SEMI-SIMPLICIAL RESOLUTIONS AND HOMOLOGY OF IMAGES AND DISCRIMINANTS OF MAPPINGS

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Introduction

In [**13**] the technique of semi-simplicial resolutions was applied to the study of the topology of the image of a stable perturbation f of a map-germ $\mathbb{C}^n \to \mathbb{C}^p$ with $n < p$. The crucial role was played there by the *multiple point sets* $D^k(f)$. These are the closures, in $(\mathbb{C}^n)^k$, of the sets of k-tuples of pairwise distinct points sharing the same image under f. The set D^k is invariant under the action of the permutation group S_k, interchanging the copies of the source. According to [**13**], there exists a spectral sequence computing the homology of the image Y of f whose E^1 term is formed by the homology of the sets D^k, anti-symmetric (*alternating*) with respect to these actions.

We get a good situation starting with a corank-1 map-germ of finite $\mathscr{A}$-codimension. Then each $D^k(f)$ is a Milnor fibre of an isolated complete intersection singularity (*icis*) and, thus, has non-trivial alternating homology only in one dimension, at least over $\mathbb{Q}$. As a consequence, the spectral sequence mentioned degenerates at E^1 and the rational homology of Y is the sum of the alternating homology of the D^k [**13**]. This decomposition is very useful in the study of the mixed Hodge structure [**13**], of the monodromy of the image [**12**], etc.

For maps $\mathbb{C}^n \to \mathbb{C}^p$ with $n \geqslant p$, the image should be replaced by the discriminant, that is, by the set of the critical values of the mapping. Indeed, the discriminant hypersurface is very much like the image of a map $\mathbb{C}^{p-1} \to \mathbb{C}^p$. For example, if we are in the nice dimensions [**4**], the discriminant of a perturbation of a map-germ of a finite $\mathscr{A}$-codimension has the homotopy type of a wedge of a finite number of middle-dimensional spheres [**8**], that is, we get the same as for the image mentioned [**18**].

In the present paper we introduce the geometric approach to the algebraic construction of the semi-simplicial resolution used in [**13**]. We extend the results, obtained in [**13**] for rational homology, to the case of integer coefficients. And we apply these results to the investigation of the topology of discriminants of corank-1 map-germs.

In § 1 we give a description of a geometric realization of the semi-simplicial resolution of the image of a finite map f. We show that the homological spectral sequence, associated to the natural filtration on this resolution, has E^1 term consisting of the homology Alt H_* of the alternating chain subcomplexes of the multiple point sets $D^k(f)$ of f.

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Alternating chains on a variety invariant under an action of a permutation group are chains twisting with respect to the sign homomorphism of this group. In § 2, following this approach, we consider a Milnor fibre F of an icis, invariant under an action of a finite subgroup G of the orthogonal group of the ambient linear space. We consider a complex of integer chains on F twisting with respect to the determinantal homomorphism of G. We prove (Theorem 2.1.2) that the homology of this complex coincides with the determinant-twisting part of the integer homology of F and, thus, is non-trivial only in one, middle, dimension.

Multiple point sets $D^k(f)$ of a stable perturbation f of a corank-1 map-germ $\mathbb{C}^n \to \mathbb{C}^p$, with $n < p$, are Milnor fibres of icises invariant under representations of permutation groups for which the determinantal and sign homomorphisms coincide (§ 3.2). Thus for the image of such an f, Theorem 2.1.2 implies degeneration of the spectral sequence of § 1 at E^1, giving a decomposition of the integer homology of the image into a sum of alternating parts of the homology groups of the sets $D^k(f)$ (Theorem 3.3.1). This is similar to the result of **[13]** on the rational homology.

In §§ 4–7 we study the discriminants of map-germs $\mathbb{C}^n \to \mathbb{C}^p$ with $n \geq p$. We treat the discriminant as the image of the critical point set and define the multiple point sets of the discriminant in the corresponding way. We fix our attention mainly on the case of map-germs of Boardman class $\Sigma^{n-p+1,1,\dots,1}$. In this nice case the multiple point sets again turn out to be icises. The discriminants, together with the whole maps, are induced from the discriminants of the stable map-germs **[7]** of the same Boardman class, that is, from the discriminants of the generalized Whitney mappings (which are versal unfoldings of function singularities A_μ). The discriminant of a Whitney mapping is a generalized swallowtail very well known in singularity theory (this is the set of non-regular orbits of the complex reflection group A_μ, or the set of polynomials of degree $\mu + 1$ in one variable with multiple roots, etc.). The explicit equations for the multiple point sets of the swallowtail (Corollary 4.3.4) show that all these sets are smooth (Theorem 4.1.1). Exactly this proves each multiple point set of the discriminant of a stable perturbation of any $\Sigma^{n-p+1,1,\dots,1}$-map of finite $\mathscr{A}$-codimension to be a Milnor fibre of an icis. The semi-simplicial resolution of the stable discriminant gives a filtration on its homology. By application of Theorem 2.1.2, the graded object of this filtration is the direct sum of the alternating homology of the fibres mentioned (Theorem 5.3.4, similar to Theorem 3.3.1 on the images).

In § 6 we consider as examples the discriminants and the multiple point sets for simple germs $\mathbb{C}^n \to \mathbb{C}^n$.

Vanishing cycles on the discriminants are defined in the traditional way (§ 5.4). Some of them arise as complex links of certain Weyl groups. In § 7 we consider their real representation, a problem pointed out by V. I. Arnold and D. Mond.

Sometimes, instead of considering mappings between smooth spaces, we consider the somewhat more general case **[11]** of projections of complete intersections onto smooth spaces. All the homologies are over the integers and we often omit $\mathbb{Z}$.

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1. *Semi-simplicial resolutions and multiple point sets*

1.1. *The geometric resolution*

Let $f\colon X \to Y$ be a finite surjective algebraic mapping of compact semialgebraic sets. The main tool of this paper is a semi-simplicial resolution of the image Y. In **[13]** we introduced an algebraic approach to this notion. Here we use geometric language to describe the same object.

Let $m < \infty$ be the maximal number of distinct preimages of a point of Y. Following **[22]**, consider an embedding i of X into some $\mathbb{R}^M$, such that the images under i of any m distinct points of X do not belong to any $(m-2)$-dimensional affine plane (say, for $X \subseteq \mathbb{R}^s = \{(z_1, \ldots, z_s)\}$ we can take an embedding into $\mathbb{R}^{sm}$ with z_j^α, where $j = 1, \ldots, s$ and $\alpha = 1, \ldots, m$, for coordinate functions). Now, let the points $x_1, \ldots, x_k \in X$ be all the distinct f-preimages of a point $y \in Y$. Consider in $y \times \mathbb{R}^M$ a closed $(k-1)$-dimensional simplex with vertices $(y, i(x_1)), \ldots, (y, i(x_k))$. Write $Y' \subset Y \times \mathbb{R}^M$ for the union of all such simplices for all points y of Y. The space Y' is homotopy equivalent to Y (see, for example, **[22]**).

The set Y' has a natural filtration

$$X \cong Y_1 \subset Y_2 \subset \ldots \subset Y_{m-1} \subset Y_m = Y', \tag{1.1}$$

where Y_k is the union of all faces of dimension less than k of all the above-mentioned simplices. To calculate the homology of Y we can consider a spectral sequence corresponding to this filtration. The E^1 term of the sequence consists of $H_*(Y_k, Y_{k-1})$ for various values of k. The relative groups are quite often easier to calculate than $H_*(Y)$ itself. Furthermore, in some nice situations, the spectral sequence collapses at E^1.

1.2. *Interpretation of $H_*(Y_k, Y_{k-1})$*

Consider the *kth multiple point set* of f:

$$D^k = \text{closure}\{(x_1, \ldots, x_k) \in X^k\colon f(x_1) = \ldots = f(x_k),\, x_i \neq x_j \text{ if } i \neq j\}.$$

Let Δ_{k-1} be the standard closed $(k-1)$-dimensional simplex oriented by a fixed order of the vertices.

Define a mapping $h\colon D^k \times \Delta_{k-1} \to Y_k$ to be linear on each simplex

$$(x_1, \ldots, x_k) \times \Delta_{k-1}$$

and send its ordered vertices to $(y, i(x_1)), \ldots, (y, i(x_k)) \in Y'$, where y is the image of all the x_i under f. Let $\text{diag}(D^k)$ be the intersection of D^k with the union of all the diagonals $x_i = x_j$ in X^k. Then

$$h\colon (D^k \times \Delta_{k-1}, (\text{diag}(D^k) \times \Delta_{k-1}) \cup (D^k \times \partial\Delta_{k-1})) \to (Y_k, Y_{k-1}).$$

There is a natural action of the permutation group S_k on $D^k \times \Delta_{k-1}$ which is a direct product of two actions: permutations of the copies of $X^k \supset D^k$ and linear automorphisms of Δ_{k-1} permuting its vertices. The mapping h is invariant with respect to this action. Note also that h is proper, $k!$-fold and locally homeomorphic on the complements to the second terms of the pairs. Thus, modulo the second term, h is a factorization by the free action of S_k.

Let us take any cell decomposition of D^k on which S_k, permuting the copies in X^k, acts by permutations of cells (so $\sigma e = e$ means that σ fixes the cell e pointwise). Consider this decomposition modulo $\text{diag}(D^k)$ only. Multiplication of the cells by Δ_{k-1} provides a cell decomposition of $D^k \times \Delta_{k-1}$ modulo

$$(\mathrm{diag}(D^k) \times \Delta_{k-1}) \cup (D^k \times \partial\Delta_{k-1}).$$

The latter provides, via h, a cell decomposition of Y_k modulo Y_{k-1}. Consider the corresponding relative chain complex of (Y_k, Y_{k-1}). This chain complex is isomorphic to the one obtained by replacement of each cell $e \times \Delta_{k-1}$ of (Y_k, Y_{k-1}) by an S_k-invariant sum of $k!$ cells in $h^{-1}(e \times \Delta_{k-1})$. The latter is divisible by Δ_{k-1} and after the division we get $\sum \mathrm{sign}(\sigma) \cdot \sigma(e')$, for $\sigma \in S_k$, for some cell e' of D^k. We denote such a sum by $\mathrm{Alt}(e')$.

The chains $\mathrm{Alt}(e')$ are contained in the chain subcomplex $\mathrm{Alt}\, C_\bullet(D^k)$ consisting of all the integer chains c on D^k such that $\sigma(c) = \mathrm{sign}(\sigma) \cdot c$ for any $\sigma \in S_k$. Moreover, $\mathrm{Alt}\, C_\bullet(D^k)$ is generated over $\mathbb{Z}$ by the various $\mathrm{Alt}\,(e')$, with e' running through all the cells of D^k. Indeed, if a cell of D^k is fixed by a permutation, it is fixed by a transposition and thus cannot appear with non-zero coefficient in any chain of $\mathrm{Alt}\, C_\bullet(D^k)$.

We denote by $\mathrm{Alt}\, H_*(D^k)$ the homology of the complex $\mathrm{Alt}\, C_\bullet(D^k)$. Since $\mathrm{Alt}\, C_\bullet(\mathrm{diag}(D^k)) = 0$, we obtain

1.2.1. Proposition. $H_i(Y_k, Y_{k-1}) = \mathrm{Alt}\, H_{i-k+1}(D^k)$.

1.2.2. Corollary. (Cf. [**13**, Proposition 2.3].) *The homological spectral sequence associated with the filtration* (1.1) *has* $E^1_{p,q} = \mathrm{Alt}\, H_q(D^{p+1})$.

When all the multiple point sets D^k are Milnor fibres of isolated complete intersection singularities of decreasing dimension, we will be able to guarantee that the spectral sequence collapses at E^1. We show this in the next section.

2. *Det-twisting vanishing homology of an icis*

2.1. *Det-twisting cycles and chains*

Consider a map-germ $f\colon \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, with $n \geq p$, such that $f = 0$ is a complete intersection with an isolated singularity (icis). A representative of the germ, defined on a sufficiently small ball $B_\eta \subset \mathbb{C}^n$, centred at the origin and of radius η, will be also denoted by f. Consider its Milnor fibre F, that is, a non-critical level $f = \varepsilon$ in B_η, with $|\varepsilon| \ll \eta$. The level F is homotopy equivalent to a wedge of a finite number of middle-dimensional spheres S^{n-p} [**14, 15**] and thus has non-trivial reduced homology only in this middle dimension. Here F could also be a non-critical level of any small perturbation of f, not necessarily a level of f itself.

All through this section we consider the slightly more complicated situation when $\mathbb{C}^n, 0$ is equipped with a complexification of a real representation ρ_G of some finite group G (we may assume the real representation to be orthogonal). We suppose $f = 0$ to be a G-invariant isolated complete intersection singularity. This means that the map-germ f must be G-invariant and have a finite G-invariant contact codimension [**7**] (thus we are requiring more than f defining an icis with G-symmetry). Then G acts on the Milnor fibre F and on its homology.

2.1.1. Definition. A cycle $c \in H_*(F; \mathbb{Z})$ is called $\det(\rho_G)$-*twisting* if $g(c) = \det(g) \cdot c$ for any $g \in G$.

If $G \subset SO(n)$, such a cycle should be invariant: $g(c) = c$ for all $g \in G$.

The set of all the $\det(\rho_G)$-twisting cycles is a sublattice $H_*^{Tw}(F)$ in the integer homology of F (we do not include ρ_G in the notation since it will be clear from the context which representation we are using).

We can also consider another twisting homological object for F. Take a cell decomposition of F on which G acts by permutations of cells (again, $ge = e$ means that the element g fixes all the points of the cell e). Consider the corresponding integer chain complex and its subcomplex of $\det(\rho_G)$-twisting chains. For example, a cell fixed by some element changing the orientation of $\mathbb{R}^n$ does not appear in the subcomplex. We denote the homology of the $\det(\rho_G)$-twisting chain subcomplex of F by $\mathrm{Tw}\, H_*(F)$. The aim of this section is to prove

2.1.2. THEOREM. *Let F be a G-invariant Milnor fibre of a G-invariant icis. Then*

$$\mathrm{Tw}\, H_*(F\,;\mathbb{Z}) = H_*^{Tw}(F\,;\mathbb{Z}).$$

2.2. *Twisting homology of homotopic spaces*

Let Y and Z be two CW-complexes equipped with actions of a finite group G. Recall that a mapping α: $Y \to Z$ is called *G-equivariant* if it commutes with the actions of G: $\alpha g = g\alpha$ for any $g \in G$.

2.2.1. DEFINITION. Two CW-complexes Y and Z are *G-equivariantly homotopy equivalent,* $Y \simeq_G Z$, if there exist two G-equivariant mappings α: $Y \to Z$ and β: $Z \to Y$, such that the compositions $\beta \circ \alpha$ and $\alpha \circ \beta$ are homotopic to the identities in the class of G-equivariant maps.

Let us fix a homomorphism χ: $G \to \{\pm 1\}$. Similar to the det-case, but in a more general setting, we can consider subcomplexes of integer chains c of Y or Z twisting with respect to χ: $g(c) = \chi(g) \cdot c$ for any $g \in G$. The homology of such a χ-twisting complex will be called the *χ-twisting homology* of the corresponding space.

2.2.2. PROPOSITION. *The χ-twisting homology groups of G-equivariantly homotopic spaces coincide.*

Indeed, χ-twisting homology of Y is the homology of the factor space Y/G with coefficients in the integer coefficient sheaf changing its orientation along the loops whose liftings to Y are paths joining points y and $g(y)$ with $\chi(g) = -1$. The G-equivariant homotopy equivalence of Y and Z lowers to a homotopy equivalence of their factor spaces inducing an isomorphism of homology with coefficients in such sheaves.

2.3. *First part of the proof of Theorem* 2.1.2

Let N be the minimal number of generators of the ring of polynomial invariants of ρ_G. Then the factor space $\mathbb{C}^n/G$ is embedded into $\mathbb{C}^N$. Since the mapping f is G-invariant, it lowers to the factor space and can be extended there to a holomorphic germ $\bar{f}$: $\mathbb{C}^N,0 \to \mathbb{C}^p,0$ such that $\bar{f} = 0$ is an icis. Adding $n - p$ generic function-germs on $\mathbb{C}^N,0$ to $\bar{f}$, we get a mapping $\hat{f}$: $\mathbb{C}^N,0 \to \mathbb{C}^n,0$ such that $\hat{f} = 0$ is an icis. On the initial $\mathbb{C}^n$ the latter mapping induces an extension

f': $\mathbb{C}^n,0 \to \mathbb{C}^n,0$ of f defining a G-invariant zero-dimensional icis. Moreover, taking generic coordinates on $\mathbb{C}^p$, we may assume that for any $k = 1, \ldots, n$ the first k coordinate functions of f' also define a G-invariant icis in $\mathbb{C}^n$.

We may assume F in the claim of the theorem to be a generic level not of f itself but of a small G-invariant perturbation of f. Let us do the above extension procedure for G-invariant perturbations of map-germs. This includes $F = F_p$ in a complete flag

$$F_n \subset F_{n-1} \subset \ldots \subset F_k \subset F_{k-1} \subset \ldots \subset F_1 \subset F_0,$$

where F_k is a non-singular, generic level of a small G-invariant perturbation f_k of a G-invariant function-germ, $F_0 = B_\eta \cap f_1^{-1}(U_1) \cap \ldots \cap f_n^{-1}(U_n)$, where the $U_k \subset \mathbb{C}$ are sufficiently small discs. As the representation is real, we may assume that for all k all the critical points of f_k on F_{k-1} are Morse points. We may also assume that only one G-orbit of these critical points lies on each critical level.

Now $F_0 \subset \mathbb{C}^n$ is G-equivariantly contractible to the origin and, thus, has $\mathrm{Tw}\, H_*$ the same as the origin has. On the other hand, for the calculation of $\mathrm{Tw}\, H_*(F_0)$ we can use a spectral sequence corresponding to the filtration of F_0 by the sets F_k. The E^1 term of this sequence is given by $\mathrm{Tw}\, H_*(F_{k-1}, F_k)$.

2.3.1. Lemma. $\mathrm{Tw}\, H_*(F_{k-1}, F_k) = H_*^{\mathrm{Tw}}(F_{k-1}, F_k)$.

Proof. The homology $H_*(F_{k-1}, F_k)$ is non-trivial in dimension $n - k + 1$ only and is freely generated by thimbles contracting the homology of F_k to the critical points of f_k **[5]**. We are going to choose these thimbles to respect the group action.

Let F_k be defined by $f_k = a$ on F_{k-1} and $c_1, \ldots, c_r$ be all the critical values of f_k on F_{k-1}. Consider a system Γ of non-intersecting paths $\gamma_1, \ldots, \gamma_r$ in $U_k \subset \mathbb{C}$ from a to $c_1, \ldots, c_r$ (Fig. 1). We shall denote by F_Y a set $F_{k-1} \cap f_k^{-1}(Y)$.

Now $(F_{k-1}, F_k) \simeq_G (F_\Gamma, F_{\Gamma'})$, where Γ' is Γ without small end segments $(p_i, c_i]$ of the paths γ_i. This G-equivariant homotopy equivalence can be obtained either by G-averaging of non-equivariant contraction fields on F_{k-1} or by lifting the corresponding contraction from the factor space $F_{k-1}/G \subset \mathbb{C}^n/G$.

Now consider a pair $(F_{[p_i,c_i]}, F_{p_i})$ for some i. By a choice of a parametrization of U_k, we may assume that $c_i = 0$, $p_i = \varepsilon > 0$, and the segment is a part of the real axis. Let b be one of the singular points of the critical level $f_k = 0$ on F_{k-1}. Denote by $H \subset G$ the isotropy subgroup of b. At b, F_{k-1} is transversal to the fixed point set of H in $\mathbb{C}^n$. Locally, near b, the action of H on F_{k-1} is equivalent to a complexification of some real orthogonal representation ρ_H of H on an $(n - k + 1)$-dimensional vector space **[20]**. The function f_k is H-invariant and has a

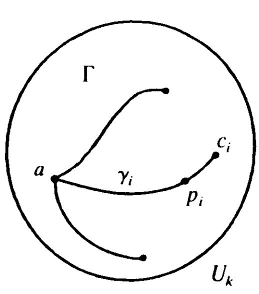

Fig. 1

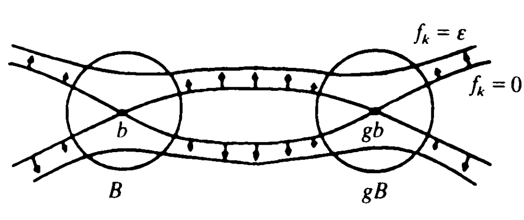

Fig. 2

Morse singularity at b. So we can introduce, in a small closed H-invariant ball $B \subset F_{k-1}$ centred at b, local coordinates on F_{k-1} for which f_k is a sum of the squares of all these coordinates.

Denote by GB the G-orbit of B. We suppose that $g_1 B$ does not intersect $g_2 B$ if g_2 is not in $g_1 H$.

We may also assume that all the fibres $f_k = \alpha$, with $0 \leqslant \alpha \leqslant \varepsilon$, intersect the boundary of GB transversally. Then, a contraction of the segment $[0, \varepsilon]$ to ε gives a contraction of $F_{[0,\varepsilon]}$ to $F_\varepsilon \cup (F_{[0,\varepsilon]} \cap GB)$ (Fig. 2). This may obviously be done G-equivariantly.

Afterwards, each level $f_k = \alpha$ in B is contractible onto its real part (in the coordinates introduced above), that is, onto a sphere S_α^{n-k} if $\alpha \neq 0$ or point b if $\alpha = 0$ [5]. The contraction may be done in an H-equivariant way for all the levels $0 \leqslant \alpha < \varepsilon$ together and may be assumed to be the identity on $f_k = \varepsilon$. The family of real spheres S_α^{n-k}, together with the critical point b, provides a real $(n-k+1)$-dimensional H-invariant disc attached to F_ε. This disc is a $\det(\rho_H)$-twisting chain, as well as its boundary S_ε^{n-k}. The G-action gives similar discs corresponding to the other critical points on this critical level of f_k. For a suitable choice of the orientations the sum of these discs is a $\det(\rho_G)$-twisting chain, since $\det(\rho_H)$ is equal to the restriction of $\det(\rho_G)$ to H due to the transversality of F_{k-1} to the fixed point set of H. This proves

2.3.2. Lemma. *For the pair* $(F_{[0,\varepsilon]}, F_\varepsilon)$,

$$\operatorname{Tw} H_i = H_i^{\mathrm{Tw}} = \begin{cases} 0 & \text{if } i \neq n-k+1, \\ \mathbb{Z} & \text{if } i = n-k+1. \end{cases}$$

From the proof of Lemma 2.3.2, we see that F_{k-1} is G-equivariantly contractible onto $F_{\Gamma'} \simeq_G F_k$ with a G-invariant set of $(n-k+1)$-dimensional cells attached. The relative $\det(\rho_G)$-twisting chain subcomplex of the pair (F_{k-1}, F_k) is thus homotopy equivalent to the $\det(\rho_G)$-twisting part of the free group generated by these cells, that is, to $H_{n-k+1}^{\mathrm{Tw}}(F_{k-1}, F_k) = \operatorname{Tw} H_*(F_{k-1}, F_k)$. This completes the proof of Lemma 2.3.1.

Each G-orbit of critical points of a Morse function f_k on F_{k-1} gives rise to exactly one G-orbit in the set of the cells mentioned and thus to one free summand for the $\det(\rho_G)$-twisting homology of the pair.

2.4. *Completion of the proof of Theorem* 2.1.2

We now return to the spectral sequence calculating $\operatorname{Tw} H_*$ for the space $F_0 \simeq_G \{0\} \in \mathbb{C}^n$. Its E^1 term reduces to

$$0 \longleftarrow H_0^{\mathrm{Tw}}(F_n) \xleftarrow{d^1} H_1^{\mathrm{Tw}}(F_{n-1}, F_n) \xleftarrow{d^1} \dots \xleftarrow{d^1} H_{n-1}^{\mathrm{Tw}}(F_1, F_2) \xleftarrow{d^1} H_n^{\mathrm{Tw}}(F_0, F_1) \longleftarrow 0.$$

As F_0 is contractible, this sequence is exact. The left-hand part of it, starting with $H_{n-k}^{\mathrm{Tw}}(F_k, F_{k+1})$, is the E^1 term of the spectral sequence for $\operatorname{Tw} H_*(F_k)$ defined by the filtration $F_n \subset F_{n-1} \subset \dots \subset F_{k+1} \subset F_k$. Thus $\operatorname{Tw} H_*(F_k)$ is non-trivial in the middle dimension $n-k$ only, where it coincides with the kernel of the differential d^1 on $H_{n-k}^{\mathrm{Tw}}(F_k, F_{k+1})$, that is, with the $\det(\rho_G)$-twisting part of the kernel of the boundary operator $H_{n-k}(F_k, F_{k+1}) \to H_{n-k-1}(F_{k+1}, F_{k+2})$ of the triad. Since the latter kernel is exactly $H_{n-k}(F_k)$, the only non-trivial homology of F_k, Theorem 2.1.2 is proved.

3. *The image of a corank-1 map of an icis, $n<p$*

3.1. *Corank-1 maps of icises*

Consider a corank-1 map-germ f_0: $\mathbb{C}^\nu,0 \to \mathbb{C}^p,0$, where $\nu \leqslant p$. Let f_o be its restriction to an icis $X_o \subset \mathbb{C}^\nu$, where $\dim X_o = n < p$. Suppose f_o to have an isolated instability, in the natural left-right sense, at the origin.

3.1.1. DEFINITION. The restriction f_o is called *a corank-1 $\mathscr{A}$-finite map-germ of an icis X_o.*

For the reasons for this terminology see § 3.6.

Let f be a restriction of a perturbation of f_0 to a Milnor fibre X of X_o. Suppose f to be $\mathscr{A}$-stable everywhere on X. Then, in the same way as in [17], we get

3.1.2. PROPOSITION. *For each $k \leqslant p/(p-n)$, the multiple point set $D^k(f)$ is a Milnor fibre of the $(p-k(n-p))$-dimensional icis $D^k(f_o)$. For $k > p/(p-n)$, $D^k(f)$ is empty.*

3.2. *The S_k-symmetric equations for $D^k(f_o)$ (cf.* [17])

Let us take coordinates in which f_0 is given by $f_0(x,u) = (\Phi_1(x,u), \ldots, \Phi_{p-\nu+1}(x,u), u)$, where $x \in \mathbb{C}$ and $u \in \mathbb{C}^{\nu-1}$. Let $g_1(x,u) = \ldots = g_{\nu-n}(x,u) = 0$ be the equations of X_o. Let $V_k = \det[x_j^{i-1}]_{i,j=1,\ldots,k}$ be the Vandermonde determinant. Take some function $\phi(x,u)$ and for a fixed i replace the terms x_j^{i-1} in the Vandermonde matrix by $\phi(x_j,u)$. Denote the determinant of the new matrix by $V_k(\phi,i)$. Then $D^k(f_o)$ is given in the space $\mathbb{C}^{k+\nu-1}$, with the coordinates $x_1, \ldots, x_k, u$ (S_k permutes the x_i), by the equations:

$$V_k(\Phi_r, i)/V_k = 0, \quad \text{where } r = 1, \ldots, p-\nu+1, \quad i = 2, \ldots, k,$$

and

$$V_k(g_s, i)/V_k = 0, \quad \text{where } s = 1, \ldots, n-\nu, \quad i = 1, \ldots, k.$$

EXAMPLE. $D^1(f_o) = X_o$.

REMARK. Since f_0 can glue together only source points with equal u-coordinates, we consider $D^k(f_o)$ here as a subset of $\mathbb{C}^{k+\nu-1}$ instead of $\mathbb{C}^{k\nu}$.

3.3. *Decomposition of the homology of the image*

In [13] there was given a description of the rational homology of the image Y of f. Now we get an identical description over the integers.

Let us return to the filtration (1.1) on the semi-simplicial resolution Y' of Y and to the expression of the E^1 term of the corresponding spectral sequence given by Corollary 1.2.2. The equations from the previous subsection have a very remarkable feature: they show that a multiple point set $D^k(f)$ is an S_k-invariant

Milnor fibre of an icis in the space of representation of the permutation group for which $\det(\sigma) = \mathrm{sign}(\sigma)$ for any permutation σ. Applying Theorem 2.1.2 to the $D^k(f)$ with such symmetries, we see that the spectral sequence collapses at E^1, since the dimensions of the multiple point sets are decreasing when k is increasing, and obtain

3.3.1. THEOREM. *Let Y be the image in $\mathbb{C}^p$ of a stable perturbation of a corank-1 map-germ of an n-dimensional icis, with $p > n$.*

(1) *If $p = n+1$, then*

$$H_n(Y;\mathbb{Z}) \cong \oplus_{k=1,\ldots,n+1} H^{\mathrm{Alt}}_{n-k+1}(D^k(f);\mathbb{Z})$$

and

$$H_q(Y;\mathbb{Z}) = 0, \quad \textit{for } 1 \leq q \neq n.$$

(2) *If $p > n+1$, then for each integer k, with $1 \leq k \leq p/(p-n)$,*

$$H_{p-k(p-n-1)-1}(Y;\mathbb{Z}) \cong H^{\mathrm{Alt}}_{p-k(p-n)}(D^k(f);\mathbb{Z})$$

and

$$H_q(Y,\mathbb{Z}) = 0 \quad \textit{for all the other values of } q.$$

3.4. *Filtration on the homology*

Let us pay a bit more attention to the case $p = n+1$ when the image Y is homotopy equivalent to a wedge of a finite number of middle-dimensional spheres S^n **[18]**.

The direct-sum decomposition of Theorem 3.3.1 is not canonical. This is a graded object for the sequence of embeddings

$$\ldots \subset H_n(Y_k) \subset H_n(Y_{k+1}) \subset \ldots \subset H_n(Y') = H_n(Y)$$

induced by the inclusions $\ldots \subset Y_k \subset Y_{k+1} \subset \ldots \subset Y'$ of the semi-simplicial resolution.

The same filtration is given by the kernels of certain boundary operators on $H_n(Y)$, expressed in the following way.

Let π^k_l: $X^k \to X^l$, for $k > l$, be the projection forgetting (the last) $k-l$ copies of X. We set $D^k_l = \pi^k_l(D^k)$, $\pi^1_0 = f$, and $D^1_0 = Y$. Consider the composition d_k of the mappings

$$H_{n-k+1}(D^k_{k-1}) \xrightarrow{i} H_{n-k+1}(D^k_{k-1}, D^{k+1}_{k-1}) \longrightarrow H_{n-k+1}(D^k, D^{k+1}_k) \xrightarrow{\partial} H_{n-k}(D^{k+1}_k).$$

Here:

i is an embedding (as D^{k+1}_{k-1} is an $(n-k)$-dimensional Stein space);

the arrow in the middle is an isomorphism induced by a homeomorphism

$$\pi^k_{k-1}\colon\ D^k \setminus D^{k+1}_k \to D^k_{k-1} \setminus D^{k+1}_{k-1};$$

∂ is the boundary operator of the pair.

We get a sequence of operators with composite

$$\delta_k = d_k \circ \ldots \circ d_1\colon\ H_n(Y) \to H_{n-k}(D^{k+1}_k).$$

Actually δ_k is a mapping onto a certain subspace of $\mathrm{Alt}\, H_{n-k}(D^{k+1}_k;\mathbb{Z})$ ('Alt' means 'with respect to the S_k action'). An easy geometric argument shows that $\operatorname{Ker} \delta_k = H_n(Y_k ; \mathbb{Z})$.

3.5. *Pairs* (D^k, D_k^{k+1})

Each corank-1 $\mathscr{A}$-finite map-germ f_o: $X_o,0 \to \mathbb{C}^p,0$ is induced from some $\mathscr{A}$-stable mapping F: $\mathbb{C}^N,0 \to \mathbb{C}^P,0$ by a suitable mapping ϕ: $\mathbb{C}^p,0 \to \mathbb{C}^P,0$ of the target spaces [**19, 7**]. Thus, the image of f_o and its multiple point sets are also induced by ϕ from the ones for F. So, we start with the $\mathscr{A}$-stable corank-1 map-germs $\mathbb{C}^N,0 \to \mathbb{C}^P,0$, with $N < P$. Any such mapping is $\mathscr{A}$-equivalent to a germ

$$\psi_{c,r}\colon \mathbb{C}^1_x \times \mathbb{C}^{N-1}_\lambda \to \mathbb{C}^P,0, \ (x, \lambda) \mapsto (u, \lambda),$$

where

$$\begin{aligned}
u_1 = q_1(x, \lambda) &= \quad x^{r+1} + \quad \lambda_1 x^{r-1} + \dots + \lambda_{r-1} x, \\
u_2 = q_2(x, \lambda) &= \quad \lambda_r x^r + \quad \lambda_{r+1} x^{r-1} + \dots + \lambda_{2r-1} x, \\
\dots \quad \dots & \quad \dots \quad \dots \quad \dots \quad \dots \\
u_c = q_c(x, \lambda) &= \lambda_{(c-1)r} x^r + \lambda_{(c-1)r+1} x^{r-1} + \dots + \lambda_{cr-1} x,
\end{aligned}$$

where $N \geqslant cr$ and $P = c + N - 1$, for some particular choice of $c > 1$ and $r \geqslant 0$.

EXAMPLES. The germ $\psi_{c,0}$ is an embedding. The image of $\psi_{2,1}$: $\mathbb{C}^2 \to \mathbb{C}^3$ is the well-known Whitney umbrella.

There is a nice correspondence between the image of $\psi_{c,r}$ and the double point set D^2 of $\psi_{c,r+1}$. This is like the discriminant of A_μ being the image of the cuspidal edge of the discriminant of $A_{\mu+1}$ under a generic projection along a line.

The set $D^2(\psi_{c,r+1}) \subset \mathbb{C}^{N+1}_{x_1,x_2,\lambda}$ is given by the equations

$$(q_i(x_1, \lambda) - q_i(x_2, \lambda))/(x_1 - x_2) = 0, \quad \text{for } i = 1, \dots, c.$$

It is smooth. The projection π_1^2 of $D^2(\psi_{c,r+1})$ onto the (x_1, λ)-space along the x_2-axis is a 1-parameter unfolding of the projection of the variety $q_i(x_2, \lambda)/x_2 = 0$, for $i = 1, \dots, c$, onto the λ-space along the x_2-axis. The latter is obviously $\mathscr{A}$-equivalent to $\psi_{c,r-1}$: $\mathbb{C}^{N-c} \to \mathbb{C}^{N-1}$ and, thus, $\mathscr{A}$-stable. Consequently, π_1^2: $D^2 \to \mathbb{C}^N$ is its trivial unfolding.

We obtain by induction

3.5.1. PROPOSITION. *For the stable mapping* $\psi_{c,r}$: $\mathbb{C}^N \to \mathbb{C}^{N+c-1}$, *the projection* π_k^{k+1}: $D^{k+1} \to D^k$, *with* $k \leqslant r$, *is* $\mathscr{A}$-*equivalent to the stable mapping* $\psi_{c,r-k}$: $\mathbb{C}^{N-(k-1)(c-1)} \to \mathbb{C}^{N-(k-2)(c-1)}$.

The spaces $D^k(\psi_{c,r})$, for $k > r + 1$, are obviously empty.

3.5.2. COROLLARY. *For a corank-1* $\mathscr{A}$-*finite map-germ* f_o: $X_o,0 \to \mathbb{C}^{n+1},0$, *of an* n-*dimensional isolated hypersurface singularity* X_o, D_k^{k+1} *is a complete intersection.*

Note that X_o can only be an icis here if it is in fact a hypersurface.

Under the conditions of the corollary, D_k^{k+1} is a hypersurface in D^k and its equation $h = 0$ in D^k is induced from the equation of the stable image of the suitable $\psi_{c,r}$. The function h can be chosen S_k-symmetric and having only Morse critical points on $D^k \backslash D_k^{k+1}$. As in § 2, using also arguments similar to [**21**], we get

3.5.3. PROPOSITION. *The group* $\operatorname{Alt} H_p(D^k, D_k^{k+1}) = H_p^{\mathrm{Alt}}(D^k, D_k^{k+1})$ *is free if* $p = \dim D^k = n-k+1$, *and is trivial otherwise.*

From the exact sequence for the homology of the S_k-alternating chain complexes of the pair (D^k, D_k^{k+1}), it follows that all the groups $\operatorname{Alt} H_p(D_k^{k+1})$ are trivial if $p \neq n-k+1,\ n-k$. Consider the remaining part of the above-mentioned exact sequence together with the S_k-alternating part of the exact sequence of the ordinary integer homology of the same pair:

$$0 \longrightarrow \operatorname{Alt} H_{n-k+1}(D_k^{k+1})$$
$$\longrightarrow \operatorname{Alt} H_{n-k+1}(D^k) \longrightarrow \operatorname{Alt} H_{n-k+1}(D^k, D_k^{k+1}) \longrightarrow \operatorname{Alt} H_{n-k}(D_k^{k+1}) \longrightarrow 0$$
$$\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad \|$$
$$0 \longrightarrow H_{n-k+1}^{\mathrm{Alt}}(D^k) \longrightarrow H_{n-k+1}^{\mathrm{Alt}}(D^k, D_k^{k+1}) \xrightarrow{\partial} H_{n-k}^{\mathrm{Alt}}(D_k^{k+1}) \longrightarrow 0$$

As the lattice $H_{n-k+1}(D^k)$ is embedded into the lattice $H_{n-k+1}(D^k, D_k^{k+1})$, the same is true for H^{Alt} and $\operatorname{Alt} H$ as well. Thus $\operatorname{Alt} H_{n-k+1}(D_k^{k+1}) = 0$.

The lower sequence need not be exact at $H_{n-k}^{\mathrm{Alt}}(D_k^{k+1}; \mathbb{Z})$, but, obviously, $\operatorname{Ker} \partial = H_{n-k+1}^{\mathrm{Alt}}(D^k; \mathbb{Z})$. Consequently,

$$\operatorname{Im} \partial = H_{n-k+1}^{\mathrm{Alt}}(D^k, D_k^{k+1}; \mathbb{Z}) / H_{n-k+1}^{\mathrm{Alt}}(D^k; \mathbb{Z})$$

is a subgroup of finite index in the free group $H_{n-k}^{\mathrm{Alt}}(D_k^{k+1}; \mathbb{Z})$. From the exactness of the upper line this factor is $\operatorname{Alt} H_{n-k}(D_k^{k+1}; \mathbb{Z})$. Thus, we have

3.5.4. PROPOSITION. *The group* $\operatorname{Alt} H_p(D_k^{k+1}; \mathbb{Z})$ *is a subgroup of finite index in the free group* $H_{n-k}^{\mathrm{Alt}}(D_k^{k+1}; \mathbb{Z})$, *if* $p = n-k$, *and is trivial otherwise.*

3.6. *Remark on $\mathscr{A}$-equivalence of map-germs of complete intersections*

The precise notion is as follows (see also [**19**]).

3.6.1. DEFINITION. Two map-germs f_i: $X_i, 0 \to \mathbb{C}^p, 0$ of complete intersections $X_i \subset \mathbb{C}^\nu, 0$ are said to be *$\mathscr{A}$-equivalent* if there exist two complex analytic diffeomorphisms, r and l, of $\mathbb{C}^\nu, 0$ and $\mathbb{C}^p, 0$ respectively, such that $f_2 = l \circ f_1 \circ r|_{X_2}$, where $r|_{X_2}$ is a restriction of r (this restriction should be a diffeomorphism between X_2 and X_1).

Actually a map-germ f: $X, 0 \to \mathbb{C}^p, 0$ of a complete intersection is a pair of mappings (F, g):

$$g\colon\ \mathbb{C}^\nu, 0 \to \mathbb{C}^s, 0 \quad \text{gives equations of } X,\ \dim X = \nu - s,$$

and

$$F\colon\ \mathbb{C}^\nu, 0 \to \mathbb{C}^p, 0 \quad \text{provides } f = F|_X.$$

By a variation of a map-germ of a complete intersection we mean an arbitrary variation of the pair (F, g). Thus, the $\mathscr{A}$-codimension of f is

$$\dim_{\mathbb{C}}(\mathcal{O}_v^p \times \mathcal{O}_v^s)/\{g^*(\mathfrak{m}_s)(\mathcal{O}_v^p \times \mathcal{O}_v^s) + \mathcal{O}_v\langle(\partial F/\partial x_1, \partial g/\partial x_1), \ldots, (\partial F/\partial x_v, \partial g/\partial x_v)\rangle + (F^*(\mathcal{O}_p))^p \times \{0\}\},$$

where $\mathcal{O}_k$ is the ring of germs of holomorphic functions on $\mathbb{C}^k,0$, $\mathfrak{m}_s \subset \mathcal{O}_s$ is the maximal ideal, $x_1, \ldots, x_v$ are coordinates on $\mathbb{C}^v,0$.

Finiteness of the $\mathscr{A}$-codimension obviously forces $X = g^{-1}(0)$ either to be smooth or to have an isolated singularity.

A traditional sheaf-theoretic argument proves

3.6.2. PROPOSITION. *The $\mathscr{A}$-codimension of a map-germ f of an icis X is finite if and only if f is $\mathscr{A}$-stable (see, for example,* [**4**]) *on* $X \backslash \{0\}$.

4. *Multiple point sets of discriminants of stable corank-1 maps between spaces of equal dimension*

Starting here and up to the end of the paper we consider discriminant hypersurfaces of maps $\mathbb{C}^n \to \mathbb{C}^p$, with $n \geqslant p$. Germs of Boardman class $\Sigma^{m,1,\ldots,1}$ [**4**] will be our particular interest, as the multiple point sets of their discriminants turn out to be icises. We begin with the $\mathscr{A}$-stable case, because the discriminant of any $\mathscr{A}$-finite germ is induced from the discriminant of the suitable stable germ.

4.1. *Whitney maps*

Consider a generalized Whitney mapping W_μ: $\mathbb{C}^\mu \to \mathbb{C}^\mu$, $(x, u_2, \ldots, u_\mu) \mapsto (\lambda_1, \ldots, \lambda_\mu)$:

$$\lambda_1 = x^{\mu+1} + u_\mu x^{\mu-1} + u_{\mu-1}x^{\mu-2} + \ldots + u_2 x, \quad \lambda_i = u_i, \ \text{for } i = 2, \ldots, \mu.$$

The *discriminant set* Δ of its critical values, called the *generalized swallowtail*, is very well known in singularity theory. Here we would like to describe the multiple point sets of this discriminant. Note a certain difference between the meaning of the notation used below and the meaning of the same notation in the $n < p$ case (actually, this difference disappears if we define the critical point set of a mapping to be the set of source points at which the differential is not submersive).

Let $D^k(W_\mu)$, abbreviated to D^k when the mapping is well understood, be the closure of the subset of points $(p_1, \ldots, p_k)$ in the kth power of the source space such that:

(1) each p_i is critical for W_μ,

(2) all these k points have the same image under W_μ,

(3) all these k points are distinct.

For example, D^1 is the critical point set $\mathscr{C}$ of the mapping. In our earlier terms, D^k is the kth multiple point set for the restriction W_μ: $\mathscr{C} \to \Delta$. We are going to prove

4.1.1. THEOREM. *The multiple point sets $D^k(W_\mu)$ are smooth $(\mu - k)$-dimensional for $k \leqslant [\frac{1}{2}(\mu + 1)]$ and empty otherwise.*

It is more convenient to change the sign of λ_1 and treat the Whitney map as the projection of the hypersurface

$$f(x, \lambda) = x^{\mu+1} + \lambda_\mu x^{\mu-1} + \lambda_{\mu-1} x^{\mu-2} + \ldots + \lambda_2 x + \lambda_1 = 0 \tag{4.1}$$

from the (x, λ)-space onto the λ. We shall also consider more general sets $D(\mu_1, \ldots, \mu_k)$, with $1 \leq \mu_1 \leq \ldots \leq \mu_k$, which are the closures of the sets of k-tuples $(p_1, \ldots, p_k)$ of distinct source points such that the germ of the mapping W_μ at p_i is $\mathscr{A}$-equivalent to W_{μ_i}. In this context $D^k = D(1, \ldots, 1)$ ('1' repeated k times).

The points p_i in a k-tuple from $D(\mu_1, \ldots, \mu_k)$ have equal λ-coordinates and only their x-coordinates $x_1, \ldots, x_k$ may be distinct. So $D(\mu_1, \ldots, \mu_k)$ will be considered as a subset of $(k+\mu)$-dimensional complex linear space with coordinates $x_1, \ldots, x_k, \lambda_1, \ldots, \lambda_\mu$. We shall write out the explicit equations for $D(\mu_1, \ldots, \mu_k)$ in this space.

4.2. *Vandermonde determinants*

Let us consider the following modification of the Vandermonde determinant. Take r monomials $1, x, x^2, \ldots, x^{r-1}$ and k points $x_1, \ldots, x_k$ on the x-axis, with $k \leq r$. Take also natural numbers $r_1, \ldots, r_k$ whose sum is equal to r. Now form the $r \times r$-matrix $M(r_1, \ldots, r_k)$ putting into its first row the values of our monomials at x_1, into the second row the values of the first derivatives of these monomials at x_1, and so on until into the r_1th row the values of their derivatives of order $r_1 - 1$ at x_1 are put, and then repeat the same procedure for the following rows with x_2 and r_2, and so on up to x_k and r_k. We shall denote the determinant of the matrix obtained by $V(r_1, \ldots, r_k)$. We get the ordinary Vandermonde determinant if all $r_i = 1$.

4.2.1. PROPOSITION.

$$V(r_1, \ldots, r_k) = \prod_{l=1}^{k} \prod_{s=0}^{r_l - 1} s! \prod_{1 \leq i < j \leq k} (x_j - x_i)^{r_i r_j}.$$

Proof. Consider the following recursive difference expressions for the derivatives of a function g in one variable. First set:

$$g^{[0]}(y_1) = g(y_1),$$
$$g^{[1]}(y_1, y_2) = \{g^{[0]}(y_2) - g^{[0]}(y_1)\}/(y_2 - y_1),$$
$$\ldots$$
$$g^{[m]}(y_1, \ldots, y_{m+1}) = m\{g^{[m-1]}(y_1, \ldots, y_{m-1}, y_{m+1}) - g^{[m-1]}(y_1, \ldots, y_{m-1}, y_m)\}/(y_{m+1} - y_m).$$

Then the limit of $g^{[m]}(y_1, \ldots, y_{m+1})$, when all $y_i \to x$ for $i \geq 1$, is the derivative of g of order m at x.

Substitute the derivatives in the matrix $M(r_1, \ldots, r_k)$ by these difference expressions using variables $y_{r_1+\ldots+r_{i-1}+1}, \ldots, y_{r_1+\ldots+r_{i-1}+r_i}$ to express the derivatives evaluated at x_i, for $i = 1, \ldots, k$. Denote the matrix obtained by M^d. Then $M(r_1, \ldots, r_k)$ is the limit of M^d when, for all $i \geq 1$, $y_{r_1+\ldots+r_{i-1}+1}, \ldots, y_{r_1+\ldots+r_{i-1}+r_i} \to x_i$.

The determinant V^d of the matrix M^d is easily seen to be equal to the ordinary Vandermonde determinant in the variables $y_1, \ldots, y_r$ multiplied by the scalar factor in the statement of the proposition and divided by the product of all the differences $y_\beta - y_\alpha$, $r_1 + \dots + r_{i-1} + 1 \leqslant \alpha < \beta \leqslant r_1 + \dots + r_{i-1} + r_i$, for $i = 1, \dots, k$. This is a polynomial in the variables y, and setting $y_{r_1+\dots+r_{i-1}+1} = \dots = y_{r_1+\dots+r_{i-1}+r_i} = x_i$ completes the proof.

Now take any holomorphic function ϕ in x depending on some parameters. Substitute x^{j-1} in the set of monomials for $M(r_1, \dots, r_k)$ by ϕ. We denote the corresponding matrix by $M(r_1, \dots, r_k; \phi, j)$ and its determinant by $V(r_1, \dots, r_k; \phi, j)$.

4.2.2. Proposition. *The function $V(r_1, \dots, r_k; \phi, j)/V(r_1, \dots, r_k)$ is holomorphic.*

Proof. We need to show that $V(r_1, \dots, r_k; \phi, j)$ has a zero of at least the same order as $V(r_1, \dots, r_k)$ on each diagonal $x_i = x_l$. So, take a derivative of $V(r_1, \dots, r_k; \phi, j)$ of a certain order m in the x_i-direction. By the differentiation rule for determinants, this is a sum of determinants of matrices like $M(r_1, \dots, r_k; \phi, j)$, but in r_i rows, where before we had the derivatives evaluated at x_i, we now have to raise the orders of these derivatives by some non-negative numbers whose sum is m. After setting $x_i = x_l$, we get at least two equal rows in such a matrix, unless the orders of all the derivatives evaluated at x_i are distinct and higher than $r_l - 1$. Thus, in the above-mentioned sum of determinants, a summand, non-vanishing on $x_i = x_l$, can appear only for $m \geqslant r_i r_l$.

4.3. *Equations of the swallowtail multiple point sets*

4.3.1. Theorem. *The multiple point set*

$$D(\mu_1, \dots, \mu_k), \quad \mu_1 + \dots + \mu_k + k = r \leqslant \mu + 1,$$

of the generalized Whitney mapping W_μ is given in $(x_1, \dots, x_k, \lambda_1, \dots, \lambda_\mu)$-space $\mathbb{C}^{k+\mu}$ by the polynomial equations

$$V(\mu_1 + 1, \dots, \mu_k + 1; f, j)/V(\mu_1 + 1, \dots, \mu_k + 1) = 0, \quad \textit{where } j = 1, \dots, r.$$

Recall that W_μ is the projection of the hypersurface

$$f(x, \lambda) = x^{\mu+1} + \lambda_\mu x^{\mu-1} + \lambda_{\mu-1} x^{\mu-2} + \dots + \lambda_2 x + \lambda_1 = 0$$

onto the λ-space.

Proof. (i) $r \leqslant \mu$. For a germ of the mapping W_μ at a point $p_i = (x_i, \lambda_1, \dots, \lambda_\mu)$ to be $\mathscr{A}$-equivalent to W_{μ_i} means that x_i is a root of f of order $\mu_i + 1$, that is, the values of f and all its derivatives in x up to the order μ_i vanish at p_i. For different $i = 1, \dots, k$ this provides $\mu_1 + \dots + \mu_k + k = r$ equations linear in $\lambda_1, \dots, \lambda_\mu$. Solving this system formally with respect to $\lambda_1, \dots, \lambda_r$, we get

$$\lambda_j = -V(\mu_1 + 1, \dots, \mu_k + 1; f_{tr}, j)/V(\mu_1 + 1, \dots, \mu_k + 1),$$

where $f_{tr} = x^{\mu+1} + \lambda_\mu x^{\mu-1} + \lambda_{\mu-1} x^{\mu-2} + \dots + \lambda_{r+1} x^r$.

Since for $l \leqslant r$, $V(\mu_1 + 1, \dots, \mu_k + 1; x^{l-1}, j)/V(\mu_1 + 1, \dots, \mu_k + 1) = \delta_{lj}$ (the Kronecker symbol) and $V(\mu_1 + 1, \dots, \mu_k + 1; \phi, j)$ is linear in ϕ, we can rewrite this as

$$V(\mu_1 + 1, \dots, \mu_k + 1; f, j)/V(\mu_1 + 1, \dots, \mu_k + 1) = 0.$$

Together with Proposition 4.2.2 this proves the theorem for this case.

(ii) $r = \mu + 1$. Consider $f_e = f + \lambda_{\mu+1}x^{\mu}$ instead of f and follow the same procedure as before. Then f provides a contact versal deformation of the fat point $x^{\mu+1} = 0$, so f_e is a trivial extension of f. Hence, the extended multiple point set is a direct product of its section $\lambda_{\mu+1} = 0$ and $\mathbb{C}$. The section mentioned is exactly the set we are looking for (for example, the rth equation for this set is $(\mu_1 + 1)x_1 + \ldots + (\mu_k + 1)x_k = 0$). Thus, we have finished.

The Vieta theorem proves

4.3.2. COROLLARY. *For* $\mu_1 + \ldots + \mu_k + k = \mu + 1$,

$$V(\mu_1 + 1, \ldots, \mu_k + 1\,; x^{\mu+1}, j)/V(\mu_1 + 1, \ldots, \mu_k + 1)$$
$$= (-1)^{\mu-j+1}\sigma_{\mu-j+2}(x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k),$$

where each x_i *appears* $\mu_i + 1$ *times as an argument of the elementary symmetric function of degree* $\mu - j + 2$ *in* $\mu + 1$ *variables.*

This statement obviously also works when some of the μ_i vanish.

The proof of Theorem 4.3.1 implies

4.3.3. COROLLARY. *The multiple point set* $D(\mu_1, \ldots, \mu_k)$ *of the generalized Whitney mapping* W_μ *is smooth* $(\mu - \mu_1 - \ldots - \mu_k)$*-dimensional if* $\mu_1 + \ldots + \mu_k + k = r \leq \mu + 1$ *and empty otherwise.*

The emptiness for $r > \mu + 1$ is evident since a polynomial of degree $\mu + 1$ in one variable cannot have k distinct roots whose sum of multiplicities $(\mu_1 + 1) + \ldots + (\mu_k + 1)$ exceeds $\mu + 1$.

Theorem 4.1.1 on the sets $D^k(W_\mu)$ is the particular case of this corollary.

4.3.4. COROLLARY. *The multiple point set* D^k *of the discriminant of the generalized Whitney mapping* W_μ, *for* $k \leq [\frac{1}{2}(\mu + 1)]$, *is given in the* $(x_1, \ldots, x_k, \lambda_1, \ldots, \lambda_\mu)$*-space* $\mathbb{C}^{k+\mu}$ *by the polynomial equations*

$$V(2, \ldots, 2\,; f, j)/V(2, \ldots, 2) = 0, \quad \textit{where } j = 1, \ldots, 2k. \tag{4.2}$$

Here f is from (4.1). In the notation of the determinants '2' appears k times. Note also that $V(2, \ldots, 2) = \prod(x_s - x_i)^4$, where $1 \leq i < s \leq k$.

5. $\Sigma^{m,1,\ldots,1}$ *Map-germs*

5.1. *Projections induced from Whitney maps*

Consider now a map-germ Φ: $\mathbb{C}^{m+p-1},0 \to \mathbb{C}^p,0$ of corank 1. We can choose coordinates $x_1, \ldots, x_m, u_2, \ldots, u_p$ in which Φ is a $(p - 1)$-parameter unfolding of some function $\phi_o(x)$:

$$\Phi\colon\ (x, u) \mapsto (\phi(x, u), u), \quad \phi(x, 0) = \phi_o(x).$$

Let us additionally require the corank of the Hessian matrix of ϕ_o at the origin not to exceed 1, that is, Φ to be of some Boardman class $\Sigma^{m,1,\dots,1}$. Then $\phi_o \in A_\mu$ (μ is the length of the Boardman index), and $\phi(x, u)$ can be written as

$$x_1^{\mu+1} + \lambda_\mu(u_2, \dots, u_p)x_1^{\mu-1} + \lambda_{\mu-1}(u_2, \dots, u_p)x_1^{\mu-2} + \dots \\ + \lambda_2(u_2, \dots, u_p)x_1 + x_2^2 + \dots + x_m^2.$$

Thus Φ is induced by a mapping $\lambda_1 = u_1$, $\lambda_2 = \lambda_2(u_2, \dots, u_p)$, $\dots$, $\lambda_\mu = \lambda_\mu(u_2, \dots, u_p)$ from the $\mathscr{A}$-stable projection $(x, \lambda) \mapsto \lambda$ of the smooth hypersurface

$$x_1^{\mu+1} + \lambda_\mu x_1^{\mu-1} + \lambda_{\mu-1}x_1^{\mu-2} + \dots + \lambda_2 x_1 + \lambda_1 + x_2^2 + \dots + x_m^2 = 0. \tag{5.1}$$

So, the discriminant $\Delta(\Phi)$ and the multiple point sets $D^k(\Phi)$ are induced by the same mapping $\lambda(u_1, \dots, u_p)$ from the corresponding objects for the Whitney mapping W_μ.

In what follows we consider the slightly more general situation when Φ is a projection $(x, u) \mapsto u$ of a hypersurface Γ induced from the hypersurface (5.1) by some map-germ $\lambda_1 = \lambda_1(u_1, \dots, u_p)$, $\dots$, $\lambda_\mu = \lambda_\mu(u_1, \dots, u_p)$ without the requirement grad $\lambda_1(0) \neq 0$.

We shall refer to the projection of the hypersurface (5.1) as a *suspended Whitney mapping.*

5.2. *Equivalence of projections*

Projections of two varieites $\Gamma_1, \Gamma_2 \subset \mathbb{C}^{n+p}$ onto $\mathbb{C}^p$ are said to be *equivalent* if and only if there exists a complex analytic diffeomorphism of $\mathbb{C}^{n+p}$ fibred over $\mathbb{C}^p$, which induces a diffeomorphism of Γ_1 and Γ_2. Variations of the equations of Γ in $\mathbb{C}^{n+p}$ provide all possible variations of its projection (since a variation of the fibration reduces to a variation of the equations, we assume the fibration $\mathbb{C}^{n+p} \to \mathbb{C}^p$ to be fixed). The codimension of the projection $(x, u) \mapsto u$ of the zero set Γ of a map-germ F: $\mathbb{C}^{n+p},0 \to \mathbb{C}^s,0$ in the space of projection-germs of all varieties is

$$\dim_{\mathbb{C}} \mathcal{O}^s_{n+p}/\{F^*(\mathrm{m}_s)\mathcal{O}^s_{n+p} + \mathcal{O}_{n+p}\langle \partial F/\partial x_1, \dots, \partial F/\partial x_n\rangle + \mathcal{O}_p\langle \partial F/\partial u_1, \dots, \partial F/\partial u_p\rangle\}.$$

The notion of equivalence of projections of varieties [**2**, **3**, **10**] coincides with the notion of fibred contact equivalence [**6**] and generalizes in a natural way the notion of $\mathscr{A}$-equivalence of maps between smooth spaces [**11**].

5.3. *Stable perturbations*

One of the necessary conditions for a projection to have finite codimension is that the variety being projected should be either smooth or an icis. For the case of a projection induced in the above-mentioned way from a suspension of a Whitney mapping W_μ one has:

5.3.1. PROPOSITION. *A map-germ* Φ: $\Gamma,0 \to \mathbb{C}^p,0$, $(x, u) \mapsto u$, *is a projection of* *finite codimension if and only if for each* $k \leqslant \min\{p, [\frac{1}{2}(\mu+1)]\}$, *the multiple point set* $D^k(\Phi)$ *of its discriminant is an icis of dimension* $p-k$.

5.3.2. PROPOSITION. *The projection* $\Phi'\colon \Gamma' \to \mathbb{C}^p$, $(x, u) \mapsto u$, *is a stable perturbation of the projection* Φ *of finite codimension if and only if all the* $D^k(\Phi')$ *are smooth for* $k \leqslant \min\{p, [\frac{1}{2}(\mu+1)]\}$.

Thus $D^k(\Phi')$ should be a Milnor fibre of the icis $D^k(\Phi)$.

The proofs of these two propositions follow the proofs of the similar statements for corank-1 maps $\mathbb{C}^n \to \mathbb{C}^{n+p}$ given in [**17**].

5.3.3. REMARK. Since Φ is induced from a suspension of W_μ, any of the two equivalent conditions of Proposition 5.3.1 implies:

(1) the sets $D^k(\Phi)$ are fat points for $p<k\leqslant[\frac{1}{2}(\mu+1)]$ and empty for $k>[\frac{1}{2}(\mu+1)]$;

(2) all the other sets $D(\mu_1, \ldots, \mu_r; \Phi)$ are icises, fat points or empty for appropriate values of $\mu_1, \ldots, \mu_r$.

Similar to this, in the setting of Proposition 5.3.2:

(1) $D^k(\Phi')$ is empty for $k>\min\{p, [\frac{1}{2}(\mu+1)]\}$;

(2) the smoothness of all non-empty $D^k(\Phi')$ implies the smoothness of all non-empty $D(\mu_1, \ldots, \mu_r; \Phi')$.

The statement of Proposition 5.3.1 is not true for projections not induced from the suspended Whitney maps. The most general situation in which Proposition 5.3.2 is valid is: all singularities of the stabilization Φ' should be equivalent to singularities of the suspended Whitney maps.

The set $D^k(\Phi')$ is an S_k-invariant Milnor fibre of the icis $D^k(\Phi)$ induced from $D^k(W_\mu)$. Due to Corollary 4.3.4, for the S_k action here $\det(\sigma)=\mathrm{sign}(\sigma)$ for any permutation σ. Thus, as in § 3.3, the filtration (1.1) on the semi-simplicial resolution of the discriminant $\Delta(\Phi')$, Corollary 1.2.2, Proposition 5.3.2 and Theorem 2.1.2 prove

5.3.4. THEOREM. *Let* $\Phi'\colon \Gamma' \to \mathbb{C}^p$ *be any small perturbation of a projection-germ of finite codimension induced from a suspension of a Whitney mapping* W_μ. *Then*

$$H_{p-1}(\Delta(\Phi'); \mathbb{Z}) = \bigoplus_{k=1,\ldots,[\frac{1}{2}(\mu+1)]} H^{\mathrm{Alt}}_{p-k}(D^k(\Phi'); \mathbb{Z})$$

and

$$H_q(Y, \mathbb{Z}) = 0, \quad \textit{for } 1 \leqslant q \neq p-1.$$

5.4. *Vanishing cycles*

Up to the end of §5.5 we consider only finite-codimensional germs of projections onto $\mathbb{C}^p$ of hypersurfaces induced from a suspension of a Whitney mapping W_μ.

The base $\mathbb{C}^\nu$ of a versal deformation of the projection contains a *bifurcation* *hypersurface* Σ of non-stable projections. For a point $\alpha \in \Sigma$ the corresponding perturbation of the inducing mapping is not transversal to some stratum of the swallowtail $\Delta(W_\mu)$.

Consider a generic line in $\mathbb{C}^\nu$ through the origin. Denote by l its generic translate. The line l intersects Σ transversally at some points $\alpha_1, \ldots, \alpha_s$. It is not difficult to show (using, say, the equivariant Morse Lemma [**1**]) that, approaching each α_i along l, we contract exactly one middle-dimensional sphere in the discriminant of a stable mapping $\Phi_{\alpha'}$, where $\alpha' \in l \backslash \Sigma$. Traditionally we call such a sphere a *vanishing cycle*. Approaching $\alpha_1, \ldots, \alpha_s$ by non-intersecting paths from a fixed α', we define a *distinguished set of vanishing cycles* on $\Delta(\Phi_{\alpha'})$.

5.4.1. THEOREM. *A distinguished set of vanishing cycles generates the integer homology of* $\Delta(\Phi_{\alpha'})$. *If all the multiple point sets* D^k *of the discriminant of the unperturbed map-germ are isolated hypersurface singularities, a distinguished set forms a basis of the homology.*

The proof of this theorem follows the proof of Theorem 1 of [**12**].

5.5. *Monodromy*

As in § 3.4, there is a filtration on $H_{p-1}(\Delta(\Phi_{\alpha'}))$ by the kernels of the sequence of the boundary operators

$$\delta_k\colon\ H_{p-1}(\Gamma(\Phi_{\alpha'})) \to H_{p-k}(D^k(\Phi_{\alpha'}), D_k^{k+1}(\Phi_{\alpha'}))$$

with successive factors isomorphic to $H_{p-k}^{\mathrm{Alt}}(D^k(\Phi_{\alpha'})\,;\mathbb{Z})$.

Take the cycle e on $\Delta(\Phi_{\alpha'})$ vanishing along a path $\gamma \subset l$, going from α' to α_i. Consider its boundaries $\delta_k e$. The non-zero boundary $\delta_{k_0} e$ of the highest order is a cycle on the smooth $D^{k_0}(\Phi_{\alpha'})$ vanishing along γ. Transfer $\delta_{k_0} e$ by the covering homotopy to $D^{k_0}(\Phi_{\alpha_*})$, where $\alpha_* \in \gamma$ is a point close to the end α_i. Following [**12**] define for $c \in H_{p-1}(\Delta(\Phi_{\alpha'}))$ the index of intersection with e to be the index of intersection on $D^{k_0}(\Phi_{\alpha_*})$ of $\delta_{k_0} c$, transferred to $H_{p-k}(D^{k_0}(\Phi_{\alpha_*}), D_{k_0}^{k_0+1}(\Phi_{\alpha_*}))$, and $\delta_{k_0} e$. In terms of these indices we can describe the Picard–Lefschetz operators on $H_{p-1}(\Delta(\Phi_{\alpha'}))$ corresponding to the vanishing cycles (cf. [**12**]). This provides a description of the monodromy group of the discriminant of the map-germ. Of course, the monodromy respects the filtration by the subspaces $\operatorname{Ker} \delta_k$ on $H_{p-1}(\Delta(\Phi_{\alpha'}))$.

In the next section we give some particular examples of the objects introduced here in a general situation.

6. *Discriminants of simple* $\Sigma^{1,\ldots,1}$ *maps*

The singularities considered here are from the list of simple (in the traditional sense) projections of complete intersections [**10**]. All of them can be induced from the suspended Whitney maps. But the suspension does not affect the discriminants. So, we may restrict ourselves to the case of mappings between spaces of equal dimension, that is, to the maps of Boardman type $\Sigma^{1,\ldots,1}$.

The map A_μ^Y *with* $\mu \geqslant 1$. This is a singularity of the projection $\mathbb{C}^{1+p} \to \mathbb{C}^p$, $(x, u) \mapsto u$ of the hypersurface

$$f(x, u) = x^{\mu+1} + g(u_\mu, \ldots, u_p)x^{\mu-1} + u_{\mu-1}x^{\mu-2} + \ldots + u_2 x + u_1 = 0,$$

where g is a function on $\mathbb{C}^{p-\mu+1}$ of the right equivalence class Y.

According to Theorem 4.1.1, the multiple point sets $D^k(A^Y_\mu) \subset \mathbb{C}^k_x \times \mathbb{C}^p_u$ of the discriminant are non-empty only if $k \leqslant [\frac{1}{2}(\mu+1)]$. All these sets, except $D^{[(\mu+1)/2]}$, that is, the highest one, are smooth: the equations (4.2) of D^k express $u_1, \dots, u_{2k}$ as functions of the independent variables $x_1, \dots, x_k, u_{2k+1}, \dots, u_p$. For the highest set we consider two cases.

(a) $\mu + 1 = 2s$. The equations (4.2) are equivalent to $f(x, u)$ being a square of a polynomial in x of the sth degree:

$$f(x, u) = (x^s - \sigma_1 x^{s-1} + \sigma_2 x^{s-2} - \dots)^2,$$

where the σ_i are the elementary symmetric functions of the coordinates $x_1, \dots, x_s$ of the point $(x_1, \dots, x_s, u) \in D^s$. Comparing the coefficients of the powers of x on the left and on the right, we get

$$0 = -2\sigma_1, \quad g = 2\sigma_2 + \sigma_1^2, \quad \text{and } u_{<\mu} \text{ are some symmetric functions in } x.$$

Thus, D^s has the singularity of the function $g - 2\sigma_2$ on the plane $\sigma_1 = 0$ in the space $\mathbb{C}^s_x \times \mathbb{C}^{p-\mu+1}_{u_\mu, \dots, u_p}$. As $\sigma_2|_{\sigma_1=0}$ is a non-degenerate quadratic form on $\mathbb{C}^{s-1}$, the stabilized discriminant of A^Y_μ is homotopy equivalent to a wedge of ν $(p-1)$-dimensional spheres, where ν is the Milnor number of the function g.

(b) $\mu = 2s$. For a point u of the s-tuple self-intersection of the discriminant, we have

$$f(x, u) = (x^s - \sigma_1 x^{s-1} + \sigma_2 x^{s-2} - \dots)^2 \cdot (x + 2\sigma_1).$$

Hence, D^s is isomorphic to a hypersurface $g = 2\sigma_2 - 3\sigma_1^2$ in the $(x_1, \dots, x_s, u_\mu, \dots, u_p)$-space and the number of the spheres in the wedge of the stabilized discriminant is the same as in (a).

We see that the operator of $[\frac{1}{2}(\mu+1)]$-order boundary (see § 3.4) is an isomorphism between the integer homology of the stabilized discriminant and of the Milnor fibre of the function singularity Y, in $p - [\frac{1}{2}(\mu-1)]$ variables. The intersection number (see § 5.5) of the cycles on the stabilized discriminant is equal to the intersection number of their $[\frac{1}{2}(\mu+1)]$-order boundaries on the regular level of the function.

In both the cases, an $\mathscr{A}$-versal deformation of our mapping is provided by the substitution for g of its $\mathscr{R}$-versal deformation. The $\mathscr{A}$-bifurcation diagram is the bifurcation diagram of zeros of g (this corresponds to the inducing mapping $\mathbb{C}^p_u \to \mathbb{C}^\mu$, into the target space of W_μ, that is, into the base of the $\mathscr{R}$-versal deformation of the function A_μ, being non-transversal to the stratum A_μ). The monodromy group of the discriminant of A^Y_μ coincides with the monodromy group of the function singularity Y in $p - [\frac{1}{2}(\mu-1)]$ variables.

The map A^k_μ with $\mu \geqslant 2$ and $k \geqslant 2$. We now consider the projection $\mathbb{C}^\mu \to \mathbb{C}^{\mu-1}$, $(x, u) \mapsto u$, of the hypersurface

$$f(x, u) = x^{\mu+1} + u_0 x^{\mu-1} + u_0^k x^{\mu-2} + u_{\mu-2} x^{\mu-3} + \dots + u_2 x + u_1 = 0.$$

(We can suspend this to a projection from $\mathbb{C}^{\mu+t}$ onto $\mathbb{C}^{\mu-1+t}$ adding a non-degenerate quadratic form in t extra coordinates on the base to the term u_0^k.)

We again have two cases.

(a) $\mu = 2s$. As for the A^Y_{2s}-singularity, the multiple point sets $D^1, \dots, D^{s-1}$ are smooth and the highest set D^s is isomorphic to the S_s-symmetric variety in the $(x_1, \dots, x_s, u_0)$-space given by the equations

$$u_0 = 2\sigma_2 - 3\sigma_1^2, \quad u_0^k = 2(\sigma_1^3 + \sigma_1\sigma_2 - \sigma_3),$$

that is, to the hypersurface

$$h = 2(\sigma_1^3 + \sigma_1\sigma_2 - \sigma_3) - (2\sigma_2 - 3\sigma_1^2)^k = 0$$

in the $(x_1, \dots, x_s)$-space (if there is a lack of the x variables, the corresponding elementary symmetric functions vanish). Using the expressions of the S_s-symmetric vector fields from [**1**], one can easily show that the main part $h_0 = 2(\sigma_1^3 + \sigma_1\sigma_2 - \sigma_3)$ of the function-germ h is non-degenerate and has symmetric $\mathscr{R}$-codimension $s+1$ (its miniversal deformation is, say, $h_0 + \lambda_0 + \lambda_1\sigma_1 + \dots + \lambda_s\sigma_s$). Thus, h is semihomogeneous and has the same codimension $s+1$. This codimension is equal to the number of S_s-orbits of Morse critical points of a generic symmetric perturbation of h [**20**]. Each $(s-1)$-cycle on a Milnor fibre of such a perturbation, vanishing at a Morse critical point, is antisymmetric with respect to the action of the stationary subgroup of this point. Hence, the rank of the S_s-alternating part of the homology of the Milnor fibre mentioned is $s+1$. Thus the stabilized discriminant of the mapping A_{2s}^k is homotopy equivalent to a wedge of $s+1$ $(2s-2)$-spheres and it does not depend on $k \geqslant 2$.

On the other hand, according to [**8, 19**], the number of the spheres in a stabilized discriminant wedge is not less than the $\mathscr{A}$-codimension of a map-germ, with the equality in the quasihomogeneous case. So, all the singularities $A_{2s}^{>s}$ are $\mathscr{A}$-equivalent to the quasihomogeneous A_{2s}^{∞} (with the term $u_0^k x^{\mu-2}$ in $f(x, u)$ omitted). All A_{2s}^k, for $k \leqslant s+1$, are $\mathscr{A}$-distinct as they have $\mathscr{A}$-codimension k. The corresponding entries in the classification tables in [**10**] should be corrected.

An $\mathscr{A}$-miniversal deformation of $A_{2s}^{s+1} = A_{2s}^{\infty}$ has $s+1$ parameters: take, say, a family of hypersurfaces

$$f_\alpha(x, u) = x^{2s+1} + u_0 x^{2s-1} + (\alpha_s u_0^s + \alpha_{s-1} u_0^{s-1} + \dots + \alpha_1 u_0 + \alpha_0) x^{2s-2}$$
$$+ u_{2s-2} x^{2s-3} + \dots + u_2 x + u_1 = 0.$$

One can easily see that this deformation does not induce a symmetric $\mathscr{R}$-versal deformation of D^s, in spite of the coincidence of the dimensions of the bases. But, consider a generic line l parallel to the α_0-axis in the base of the deformation of the mapping. This corresponds to a line of values of a generic symmetric perturbation h' of the function h_0. The points of the intersection of l with the $\mathscr{A}$-bifurcation diagram give the critical values of h'. One can show that the isotropy subgroups of the critical points on these $s+1$ critical levels of h' are as follows:

S_s, on two levels;

$S_i \times S_{s-i}$, also on two levels for each $i = 1, \dots, [\frac{1}{2}(s-1)]$;

$S_{s/2} \times S_{s/2}$, for s even, on one level.

Consider the mapping $\mathbb{C}_u^{\mu-1} \to \mathbb{C}^\mu$, $\mu = 2s$, into the base of $\mathscr{R}$-versal deformation of A_{2s}, depending on α and inducing the projection of the perturbed hypersurface. The points $p_0, ..., p_s$ of l, corresponding to the above-mentioned critical levels of h', are responsible for the non-transversality of this mapping to the strata A_{2s} and A_{2s-1}, (A_{2i-1}, A_{2s-2i}) and $(A_{2i}, A_{2s-2i-1})$, (A_{s-1}, A_s) respectively. Thus there is a one-to-one correspondence between the points $p_0, ..., p_s$ and all zero- and one-dimensional strata in the base of $\mathcal{R}$-versal deformation of A_{2s}.

(b) $\mu + 1 = 2s$, with $s \geqslant 2$. Now we have two singular multiple point sets, D^s and D^{s-1}. The first, D^s, is given by S_s-symmetric equations

$$\sigma_1 = 0, \quad \sigma_3 + 2^{k-1}\sigma_2^k = 0$$

in the $(x_1, ..., x_s)$-space. Using [1], we calculate the symmetric $\mathcal{R}$-codimension of this singularity. This is $p + 1$ for $s = 2p + 1$, and $k - p + 1$ for $s = 2p$. This will be the rank of the S_s-alternating homology of a symmetric Milnor fibre of D^s.

The second, D^{s-1}, is a hypersurface in the $(x_1, ..., x_{s-1}, u_0)$-space defined by the S_{s-1}-symmetric equation

$$2\rho_3 - 4\rho_1\rho_2 + 4\rho_1^3 + 2\rho_1 u_0 + u_0^k = 0,$$

where the ρ_i are elementary symmetric functions in $x_1, ..., x_{s-1}$. The symmetric $\mathcal{R}$-codimension of this function, that is, the rank of the S_{s-1}-alternating homology of its Milnor fibre, is $p + k - 1$ for $s = 2p + 1$, and p for $s = 2p$.

Adding the ranks of the alternating homology groups of D^s and D^{s-1}, we get that the stabilized discriminant of A_{2s-1}^k is homotopy equivalent to a wedge of $s + k - 1$ $(2s-2)$-dimensional spheres. Thus all the members of the series $A_{2s-1}^{\cdot}$ are distinct. So, the classification tables in [**10**] should be corrected again.

By Theorem 5.4.1, in both cases (a) and (b) a distinguished set of vanishing cycles forms a basis of the homology of the stable discriminant.

7. *The complex link of a Coxeter group*

7.1.

The classification theorems of [**10**] say that, in addition to A_μ^Y and A_μ^k, the only simple projections (that is, $\mathcal{A}$-simple map-germs) Φ of hypersurfaces onto spaces of dimension greater than 2 are the ones induced from $\mathcal{R}$-versal deformations of simple functions in the same way as A_μ^Y is induced from A_μ. Namely, take a function $\phi(x)$ of type $X = A_\mu, D_\mu, E_6, E_7$ or E_8 [4]. Consider its $\mathcal{R}$-miniversal deformation $F(x, \lambda) = \phi(x) + \lambda_\mu e_\mu(x) + ... + \lambda_1 e_1(x)$, where the e_i form a monomial basis of the local ring $\mathcal{O}_x/\mathcal{O}_x\langle\partial\phi/\partial x_j\rangle$ and $e_\mu = \operatorname{Hess}\phi$. Then a simple projection Φ, $(x, u) \mapsto u$, is induced from a projection $(x, \lambda) \mapsto \lambda$ of a hypersurface $F(x, \lambda) = 0$ by setting $\lambda_i = u_i$ for $i = 1, ..., \mu - 1$, $\lambda_\mu = h(u_\mu, ..., u_p)$, where h is again an $\mathcal{R}$-simple function of some other type Y. We denote the induced projection Φ by X^Y. For $X = A_\mu$ we get exactly the singularities studied in the first half of the previous section.

An $\mathcal{R}$-miniversal deformation of Y provides a miniversal deformation of the projection X^Y (that is, an $\mathcal{A}$-miniversal deformation of the map-germ). As X^Y is quasihomogeneous, the number of middle-dimensional spheres S^{p-1} in the wedge of its stable discriminant is equal to $\mathcal{A}\text{-codim}\, X^Y = \mathcal{R}\text{-codim}\, Y$ [**8**]. This number will be denoted by ν. The bifurcation set $\Sigma \subset \mathbb{C}^\nu$ of the projection X^Y coincides with the bifurcation diagram of zeros of the function Y and corresponds to non-transversality of a perturbation of the inducing mapping to the stratum X of the bifurcation diagram of zeros $\Delta(X)$ of X (this is the discriminant of the stable projection $(x, \lambda) \mapsto \lambda$ of $F(x, \lambda) = 0$). A generic non-transversality provides a projection which is stable everywhere except at one point at which it has an X^{A_1} singularity ($\lambda_\mu = u_\mu^2 + \ldots + u_p^2$). This means that at a regular point of Σ there is exactly one copy of S^{p-1} vanishing on the stable discriminant of X^Y. Consequently, as a generic 1-parameter unfolding of X^Y is a stable mapping onto $\mathbb{C}^{p+1}$, a distinguished set of vanishing cycles forms a basis of the homology of the stabilization of $\Delta(X^Y)$.

The vanishing cycles here have a very nice real representation, which we are going to describe.

A stabilization of $\Delta(X^{A_1})$ is a $(p-\mu+1)$th suspension of a section of $\Delta(X)$ by $\lambda_\mu = c$, where c is a non-zero constant. According to [**1**], $\lambda_\mu = 0$ is a generic section of our $\Delta(X)$, which is the discriminant of the corresponding Weyl group. Thus $\lambda_\mu = c \neq 0$ is the complex link of the group discriminant [**9, 16**] and this link is homotopy equivalent to $S^{\mu-2}$, vanishing as $c \to 0$. We consider the nature of this vanishing cycle not only for the A, D, E-cases but for any Coxeter group.

7.2.

So, consider a finite irreducible group W generated by reflections in $\mathbb{R}^\mu$. Consider the complexification of this representation and the orbit space $\mathbb{C}^\mu/W \cong \mathbb{C}^\mu$ containing the discriminant $\Delta(W)$, the set of the irregular orbits. Take a generic function on the orbit space vanishing at the origin. A diffeomorphism of the pair $(\mathbb{C}^\mu, \Delta(W))$ reduces it to the basic invariant λ_μ of the second order, positive on $\mathbb{R}^\mu$ [**1**]. A section L_W of $\Delta(W)$ by $\lambda_\mu = c > 0$ represents the complex link of the discriminant.

On the other hand, the real part $\mathbb{R}^\mu$ of the orbit space contains a (quasi-)cone $\mathbb{R}^\mu/W$. It has a boundary δ which is a factor space of the real mirrors. Set $l_W = \delta \cap \{\lambda_\mu = c\} \subset L_W$ taking $c > 0$.

7.2.1. THEOREM. *The section L_W is homotopy equivalent to the section l_W which is homeomorphic to $S^{\mu-2}$.*

Proof. Note that λ_μ is a positive quadratic form on the configuration space $\mathbb{R}^\mu$. Thus any positive level of λ_μ in $\mathbb{C}^\mu$ is contractible onto its real part which is $S^{\mu-1}$. A W-equivariant contraction is provided by W-averaging of the contraction field. This lowers to the orbit space and contracts the pair $(\{\lambda_\mu = c\}, L_W)$ onto $(\mathbb{R}^\mu/W \cap \{\lambda_\mu = c\}, l_W)$. Factorization by W is a homeomorphism of a Weyl chamber C onto its image. The intersection of the real sphere $S^{\mu-1} = \{\lambda_\mu = c\}$ with C is homeomorphic to a ball. So l_W is homeomorphic to the boundary $S^{\mu-2}$ of this ball.

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