumps of local values of the linking selftangency invariants*

This shows why there are two integer selftangency invariants, Arnold's $J^{\pm}$ [1, 2, 3], for plane curves, and only one, ℓ^{-}, for surfaces in 3-space. The fact of (3) reflects existence of locally noncoorientable direct selftangency substratum, like H^{+}, in higher dimensions.

As in the case of plane curve invariants, we can coorient the inverse selftangency stratum in the direction of decrease of ℓ^{-} and the direct selftangency stratum, for $\mathbf{R}^{even}$, in the direction of increase of ℓ^{+}. For plane curves this gives us $J^{\pm}$ up to additive constants [2, 3]. For ambient $\mathbf{R}^3$, the first choice gives exactly $E^2 + H^- - E^0$.

Example 6.3 On the standard $(n-1)$-sphere in $\mathbf{R}^n$, with the outer coorientation, $\ell^+ = (-1)^{n-1}$ and $\ell^- = -1$. On the sphere with the inner coorientation $\ell^+ = \ell^- = (-1)^{n-1}$.

Since the 6-sphere in $\mathbf{R}^7$ is evertible [16], we get

Corollary 6.4 *During a generic eversion of a 6-sphere, the number of opposite selftangencies is odd.*

Counting the signs of crossings of the discriminant, this number is -1.

7 Proofs

7.1 Proofs of Theorems 2.2 and 2.4 on enumeration of local invariants of generic maps

Consider a function ϕ on the set of the top strata S of the discriminant. Let us *integrate* ϕ along a generic loop γ in Ω. For this, taking zero as the initial sum, we follow γ and, every time when it intersects a stratum S, add either $\phi(S)$ (if the intersection is done in the direction of the coorientation of the stratum) or $-\phi(S)$ (if the intersection is done in the opposite direction).

$\{\phi(S)\}$ is the set of jumps of a local invariant along the strata if and only if the integral of ϕ along any generic loop is zero (then the invariant is uniquely defined by $\{\phi(S)\}$ up to an additive constant). This gives a system of linear equations on the values $\phi(S)$.

Since Ω is contractible, the equations should only express vanishing of the integrals along small loops in Ω around the set of nongeneric points of the discriminant (the latter has codimension 2 in Ω). Such a loop is realized as a loop around the origin in the parameter space of a germ of a generic 2-parameter family of mappings from our oriented surface to $\mathbf{R}^3$. So we will look through all such deformation and write out the linear system on the jumps.

The jump along a stratum S will be denoted by s. The coorientations of the strata are as in Sect.1. Unknowns h^+ and q^2, the jumps along the noncoorientable strata, may be nonzero only for $\mathbf{Z}_2$-invariants. The complete list of the 2-parameter families, particular normal forms and bifurcation diagrams are provided by general machinery of the Theory of Singularities [10, 5, 6, 7], and in what follows we do not consider any details of the classification procedure.

7.1.1 Uni-germs

The following is the complete list of generic 2-parameter families of map-germs $(\mathbf{R}^2, 0) \to \mathbf{R}^3$ [10]:

$$\begin{array}{ll} A_2: & x, y^2, y(y^2 + x^3 + \lambda x + \mu) \\ B_2^{\pm}: & x, y^2, y(x^2 \pm y^4 + \lambda y^2 + \mu) \\ H_2: & x, xy + y^2(y^3 + \lambda y + \mu), y^3 + \lambda y \end{array}$$

The bifurcation diagrams for the two different coorientations of the sheet, along with generic members of the families, are shown in Fig.14. Walking counterclockwise around the origins in the parameter planes we read the equations:

$$\begin{array}{llll} A_2: & (1) \quad b^+ - k^+ = 0, & (2) & b^- - k^- = 0 \\ B_2^-: & (3) \quad k^+ - h^- - b^- = 0, & (4) & k^- + h^- - b^+ = 0 \\ B_2^+: & (5) \quad e^2 + k^- - b^+ = 0, & (6) & e^0 + k^+ - b^- = 0 \\ H_2: & (7) \quad b^+ + c^{--} - c^{++} - b^- = 0 & & \end{array}$$

7.1.2 Bi-germs

Degenerate tangency of two smooth sheets

$$z = 0 \qquad \text{and} \qquad z = x^2 + y^3 + \lambda y + \mu$$

provides (Fig.15):

$$(8) \quad e^2 = h^-, \qquad (9) \quad e^0 = -h^-, \qquad (10) \quad e^1 = h^+.$$

Interaction of a smooth sheet with an umbrella. At the pinch point, the smooth sheet may be nontransversal either to the tangent line to the handle or to the image of the differential. The latter case has two subcases. All this is shown in Fig.15. The families are families of parallel translations of the umbrella. (λ, μ) are the coordinates of the pinch point. We get 8 equations:

$$\begin{array}{llll} (11) & c^{++} + c^{--} = t^3 & (12) & c^{+-} + c^{-+} = t^2 \\ (13) & c^{-+} + c^{+-} = t^1 & (14) & c^{--} + c^{++} = t^0 \\ (15) & c^{++} + e^1 = c^{-+} + e^2 & (16) & c^{+-} + e^0 = c^{--} + e^1 \\ (17) & c^{++} = c^{-+} + h^+ + h^- & (18) & c^{+-} = c^{--} + h^+ + h^-. \end{array}$$

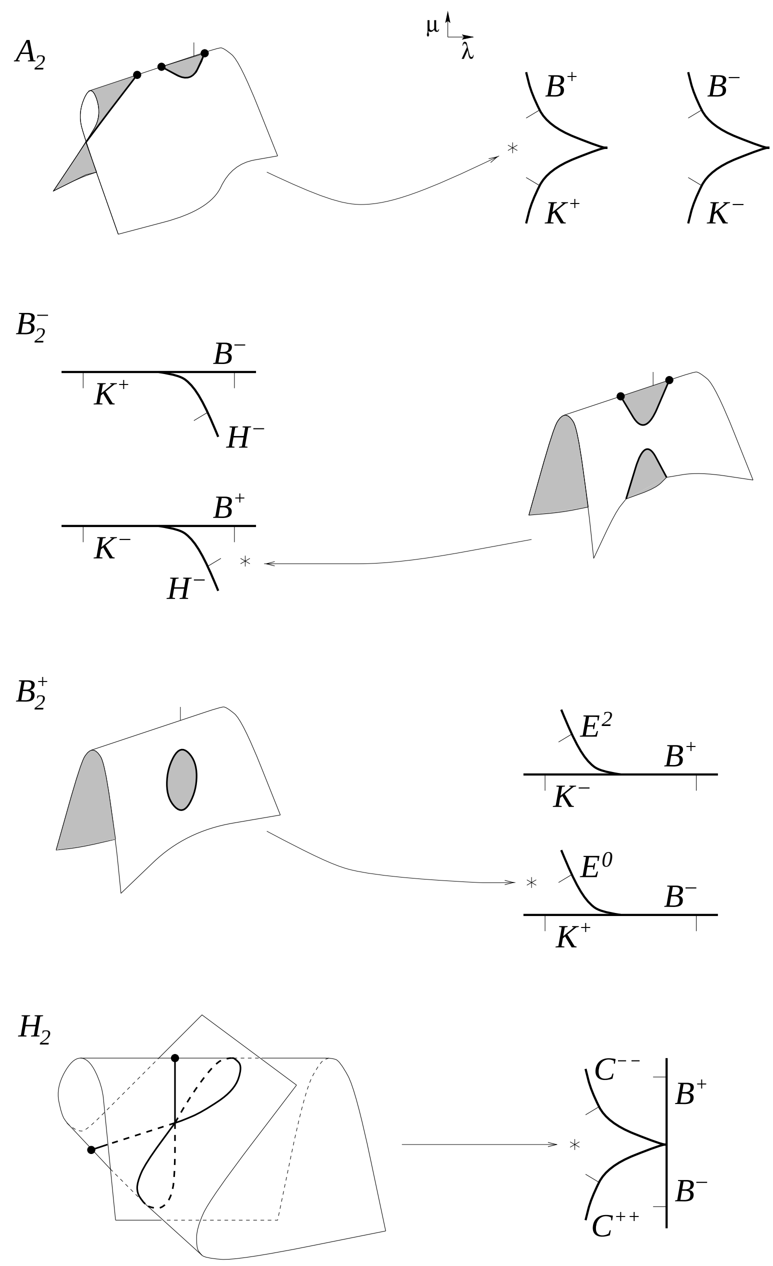

Figure 14: *Bifurcations in generic 2-parameter families of uni-germs*

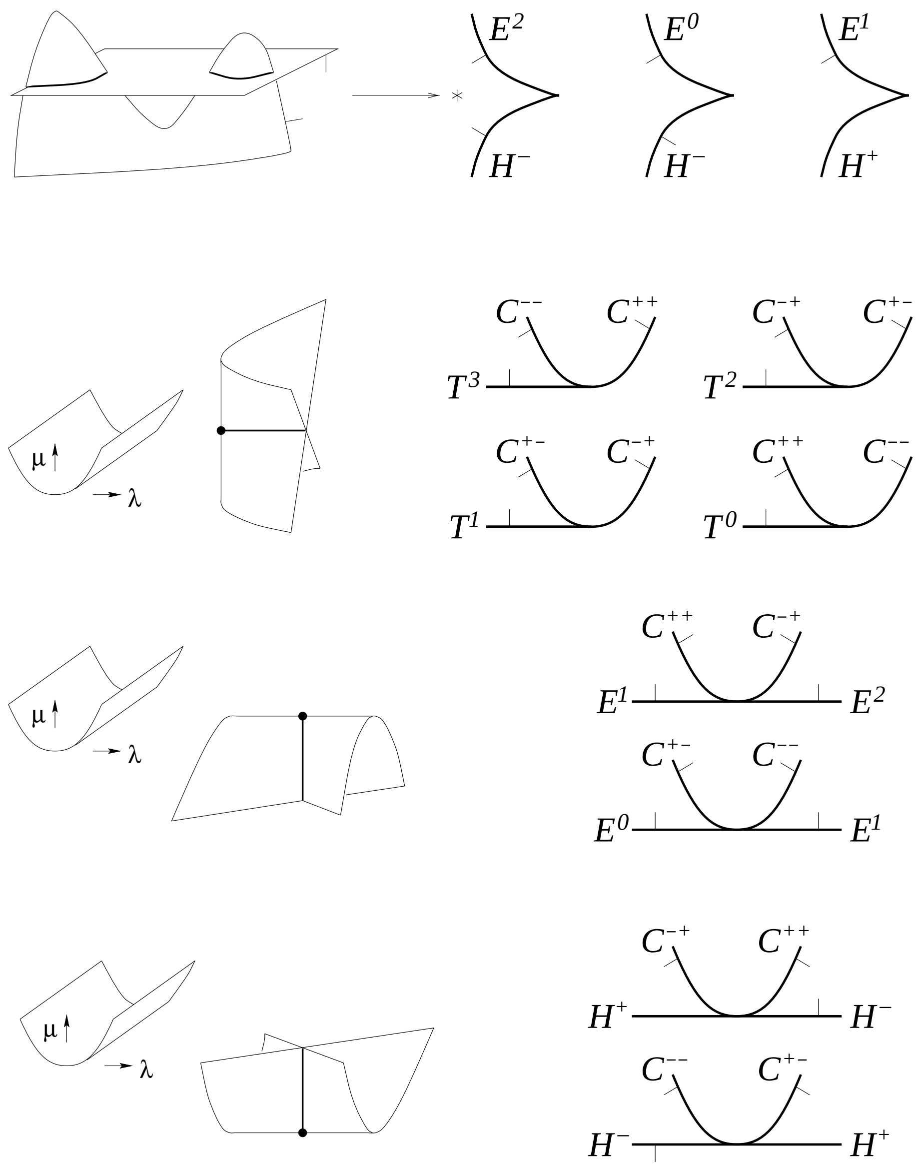

Figure 15: *Bifurcations in generic 2-parameter families of bi-germs*

7.1.3 Some 3-germs

Lemma 7.1 (20) $t^0 = t^1 = t^2 = t^3$.

Proof. Consider the interaction of three smooth sheets in Fig.16. λ and μ are the coordinates of the vertex of the parabolic sheet. From the bifurcation diagram we get $t^3 = t^2$. Changing the coorientations of the sheets we obtain $t^2 = t^1 = t^0$.

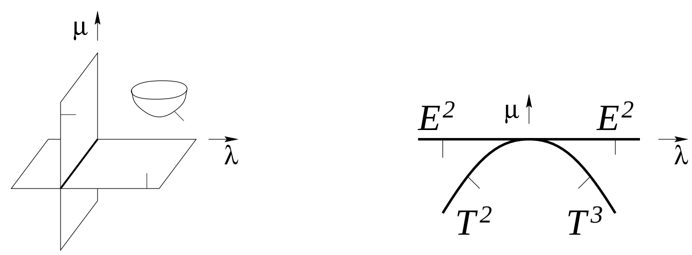

Figure 16: *Bifurcation showing that* $t^3 = t^2$

Lemma 7.2 (19) $q = 0$.

Proof. Consider the interaction of an umbrella and two transversal smooth sheets (Fig.17, parameters λ and μ are the coordinates of the pinch point). Walking counterclockwise around the origin in the parameter plane we read the equation:

$$c^{-+} + t^2 + c^{++} + q^3 - c^{-+} - t^3 - c^{++} = 0.$$

Thus, by the previous Lemma, $q^3 = 0$. Changing the coorientations of the smooth sheets, we get $q^4 = q^2 = 0$.

Remark. We have distinguished two directions at the pinch point: the tangent line to the handle and the image of the differential. In the proof of Lemma 7.2 we could take a different umbrella: with these two directions, for $\lambda = \mu = 0$, not separated by the smooth sheets. Similarly, in the proof of Lemma 7.1 we could take a hyperbolic sheet instead of the elliptic one. These changes would have no influence on the final results.

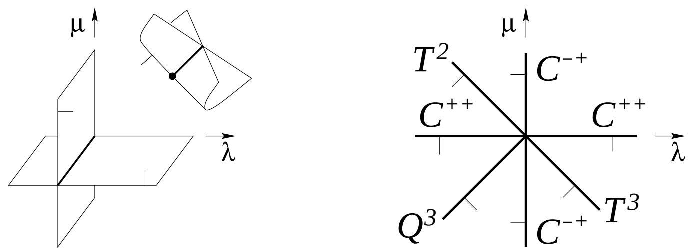

Figure 17: *Interaction of an umbrella and two transversal smooth sheets*

There are some more generic 2-parameter families of mappings involving only smooth sheets: quintuple point, a sheet passing through three coordinate planes with tangency to one of the coordinate lines, second order (degenerate) tangency of a smooth sheet and the line of transversal intersection of two others. But, it follows from the two lemmas above that these families provide no further independent equations.

Solving the system (1–20) over $\mathbf{Z}$ and over $\mathbf{Z}_2$ we get the systems of jumps claimed in Theorems 2.2 and 2.4 respectively.

7.2 Proofs of Theorems 3.1 and 3.2 on enumeration of local invariants of immersions

Due to the comments made after the formulation of Theorem 3.2 in Sect.3, we have to consider only local events in generic 2-parameter families of immersions. This leaves us with the twelve 'smooth' unknowns and the above equations (8–10) and (20). But now we will obtain extra information from some bifurcations that we did not need to study attentively at the end of the previous subsection.

Lemma 7.3 (21) $q^2 = q^3 = q^4$.

Proof. Fig.18 shows bifurcations of a sheet $u + w = (v + \lambda)^2 + \mu$ with respect to the coordinate planes in the (u, v, w)-space (for $\lambda = \mu = 0$, the sheet passes through the origin being tangent to the v-axis). This implies $q^3 = q^4$. Changing the coorientation of, say, the curved sheet we get the other equality.

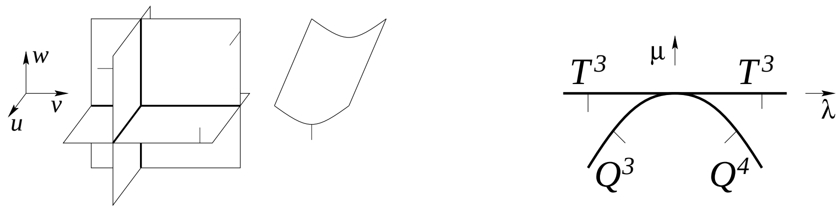

Figure 18: *Bifurcation showing that* $q^3 = q^4$

Since Q^2 cannot be cooriented by local means, an integer invariant has no jumps along the whole of the Q-stratum. Jumps of a $\mathbf{Z}_2$-invariant must be the same along all three substrata of Q.

The bifurcation diagram of a quintuple point (five plane passing through one point) consists of 5 Q-lines. This introduces no new equations.

The bifurcation diagram of a smooth sheet $u = v^3 + \lambda v + \mu$ with respect to two sheets $u^2 = w^2$ (for $\lambda = \mu = 0$, the first sheet has second order tangency with the line of intersection of the two others) is a cusp of T. Following the coorientations of the sheets we get either $t^0 = t^3$ or $t^1 = t^2$. Once again, nothing is new.

Now Theorems 3.1 and 3.2 follow as solutions to the system (8–10), (20), (21).

7.3 Proof of Theorem 4.2 about the integral invariant

The equality $I_\int = 2I_3 - I_t - I_p$ up to an additive constant is equivalent to the equality of jumps of both sides along all the strata. So, let us compare the jumps. We use the obvious

Lemma 7.4 *Crossing a sheet against its coorientation increases by 1 the degree of a connected component of the complement of the image.*

B^+, Fig.19. Here d is the degree of one of the components of the complement to the image. It determines the degrees of the other components. We calculate the difference of values of $I_\int$ on two mappings whose images differ only inside a small ball, where we substitute the right picture of Fig.19 for the left one. Due to the additivity of Euler characteristics, the calculations can be done locally [19, 20]:

$$b^+(I_\int) = ((d-1)+d+(d+1) - \frac{1}{2}(d+d)) - ((d-1)+d) = 1.$$

Similarly, $b^-(I_\int) = -1$.

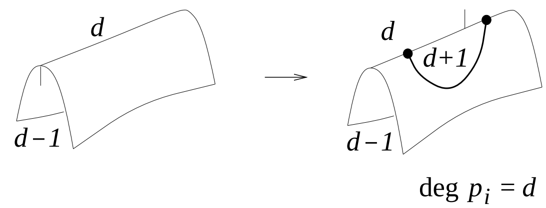

Figure 19: *Distribution of degrees for B^+ bifurcation*

$C^{\pm,+}$, Fig.20. We have:

$$c^{\pm,+}(I_\int) = ((d \pm 1 + 1) - (d + \frac{1}{2}) - \frac{1}{2}((d+1)) - \frac{1}{2}d = \pm 1.$$

Similarly, $c^{\pm,-}(I_\int) = \pm 1$.

All the calculated jumps coincide with the jumps of $2I_3 - I_t - I_p$. Note that jumps $c^{\alpha,+}$ and b^+ specify a local invariant on Ω uniquely. Thus it only remains to prove locality of I.

There are no more direct calculations needed.

Indeed, assume we need to calculate a jump of $I_\int$ along K^+, with a given distribution of degrees for the components of $\mathbf{R}^3 \setminus \mathrm{Im} f$ (the distribution is completely defined by the degree of any of the components). We can realize such a jump within the 2-parameter A_2-family (Fig.14) with the appropriate distribution of the degrees (Fig.21). The difference of the values of $I_\int$ on two particular mappings is determined independently of any path connecting the mappings in Ω. So, the jumps of $I_\int$ in the A_2-family along K^+ and B^+ are equal (see the bifurcation diagram of A_2 in Fig.14). This means that the jumps of $I_\int$ and $2I_3 - I_t - I_p$ along K^+ coincide.

In the similar way, jumps of $I_\int$ known by now successively determine the following jumps of $I_\int$ (for any possible distribution of the degrees).

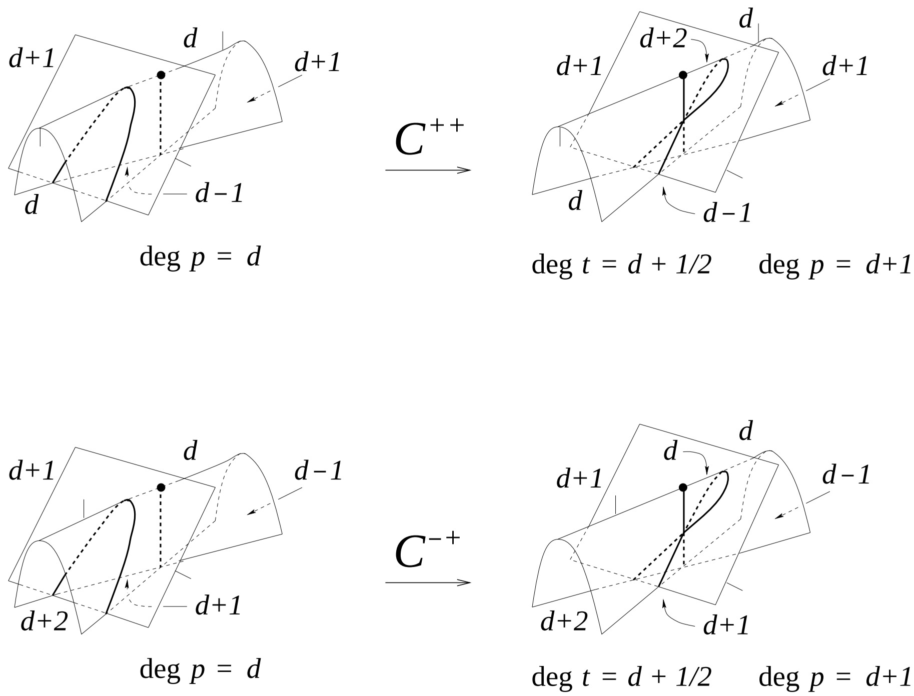

Figure 20: *Distribution of degrees for* $C^{\pm,+}$ *bifurcations*

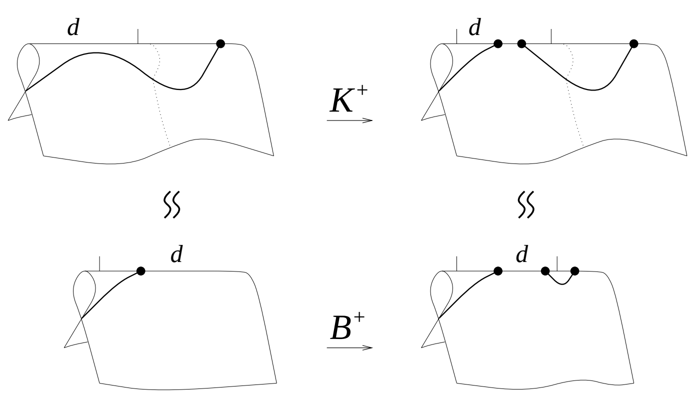

Figure 21: *Embedding of* K^+ *bifurcation into* A_2*-family*

(1) from Fig.14: $k^- = -1, \quad h^- = 2, \quad e^2 = 2, \quad e^0 = -2;$

(2) from Fig.15: $t^j = 0,\ j = 0, 1, 2, 3, \quad e^1 = 0, \quad h^+ = 0;$

(3) from Fig.17: $q^4 = q^3 = q^2 = 0.$

Thus, $I_\int$ is a local invariant.

7.4 Proof of Theorem 4.4 about the smoothed form of the integral invariant

Values of $I_\int$ and $\tilde{I}_\int = \sum \widetilde{deg}(\widetilde{D})\chi(\widetilde{D}) - \sum deg(p)$ on embeddings coincide. So, as in the proof of Theorem 4.2, it is sufficient to show that the jumps of $I_\int$ and $\tilde{I}_\int$ coincide along $B^\pm$ and $C^{\alpha,\beta}$. Fig.22, that shows smoothed B^+ and $C^{\pm,+}$ transformations, gives the following jumps of $\tilde{I}_\int$:

$$b^+ = ((d-1)\cdot 1 + d\cdot 2 + (d+1)\cdot 1 - (d+d)) - \\ -((d-1)\cdot 1 + d\cdot 1) = 1,$$

$$c^{++} = ((d-1)\cdot 1 + d\cdot 0 + (d+1)\cdot 2 + (d+2)\cdot 1 - (d+1)) - \\ -((d-1)\cdot 1 + d\cdot 1 + (d+1)\cdot 2 - d) = 1,$$

$$c^{-+} = ((d-1)\cdot 1 + d\cdot 1 + (d+1)\cdot 1 + (d+2)\cdot 1 - (d+1)) - \\ -((d-1)\cdot 1 + d\cdot 1 + (d+1)\cdot 1 + (d+2)\cdot 1 - d) = -1.$$

Changing the coorientations of the sheets to the opposite ones interchanges the signs $+$ and $-$ in the expressions of the degrees. This gives for $\tilde{I}_\int$: $b^- = -1$, $c^{--} = -1$, $c^{+-} = 1$.

All the calculated jumps are the same as for $I_\int$. Thus, $\tilde{I}_\int = I_\int$.

7.5 Sketch of the proof of Theorem 5.5 about the linking invariant

The invariant ℓ^- certainly has zero jumps along the strata T, Q, E^1, H^+. Direct computations show:

Lemma 7.5 $\quad c^{\pm,+}(\ell^-) = \mp 1, \quad b^+(\ell^-) = 1.$

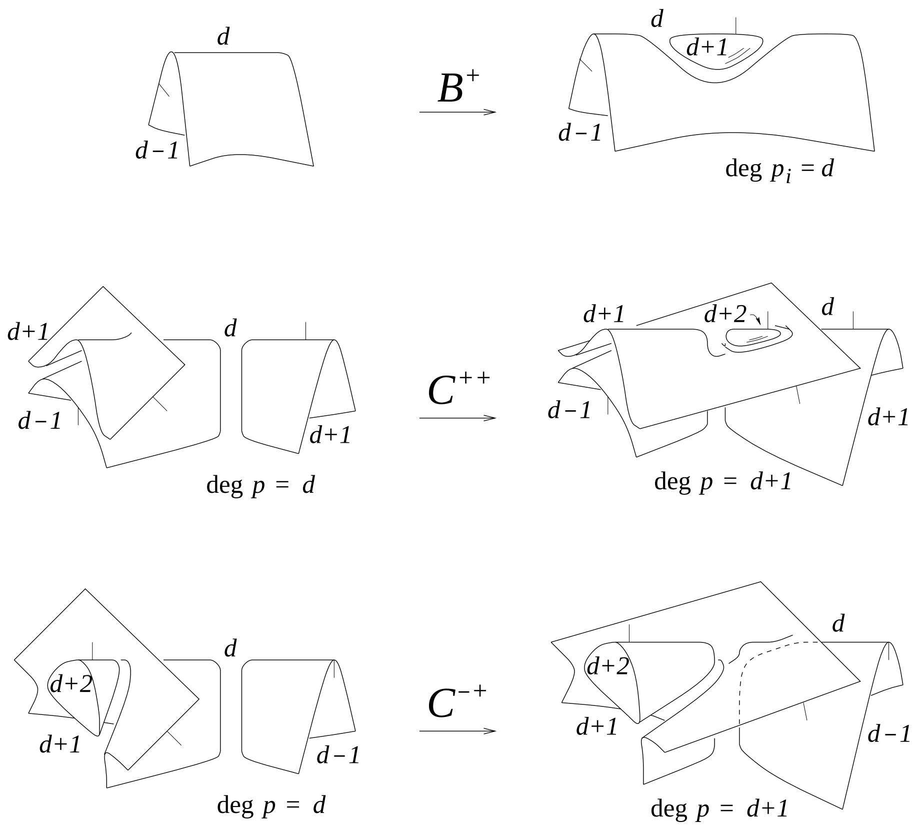

Figure 22: *Bifurcations of smoothed images*

As in subsection 7.3, Lemma 7.5 defines all the other jumps of ℓ^- via 2-parameter bifurcations. Thus ℓ^- is local. The values of t, $c^{+,+}$ and b^+, together with the initial conditions checked on standard embeddings of surfaces of genus g (subsections 4.2 and 5.2), identify the invariant ℓ^- as $-I_\int + 2I_p$.

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