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SIMPLE PROJECTING MAPS

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UDC 517.27

This paper provides a complete list of simple germs of projecting maps of submanifolds (some of them singular) on a space of the same or smaller dimension and enumerates all contiguities of simple projecting maps of hypersurfaces. Two particular cases — the classification of projections of surfaces in general position in $\mathbf{R}^3$ on $\mathbf{R}^2$ and the classification of simple projecting maps in $\mathbf{C}^1$ — were published earlier in [1, 2].

Definition. A *projecting map* of a submanifold V of a total space of a bundle E on base B is a triple $V \to E \to B$ consisting of an embedding and a projection. An *equivalence* of projecting maps $V_i \to E_i \to B_i$, i = 1, 2, is a commutative 3 × 2 diagram whose vertical arrows are diffeomorphisms, $h: E_1 \to E_2$ and $k: B_1 \to B_2$, such that $hV_1 = V_2$. A projecting map $V \to E' \to B$ is said to be *stably equivalent* to the projecting map $V \to E \to B$ if V is embedded in E' as a submanifold of the total space of subbundle $E \subset E'$.

Analogous definitions are given for germs.

We denote by n and p the dimensions of the fiber and, respectively, the base of the bundle, and use m for the codimension of V in E. Locally $E \simeq \mathbf{K}^n \times \mathbf{K}^p, B \simeq \mathbf{K}^p$, where K = R, C, and the projection $E \to B$ is the projection on the second factor. Let $V = f^{-1}(0)$, where f belongs to the space $\mathscr{E}^m(n+p)$ of germs at zero of infinitely differentiable, analytic (holomorphic), or formal mappings from $\mathbf{K}^n \times \mathbf{K}^p$ into $\mathbf{K}^m$, and f(0) = 0. We denote by $x = (x_1, \ldots, x_n)$ the points of the fiber $\mathbf{K}^n$ and by $u = (u_1, \ldots, u_p)$ the points of the base $\mathbf{K}^p$. For simplicity, the germ of the projecting map $(x, u) \to u$ of the surface given by the equations $f_1(x, u) = 0, \ldots, f_m(x, u) = 0$ will be referred to as the projecting map f.

Definition. Two germs of projecting maps at the points τ_1 and τ_2 are said to be *t-equivalent* if they become equivalent after shifting τ_1 and τ_2 to 0. The *codimension* of the projecting map f is the codimension of its equivalence class in the space $\mathscr{E}^m(n+p)$.

Definition. A germ of projecting map is *simple* if it has no modules (continuous invariants with respect to t-equivalence).

One can show that a projecting map of finite codimension has a sufficient jet of finite order. This reduces the classification of simple projecting maps to the formal case. From

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TABLE 1

p	f_0	$\Delta(x, u)$	Codimension	Notation
Smooth hypersurfaces (grad $f\|_{x=0, u=0} \neq 0$)				
$\geqslant 1$	A_0	0	0	A_0
1	X_μ, $\mu > 0$	u	$\mu-1$	X_μ
$\geqslant \mu$	X_μ, $\mu > 0$	$\sum_{i=1}^{\mu} u_i e_i(x)$	0	X_μ^V
$\geqslant \mu - 1$	X_μ, $\mu > 1$	$\sum_{i=1}^{\mu-1} u_i e_i(x) + g(u_\mu, \ldots, u_p) e_\mu(x)$, $g \in Y_\nu$, $\nu > 0$	ν	$X_\mu^{Y_\nu}$
	A_μ, $\mu \geqslant 3$	$u_{\mu-1} x_1^{\mu-1} + (u_{\mu-1}^k + q) x_1^{\mu-2} + \sum_{i=1}^{\mu-2} (u_i x_1^{i-1})$, $1 < k < \varkappa(\mu)$	k	A_μ^k
	A_μ, $\mu \geqslant 4$	$u_{\mu-1} x_1^{\mu-1} + q x_1^{\mu-2} + \sum_{i=1}^{\mu-2} (u_i x_1^{i-1})$	$\varkappa$	$A_\mu^\varkappa$
2	A_3	$u_1^2 x_1 \pm u_1^2 x_1^2 + u_2$	3	${}^2A_3^\pm$
		$u_1^2 x_1 + u_2$	4	${}^2A_3^0$
	A_4	$u_1 x_1 \pm u_1 x_1^3 + u_2$	2	${}^2A_4^\pm$
		$u_1 x_1 + u_2$	3	${}^2A_4^0$
		$u_1 x_1^2 + u_1 x_1^3 + u_2$	3	${}^2A_4^1$
		$u_1 x_1^2 \pm u_1^2 x_1^3 + u_2$	4	${}^2A_4^{2\pm}$
		$u_1 x_1^2 + u_2$	5	${}^2A_4^3$
Singular hypersurfaces (grad $f\|_{x=0, u=0} = 0$)				
1	A_1	$\pm u^\mu$	$\mu-1$	B_μ
> 1	A_1	$g(u_1, \ldots, u_p)$, $g \in Y_\nu$	ν	$A_1^{Y_\nu}$
1	A_μ, $\mu > 1$	$u x_1$	μ	$C_{\mu+1}$
	A_2	u^2	3	F_4
$\geqslant 2$		$u_1 x_1 + q$	2	A_2^2
2		$u_1^2 + u_2^2 \pm u_1^k x_1$, $k \geqslant 2$	$2k$	${}^2A_2^{k\pm}$

Giusti's work [3] one extracts the following result.

Proposition. For $n \geq m$ every simple projecting map is stably equivalent to some projecting map with one of the following three values for the dimension parameters: 1) $m = 1$ (hypersurface); 2) $n = 2$, $m = 3$; 3) $n = 2$, $m = 2$.

We next formulate the classification results for each of these three cases.

1. In Table 1 X_μ (or Y_ν) denotes one of the normal forms of simple singularities of functions of n (or $p - \mu + 1$) variables, i.e., $X, Y = A, D, E$ (see [4]; for example, A_μ: $\pm x_1^{\mu-1} \pm x_2^2 \pm \ldots \pm x_n^2$); if the collection of arguments of function g appearing in Table 1 is empty, we consider that $g \in A_1$, $g = 0$; $e_1, \ldots, e_\mu$ is a monomial basis of the local ring of the

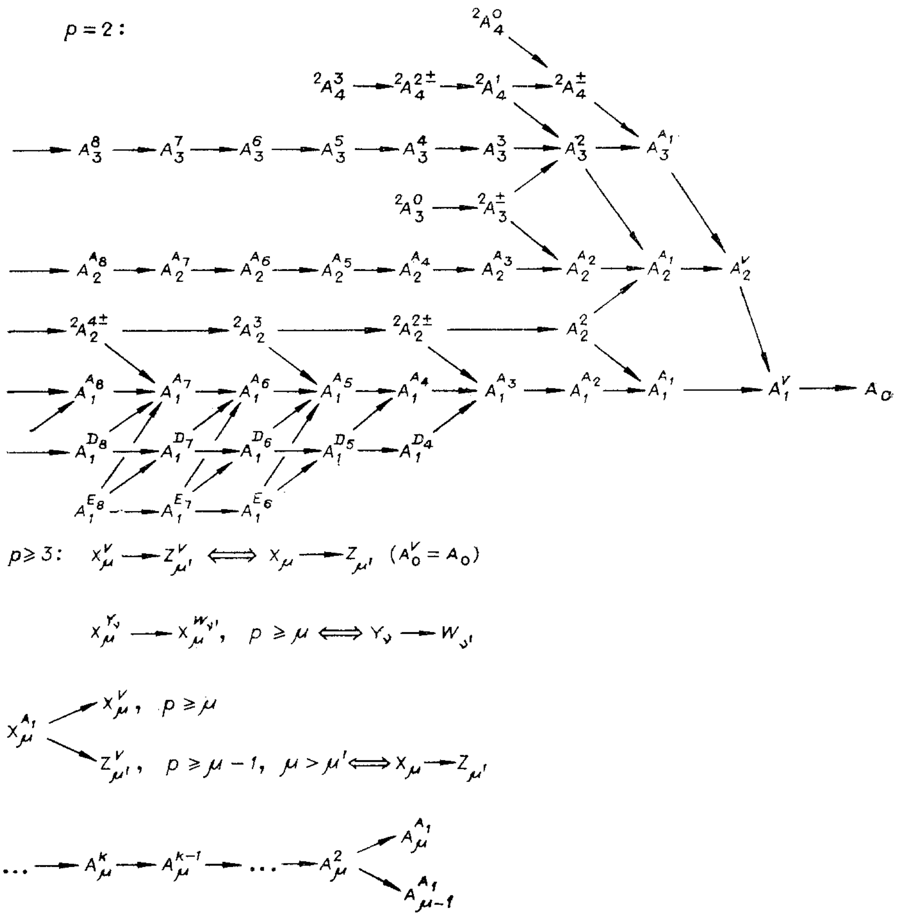

Fig. 1

function f_0, with $e_\mu = \operatorname{Hess} f_0$; $q = \pm u_\mu^2 \pm \ldots \pm u_p^2$; and function $\varkappa(\mu)$ assumes the following values:

μ	2	3	4	5	6	7	8	9	10	≥11
$\varkappa$	2	∞	3	5	4	6	5	7	6	7

THEOREM 1. A germ of projecting map of real hypersurface is simple if and only if it is equivalent to the germ at zero of the projecting map $(x, u) \to u$ of a manifold $f = 0$, where $f(x, u) = f_0(x) + \Delta(x, u)$ is one of the functions appearing in Table 1.

The projecting maps displayed in Table 1 are pairwise nonequivalent up to the arrangement of signs in the form q and the normal forms X_μ and Y_ν. For $p = 1$, $A_1^V = A_1$.

For all simple projecting maps of singular hypersurfaces the given hypersurfaces have isolated singularities.

Remark. The projecting maps of smooth hypersurfaces were classified by Arnol'd [1] for $(n, p) = (1, 2)$ and for codimension at most 2. They are all simple.

Definition. We say that the projecting map f is *contiguous* to the projecting map g, and write $f \to g$, if one can produce a projecting map equivalent to g by an arbitrarily small perturbation of f.

Arnol'd observed that the list of simple projecting maps of hypersurfaces on the line ($p = 1$) is identical to the list of simple functions on manifolds with boundary (A_μ, B_μ, C_μ, D_μ, E_μ, F_4; see [5]): the boundary is the preimage of zero under the projection $(x, u) \to u$. Accordingly, the contiguity diagrams of these objects coincide too.

One can show that for $p \geq 2$, up to transitivity ($X \to Y$, $Y \to Z \Rightarrow X \to Z$), there exist only the contiguities of simple projecting maps of surfaces indicated in Fig. 1.

The complex list of germs of simple projecting maps of hypersurfaces differs from the real list by the absence of series $\{{}^2A_2^{k\pm}, k \geq 2\}$ (in the $\mathbf{C}$-case the latter is contiguous to

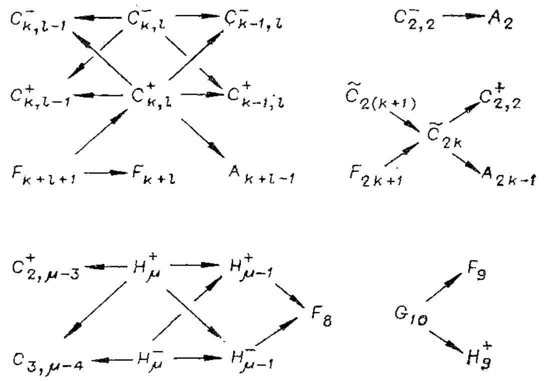

Fig. 2

a projecting map containing a module in the normal form) and by the fact that everywhere in Table 1 the sign ± is replaced by +.

2. THEOREM 2. For $n = m + 1 \geqslant 3$ a germ of projecting map is simple if and only if it is stably equivalent to either a simple projecting map of hypersurface or to the versal deformation of some simple curve in $\mathbf{K}^3$.

The list of simple curves in $\mathbf{C}^3$ not equivalent to plane curves is given in Giusti's works [3, 5]. The corresponding real list differs from Giusti's list by the arrangement of the signs + and − in certain normal forms (the curves S and T_8) and by the addition of a new curve $\tilde{T}_7$ ($x_1^2 + x_3^3 = 0$, $x_2^2 + x_3^2 = 0$) which is **C**-equivalent to T_7.

The list of contiguities of those simple projecting maps which are described by Theorem 2 is the same as in the case of hypersurfaces, except for the fact that $X^\vee_\mu$ and $Z^\vee_\mu$, are now versal deformations of simple curves in $\mathbf{K}^3$ (which are no longer necessarily plane).

3. The simple germs of projecting maps of surfaces on manifolds of the same dimension are deformations of simple 0-dimensional complete intersections in $\mathbf{K}^2$. Therefore, before carrying out the classification of simple projecting maps for $n = m = 2$ we must write the list of the indicated complete intersections.

The **C**-case is treated in Giusti's work [3].

The real list consists of the following maps from $(\mathbf{R}^2, 0)$ into $(\mathbf{R}^2, 0)$ (our notation differs from Giusti's):

A_μ, $\mu \geqslant 0$	$(x_1, x_2^{\mu+1})$
$C^\pm_{k,l}$, $2 \leqslant k \leqslant l$	$(x_1x_2, x_1^k \pm x_2^l)$
$\tilde{C}_{2k}$, $k \geqslant 3$	$(x_1^2 + x_2^2, x_2^k)$
$H^\pm_{m+5}$, $m \geqslant 4$	$(x_1^2 \pm x_2^m, x_1x_2^2)$
F_{2m+1}, $m \geqslant 3$	$(x_1^2 + x_2^3, x_2^m)$
F_{2m+4}, $m \geqslant 2$	$(x_1^2 + x_2^3, x_1x_2^m)$
G_{10}	(x_1^2, x_2^4).

If at least one of the numbers k and l is odd, then the singularities $C^+_{k,l}$ and $C^-_{k,l}$ are equivalent. Singularities H^+_{2m}, and H^-_{2m}, are also equivalent. For the field **C** we have that for any k, l, and μ, $C^+_{k,l} \sim C^-_{k,l}$, $\tilde{C}_{2k} \sim C^+_{k,k}$, and $H^+_\mu \sim H^-_\mu$. In all these cases the upper indices are omitted.

Using the list of contiguities of simple 0-dimensional complete intersections in $\mathbf{C}^2$ (see [3]) it was not hard to show that all the contiguities of the corresponding real singularities appear in Fig. 2.

We next formulate a theorem on the classification of simple projecting maps of surfaces on manifolds of the same dimension. We shall use the notations: $f(x, u) = f_0(x) + \Delta(x, u)$; $e_1, \ldots, e_\mu$ for the monomial basis of the space $\mathscr{E}^2(2)/\{\langle f_{01}, f_{02}\rangle \cdot \mathscr{E}^2(2) + \mathscr{E}^1(2)\langle \partial f_0/\partial x\rangle\}$, $x = (x_1, x_2)$, ordered according to weight: $\mathrm{wt}\, e_i \leqslant \mathrm{wt}\, e_{i+1}$ (see [3]); g for a real simple function of class Y_ν, $\nu > 0$, i.e., Y = A, D, E (as in Theorem 1, if the collection of arguments of g is empty,

TABLE 2

p	f_0	$\Delta(x, u)$	Codimension	Notation
1	$C^{\pm}_{k,l}, \widetilde{C}_{2r}$, F_μ, $2 \leqslant k \leqslant l$, $r \geqslant 2,\ \mu \geqslant 7$	$(0,\ u)$	$\mu-1$	X_μ
	$C^{+}_{2,2}$ $(x_1^2,\ x_2^2)$	$(x_2^3,\ u)$	4	F_5
	$C_{2,3}$	$(u,\ 0)$	5	F_6
2	$C^{-}_{2,2}$	$(u_1,\ u_2 + u_1^k x_1),\quad k \geqslant 1$	$2k$	${}^2C^{2k}_{2,2}$
$\geqslant 3$	$C^{+}_{2,2}$ $(x_1^2,\ x_2^2)$	$(u_1 + u_2^r x_2,\ u_2 + u_1^s x_1),\quad 1 \leqslant r \leqslant s$	$r+s$	${}^2C^{r,s}_{2,2}$
		$(u_1 + u_3 x_2,\quad u_2 + h(u)\, x_1)$ $h(u) = u_3 + g(u_4, \ldots, u_p)$	ν	$C^{+,Y_\nu}_{2,2}$
		$h(u) = u_1^k \pm u_3^2 + q,\quad k \geqslant 1$	$k+1$	$C^{2,k}_{2,2}$
		$h(u) = u_1 + u_3^3 + q$	3	$C^{3}_{2,2}$
$\geqslant 4$		$h(u) = u_1^k + u_3 u_4 \pm u_4^l + \sum\limits_{i=5}^{p} \pm u_i^2$, $k \geqslant 1,\quad l \geqslant 3$	$k+l-1$	$C^{l,k}_{2,2}$
		$h(u) = u_1 \pm u_3^2 + u_4^3 + \sum\limits_{i=5}^{p} u_i^2$	4	$C^{3,1,2}_{2,2}$
$\geqslant \mu - 1$	$C^{-}_{2,2}, F_8$, $C^{\pm}_{2,l},\quad l > 2$	$\sum\limits_{i=1}^{\mu-1} u_i e_i(x) + g(u_\mu, \ldots, u_p)\, e_\mu(x)$	ν	$X^{Y_\nu}_\mu$
	$C_{2,3}$	$(u_2,\ u_1 + u_3 x_2 + u_4 x_2^2 + (u_4^r + q)\, x_1)$, $r > 1$	r	$C^{r}_{2,3}$
	$C_{3,l}, l > 3$	$\Bigl(u_l,\ \sum\limits_{i=1}^{l-1} u_i x_2^{i-1} + u_{l+1} x_1 + u_{l+2} x_1^2 +$ $+ h(u)\, x_2^{l-1}\Bigr)$ $h(u) = \pm u_{l+2} + g(u_\mu, \ldots, u_p)$	ν	$C^{Y_\nu}_{3,l}$
		$h(u) = \pm u^r_{l+2} + q,\quad r > 1$	r	$C^{r}_{3,l}$
	F_7, F_9, F_{10}	$\sum\limits_{i=1}^{\mu-1} u_i e_i(x) + (u_{\mu-1} + g(u_\mu, \ldots, u_p))\times$ $\times e_\mu(x)$	ν	$F^{Y_\nu}_\mu$
	F_{10}	$\sum\limits_{i=1}^{\mu-1} u_i e_i(x) + (u_9^r + q)\, e_{10}(x)$, $1 < r < \varkappa > 4$	r	F^{r}_{10}
	G_{10}	$\sum\limits_{i=1}^{\mu-1} u_i e_i(x) + (u_9 + g(u_{10}, \ldots, u_p))\times$ $\times\ e_{10}(x)$, $e_9 = (x_2^3,\ 0)$ $e_{10} = (0,\ x_1 x_2^2)$	ν	$G^{Y_\nu}_{10}$
$\geqslant \mu$	X_μ	$\sum\limits_{i=1}^{\mu} u_i e_i(x)$	0	X^{V}_{μ}

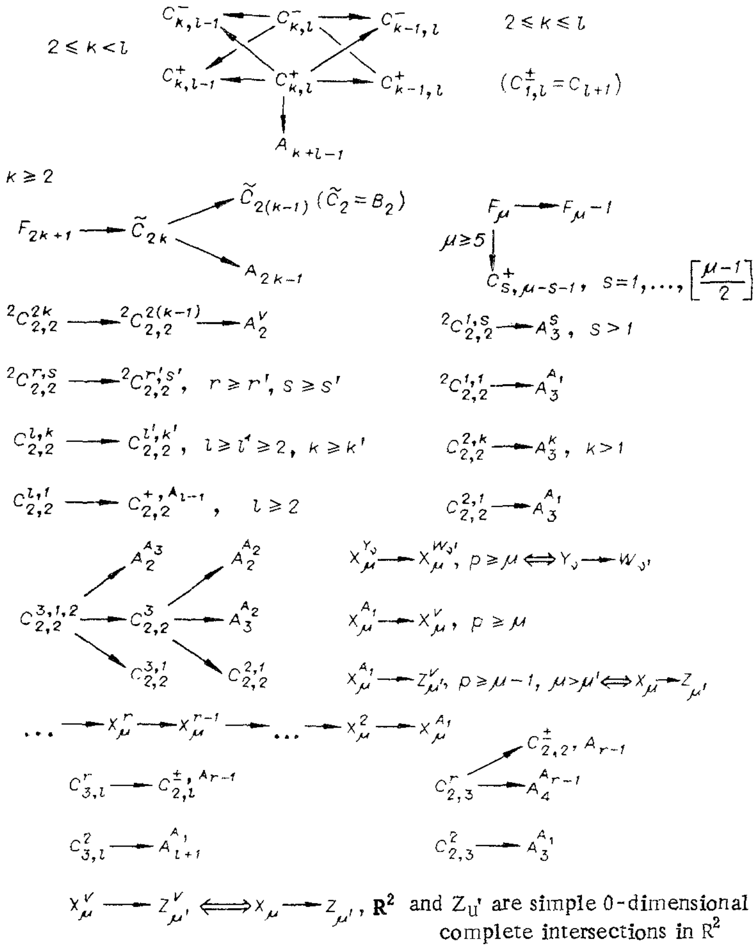

Fig. 3

we consider that $g\in A_1$, $g=0$); $q=\pm u_\mu^2\pm\dots+u_p^2$; X_μ for the class of f_0. If not otherwise stipulated, we write the normal form $C^+_{2,2}$ as (x_1^2, x_2^2). The complete intersection $\tilde C_4(x_1^2+x_2^2, x_2^2)$ is equivalent to $C^+_{2,2}$.

THEOREM 3. A germ of projecting map of real surface on a manifold of the same dimension is simple if and only if it is stably equivalent to either a simple projecting map of hypersurface for n = 1, or to the germ at zero of the projecting map $(x, u)\to u$ of the manifold $f = 0$, where f is one of the maps appearing in Table 2.

Remarks on Table 2.

a) The value $\varkappa$ (in series F^r_{10}) is not known exactly (possibly, $\varkappa=\infty$).

b) In the last row of the table X_μ designates one of the singularities $C^{\pm}_{k,l}$-G_{10}.

All surfaces appearing in this table are singular for p = 1 (with isolated singularity) and smooth for p > 1.

One has the following equivalences of projecting maps over the field $\mathbf{C}$:

$$C^+_{k,l}\sim C^-_{k,l},\quad 2\leqslant k\leqslant l;$$
$$\tilde C_{2k}\sim\tilde C_{k,k},\quad 2\leqslant k;\quad {}^2C^{2r}_{2,2}\sim{}^2C^{r,r}_{2,2},\quad r\geqslant 1;$$
$$C^{+,Y_\nu}_{k,l}\sim C^{-,Y_\nu}_{k,l},\quad C^{+,V}_{k,l}\sim C^{-,V}_{k,l}\quad 2\leqslant k\leqslant l;$$
$$\tilde C^V_{2k}\sim C^{+,V}_{k,k},\quad 3\leqslant k;\quad H^{+,V}_\mu\sim H^{-,V}_\mu,\quad \mu\geqslant 9.$$

The complex list of simple projecting maps for n = m is obtained from the list of $\mathbf{C}$-simple projecting maps of hypersurfaces for n = 1 and Table 2, in which the projecting maps indicated above are identified and the sign ± is replaced by + in all normals forms.

Some contiguities of projecting maps appear in Fig. 3 (where projecting maps stably equivalent to projecting maps of hypersurfaces are denoted by the same symbols).

For the field **R** and for p = 1, and for the field **C** and p = 1, 2, the list of contiguities given here is exhaustive modulo transitivity. For the remaining cases this is possibly not true.

4. In conclusion, we indicate the codimensions C and C_0 of the sets of simple projecting maps of nonsingular and, respectively, not necessarily singular surfaces, depending on the values of the triple (n, m, p):

a) n = m = 1

$$\begin{array}{c|c|c|c} p & 1 & 2 & \geqslant 3 \\ \hline C & \infty & 3 & 2 \end{array}$$

n = m = 2

$$\begin{array}{c|c|c|c|c|c} p & 1 & 2 & 3,4 & 5-9 & \geqslant 10 \\ \hline C & \infty & 3 & 2 & 1 & 0 \end{array}$$

n = m ⩾ 3 — the same as for n = m = 2, except for the case p = 9: C(9) = 0;

b) n = m + 1 = 2

$$\begin{array}{c|c|c|c|c} p & 1 & 2-6 & 7 & \geqslant 8 \\ \hline C & 7 & 2 & 1 & 0 \end{array}$$

n = m + 1 ⩾ 3 — the same, if p ≠ 4, 5, 6, otherwise C = 1;

c) n = m + 2 = 3 or n > m + 2 ⩾ 4:

$$\begin{array}{c|c|c|c|c} p & 1 & 2-5 & 6 & \geqslant 7 \\ \hline C & 6 & 2 & 1 & 0 \end{array}$$

n = m + 2 ⩾ 4 — the same, if p ≠ 5, 6: C(5) = 1, C(6) = 0;

d) for all n and m, n ⩾ m, $C_0(1) = 4$ and $C_0(p) = C(p)$ for p ⩾ 2.

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