To V. I. Arnold on the occasion
of his 60th birthday

Polynomial invariants of Legendrian links and plane fronts

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Abstract

We show that the framed versions of the Kauffman and HOMFLY polynomials of a Legendrian link in the standard contact 3-space and solid torus are genuine polynomials in the framing variable. This proves a series of conjectures of [5] and provides estimates on the Bennequin-Tabachnikov numbers of such links.

In a series of recent papers [1, 2, 3], Arnold revived attraction of the study of plane curve invariants. The topic whose name sounds extremely elementary is highly complicated. For example, it contains the entire knot theory as its proper subset.

Many definitions and problems of knot theory can be reformulated in terms of plane curves. The language of generic plane fronts, that is cooriented curves with transverse double points and cusps, provides a convenient tool to study invariants of Legendrian links in the standard contact solid torus $ST^*\mathbf{R}^2$ and 3-space. In terms of fronts, one can define restrictions of classical framed link invariants to the subset of all Legendrian links. Considering, say, polynomial invariants in this context it is rather easy to write out the corresponding skein relations for fronts. The obtained relations often clearly indicate that there are rather obvious bounds on the values of some basic knot

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invariants, like the writhe, restricted to Legendrian curves. Unfortunately, it is not so easy to show that the obtained system of axioms defines the polynomials inambiguously: the ordinary cross-changings in link diagrams and other moves are not Legendrian.

The programme of evaluation of the polynomial link invariants on Legendrian knots and links was formulated in a series of conjectures in [5]. Partly, for the HOMFLY polynomials of Legendrian links whose fronts are regular curves, it has been carried out in [6].

In the present paper we give up the condition of regularity of a front and work with arbitrary Legendrian knots and links in the standard contact solid torus and $\mathbf{R}^3$. We prove a series of conjectures of [5] on certain non-Laurent polynomiality of the framed versions of the Kauffman and HOMFLY polynomials of canonically framed Legendrain links. This implies a series of estimates on the Bennequin-Tabachnikov numbers of Legendrian knots. The one coming from the HOMFLY polynomial for $\mathbf{R}^3$ was recently obtained by Fuchs and Tabachnikov [8] by comparison of the results of [4] and [14, 7].

The Bennequin-Tabachnikov number bounds we obtain for the solid torus are stronger than that induced by an inclusion of the solid torus into the standard contact 3-sphere. For example, for a knot topologically equivalent to the ascending one running around the solid torus $i > 0$ times, our estimates immediately give the exact bound of $2i - 1$.

The paper is organised as follows. In Section 1 we recall the definitions we are using. Section 2 contains the statements of our main results on the Kauffman polynomials and the corresponding bounds for the Bennequin-Tabachnikov numbers. In Sections 3 and 4 we prove that our system of axioms for the Legendrian Kauffman polynomial is complete. Section 5 treats another realisation of the standard contact 3-space as the 1-jet bundle of functions on a line. Section 6 is devoted to the HOMFLY polynomial and corresponding estimates. In Section 7 we conjecture on existence of certain modifications of the classical link polynomials in Legendrian case.

Very often we treat the solid torus $ST^*\mathbf{R}^2$ and its universal covering 3-space in parallel: the solid torus considerations are much more natural when one studies plane fronts, while the $\mathbf{R}^3$ case is more reasonable from the point of view of contact geometry. However, the main statement is proved only for the Kauffman polynomial of links in the solid torus: all the other cases are absolutely analogous. The proof uses certain ideas of Hoste's proof [10] of well-definedness of the HOMFLY polynomial. The exposition is self-contained (modulo the existence of the classical knot polynomial invariants).

1 Legendrian links and their fronts

1.1 Standard contact spaces

We recall a few basic notions.

A contact element at a point of a plane is a line in the tangent plane. Its *coorientation* is a choice of one of two half-planes into which it divides the tangent plane. The manifold M of all cooriented contact elements of the plane is the spherisation $ST^*\mathbf{R}^2$ of the cotangent bundle of the plane. It is diffeomorphic to the solid torus $\mathbf{R}^2 \times S^1$: the coorienting normal vector of a contact element is defined by the angle $\varphi \bmod 2\pi$ which it makes with a fixed direction on the plane. Manifold M has the standard contact structure defined as zeros of the 1-form $\alpha = (\cos\varphi)dx + (\sin\varphi)dy$, where (x, y) are coordinates on $\mathbf{R}^2$ with the positive direction of the x-axis being that fixed above.

We equip M with the orientation $dx \wedge dy \wedge d\varphi = -\alpha \wedge d\alpha$. It is opposite to the orientation usually taken in the contact geometry.

Along with the solid torus M we will also be considering its universal cover $\widetilde{M} \simeq \mathbf{R}^3$, with the orientation induced from that of M. Its standard contact form is given by the same formula as α with the only difference that now the angular coordinate φ is not reduced $\bmod 2\pi$.

1.2 Fronts

Definition 1.1 A *Legendrian curve* in a contact 3-manifold is a mapping of a disjoint union of a finite number of circles for which the pull-back of the contact form vanishes. A *Legendrian link* is an embedded Legendrian curve.

The image of the canonical projection of a Legendrian link L from M or $\widetilde{M}$ to the plane is called *the front of* L. An arbitrary small perturbation in the class of Legendrian links is able to bring a link in general position with respect to the canonical projection. The front of such a generic Legendrian link has only transverse double points and semi-cubical cusps as its singularities.

At any point of a front there is a natural coorientions by the coorienting normal of the contact element $a \in L$ whose projection this point is.

A cooriented multi-component plane curve is a front of a unique Legendrian curve in M. So such a curve will be called *a front*, with no reference to the corresponding Legendrian curve.

A necessary and sufficient condition for an above plane curve to be the front of a Legendrian curve in $\widetilde{M}$ is vanishing of the winding numbers of all of its components. The winding number is the number of full rotations made by the coorienting normal as we trace the component once. The Legendrian link in $\widetilde{M}$ is well-defined by its front only if there is chosen a point on each component of the front and there is an indication to which φ-level of $\mathbf{Z}$-many possible ones this point should be rised. We call such fronts *marked winding-free*.

A generic homotopy in the class of Legendrian *immersions* produces generic perestroikas of the front (see Fig.1; the non-immersive case is briefly discussed in Section 7). Only *dangerous self-tangencies* of fronts, when the coorientations of the two tangent branches coincide, correspond to topological changes in the links. A Legendrian link in the solid torus $ST^*\mathbf{R}^2$ gets the double point and experiences the change-crossing at each of these instants (Fig.2).

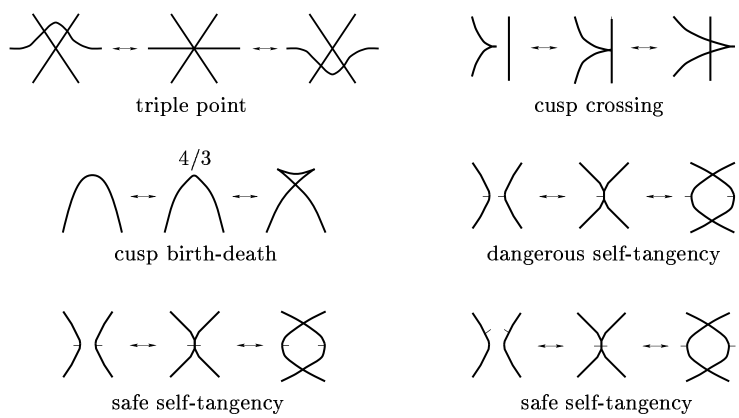


Figure 1: *Perestroikas of generic fronts.*

Definition 1.2 An invariant of fronts is called a *J^+-type invariant* if it does not change under homotopies which involve no dangerous self-tangencies.

Our terminology follows the name of the first invariant of this type introduced by Arnold in [1].

From the above discussion, we see that the theory of invariants of Legendrian links in $ST^*\mathbf{R}^2$ is isomorphic to that of J^+-type invariants of fronts. So in what follows we will make no distinction between an invariant of Legendrian links in the standard contact solid torus and its lowering to the J^+-type invariant of fronts.

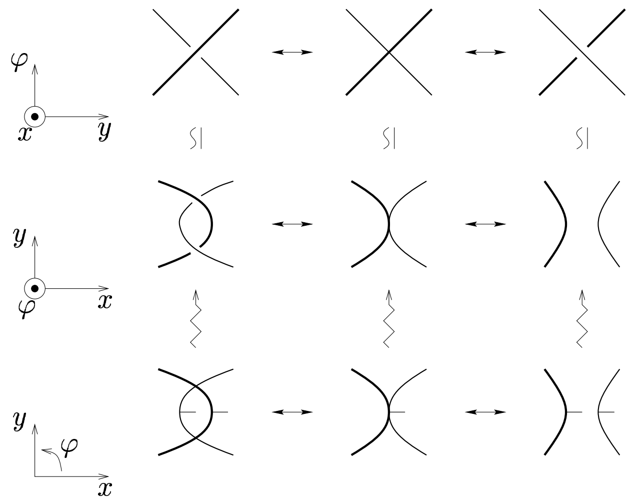
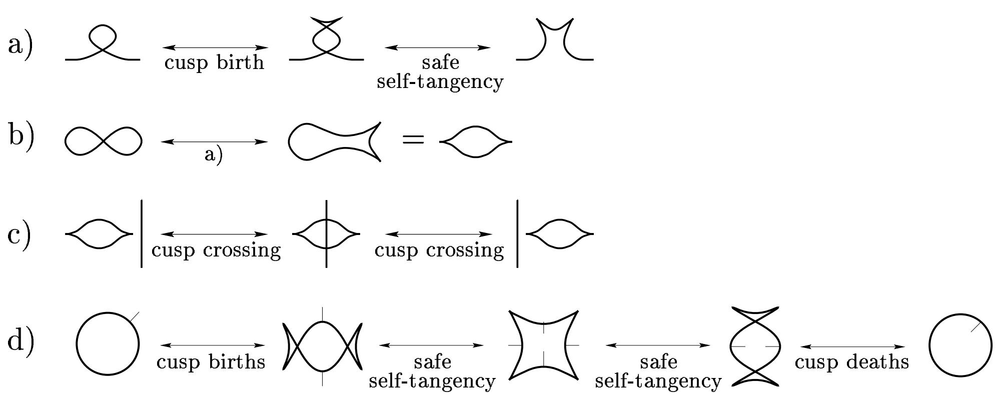


Figure 2: *A dangerous self-tangency of a front rises to the change-crossing of the Legendrian link.*

Example 1.3 The following local homotopies with no dangerous self-tangencies will be very useful for our further considerations:



Definition 1.4 An invariant of marked winding-free fronts is called a J_0^+-*type invariant* if it does not change under homotopies which involve no direct self-tangencies with coinciding phases $\varphi \in \mathbf{R}$ of the branches at the points of tangency.

The theory of such invariants coincides with the theory of invariants of Legendrian knots in $\widetilde{M} \simeq \mathbf{R}^3$.

1.3 The Bennequin-Tabachnikov number

Any unframed link type in M and $\widetilde{M}$ has a Legendrian representative (see, e.g. [9]). This is not the case for the framed setting.

A Legendrian link in a 3-manifold with a cooriented contact structure has a canonical framing by the coorienting vectors of the contact planes. In the cases of $ST^*\mathbf{R}^2$ and its universal cover, this is isomorphic to the framing by the Legendrian lift of the front of the initial link slightly shifted in the direction of its coorientation (Fig.3).

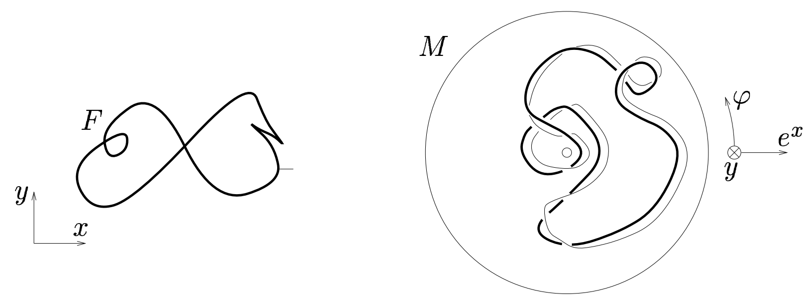


Figure 3: *Legendrian lifting of a plane front to a canonically framed knot in the solid torus $M = ST^*\mathbf{R}^2$.*

Definition 1.5 The writhe β of the canonical framing of a Legendrian knot in $\widetilde{M} \simeq \mathbf{R}^3$ is called the *Bennequin number* of the knot.

The Bennequin number can be calculated as follows. The front, with its double points correctly resolved, is in fact a knot diagram of its Legendrian knot. This assigns "+" or "−" to each of the double points, as if we were calculating the writhe of the blackboard framing. Now β is the sum of all these signs plus half the number of the cusps.

One of the main results of [4] tells that the Bennequin number is bounded from one side on the set of all Legendrian knots representing the same unframed knot type in $\mathbf{R}^3$. For our choice of orientation the numbers are bounded from below.

Example 1.6 a) For an unknot $\beta \geq 1$ [4].

b) The original estimate of [4] provides the same bounds for a knot and its mirror image. For example, for both the right- and left-handed trefoils it gives $\beta \geq -1$. This bound is exact for the left-handed trefoil. In [12, 8] it was shown that for the right-handed trefoil the exact lower bound is 6.

An analog of the Bennequin number for knots in $ST^*\mathbf{R}^2$ was introduced by Tabachnikov in [16]. He set it to be the index of intersection of the knot shifted in the direction of the framing and a 2-film realising homology between the initial knot and a multiple of the fibre of the projection $ST^*\mathbf{R}^2 \to \mathbf{R}^2$ over a sufficiently distant point. According to one of equivalent definitions, this is also the writhe of the canonically framed Legendrian knot with respect to the projection $(x, y, \varphi) \mapsto (e^x, \varphi)$ to the annulus $\mathbf{R}^2 \setminus 0$ with polar coordinates (ρ, φ) (see Fig.3 in which $\beta = 4$).

The mentioned boundedness of the Bennequin numbers implies similar boundedness of the Tabachnikov numbers.

Remark 1.7 By the Bennequin and Tabachnikov numbers of oriented Legendrian links we will mean the corresponding writhes of the canonically framed links.

2 Kauffman polynomials of fronts

2.1 The polynomial of Legendrian links in the standard solid torus

In [19] Turaev introduced the Kauffman polynomial of a framed non-oriented link in a solid torus. This is an element of $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \xi_1, \xi_2, \ldots]$ uniquely defined by the relations and initial data of Fig.4. The links L_1 and L_2 there are mutually unlinked. All the links are equipped with the framing which is blackboard with respect to a fixed projection of the solid torus to the annulus. The knot Ξ_3 is a pattern for the whole series Ξ_i.

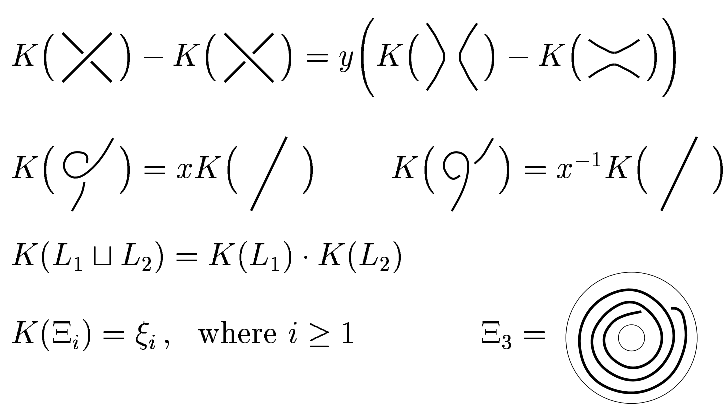


Figure 4: *Definition of the framed version of the Kauffman polynomial for links with the blackboard framing in a solid torus.*

Example 2.1 On an unknot with the trivial framing $K = \frac{x - x^{-1}}{y} + 1$.

The Legendrian lifting lowers the polynomial to generic plane fronts. Translation of the rules of Fig.4 to fronts gives rise to the rules of Fig.5. The fronts F_1 and F_2 of the third line are lying in disjoint half-planes. The relation between the Legendrian generators z_i and the blackboard generators ξ_i is easily seen to be $z_i = x^i \xi_i$ (cf.[6]).

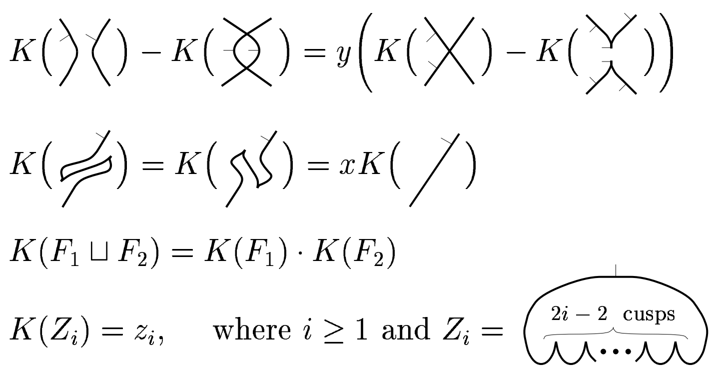


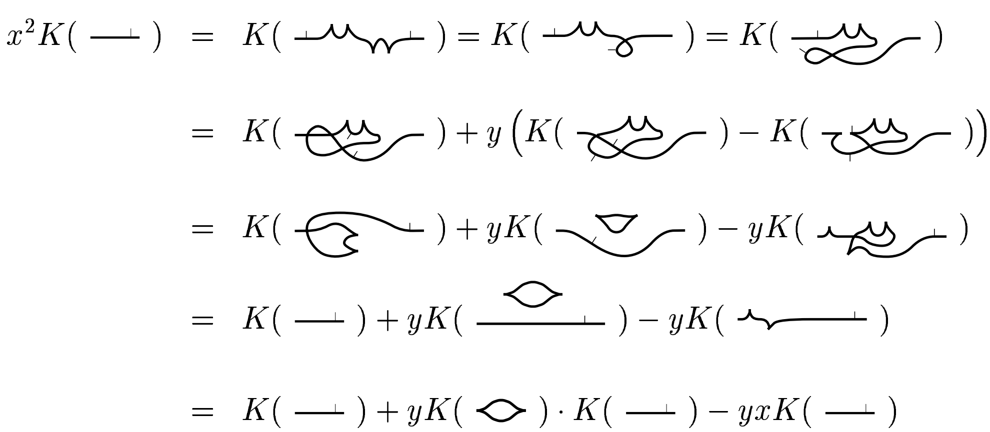
Figure 5: *Definition of the Kauffman polynomial for fronts.*

Theorem 2.2 *There exists a unique J^+-type invariant $K(F) \in \mathbf{Z}[x, y^{\pm 1}, z_1, z_2, \ldots]$ of a generic front F satisfying the relations and initial data of Fig.5.*

Note that there are no negative powers of the framing variable x now.

The existence of a polynomial invariant satisfying the rules of Fig.5 is guaranteed by [19]. Basicly we only need to show that our system of axioms defines the polynomial inambiguously. This will be done in Section 4. The non-Laurent polynomiality in x will follow from the reduction procedure we will be using in the proof: the zigzag relations will be applied only to reduce the number of cusps.

Example 2.3 The lips front, with two cusps and no double points, is the simplest possible. To calculate its Kauffman polynomial one can proceed as follows (making use of Example 1.3):



So, $K($  $) = \frac{x^2-1}{y} + x$. Indeed, the lips front lifts to the Legendrian unknot in M with $\beta = 1$, so its Kauffman polynomial should be that of an unknot with the trivial framing times x.

In what follows, we will denote the lips front by Z_0.

2.2 The polynomial of Legendrain links in the standard $\mathbf{R}^3$

The Kauffman polynomial $K_0 \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ of framed links in $\mathbf{R}^3$ is defined by the rules of Fig.4 with all the information about the curves Ξ_i omitted [13]. Its Legendrian version in terms of fronts is respectively given by Fig.5 without mentioning the fronts Z_i. Only generic marked winding-free fronts are now considered. The phases of the two interacting branches in the main skein relation must coincide. We call this modification of Fig.5 its J_0^+*-version.*

Theorem 2.4 *There exists a unique* J_0^+*-type invariant* $K_0(F_0) \in \mathbf{Z}[x, y^{\pm 1}]$ *of a generic marked winding-free plane front* F_0 *satisfying the relations and initial data of the* J_0^+*-version of Fig.5.*

Since the proof of this Theorem simply repeats that of Theorem 2.2, we do not give it in the paper.

2.3 The Bennequin-Tabachnikov number estimates

Due to Theorems 2.2 and 2.4, the Kauffman polynomial of a (marked winding-free) plane front is a genuine polynomial in x, not a Laurent one. This implies the following restriction on the values of the Bennequin-Tabachnikov numbers of oriented Legendrian links in the solid torus and 3-space.

Let $wr(L)$ be the writhe of an oriented link L either in the solid torus or $\mathbf{R}^3$. Define the unframed versions of the Kauffman polynomial as

$$K_u(L) = x^{-wr(L)}K(L) \quad \text{and} \quad K_{0,u}(L) = x^{-wr(L)}K_0(L)\,.$$

Following [19, 13], these polynomials depend only on unframed topological type of L. Thus we can speak about the polynomials K_u and $K_{0,u}$ of unframed oriented links. In the case of knots the orientation does not matter.

Theorem 2.5 *Let $\mathcal{L}$ be an unframed oriented link in the standard contact manifold $\widetilde{M} \simeq \mathbf{R}^3$ or $M = ST^*\mathbf{R}^2$. Let x^k be the minimal power of the framing variable x in the corresponding unframed version of the Kauffman polynomial of $\mathcal{L}$. Then the Bennequin-Tabachnikov number of any Legendrian representative of $\mathcal{L}$ is at least $-k$.*

Example 2.6 For an unknot this coincides with the classical bound $\beta \geq 1$ [4].

Example 2.7 The Theorem implies that the minimal Bennequin-Tabachnikov number of a Legendrian representative of the basic knot Ξ_i in the solid torus (Fig.4) is that of the Legendrian lifting of the front Z_i, which is $2i-1$. Note that inclusion of the standard contact solid torus into the standard contact 3-sphere gives only $\beta \geq 1$ for any i: all the Ξ_i get unknotted in S^3.

Example 2.8 For the left- and right-handed $(2,q)$-torus links in $\widetilde{M} \simeq \mathbf{R}^3$, Theorem 2.5 gives $\beta \geq 2-q$ and $\beta \geq 2q$ respectively, $q \geq 2$. The exactness of these estimates in all these cases follows from the examples of Figs. 6 and 7. The double points of the (marked) winding-free fronts in these Figures are resolved respecting the phases φ of the branches.

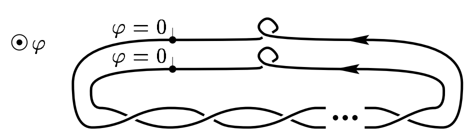


Figure 6: *Legendrian representative of the left-handed $(2,q)$-torus link in $\mathbf{R}^3$ with the minimal possible Bennequin number $2-q$.*

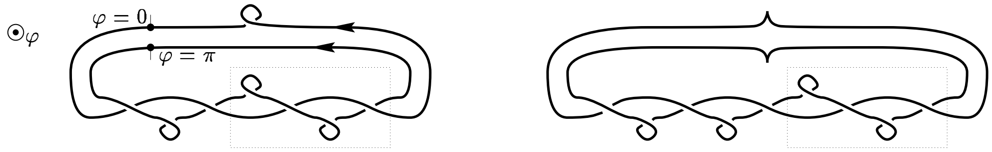


Figure 7: *Legendrian representatives of the right-handed* $(2, 4)$*-torus link and* $(2, 5)$*-torus knot in* $\mathbf{R}^3$ *with the minimal possible Bennequin numbers 8 and 10. Minimal representatives, with* $\beta = 2q$*, of all the other right-handed* $(2, q)$*-torus links are obtained by either omitting the distinguished fragments (for* $q = 2$ *and* $q = 3$*) or by their consecutive repetition (for* $q \geq 6$*).*

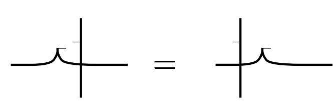
Remark 2.9 A similar to ours estimate of the Bennequin number for knots in $\mathbf{R}^3$ by the lowest degree of the framing variable in the mod 2 Kauffman polynomial was derived in [8] from the results of [17]. It is not known if the lowerest degrees in the integer and mod 2 Kauffman polynomials for $\mathbf{R}^3$ may differ. In all the examples we know they coincide. The work [18] implies that for alternating knots they are equal. See also [15].

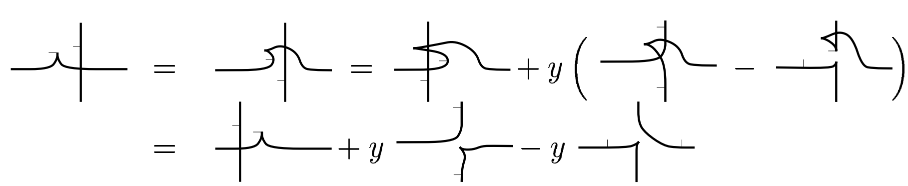
3 Supplementary moves

In this section we collect a series of facts which will be used as blocks in the proof of Theorem 2.2. In all the formulas of this section the relations on the fronts are in fact the relations on their Kauffman polynomials K.

3.1 Pulling a cusp through a line

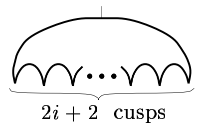
A cusp can be pulled through a line either with no affect on the polynomial or modulo the polynomials of fronts with one double point less:

a) 

b) 

3.2 Another series of basic curves

Consider the curves

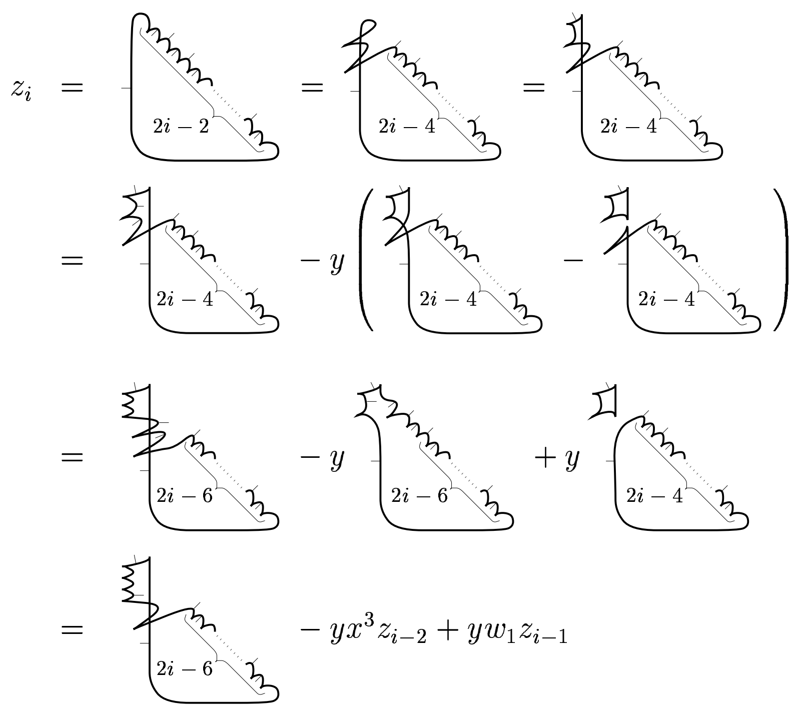
$W_i =$ , $\quad i \geq 1.$

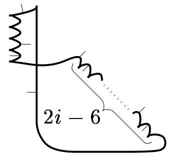
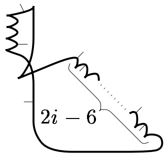
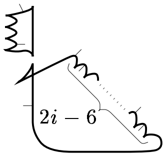
Let w_i be the Kauffman polynomial of W_i.

Lemma 3.1 *The w_i, $i \geq 1$, may be taken as independent variables in the definition of the Kauffman polynomial of fronts instead of the z_i, $i \geq 1$.*

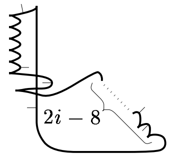
Proof. Example 1.3.d shows that $w_1 = K(W_1) = K(Z_1) = z_1$.

For the other curves, a rather long chain of applications of the elementary rules and Example 1.3.a provides a recursive relation between the z_i and w_i:

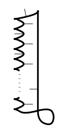


$$= \quad$$  $$- y \Bigg(\quad$$  $$- \quad$$  $$\Bigg)$$

$$-yx^3 z_{i-2} + y w_1 z_{i-1}$$

$$= \quad$$  $$- yx^5 z_{i-4} + y w_2 z_{i-2} - yx^3 z_{i-2} + y w_1 z_{i-1}$$

$$= \quad \ldots$$

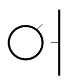
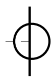
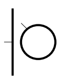
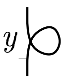
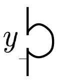
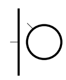
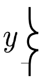
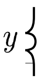
$$= \quad$$  $$- y \sum_{t=1}^{[\frac{i}{2}]} x^{2t-1} w_{i-2t} - y \sum_{s=1}^{[\frac{i-1}{2}]} x^{2s+1} z_{i-2s} - y \sum_{u=1}^{i-1} w_u z_{i-u}$$

$$= \quad w_i - y \sum_{t=1}^{[\frac{i}{2}]} x^{2t-1} w_{i-2t} - y \sum_{s=1}^{[\frac{i-1}{2}]} x^{2s+1} z_{i-2s} - y \sum_{u=1}^{i-1} w_u z_{i-u}$$

3.3 Pulling basic curves through a line

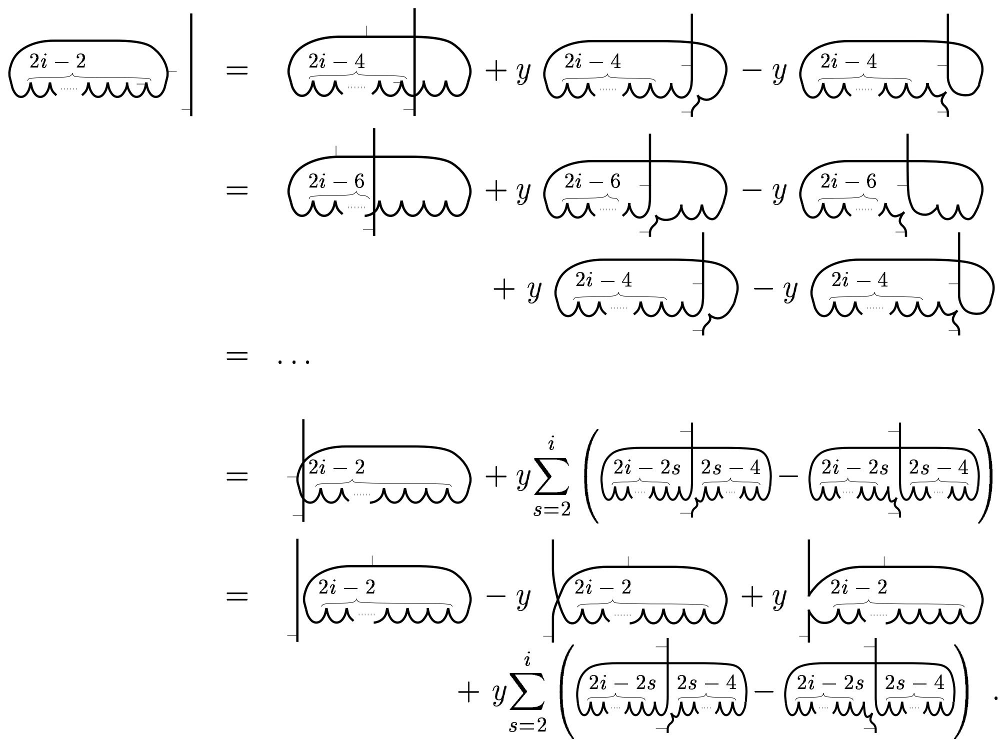
Example 1.3.c shows that the lips front Z_0 passes through a line with no change in the Kauffman polynomial.

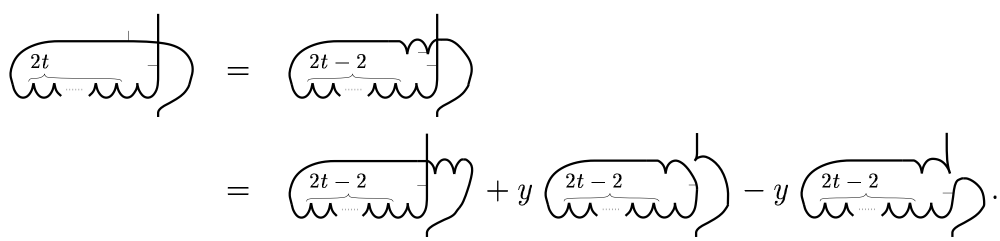
The similar move for the circle Z_1 is not trivial:

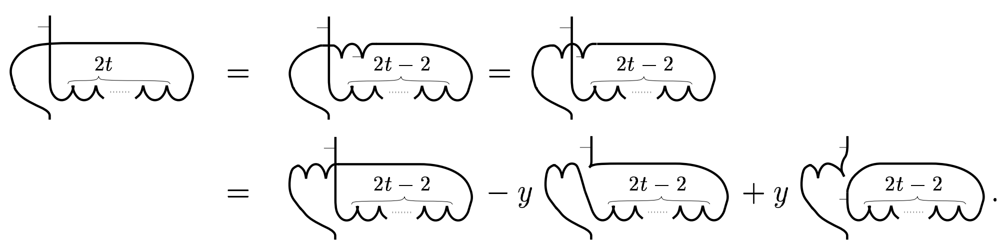
 $=$  $=$  $-y$  $+y$  $=$  $-y$  $+y$  .

Due to Example 1.3.d, the coorientation of the circle does not matter.

The polynomial of a front containing a generic curve Z_i to one side of a line can be similarly expressed in terms of the polynomials of fronts containing various curves $Z_{\leq i}$ to the other side of the line. This is an inductive implication of the following relations which are based on the observations of Section 3.1:


2i − 2
2i − 4
2i − 6
2i − 2s
2s − 4
s=2
i


2t
2t − 2


2t
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4 Proof of Theorem 2.2

We prove that the system of axioms of Theorem 2.2 uniquely defines the polynomial. We do this by induction on the number of double points of a front. Our method is rather similar to that of [6].

All the fronts we are considering are generic. We say that *a front reduces to a number of some others* if the calculations of the Kauffman polynomial of this front can be reduced to calculations for these others.

4.1 Fronts with no double points

Due to the zigzag and multiplicativity rules of Fig.5, we assume we are dealing with a nest of curves Z_i, $i = 0, 1, 2, \ldots$, and W_j, $j = 1, 2, \ldots$. According to Section 3.2, we may assume all the innermost curves to be the Z_i. Now Section 3.3 reduces our nest to the nests of depth one lower. Iterating the procedure we end up with collections of curves Z_i bounding disjoint regions. This provides the base for our induction.

4.2 Reduction of the number of double points

4.2.1 Easy cases

There are several situations in which it is very easy to make the induction step and reduce a front to fronts with fewer double points.

Definition 4.1 A *simple curl* of a front is a cusp-free α-shaped loop that contains no fragments of the collection in its interior. A *semi-simple curl* is a similar α-shaped loop with a number of cusps on it.

Lemma 4.2 *A front with a simple curl reduces to a front with one double point less.*

Proof. This is Example 1.3.a.

Lemma 4.3 *A front with a semi-simple curl reduces to fronts with fewer double points.*

Proof. The observations of Section 3.1 reduce the situation to that of the previous lemma.

Definition 4.4 A closed disc D is called an *n-gon* of a front F if its boundary ∂D is contained in F and has exactly n vertices, that is double points of F where ∂D fails to be differentiable.

Note that cusps are not considered as vertices of an n-gon.

Adjective *semi-simple* (*simple*) will be applied to an n-gon whose interior contains no other fragments of the front (and whose boundary has no cusps). For example, a simple curl bounds a simple 1-gon.

A simple 2-gon is one more configuration for which the reduction of the number of double points is obvious. The moves of Section 3.1 allow similar reduction for a semi-simple 2-gon.

Our reduction tactics in more complicated situations is either to find or create a simple 2-gon within certain minimal configurations.

4.2.2 Minimal 0- or 1-gons

Definition 4.5 A 0- or 1-gon D of a front F is called *minimal* if there are neither 0- nor 1-gons of F inside D.

Intersection of the interior of D with F may be non-empty.

A 0-gon will be called *isolated* if it is lying apart from the rest of the front.

Search for a minimal 0- or 1-gon. To find a minimal 0- or 1-gon we can proceed as follows.

Start at any generic point of F and walk along F until the first second-time visit to some point. Take the closed path we have traced. It bounds a closed 0- or 1-gon D. Let us try to reduce it.

Start a similar walk as before from a point of $F \cap D$, but with a restriction not to leave D. If, during this trip, we are able to make a closed loop different from ∂D it will provide us with a 0- or 1-gon contained in D. Take it for a new, reduced disc.

If we are not able to make a closed loop, do similar try starting at another topologically different point of $F \cap D$.

In a finite number of steps we will not be able to make any further reduction.

It is easy to see that if we end up with a minimal 1-gon it must be α-shaped, not heart-shaped. Of course, cusps on its boundary are allowed.

A minimal 0-gon should be assumed non-isolated. Otherwise, in absense of minimal 1-gons, the entire front is an unnested disjoint union of curves with no double points, and we know how to calculate the polynomial of such a configuration.

Now we are going to reduce our curve within a minimal 0- or 1-gon. Since we have already shown that it is possible for a semi-simple 1-gon, we are left with three options to consider:

a) D is a non-isolated semi-simple 0-gon;

b) D is a non-semi-simple 1-gon;

c) D is a non-semi-simple 0-gon.

We treat these cases one after another.

A non-isolated minimal 0-gon is semi-simple. As earlier, we may assume its boundary to be a basic curve Z_i, $i = 0, 1, 2, \ldots$. Following Section 3.3, we pull this curve in a fixed direction through the branches away from the rest of the front. Using the same direction, we do the same with all the cloud of additional basic curves created by our pulling-away.

From this moment on we assume no minimal semi-simple 0-gons in our front.

A minimal 1-gon D is not semi-simple. According to Section 3.1, we may assume the disc D, along with its boundary, to be cusp-free. Due to the minimality, any branch of the front visiting D has no self-intersection. All the options are covered by the following three cases.

a) There are no double points of the front F inside D. This means that we are able to kill a simple 2-gon adjacent to the boundary ∂D of D either with no change of the polynomial (if the killing move is a safe self-tangency) or applying the main skein relation (if the killing self-tangency is dangerous).

b) Each pair of branches of F visiting D has at most one point of intersection. We assume there are no simple 2-gons adjacent to ∂D.

Lemma 4.6 *The disc D contains a simple 3-gon Δ with exactly one of its sides on ∂D.*

Pushing the inner vertex of Δ through ∂D by the triple-point move we reduce the number of double points of F inside D. Iteration of the process finally reduces the situation to that of a).

Proof of the Lemma. Let B_1 be a branch of $F \cap D$ that intersects some other branches inside D. We may assume the double point of ∂D and all the branches inside D which do not intersect B_1 are on the same side of B_1. This is a sort of a minimality condition on B_1.

Let $P \in B_1$ be the double point closest to an endpoint N of B_1 (see Fig.8). Let B_2 be the other branch passing through P. One of its endpoints, Q, is a vertex of a 3-gon NPQ based on ∂D. This 3-gon may be non-simple: there can be some other double points of F on the side PQ (due to the minimality of B_1 this is the only possible obstruction to the simplicity of NPQ). Choose the one, R, closest to Q. Consider the branch B_3 through R. It cuts the corner piece QRS of NPQ. This is guaranteed by the fact that neither pair of the branches has more than one point of intersection.

Now, if QRS is still non-simple, we iterate the descending procedure.

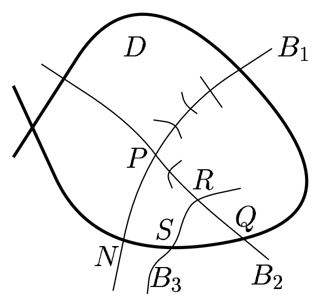


Figure 8: *Search for a minimal 3-gon.*

c) There are two branches of $F \cap D$ having at least two points of intersection inside D.

We again assume there is no simple 2-gon adjacent to ∂D.

Let B^1 and B^2 be branches with more than one common point. There exists a 2-gon $T \subset D$ whose boundary is lying on these branches and whose vertices are two successive intersections of B^1 and B^2. We may assume the following minimality properties of T:

1) endpoints of any branch of $F \cap T$ are on the different sides of T;

2) any pair of branches of $F \cap T$ has at most one common point.

If there are any double points of F inside T, we remove them out using the triple-point moves as in *b*). After this we remove all the branches of F out from T by the triple-point moves across the vertices of T. Now the 2-gon T is simple and we apply a self-tangency move to kill it.

A minimal 0-gon D is not semi-simple. Following Section 3.1, we collect all the cusps of ∂D in a small neighbourhood of its arbitrarily distinguished generic point. This reduces the case to the previous one with this distinguished point playing the role of the only vertex of the 1-gon.

This completes the proof of Theorem 2.2.

5 Another approach to Legendrian links in $\mathbf{R}^3$

The standard contact 3-space can also be treated as the space $J^1(\mathbf{R},\mathbf{R})$ of 1-jets of functions on a line. The contact form α is then $dy - p\,dx$, where y corresponds to values of a function, x to its argument, and p to its derivative. We again orient $\mathbf{R}^3$ with the form $-\alpha \wedge d\alpha = dx \wedge dy \wedge dp$.

Now Legendrian links are represented by their projections to the (x,y)-plane. A generic front in this plane is a curve collection whose only singularities are cusps and transverse double points and which has no tangents parallel to the y-axis. We call such a collection *a front with no vertical tangents*. In order to restore the Legendrian link in $J^1(\mathbf{R},\mathbf{R})$ from a generic front one resolves each double point putting the branch with the greater slope $\partial y/\partial x$ to the higher p-level (Fig.9). The canonical Legendrian framing now is that by the positive y-direction.

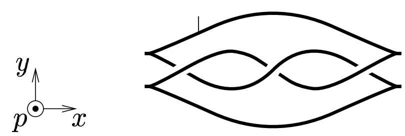


Figure 9: *Lifting of a front with no vertical tangents to the left-handed trefoil in* $J^1(\mathbf{R},\mathbf{R})$.

Generic homotopies in the class of Legendrian immersions in $J^1(\mathbf{R},\mathbf{R})$ provide the same list of generic perestroikas of fronts in the (x,y)-plane as earlier (see Fig.1), except for safe self-tangencies. Of course, no vertical tangents are allowed in any of these perestroikas.

Definition 5.1 An invariant of generic fronts with no vertical tangents is called a *J^+_{jet}-type invariant* if it does not change under homotopies which involve no self-tangencies.

The theory of J^+_{jet}-type invariants is that of invariants of Legendrian knots in $J^1(\mathbf{R},\mathbf{R})$ (and, thus, is isomorphic to the theory of J^+_0-type invariants).

In terms of fronts with no vertical tangents, the rules of the Kauffman polynomial in $\mathbf{R}^3$ are those of Fig.5 with the curves Z_i omitted and all the front fragments rotated by 90 degrees clockwise to avoid vertical tangents. Such a modification of Fig.5 will be called its *J^+_{jet}-version.*

Theorem 5.2 *There exists a unique J^+_{jet}-type invariant $K(F_{jet}) \in \mathbf{Z}[x, y^{\pm 1}]$ of a generic front F_{jet} with no vertical tangents satisfying the relations and initial data of the J^+_{jet}-version of Fig.5.*

The proof is very similar to that of Theorem 2.2, and there is no reason to give it here. The major difference is that now, instead of pulling all cusps away from the minimal domains, one should reduce the number of cusps by applying the zigzag skeins. This reduces the considerations to a certain number of elementary situations each of which can be easily treated.

For example, consider a semi-simple curl of a front with no vertical tangents. It is easy to see that, if there are at least 3 cusps of the curl, there must be a zigzag on it (see [8]). Take a minimal zigzag, that is a one for which the strip between the vertical lines through its cusps contains no other zigzags of the curl. By an obvious homotopy of the front, we can make such a zigzag arbitrarily small comparing with the other elements of the front, and eliminate the zigzag.

Of course, the concept of simplicity of the fragments should also be changed so that the simple fragments now would be those with minimal possible numbers of cusps. Among curls these are swallowtails and one-cusp curles. There is a similar, and even more particular, splitting of the other minimal configurations.

6 HOMFLY polynomials

6.1 The Legendrian versions

Fig.10 recalls the definition of the framed version of the HOMFLY polynomial of oriented links in the solid torus [19].

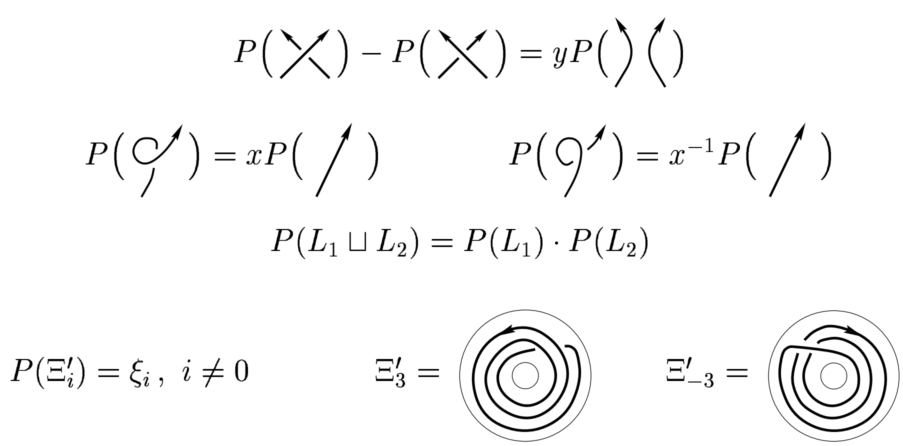


Figure 10: *The definition of the HOMFLY polynomial for oriented links with the blackboard framing in the solid torus.*

Omitting in Fig.10 all the information about the knots Ξ'_i and corresponding variables one gets the definition of the HOMFLY polynomial for knots in the 3-space [11, 13].

Fig.11 translates the rules of Fig.10 to fronts. Relations of its first three lines are also valid for the fragments with all the orientations reversed; $F_1 \sqcup F_2$ is the disjoint union of the two fronts on different sides of a certain straight line.

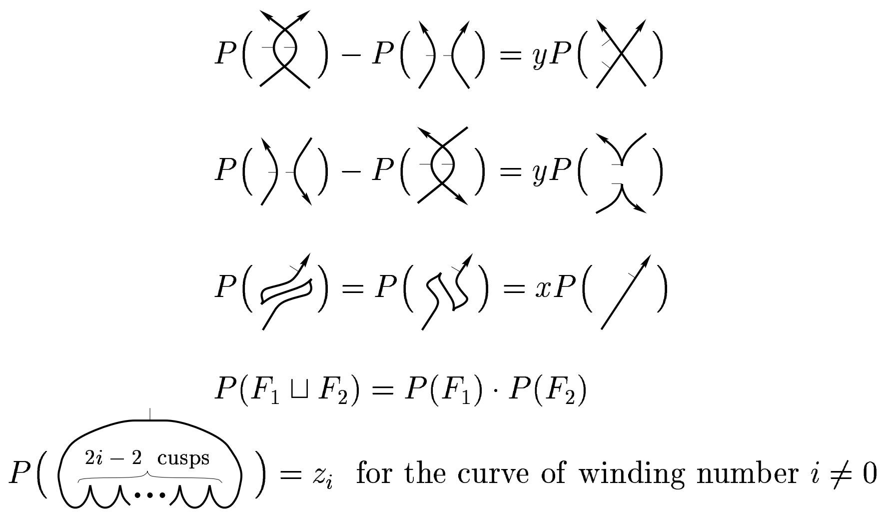


Figure 11: *The definition of the HOMFLY polynomial for oriented plane fronts.*

The definitions of the J_0^+- and J_{jet}^+-versions of Fig.11 are obvious (cf. Section 2.2 and Section 5).

As earlier, one proves (cf. [6])

Theorem 6.1 *There exist*

1) *a unique J^+-type invariant $P(F) \in \mathbf{Z}[x, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$ of a generic plane front F;*

2) *a unique J_0^+-type invariant $P_0(F_0) \in \mathbf{Z}[x, y^{\pm 1}]$ of a generic marked winding-free plane front F_0;*

3) *a unique J_{jet}^+-type invariant $P_0(F_{jet}) \in \mathbf{Z}[x, y^{\pm 1}]$ of a generic plane front F_{jet} with no vertical tangents*

satisfying the relations and initial data of Fig.11, and of its J_0^+- and J_{jet}^+-versions respectively.

6.2 Maslov index

It is easy to strengthen the last theorem and establish divisibility of the HOMFLY polynomials by certain powers of the framing variable.

Consider an oriented and cooriented plane front. A cusp of such a front is called *positive* if the velocity vectors of its outgoing branch have positive projections to the normal of the coorientation at the cusp point. Otherwise the cusp is called *negative*.

Definition 6.2 Half a difference $\mu = \frac{1}{2}(\mu_+ - \mu_-)$ between the numbers of positive and negative cusps is called *the Maslov index* of the front or of the corresponding Legendrian link.

The Maslov index is easily seen to be integer.

All the basic fronts of Fig.11 have zero Maslov index. The Maslov indices of all the three fronts participating in both versions of the main skein relation coincide. The zigzag skeins relate the change of the Maslov index by ± 1 to the divisibility of the polynomial by x. In the chain of calculations of the polynomial of a particular front, the zigzag skeins are used only to reduce the number of cusps. Thus, for each of the three theories of plane fronts under consideration, we get

Corollary 6.3 *In the ring of genuine polynomials in the framing variable, the HOMFLY polynomial of a front is divisible by $x^{|\mu|}$, where μ is the Maslov index of the front.*

6.3 The Bennequin-Tabachnikov number estimates in terms of the HOMFLY polynomial

The unframed analogs of the HOMFLY polynomials are

$$P_u(L) = x^{-wr(L)}P(L) \quad \text{and} \quad P_{0,u}(L) = x^{-wr(L)}P_0(L)\,,$$

where, as in Section 2.3, $wr(L)$ is the writhe of a framed link L either in the solid torus or in $\mathbf{R}^3$.

The non-Laurent polynomiality of the framed versions of the polynomials, in the strengthening of Corollary 6.3, implies (cf. [6])

Theorem 6.4 *Let $\mathcal{L}$ be an oriented unframed link in the standard contact manifold $\widetilde{M} \simeq \mathbf{R}^3$ or $M = ST^*\mathbf{R}^2$. Let x^r be the minimal power of the* *framing variable x in the corresponding unframed version of the HOMFLY polynomial of $\mathcal{L}$. Then, for any Legendrian representative L of $\mathcal{L}$,*

$$\beta + |\mu| \geq -r \, ,$$

where β and μ are the Bennequin-Tabachnikov number and Maslov index of L.

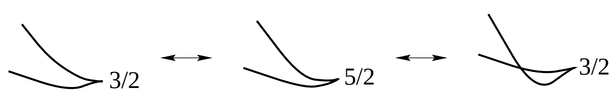
For $\mathbf{R}^3$ this is the theorem of Fuchs-Tabachnikov [8].

Example 6.5 (cf. Example 2.8) With no information on μ, for the left-handed $(2, q)$-torus links in $\mathbf{R}^3$ the estimates of Theorems 2.5 and 6.4 are the same: $\beta \geq 2 - q$. For the right-handed series the estimate of Theorem 2.5 is stronger than that of Theorem 6.4: $\beta \geq 2q$ instead of $\beta \geq 2 + q$.

For a generic non-oriented Legendrian knot, the number $|\mu|$ is well-defined. Nevertheless, one cannot sthrengthen Theorem 2.5 on the Kauffman polynomial estimate to include $|\mu|$ similarly to Theorem 6.4: the $(2, 5)$-torus knot of Fig.7 has $|\mu| = 1$ and its Bennequin number is equal to the negative of the lowest power of the framing variable in the unframed version of the Kauffman polynomial.

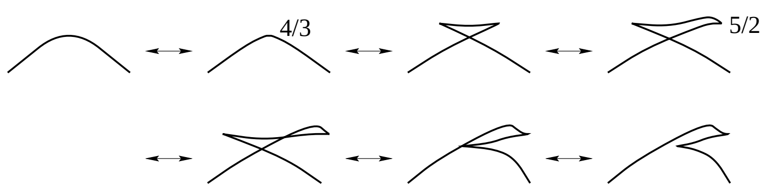
7 Extra variable?

A generic homotopy in the space of all Legendrian curves which are not necessary immersions involves one more front perestrioka in addition to those listed in Section 1.2. This is passing through a ramphoid cusp which is locally a parabola of degree 5/2:



At the bifurcation instant the Legendrian curve is not an immersed one.

The existence of the ramphoid cusp perestroika implies that the zigzag surgery used in all the definitions of all the front polynomials in this paper is in fact a multi-step local homotopy in the space of all fronts:



In this sequence, the change of topology of the Legendrian link takes place only at the ramphoid cusp bifurcation. Here both the Bennequin-Tabachnikov number and Maslov index are changing by ± 1.

Unfortunately, all our definitions respect only the change in the canonical framing and do not respect the change of the Maslov index at all. Apparently there should exist certain refinements of the Kauffman and/or HOMFLY polynomials for Legendrian links which contain at least one more variable responsible for the Maslov index.

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