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Symmetric Quartics with Many Nodes

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§0. Introduction

Let $N_n(d)$ be the maximal number of Morse singular points (*nodes*, in our terminology) which a projective $(n-1)$-dimensional hypersurface of degree d can have. The exact value of the number is known for very small range of the two parameters, namely, for

(1) points on a line, $N_1(d) = [d/2]$;
(2) plane curves, $N_2(d) = d(d-1)/2$;
(3) cubics in any dimension, $N_n(3) = \binom{n+1}{[n/2]}$ (see 4.3);
(4) quartics in 3- and 4-spaces, $N_3(4) = 16$ (Kummer, 1864) and $N_4(4) = 45$ (Burkhardt, 1891 [6]);
(5) quintic surfaces, $N_3(5) = 31$ (Togliatti [15], Beauville [4]).

For the other dimensions and degrees only estimates exists. Thus, for $n = 3$ the degree-asymptotically best upper bound is given by the Bogomolov-Miyaoka inequality on the Chern numbers of surfaces [11].

In higher dimensions the best known upper bound is so-called Varchenko's spectral bound [16, 17]. The corresponding inequality was initially conjectured by V. I. Arnold. It arises from the analysis of mixed Hodge structure on the vanishing cohomology of an isolated hypersurface singularity. This is the second refinement, after the first one by Givental [8], of the result by Bruce [5].

The lower bounds are provided by numerous examples. The main efforts have been applied to the surface case. The degree-asymptotically best series of surfaces was built up by Chmutov [7]. He introduced also another, higher-dimensional series [3, 17] having a reasonable deviation from the Varchenko bound. See [3, 7, 13, 17] and references there for the other estimates.

Many hypersurfaces giving the best known lower bound for the number $N_n(d)$ are invariant under certain actions of permutation groups. Among them there are the Segre cubic [12], the Burkhardt quartic [6], van Straten's

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quintic [13], the both Chmutov's series.

This suggests looking for new, improved examples not in the pencils of all hypersurfaces of fixed degree, but in the pencils of hypersufaces with symmetries. This reduces the dimension of the pencil (and all the calculations as well) drastically. But, surprisingly, it does not eliminate hypersurfaces with a lot of singularities. For example, the Segre cubic is the only possible cubic with A_5-reflection symmetry in $\mathbb{C}P^4$.

In this note we use the above idea to find good examples of quartic hypersurfaces in $\mathbb{C}P^n$ among the ones with finite reflection group symmetry. As the measure of goodness one can take the ratio between the numbers $L_n(4)$ of the nodes in our examples and Varchenko's spectral bound $Ar_n(4)$ [3, 15, 17] for $N_n(4)$. Asymptotically, for n tending to infinity, this ratio is $\sqrt{3}/2$. For small dimensions it is even better:

n	2	3	4	5	6	7	8	9	10	11	12
$Ar_n(4)$	6	16	45	126	357	1016	2907	8350	24068	69576	201643
$L_n(4)$	6	16	45	120	336	938	2688	7680	21120	63360	183040
$Chm_n(4)$	4	12	24	80	160	560	1120	4032	8064	29658	59316

Certainly, for $n = 3$ and $n = 4$ we get a Kummer surface and the Burkhardt quartic included into our series. We have also put into the table the numbers $Chm_n(4)$ of nodes of Chmutov's quartics [3, 17]. They do not look very impressive here, but one must remember that Chmutov's series gives good asymptotical coincidence with the spectral bound only for the dimension fixed and the degree increasing. Moreover, his series serves all dimensions.

The structure of the note is as follows.

In §1 we recall Varchenko's spectral bound for the numbers $N_n(d)$. We also calculate the asymptotical growth of the bound.

In §2, in the case of a reflection group symmetry, we reformulate the condition that a hypersurface in $\mathbb{C}P^n$ has a singularity as the condition that the corresponding hypersurface in the orbit space is nontransversal to the discriminant of the group. Thus, to produce examples of hypersurfaces with more nodes, we must find, in the orbit space, a hypersurface that is nontransversal to the strata of orbits as long as possible for the given degree.

In §3 we consider quartics in $\mathbb{C}P^n$ invariant under the standard representation of the reflection group B_{n+1}. Using the ideology of the previous section, for any n we study the pencil of all such hypersurfaces. We point out all its nonsmooth members and choose the one with the maximal number of nodes. The calculations, giving the quartics with fairly high numbers of singular points, are rather simple, since the pencil is one-dimensional.

In §4, the last one, we carry out similar calculations for the groups A_{n+1}. The calculations show that the lower bounds for $N_n(4)$ in dimensions $n = 4$

and $n = 7$ given by A-invariant quartics are better than the ones obtained in the B-case.

We conclude §4 with A_{n+1}-invariant cubics. Actually, this could be the most natural point at which to start the study of nodal hypersurfaces with symmetries. In spite of their elementary nature, considering these cubics (generalizing the Segre cubic to higher dimensions), one immediately gets the exactness of the spectral bound for cubics given by the examples much less sophisticated than Givental's [17].

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REMARK. D. van Straten informed me that the same extremal cubics we represent here were discovered by T. Kalker [10]. It has also been shown that the extremal cubics in $\mathbb{C}P^5$ are the only ones having moduli [14]. A similar rigidity question arises naturally for our series of the symmetric quartics.

§1. The spectral bound and its asymptotics

DEFINITION. Arnold's number $Ar_n(d)$ is the number of integer points $(k_0, \dots, k_n)$ in the open cube $(0, d)^{n+1} \subset \mathbb{R}^{n+1}$ lying in the hyperplane

$$\sum_{i=0}^{n} k_i = [nd/2] + 1.$$

EXAMPLE. $Ar_n(3) = \binom{n+1}{[n/2]}$.

REMARK. Usually (see [2, 3, 16, 17]) the number $Ar_n(d)$ is expressed as the number of integer points $(k_1, \dots, k_n)$ in the open cube $(0, d)^n$ such that

$$(n-2)d/2 + 1 < \sum_{i=1}^{n} k_i \le nd/2.$$

THEOREM 1 (Varchenko's spectral bound [3, 16, 17]). *Let $V \subset \mathbb{C}P^n$ be a hypersurface of degree d with isolated Morse singularities only. Then the number of singular points of V cannot exceed $Ar_n(d)$.*

In what follows, writing $c_n \approx \varphi(n)$ for a sequence $c.$ and a function φ, we mean that the ratio $c_n/\varphi(n)$ tends to 1 as n tends to infinity.

According to [3, 17], for n fixed,

$$Ar_n(d) = a_n d^n + \text{terms of lower order in } d,$$

where $a_n \approx \sqrt{6/\pi n}$.

Here we calculate the asymptotic behavior of $Ar_n(d)$ in the "perpendicular direction" and correct the base of the exponent in the above formula:

Theorem 2. *For d fixed and n tending to infinity, we have*

$$Ar_n(d) \approx \sqrt{\frac{6}{d(d-2)}} \cdot \frac{(d-1)^{n+1}}{\sqrt{\pi n}}.$$

Example. $Ar_n(4) \approx \frac{\sqrt{3}}{2} \cdot \frac{3^{n+1}}{\sqrt{\pi n}}.$

Proof. Consider a particular solution to the equation

$$\sum_{i=0}^{n} k_i = [nd/2] + 1,$$

subject to the required restrictions. For $j = 1, \ldots, d-1$, set $r_j = \sharp\{k_i = j\}$. This gives the precise expression

$$Ar_n(d) = \sum \frac{(n+1)!}{\prod_{j=1}^{d-1} r_j!},$$

where the summation is taken over all $(d-1)$-tuples $(r_1, \ldots, r_{d-1})$ of non-negative numbers such that

$$\sum_1^{d-1} r_j = n+1 \quad \text{and} \quad \sum_1^{d-1} j r_j = [nd/2] + 1.$$

We rewrite the sum as

$$Ar_n(d) = \frac{(n+1)!(\sqrt{n+1})^{d-3}}{\Gamma^{d-1}(\frac{n+d}{d-1})} \sum \left(\frac{\Gamma^{d-1}(\frac{n+d}{d-1})}{\prod_{j=1}^{d-1} r_j!} \right) \cdot \left(\frac{1}{(\sqrt{n+1})^{d-3}} \right),$$

where Γ is the Gamma-function.

Applying Stirling's formula, we see that the factor in front of the sum is

$$\approx \frac{(d-1)^{n+\frac{d+1}{2}}}{\sqrt{n}(\sqrt{2\pi})^{d-2}}.$$

Setting

$$r_j = \frac{n+1}{d-1} + u_j\sqrt{n+1}, \; j = 1, \ldots, d-1,$$

and applying Stirling's formula again, we see that the first factor under the sum tends to

$$\exp\left(-\frac{1}{2}(d-1) \sum_1^{d-1} u_j^2 \right).$$

Thus the whole sum tends to the integral of this function along the whole real $(u_1, \ldots, u_{d-3})$-coordinate space with the two remaining u_j's expressed via the relations

$$\sum_1^{d-1} u_j = 0 \quad \text{and} \quad \sum_1^{d-1} j u_j = 0.$$

Evaluation of the integral yields a constant providing the asymptotics claimed in the Theorem.

§2. Reflection group symmetry

Suppose that the ambient space $\mathbb{C}P^n$ of a hypersurface V is a projectivization of a linear space $\mathbb{C}^{n+1}$ equipped with a standard action of an irreducible finite reflection group G. Then G acts on $\mathbb{C}P^n$ as well. Suppose that V is G-invariant, i.e., is a zero set of a G-invariant function on $\mathbb{C}^{n+1}$. Let us lower this function to the quotient space $\mathbb{C}^{n+1}/G \cong \mathbb{C}^{n+1}$. There its zeros form a certain quasi-homogeneous hypersurface W. Let us consider the mutual position of W and the discriminant $\Delta \subset \mathbb{C}^{n+1}/G$ of the group G. The hypersurface W is always nontransversal to the stratum $\{0\}$ of the canonical orbit-type stratification of Δ, and it may be nontransversal to some other strata of this stratification as well. The set of points of nontransversality of W to strata of positive dimension is invariant under the quasi-homogeneous contraction of $\mathbb{C}^{n+1}/G$. The following lemma easily follows from [1].

LEMMA 3. *Assume that W is smooth. Then the quasi-homogeneous orbits of points of nontransversality of W to the canonical stratification of $\Delta \setminus \{0\}$ are in one-to-one correspondence with the G-orbits of singular points of the initial hypersurface $V \subset \mathbb{C}P^n$.*

Some orbits of the reflection groups are of considerable length. This suggests that for constructing examples of projective hypersurfaces V of fixed degree d with many singular points, one can try to build up a hypersurface W of weighted degree d in the quotient space $\mathbb{C}^{n+1}/G$ of a certain reflection group, so that W would be nontransversal to strata of the orbits of the greatest sum of lengths possible for this degree. Thus, instead of doing calculations for a high-dimensional pencil of arbitrary hypersurfaces, we reduce our attention to a very small pencil of symmetric ones. The idea turns out to be rather effective, at least in low degrees.

§3. B-invariant hypersurfaces

It would be very natural to start with hypersurfaces invariant under the actions of reflection groups of the A-series. But since they give (this time, for quartics) not too many good examples, we postpone this to the next section and start with the series B.

3.1. Stratification of the discriminant and limits of tangent spaces. The standard representation of the reflection group $G = B_{n+1}$ is the action on the $(x_0, \dots, x_n)$-coordinate linear space by permutations of the coordinates and multiplication of some of them by -1. The orbit space $\mathbb{C}^{n+1}/B_{n+1}$ of the standard representation is the $(n+1)$-dimensional space of polynomials

$$p_\lambda(z) = z^{n+1} + \lambda_1 z^n + \dots + \lambda_{n+1}.$$

Here λ_i is the $(-1)^i$ times the ith elementary symmetric function in the x_j^2's. Thus the roots of $p_\lambda(z)$ are the x_j^2's.

The discriminant $\Delta(B_{n+1})$ is the set of polynomials $p_\lambda(z)$ with multiple or zero roots. It is stratified by the sets

$$A_{a_1}\dots A_{a_k}B_b, \qquad (a_1+1)+\dots+(a_k+1)+b=n+1,$$

of polynomials with the root 0 of multiplicity $b\geq 0$ and $k\geq 0$ distinct nonzero roots of multiplicities $a_j+1\geq 1$. The dimension of such a stratum is k. The origin $\lambda=0$ is in the closure of any stratum.

REMARK. The name of a stratum of the discriminant is the name of the subgroup of the reflection group fixing a point of the corresponding orbit.

LEMMA 4. *The limit at the origin of tangent spaces to any k-dimensional stratum of $\Delta(B_{n+1})$ is well defined and is equal to the linear space spanned by the vectors $\partial_{\lambda_1},\dots,\partial_{\lambda_k}$.*

PROOF. Let us consider the set S in the $(z_1,\dots,z_k,\lambda_1,\dots,\lambda_{n+1})$-coordinate space defined by the following equations and inequalities:

(1) $p_\lambda(z_j)=p_\lambda'(z_j)=\dots=p_\lambda^{(a_j)}(z_j)=0, \quad j=1,\dots,k;$
(2) $p_\lambda^{(a_j+1)}(z_j)\neq 0, \quad j=1,\dots,k;$
(3) $\lambda_{n+1}=\dots=\lambda_{n+2-b}=0;$
(4) $\lambda_{n+1-b}\neq 0;$
(5) $z_j\neq z_i, \quad j\neq i;$
(6) $z_j\neq 0.$

The image of S under the projection into the λ-space is the stratum $A_{a_1}\dots A_{a_k}B_b$. The inequalities (2) guarantee that the tangent space to S projects isomorphically onto the tangent space to the stratum. Thus the latter is the kernel of the $(n+1-b-k)\times(n+1-b)$-matrix

$$\begin{pmatrix}
z_1^n & z_1^{n-1} & \dots & z_1^b\\
(z_1^n)' & (z_1^{n-1})' & \dots & (z_1^b)'\\
\dots & \dots & \dots & \dots\\
(z_1^n)^{(a_1-1)} & (z_1^{n-1})^{(a_1-1)} & \dots & (z_1^b)^{(a_1-1)}\\
\dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots\\
z_k^n & z_k^{n-1} & \dots & z_k^b\\
(z_k^n)' & (z_k^{n-1})' & \dots & (z_k^b)'\\
\dots & \dots & \dots & \dots\\
(z_k^n)^{(a_k-1)} & (z_k^{n-1})^{(a_k-1)} & \dots & (z_k^b)^{(a_k-1)}
\end{pmatrix}.$$

The ith column from the left corresponds here to the λ_i-component of the tangent vector.

Since $z_j\neq z_i\neq 0$, the right maximal minor of the matrix does not vanish (see e.g. [9]). Therefore, the k-dimensional tangent space to the stratum projects isomorphically onto the $(\lambda_1,\dots,\lambda_k)$-plane.

The ratio between any maximal minor and the right one is a holomorphic function on the whole $(z_1, \dots, z_k)$-space [9] vanishing at the origin. Thus the $\lambda_{>k}$-components of a vector tangent to the stratum are uniquely determined by the $\lambda_{\leq k}$-ones and tend to zero with $(z_1, \dots, z_k)$, i.e., λ tending to zero.

3.2. B-invariant quartics. Almost all of our best examples of quartics with many singular points are contained in one infinite series of B_{n+1}-invariant quartics in $\mathbb{C}P^n$.

For n fixed, a generic hypersurface in the pencil of all B_{n+1}-invariant quartics in $\mathbb{C}P^n$ is smooth. Exceptional members of the pencil have singularities. So we point out these exceptional members and choose one with the greatest number of nodes.

When written in terms of the invariants λ_i, the pencil of all B_{n+1}-symmetric quartics in $\mathbb{C}P^n$ gives a pencil of hypersurfaces

$$\alpha\lambda_2 + \beta\lambda_1^2 = 0, \qquad (\alpha : \beta) \in \mathbb{C}P^1,$$

in $\mathbb{C}^{n+1}/B_{n+1} \cong \mathbb{C}^{n+1}$.

Since we are looking for projective hypersurfaces with isolated singularities, we set $\alpha = 1$. The corresponding hypersurface in $\mathbb{C}^{n+1}$ will be denoted by W_β. For which β is W_β nontransversal to the discriminant, so that the projective hypersurface V_β has singularities?

According to Lemma 4, W_β may be nontransversal to one-dimensional strata only. This means that a stratum A_aB_b, $a, b \geq 0$, $a + b + 1 = n + 1$, has to lie in W_β. So, we write out a parametrization for the stratum and check when the stratum turns out to be in W_β.

The parametrization is

$$p_\lambda(z) = (z - t)^{a+1} z^b,$$

i.e. $\lambda_1 = -(a+1)t$, $\lambda_2 = a(a+1)t^2/2$.

So, the stratum A_aB_b is in W_β iff

$$\beta = -\frac{a}{2(a+1)}.$$

Then $V_\beta \subset \mathbb{C}P^n$ has singularities along the B_{n+1}-orbit of the point

$$(\underbrace{1 : 1 : \cdots : 1}_{a+1 \text{ times}} : \underbrace{0 : 0 : \cdots : 0}_{b \text{ times}}).$$

From the proof of Lemma 4, it follows that, at any point of the stratum A_aB_b, the hypersurface W_β is transversal to the limit tangent space of any higher-dimensional stratum. Thus, [1] implies that all the above singular points are Morse.

Counting the length of the projective orbit, we get

THEOREM 5. *The hypersurface*

$$2(a+1)\sum_{0\le i<j\le n} x_i^2x_j^2 - a\Big(\sum_{0\le j\le n} x_j^2\Big)^2 = 0$$

in $\mathbb{C}P^n$ *has only Morse singular points, and exactly* $2^a\binom{n+1}{a+1}$ *of them.*

The number obtained is maximal for $a = [2n/3]$ and $a = [(2n+1)/3]$. These integer parts differ for $n \equiv 1 \pmod 3$, but produce critical orbits of the same length. Thus we get

COROLLARY 6. *The maximal number of nodes a quartic hypersurface in* $\mathbb{C}P^n$ *may have is at least*

$$2^{[2n/3]}\binom{n+1}{[2n/3]+1}.$$

These are the lower bounds presented as $L_n(4)$ in the table in the Introduction, with the two exceptions: $n = 4$ and $n = 7$. For these two dimensions it is possible to raise the lower bound of the Corollary from 40 to $45 = Ar_4(4)$ and from 896 to 938 respectively, by considering A_{n+1}-invariant hypersurfaces. This is one of the topics of the next section.

We finish the present section with the description of the asymptotic behavior of the numbers $L_n(4)$ following from Stirling's formula:

COROLLARY 7.

$$L_n(4) \approx \frac{3}{4}\cdot\frac{3^{n+1}}{\sqrt{\pi n}}.$$

Recalling from §1 that

$$Ar_n(4) \approx \frac{\sqrt{3}}{2}\cdot\frac{3^{n+1}}{\sqrt{\pi n}},$$

we get the following asymptotic ratio.

COROLLARY 8. $L_n(4)/Ar_n(4) \approx \sqrt{3}/2$.

REMARK. The number of nodes on Chmutov's quartic is

$$2^{[(n+1)/2]}\binom{n}{[(n+1)/2]}.$$

This is asymptotically

$$\approx \frac{(2\sqrt{2})^{n+1}}{\sqrt{2\pi n}} \text{ for } n \text{ odd and } \approx \frac{(2\sqrt{2})^{n+1}}{2\sqrt{\pi n}} \text{ for } n \text{ even.}$$

§4. A-invariant hypersurfaces

4.1. Discriminants and tangent limits. Again we begin by recalling the standard representation of the reflection group A_{n+1}. This is the action by all possible permutations of coordinates on the hyperplane $\mathbb{C}^{n+1} = \{x_0+\dots+x_{n+1} = 0\}$ in a linear $(x_0, \dots, x_{n+1})$-coordinate space.

The orbit space is the space of polynomials $q_\sigma(y) = y^{n+2} + \sigma_2 y^n + \cdots + \sigma_{n+2}$. The factorization mapping is the Vieta mapping relating a point $(x_0, \ldots, x_{n+1})$ to the set of elementary symmetric functions $(-1)^i\sigma_i(x)$ of the coordinates.

The discriminant is the set of polynomials with multiple roots. It is stratified by the sets

$$A_{a_0} \cdots A_{a_k}, \quad a_0 \geq \cdots \geq a_k \geq 0, \quad (a_0 + 1) + \cdots + (a_k + 1) = n + 2,$$

of polynomials with exactly $k + 1$ distinct roots of positive multiplicities $a_0 + 1, \ldots, a_k + 1$. The dimension of such a stratum is k.

One can prove the following statement in the same way as Lemma 4.

LEMMA 9. *The limit at the origin $\sigma = 0$ of tangent spaces to any k-dimensional stratum of the discriminant $\Delta(A_{n+1})$ is well defined and is equal to the linear space spanned by the vectors $\partial_{\sigma_2}, \ldots, \partial_{\sigma_{k+1}}$.*

4.2. A-invariant quartics. Let us consider a family

$$\sigma_4(x) + \delta\sigma_2^2(x) = 0, \quad \delta \in \mathbb{C},$$

of A_{n+1}-invariant quartics V_δ in $\mathbb{C}P^n$. This family contains all such quartics with isolated singularities. As usual, W_δ is the hypersurface in $\mathbb{C}^{n+1}$ corresponding to V_δ.

Lemma 9 does not allow any W_δ to be nontransversal to any stratum of dimension greater than 2. Looking for all possible W_δ that are not transversal to the 1-dimensional strata $A_{a-1}A_{b-1}$, in a way similar to 3.2, we get

PROPOSITION 10. *Let a and b be positive integers with sum $n + 2$. Set*

$$\delta = \frac{a+b}{ab} - \frac{3}{a+b} - \frac{1}{2}.$$

Then the quartic V_δ in $\mathbb{C}P^n$ has Morse singular points along the projective A_{n+1}-orbit of the point $(\underbrace{b : \cdots : b}_{a \text{ times}} : \underbrace{-a : \cdots : -a}_{b \text{ times}})$.

Thus in these cases V_δ has $\binom{a+b}{a}$ nodes if $a \neq b$, and $\frac{1}{2}\binom{2a}{a}$ nodes if $a = b$. So the number of nodes appearing because of the nontransversality to 1-dimensional strata, is maximal for $a = [n/2] + 1$.

Consideration of all possible nontransversalities to 2-dimensional strata $A_{a-1}A_{b-1}A_{c-1}$ gives

PROPOSITION 11. *Let $a \geq b \geq c$ be positive integers with sum $n + 2$.*

(1) *Suppose that at least two of these numbers are distinct and $b \neq (a + c)/2$. Set*

$$s(a, b, c) = a(b - c)^2 + b(c - a)^2 + c(a - b)^2,$$
$$\delta = -s(a - 1, b - 1, c - 1)/(2s(a, b, c)).$$

Then the quartic V_δ in $\mathbb{C}P^n$ has nodes along the projective A_{n+1}-orbit of the point

$$(\underbrace{b-c:\cdots:b-c}_{a\ \text{times}}:\underbrace{c-a:\cdots:c-a}_{b\ \text{times}}:\underbrace{a-b:\cdots:a-b}_{c\ \text{times}}).$$

(2) *Let $a=b=c$. Then for any $\delta \neq (1-a)/2a$ the quartic V_δ has nodes along the projective A_{n+1}-orbit of the point*

$$(\underbrace{1:\cdots:1}_{a\ \text{times}}:\underbrace{\varepsilon:\cdots:\varepsilon}_{a\ \text{times}}:\underbrace{\varepsilon^2:\cdots:\varepsilon^2}_{a\ \text{times}}),$$

where $\varepsilon^3=1$. For the above exceptional value of δ, the quartic V_δ has nonisolated singularities along the projective A_{n+1}-orbits of the points

$$(\underbrace{1:\cdots:1}_{a\ \text{times}}:\underbrace{y_2:\cdots:y_2}_{a\ \text{times}}:\underbrace{y_3:\cdots:y_3}_{a\ \text{times}}).$$

PROOF (sketch). The nontransversality of W_δ to the stratum $A_{a-1}A_{b-1}A_{c-1}$ at a point σ corresponding to the polynomial $q_\sigma(y)$ with three distinct roots y_1, y_2, y_3 of multiplicities a, b, c respectively is equivalent to the rank of the $n\times(n+1)$-matrix

$$\begin{pmatrix}
2\delta\sigma_2 & 0 & 1 & 0 & \dots & 0 & 0\\
y_1^n & y_1^{n-1} & y_1^{n-2} & y_1^{n-3} & \dots & y_1 & 1\\
(y_1^n)' & (y_1^{n-1})' & (y_1^{n-2})' & (y_1^{n-3})' & \dots & (y_1)' & (1)'\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
(y_1^n)^{(a-2)} & (y_1^{n-1})^{(a-2)} & (y_1^{n-2})^{(a-2)} & (y_1^{n-3})^{(a-2)} & \dots & (y_1)^{(a-2)} & (1)^{(a-2)}\\
y_2^n & y_2^{n-1} & y_2^{n-2} & y_2^{n-3} & \dots & y_2 & 1\\
(y_2^n)' & (y_2^{n-1})' & (y_2^{n-2})' & (y_2^{n-3})' & \dots & (y_2)' & (1)'\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
(y_2^n)^{(b-2)} & (y_2^{n-1})^{(b-2)} & (y_2^{n-2})^{(b-2)} & (y_2^{n-3})^{(b-2)} & \dots & (y_2)^{(b-2)} & (1)^{(b-2)}\\
y_3^n & y_3^{n-1} & y_3^{n-2} & y_3^{n-3} & \dots & y_3 & 1\\
(y_3^n)' & (y_3^{n-1})' & (y_3^{n-2})' & (y_3^{n-3})' & \dots & (y_3)' & (1)'\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
(y_3^n)^{(c-2)} & (y_3^{n-1})^{(c-2)} & (y_3^{n-2})^{(c-2)} & (y_3^{n-3})^{(c-2)} & \dots & (y_3)^{(c-2)} & (1)^{(c-2)}
\end{pmatrix}$$

being nonmaximal. Since the y_i's are distinct, the lower right minor of order $n-1$ does not vanish. This reduces the nontransversality condition to the condition that the matrix

$$\begin{pmatrix} 2\delta\sigma_2 & 0 & 1\\ \hat\sigma_1^2-\hat\sigma_2 & \hat\sigma_1 & 1\end{pmatrix}$$

have rank 1. Here σ_i (resp. $\hat\sigma_i$) is the ith elementary symmetric function in $n+2$ (resp. $n-1$) arguments among which there are a (resp. $a-1$) y_1's, b (resp. $b-1$) y_2's and c (resp. $c-1$) y_3's. Recalling the condition $\sigma_1=0$, elementary calculations lead to the assertion of the Proposition.

Thus the number of the nodes appearing because of the nontransversality to the 2-dimensional strata is

$$\frac{(n+2)!}{a!\,b!\,c!}, \text{ if } a>b>c,\ b\neq(a+c)/2,$$
$$\frac{(n+2)!}{2a!\,b!\,c!}, \text{ if } a=b>c \text{ or } a>b=c,$$
$$\frac{(3a)!}{3(a!)^3}, \text{ if } a=b=c=(n+2)/3.$$

Choosing the maximal one among the numbers of the first two cases, we get the following list of the triples (a, b, c) corresponding to the longest orbits of the nodes:

$n=3m-1$		$n=3m$	
$m=1,2:$	$(m+1, m, m)$	$m=1,2:$	$(m+1, m+1, m)$
$m>1:$	$(m+2, m, m-1)$	$m>1:$	$(m+2, m+1, m-1)$

$n=3m+1$	
$1\leq m\leq 4:$	$(m+3, m, m)$
$m>3:$	$(m+4, m, m-1)$

One can easily check that the numbers of nodes corresponding to these triples are greater than the numbers coming from the nontransversality to the one-dimensional strata.

Assuming that for any exceptional value of δ the hypersurface W_δ is not transversal to only one stratum to which there is transversality for a generic value of the parameter (this is the case for dimensions $n\leq 10$ at least), we obtain

THEOREM 12. *The maximal number of nodes an A_{n+1}-invariant quartic in $\mathbb{C}P^n$ can have is at least*

(1) *for small n:*

n	2	3	4	5	6	7	8	9	10
	6	15	45	105	280	938	2520	6930	20790

(2) *for $n>10$:*

$$\frac{(3m+1)!}{(m+2)!\,m!\,(m-1)!}, \quad \textit{if } n=3m-1;$$
$$\frac{(3m+2)!}{(m+2)!\,(m+1)!\,(m-1)!}, \quad \textit{if } n=3m;$$
$$\frac{(3m+3)!}{(m+4)!\,m!\,(m-1)!}+\frac{(3m+3)!}{3((m+1)!)^3}, \quad \textit{if } n=3m+1.$$

COROLLARY 13. *The maximal numbers of nodes that quartics in $\mathbb{C}P^4$ and $\mathbb{C}P^7$ can have are $15+30=45$ and at least $378+560=938$ respectively. This many nodes appear on the A_5- and A_8-invariant quartics*

$$\sigma_4(x)=0 \textit{ and } 4\sigma_4(x)-\sigma_2^2(x)=0$$

respectively.

This exceeds the numbers of nodes obtained in the B_5- and B_8-invariant cases. For all the other dimensions the B-examples are better.

REMARKS.

(1) $\sigma_4(x) = 0$ in $\mathbb{C}P^4$ is the Burkhardt quartic.

(2) The asymptotic behavior of the numbers obtained in Theorem 12 is

$$\approx \frac{6\sqrt{3}3^{n+1}}{\pi n} \text{ for } n \equiv 1 (\text{mod } 3) \text{ and } \approx \frac{9\sqrt{3}3^{n+1}}{2\pi n} \text{ otherwise.}$$

4.3. Cubics.

THEOREM 14. *The spectral bound for the number of nodes of a projective cubic hypersurface is exact.*

This theorem follows from

PROPOSITION 15. *The spectral bound for cubics is achieved on the following hypersurfaces* $C_n \subset \mathbb{C}P^n$:

$$C_{2m} = \{\sigma_3(x) = 0\} \subset \{(x_0 : \cdots : x_{2m+1}),\ \sigma_1(x) = 0\},$$

$$C_{2m+1} = \{\sigma_3(x) + z\sigma_2(x) + \tfrac{m(m+1)(m+2)}{12} z^3 = 0\} \subset \{(x_0 : \cdots : x_{2m+1} : z),\ \sigma_1(x) = 0\},$$

where $(-1)^i\sigma_i(x)$ *is the ith elementary symmetric function in* $x_0, \dots, x_{2m+1}$.

REMARK. C_4 is the Segre cubic [12].

PROOF. C_{2m}. This is an A_{2m+1}-invariant hypersurface. According to Lemma 9, the corresponding hypersurface $W = \{\sigma_3 = 0\}$ in $\mathbb{C}^{2m+1}$ may be nontransversal only to (i.e. contain) one-dimensional strata of the discriminant of A_{2m+1}. A parametrization of the closure of such a stratum $A_{a-1}A_{b-1}$, $a + b = 2m + 2$, is given by

$$p_\sigma(x) = (x - bt)^a(x + at)^b.$$

We get $\sigma_3 = 0$ here iff $a = b$. This implies that C_{2m} has singularities only along the A_{2m+1}-orbit of the point

$$(\underbrace{1 : \cdots : 1}_{m+1 \text{ times}} : \underbrace{-1 : \cdots : -1}_{m+1 \text{ times}}).$$

The length of the projective orbit is $\frac{1}{2}\binom{2m+2}{m+1} = Ar_{2m}(3)$.

As in subsection 3.2, it follows that, at any point of the stratum A_mA_m, the hypersurface W is transversal to the limit tangent space of any higher-dimensional stratum, and [1] implies that all the above singular points are nondegenerate.

C_{2m+1}. This is an A_{2m+1}-invariant hypersurface. All the cubics with isolated singularities having this type of symmetry are contained in the 2-parameter family of cubics

$$V_{\alpha,\beta} = \{\sigma_3(x) + \alpha z \sigma_2(x) + \beta z^3 = 0\} \subset \{(x_0 : \cdots : x_{2m+1} : z),\ \sigma_1(x) = 0\}.$$

Let us determine which of these cubics have singularities.

The discriminant in the quotient space

$$\mathbb{C}^{2m+2}/A_{2m+1} = \{(\sigma_2(x), \ldots, \sigma_{2m+2}(x), z)\}$$

is a product of the discriminant of A_{2m+1} in its standard representation and the z-coordinate line. By Lemma 9 all the hypersurfaces $W_{\alpha,\beta}$ in the quotient space corresponding to the cubics of the family are transversal to all the strata of dimension at least 3. Thus we have to look for exceptional hypersurfaces nontransversal to the lower-dimensional strata. We denote the product strata of the cylindrical discriminant in the same way as the corresponding strata of the standard discriminant.

Nontransversality to the A_{2m+1}-stratum $\{\sigma = 0\}$ means that the stratum must be contained in $W_{\alpha,\beta}$. This happens for $\beta = 0$.

The parametrization of the closure of a 2-dimensional stratum $A_{a-1}A_{b-1}$, $a + b = 2m + 2$, comes from the one mentioned in the consideration of the C_{even}-case. The trace of $W_{\alpha,\beta}$ in the (t, z)-plane of the parameters on the stratum is the curve

$$\frac{1}{3}ab(a+b)(b-a)t^3 - \frac{1}{2}ab(a+b)\alpha z t^2 + \beta z^3 = 0.$$

$H_{\alpha,\beta}$ is nontransversal to the stratum iff this curve has multiple components different from $t = 0$. The latter takes place iff

$$6(a-b)^2\beta = ab(a+b)\alpha^3.$$

Rescaling z, we can assume $\alpha = 1$, if $\alpha \neq 0$, and $\beta = 1$, if $\alpha = 0 \neq \beta$. Then the nontransversality data obtained gives

Proposition 16. *The following is the complete list of singularities in the family $\{V_{\alpha,\beta}\}$ of the cubics:*

(1) $V_{0,0}$ *has nonisolated singularities;*
(2) $V_{0,1}$ *has isolated non-Morse critical points;*
(3) $V_{1,0}$ *has a node at the point* $(0 : \cdots : 0 : 1)$;
(4) $V_{1,\beta}$ *has nodes along the* A_{2m+1}*-orbit of the point*

$$(\underbrace{b : \cdots : b}_{a \text{ times}} : \underbrace{-a : \cdots : -a}_{b \text{ times}} : b - a),$$

where for any pair (a, b) *of natural numbers,* $a > b$, $a+b = 2m+2$, *we set*

$$\beta = \frac{ab(a+b)}{6(a-b)^2}.$$

The length of the orbit in the latter case is $\binom{2m+2}{a}$. It is maximal, equal to $Ar_{2m+1}(3)$, for $a = m+2$, i.e. for $\beta = m(m+1)(m+2)/12$.

REMARK. Kalker's odd-dimensional cubics [10] are given by the same equations as ours. His even-dimensional ones are

$$\sigma_1(u) = \sigma_3(u) = u_1 - u_2 = 0 \text{ in } (u_1 : \cdots : u_{2m+4})\text{-space.}$$

The nodes are at the point

$$c = (\underbrace{1 : \cdots : 1}_{m+2 \text{ times}} : \underbrace{-1 : \cdots : -1}_{m+2 \text{ times}})$$

and all the points obtained by permutations of the last $2m+2$ coordinates of c. It is easy to show that Kalker's even-dimensional cubic is isomorphic to our C_{2m+1}.

REFERENCES

1. V. I. Arnold, *Wave front evolution and equivariant Morse lemma,* Commun. Pure Appl. Math. **29** (1976), no. 6, 557–582.
2. ———, *Index of a singular point of a vector field, the Petrovskii-Oleinik inequality and mixed Hodge structures,* Functional Anal. Appl. **12** (1978), no. 1, 1–12.
3. V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable mappings,* vol. 2, Birkhäuser, Boston, 1988.
4. A. Beauville, *Sur le nombre maximum de points doubles d'une surface dans* P^3 $(\mu(5) = 31)$, J. Geométrie Algébrique d'Angers, Algebraic Geometry (Angers 1979), Sijthoff and Noordhoff (1980), 207–215.
5. J. W. Bruce, *An upper bound for the number of singularities on a projective hypersurface,* Bull. London Math. Soc. **13** (1981), 47–50.
6. H. Burkhardt, *Untersuchungen aus dem Gebiete der hiperelliptichen Modulfunctionen, Zweiter Theil,* Math. Ann **38** (1891), 161–224.
7. S. V. Chmutov, *Examples of projective surfaces with many singularities,* J. Alg. Geom. **1** (1992), 191–196.
8. A. B. Givental, *Maximum number of singular points on a projective hypersurface,* Functional Anal. Appl. **17** (1983), no. 3, 223–225.
9. V. V. Goryunov, *Semi-simplicial resolutions and homology of images and discriminants of mappings,* Proc. London Math. Soc. (to appear).
10. T. Kalker, *Cubic fourfolds with fifteen ordinary double points,* Thesis, Leiden Univ., 1986.
11. Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants,* Math. Ann. **268** (1984), 159–171.
12. C. Segre, *Sulla varieta cubica con dieci punti doppi dello spazio a quattro dimensioni,* Atti Acc. Torino **XXII** (1887), 791–801.
13. D. van Straten, *A quintic hypersurface in* $\mathbb{P}^4$ *with* 130 *nodes,* Topology **32** (1993), 857–864.
14. ———, *The Jacobian module of nodal hypersurfaces,* in preparation.
15. E. G. Togliatti, *Sulle superficie algebriche col massimo numero di punti doppi,* Rend. Sem. Mat. Torino **9** (1950), 47–59.
16. A. N. Varchenko, *On semicontinuity of the spectrum and an upper estimate for the number of singular points of a projective hypersurface,* Soviet Math. Dokl. **27** (1983), 735–739.

17. ———, *Asymptotics of integrals and Hodge structures*, J. Sov. Math. **27** (1984), 2759–2785.

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