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Morsifications of Rational Functions

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ABSTRACT. The paper is devoted to enumerative problems of the theory of real singularities. We calculate the number of connected components of the set of rational functions on a real line having as many poles and critical points as possible. In the Appendix, in terms of snakes on Dynkin diagrams, we obtain the numbers of topologically different real morsifications of simple function-germs E_μ with μ distinct critical values.

A series of recent papers [1–3] by Arnold was devoted to the study of a new invariant of a real isolated function singularity, the number of its topologically different *very nice morsifications*. These are morsifications with as many real critical values as possible. Calculations done by Vakulenko [6] show that for an A_μ singularity the value of the invariant is equal either to an Euler number or to a tangent number. The values for the other infinite series, D, of the simple function-germs were obtained in [16] (see also [12, 13]). They turned out to be closely related to similar invariants for Laurent polynomials with a simple pole. In the present note, extending the area of study, we consider the space of rational functions on the real line. We calculate the number of connected components of a set of M-functions, the ones with as many real critical points and poles as possible. This numerical invariant is rougher than the invariant introduced by Arnold. Say a set of M-morsifications of an A_μ singularity, i.e., the space of M-polynomials of fixed degree, is connected. But for rational functions the situation is far from such simplicity.

In the Appendix we calculate the numbers of connected components of the set of very nice morsifications of E_6, E_7, and E_8 function singularities. These numbers are 82, 768, and 4056 respectively. The calculation is based on the bijection between the set of connected components of space of M-morsifications and different R-diagrams of a simple singularity.

Let $\Lambda_{\mu,\nu} = \mathbb{R}^{\mu+\nu}$ be the space of rational functions in one variable

$$r(x) = p(x)/q(x) = (x^\mu + \lambda_1 x^{\mu-1} + \cdots + \lambda_\mu)/(x^\nu + \lambda_{\mu+1}x^{\nu-1} + \cdots + \lambda_{\mu+\nu}).$$

For generic values of the parameters, the number of distinct complex critical points of the function is either $\mu+\nu-1$ if $\mu \neq \nu$, or $2\mu - 2$ if $\mu = \nu$. This is the maximal

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possible number of critical points. In this case all of them are Morse, p and q are relatively prime and all the ν poles are simple.

DEFINITION. A rational function $r(x)$ is called a *rational M-function* if
(1) it has the above maximal number of critical points;
(2) all the critical points are real;
(3) all the poles of r are real too.

We denote the set of all rational M-functions in $\Lambda_{\mu,\nu}$ by $M_{\mu,\nu}$. Our goal is to calculate the number $m_{\mu,\nu}$ of connected components of this set.

EXAMPLES. 1. The set of all M-polynomials of fixed degree is connected, i.e., we have $m_{\mu,0} = 1$ [1].

2. The set of all Laurent M-polynomials with a single pole of order 1 has $m_{\mu,1} = \mu - 1$ components [16]. In this case, shifting the pole to the origin, we get the αth component formed by the functions with α critical points positive and $\mu - \alpha$ critical points negative, $0 < \alpha < \mu$.

For the general case we prove the following

THEOREM. *The number of connected components of the set $M_{\mu,\nu}$ of rational M-functions is equal to*

$$m_{\mu,\nu} = \begin{cases} \dfrac{(\mu-\nu)(\mu+2\nu-1)!}{(\mu+\nu)!\,\nu!}, & \text{if } \mu > \nu, \\ \dfrac{2(\nu-\mu)(\mu+2\nu-1)!}{(2\nu)!\,\mu!}, & \text{if } \mu < \nu, \\ \dfrac{4(3\nu+1)!}{(2\nu+2)!\,\nu!}, & \text{if } \mu = \nu. \end{cases}$$

Our enumeration of the connected components is based on the consideration of the curves $\gamma(r) = \{\operatorname{Im} r(x) = 0\}$ in $\mathbb{C}$ up to orientation-preserving diffeomorphisms of $\mathbb{C} \simeq \mathbb{R}^2$, $\mathbb{R}$. In the simplest cases, for $\nu = 0$ or $\mu = 0$, all the rational M-functions r have equivalent curves $\gamma(r)$. We show them in Figure 1. The nodal points there are the critical points and the crosses are the poles.

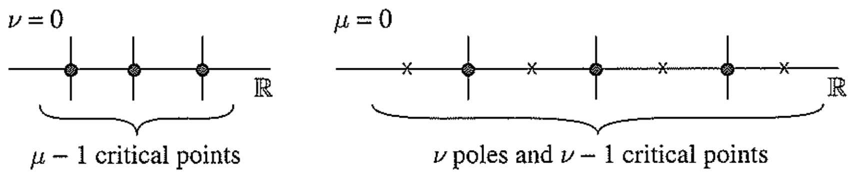
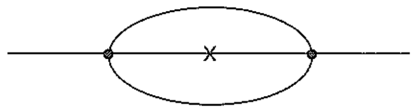


FIGURE 1. Inverse image $\gamma(r)$ of the real axis under a rational M-function r

In the other cases, up to the above diffeomorphisms, the following inductive surgery procedure is valid.

LEMMA 1. *The curve $\gamma(r) = \{\operatorname{Im} r(x) = 0\}$ of a rational M-function $r \in M_{\mu,\nu}$, $\mu \neq \nu$, is obtained from the curve $\gamma(r') = \{\operatorname{Im} r'(x) = 0\}$ of some rational M-function $r' \in M_{\mu-1,\nu-1}$ by replacing one of the $\mu + 2\nu - 3$ intervals into which the poles and the* *critical points of r' cut the real axis, by an elementary block with the following circular component*:



If $(\mu, \nu) \neq (2, 1)$, then among the intervals mentioned in Lemma 1 there are two infinite ones. If $(\mu, \nu) = (2, 1)$, the only interval we have is the whole real axis.

REMARK. The case $\mu = \nu$ is easily seen to give the same curves as $\nu = \mu + 1$.

EXAMPLES. The families represented in Figure 2 (see p. 88) have less than $\mu + \nu$ parameters due to linear transformations of the source and the target. The reduced parameter space is subdivided into several open regions (some of them are connected components of the set $M_{\mu,\nu}$ of rational M-functions) by a *bifurcation diagram*. The diagram is formed by the values of the parameters corresponding to functions having:

degenerate critical points (Σ_c),
nonsimple poles (Σ_p),
numerator and denominator with common roots (Σ_0),
a critical point at $x = \infty$ (Σ_∞, for $\mu = \nu$ only).

Each connected component of $M_{\mu,\nu}$ is marked with the corresponding curve γ.

The proof of Lemma 1 is very close to the proofs of similar statements in **[2, 16]**. We only point out the facts the proof is based on.

(1) The closure $\bar{\gamma}$ of the curve $\gamma(r)$ is smooth except for the nodes on $\mathbb{R}$ at the critical points of r.
(2) Exactly $2|\mu - \nu|$ branches of the curve $\gamma(r)$ go to infinity.
(3) Circulating by the gradient flow of r along $\bar{\gamma} \cup \{x = \infty\}$, we can make a cycle only having passed through a pole. Here $x = \infty$ is regarded as a pole if $\mu > \nu$.
(4) All the equivalence classes of the curves are realizable.

For (4) one constructs the corresponding mapping between the Riemann spheres as in **[2, §2]**.

REMARK. The surgery of the Lemma relates a particular pole to each circular component of γ.

The mapping in (4), i.e., a rational M-function, depends continuously on the possible choice of the critical values and on the positions of the critical points and the poles while we stay in the same equivalence class of the curves γ. As in the polynomial case **[2, §2]** and the Laurent case **[16]**, the set of M-functions r having equivalent curves $\gamma(r)$ is easily seen to be connected and, moreover, contractible (see the end of the main part of this note). Thus, we get

COROLLARY 2. *The number $m_{\mu,\nu}$, $\mu \neq \nu$, of connected components of the set of rational functions is equal to the number of equivalence classes of the curves $\gamma(r)$ of Lemma* 1.

Now we consider the three cases of the Theorem. In what follows by "a curve γ" we mean its equivalence class. We say that a node (resp. pole) is *free* if it has two infinite whiskers (resp. is not contained in a disk bounded by a circular component of γ). The curve γ has

for $\mu > \nu$: ν circles, $\mu - \nu - 1$ free nodes and no free poles;
for $\mu < \nu$: μ circles, $\nu - \mu - 1$ free nodes and $\nu - \mu$ free poles.

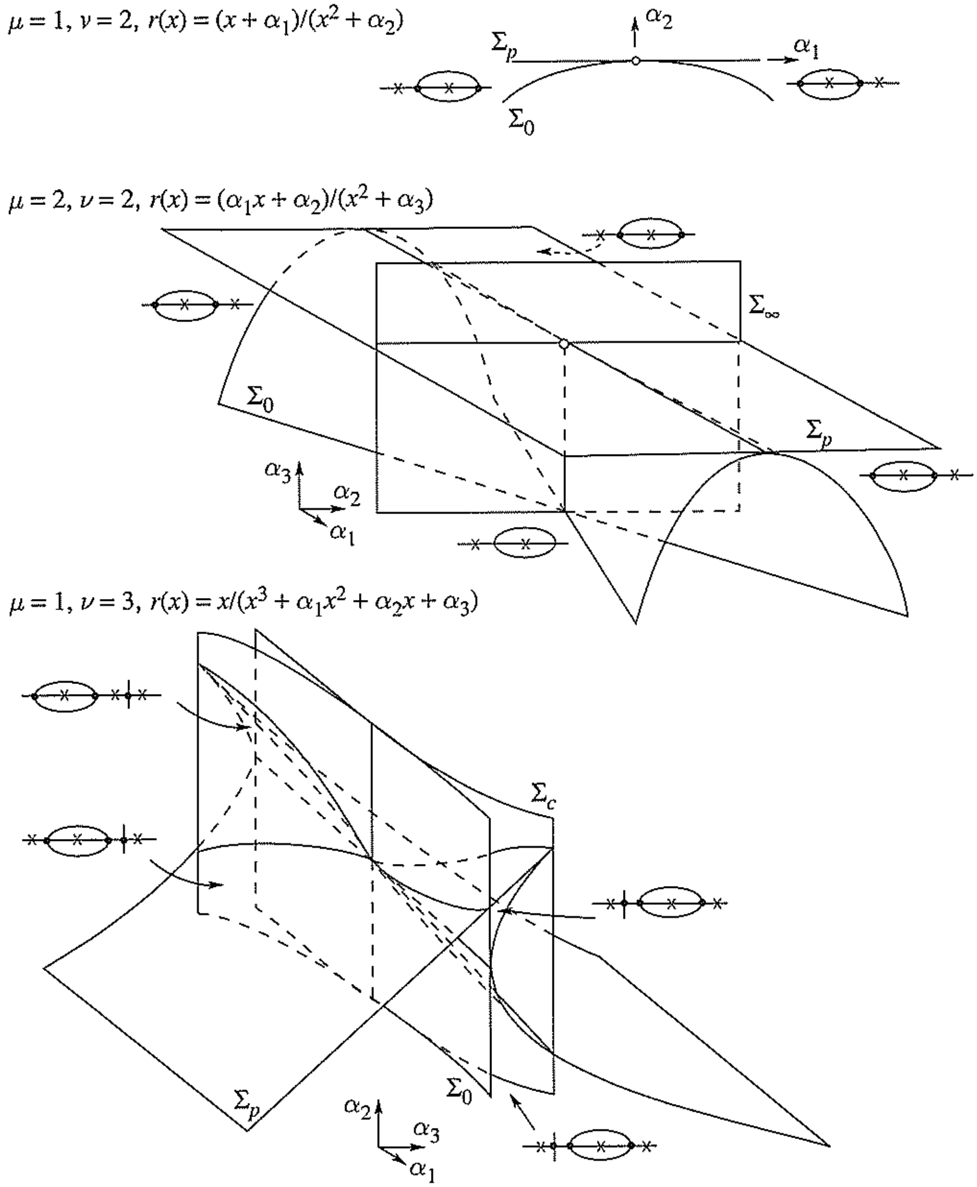


FIGURE 2. Bifurcation diagrams and sets of rational M-functions for low values of μ

An *exterior circle* of γ is its circular component not contained in the disk bounded by any other circular component.

$\mu > \nu$. Let us slightly change the notation to indicate the number $\beta = \mu - \nu - 1$ of free nodes:

$$\phi_{\beta,\nu} = m_{\nu+\beta+1,\nu}, \qquad \beta, \nu \geqslant 0.$$

Thus, we need to show that

$$\phi_{\beta,\nu} = \frac{(\beta+1)(3\nu+\beta)!}{(2\nu+\beta+1)!\,\nu!}\,.$$

Let us start with $\beta = 0$: no free poles, no free nodes, all the 2ν critical points participate in the ν circles. Denote by $n_{\nu,s}$ the number of curves γ with exactly s exterior circles.

LEMMA 3. *We have*

$$n_{\nu,s} = \sum_{\alpha\geqslant 1}\sum_{k\geqslant 0}(k+1)n_{\alpha-1,k}n_{\nu-\alpha,s-1}, \qquad \nu \geqslant 1\,.$$

PROOF. Take an exterior circle of γ, the first from the right on the real axis. Let $\alpha - 1$ be the number of circles inside it, and k the number of exterior circles in this subconfiguration of $\alpha - 1$ circles. There are $k+1$ possibilities for the position of the pole corresponding to the above circle of γ (see our latter remark) with respect to the subconfiguration.

Let us introduce the generating function

$$N(x,y) = \sum_{\nu,s\geqslant 0} n_{\nu,s}x^{\nu}y^{s+1}.$$

COROLLARY 4. *We have* $N(x,y) = xy\,\dfrac{\partial N}{\partial y}(x,1)N(x,y) + y$.

Let us take the partial y-derivative of this relation and evaluate both the derivative and the initial relation at $y=1$. Introducing

$$\phi(x) = N(x,1) = \sum_{\nu\geqslant 0}\phi_{0,\nu}x^{\nu}, \qquad \psi(x) = \frac{\partial N}{\partial y}(x,1),$$

we obtain

$$\phi = x\psi\phi + 1, \qquad \psi = x\psi(\phi+\psi) + 1\,.$$

Excluding ψ, we obtain

COROLLARY 5. $x\phi^3 - \phi + 1 = 0$.

REMARK. Replacing here the degree 3 by 2, we get the generating function for the Catalan numbers.

The assertion of the Theorem for $\mu = \nu + 1$ is equivalent to the recurrence relation

$$\phi_{0,\nu+1} = \frac{3(3\nu+1)(3\nu+2)}{2(\nu+1)(2\nu+3)}\,\phi_{0,\nu} \quad \text{with } \phi_{0,0} = 1\,.$$

This means that the function $\phi(x)$ must satisfy the differential equation

$$2(2x\partial_x + 1)x\partial_x\phi = 3x(3x\partial_x+1)(3x\partial_x+2)\phi$$

being its unique solution at $x = 0$ with $\phi(0) = 1$. Elementary calculations show that this follows from the last Corollary.

Now let $\mu - \nu - 1 = \beta > 0$. Let

$$\Phi(x, z) = \sum_{\beta\geqslant 0}\sum_{\nu\geqslant 0} \phi_{\beta,\nu} x^\nu z^\beta$$

be a generating function. Then

LEMMA 6. $\Phi = \phi/(1 - z\phi)$.

PROOF. For $\beta > 0$ let us consider the free node of γ, the first from the right on the real axis. Let α be the number of circles to the right of this node. Then

$$\phi_{\beta,\nu} = \sum_{\alpha\geqslant 0} \phi_{0,\alpha}\phi_{\beta-1,\nu-\alpha} \,.$$

For the generating functions, this means $\Phi(x, z) = z\phi(x)\Phi(x, z) + \phi(x)$.

Thus

$$\Phi(x, z) = \sum_{\beta\geqslant 0} z^\beta \phi^{\beta+1}(x) \,.$$

To prove the theorem for $\mu > \nu$, it remains to show that

$$\phi^{\beta+1}(x) = \sum_{\nu\geqslant 0} \frac{(\beta + 1)(3\nu + \beta)!}{(2\nu + \beta + 1)!\,\nu!} x^\nu.$$

This is equivalent to the requirement that the function $\phi^{\beta+1}(x)$ is analytic at the origin and satisfies the differential equation

$$\begin{aligned}&(2x\partial_x + \beta + 1)(2x\partial_x + \beta)x\partial_x \psi \\ &\quad = x(3x\partial_x + \beta + 3)(3x\partial_x + \beta + 2)(3x\partial_x + \beta + 1)\psi, \qquad \psi(0) = 1 \,.\end{aligned}$$

Again, the fact that this is exactly the case follows from the cubic equation on ϕ.

$\boldsymbol{\mu < \nu}$. Let $\varepsilon = \nu - \mu > 0$ be the number of free poles of γ. To each of the free poles we attach a complex-conjugate-symmetric pair of whiskers going to infinity and intersecting neither γ nor the other pairs added. We get a curve γ' of some rational M-function $(x^{\nu+\varepsilon} + \dots)/(x^{\nu-\varepsilon} + \dots)$.

The mapping $\gamma \mapsto \gamma'$ is obviously one-to-one. Indeed, the inverse mapping is constructed as follows. Each curve γ' has exactly $2\varepsilon - 1$ free nodes. Order these nodes: $x_1 > x_2 > \dots > x_{2\varepsilon-1}$. Now omit the whiskers starting at all x_{odd} and declare all x_{odd} to be poles. We obtain

LEMMA 7. $m_{\nu-\varepsilon,\nu} = m_{\nu+\varepsilon,\nu-\varepsilon}$.

This is exactly the statement of the Theorem for $\mu < \nu$.

REMARK. It is not too difficult to see that the function $\psi(x) = (\partial N/\partial y)(x, 1)$ above is the generating function $\sum_{\mu\geqslant 0} m_{\mu,\mu+1}x^\mu$.

$\boldsymbol{\mu = \nu}$. The assertion of the Theorem for this case follows from

LEMMA 8. $m_{\nu,\nu} = 2m_{\nu-1,\nu}$.

PROOF. The function

$$r(x) = (x^\nu + \lambda_1 x^{\nu-1} + \dots)/(x^\nu + \lambda_{\nu+1} x^{\nu-1} + \dots)$$

is a rational M-function if and only if $\lambda_1 \neq \lambda_{\nu+1}$ and the function

$$(r(x) - 1)/(\lambda_1 - \lambda_{\nu+1}) = (x^{\nu-1} + \dots)/(x^\nu + \dots)$$

is a rational M-function. According to the two possible signs of $(\lambda_1 - \lambda_{\nu+1})$, we obtain $m_{\nu,\nu} = 2m_{\nu-1,\nu}$.

Thus the Theorem is proved.

The decreasing order of critical points of a function on the real line induces an ordering of the critical values. Suppose $\mu \neq \nu$. Each equivalence class of γ imposes a certain system of $\mu + \nu - 2$ inequalities on the ordered set of the $\mu + \nu - 1$ critical values of a rational M-function r from the corresponding connected component R_γ. The same inequalities define a simplicial cone $\mathcal{R}_\gamma$ in Euclidean space $\mathbb{R}^{\mu+\nu-2} = \{z_1 + \dots + z_{\mu+\nu-1} = 0\} \subset \mathbb{R}^{\mu+\nu-1}$ equipped with the set of diagonals $z_i = z_j$ (mirrors of the reflection group $A_{\mu+\nu-2}$). As in [2, 3, 16], we have

(1) R_γ is homeomorphic to $\mathcal{R}_\gamma \times \mathbb{R}^2$, in particular R_γ is contractible;
(2) the number of chambers in the cone R_γ coincides with the number of connected components of the set of very nice (= with all critical values different) rational functions contained in R_γ, and each of these components is contractible too.

For $\mu = \nu$, we must replace $\mu + \nu$ by $2\nu - 1$ everywhere and $\mathbb{R}^2$ for $\mathbb{R}^3$ in (1).

OPEN COMBINATORIAL QUESTION. Calculate the number of chambers in $\mathcal{R}_\gamma$.

Related problems. For recent progress in problems closely related to the topic of the present paper, see [4, 5, 7–15, 18]. For example, [10] establishes a direct correspondence between very nice morsifications of A_μ function singularities and lemniscate configurations of complex polynomials. The existence of such a relation has been suggested by the observation that in both situations the sequence of numbers of distinct objects was one and the same, namely, the Euler-tangent sequence [1, 11]

$$1,\ 1,\ 2,\ 5,\ 16,\ 61,\ 272,\ 1385,\ 7936,\ 50521, \dots .$$

Probably a similar correspondence exists for rational functions as well.

In the quantum settings, two sequences starting like the Euler–tangent one were recently discovered. According to Kirillov [17], the number of conjugacy classes in the group of nondegenerate upper-triangular $n \times n$-matrices over $\mathbb{Z}_2$ is

$$1,\ 2,\ 5,\ 16,\ 61,\ 275,\ 1430,\ 8506,\ 57205, \dots .$$

Zograf's calculations [19] of volumes of moduli spaces of punctured spheres produced

$$1,\ 5,\ 61,\ 1379,\ 49946, \dots .$$

It would be very exciting to find, as suggested by Arnold, a direct relation between these quantum objects and, say, classes of very nice morsifications of function-germs.

Appendix. Very nice morsifications of E_μ singularities

In the introduction to the main part of this note, we mentioned two invariants of a real function singularity: the number of connected components of the set of M-morsifications and of very nice morsifications. For simple singularities of the series A and D, these invariants have been calculated in [1–5]. Here we present a different approach, universal for at least all the simple functions, and carry out the calculations for the remaining simple germs E_6, E_7, and E_8.

1. **M-components.** Consider any real simple function-germ X_μ, $X = A, D, E$, on a plane, with Milnor number μ. The base of its truncated miniversal deformation [4] contains an open set of *M-morsifications* of X_μ (functions with μ real critical points). According to [7], each connected component of this set (shortly, *M-component*) contains *sabirifications* of X_μ, functions with all saddle points on the same level.

A sabirification f defines an *R-diagram* of X_μ. This is an analog of a Dynkin diagram with extra information about the indices of the critical points of f. Each critical point of f corresponds to a vertex of the R-diagram marked by a plus (maximum of f) or by a minus (minimum) or considered neutral (saddle). The minima and maxima are taken on in the regions of the plane bounded by a saddle level curve. We join a plus-vertex and a minus-vertex by an edge if the corresponding regions are separated by an interval of the saddle level. We join a plus-vertex or a minus-vertex with a neutral vertex if the closure of the region contains the saddle point.

All the possible R-diagrams of the simple functions are listed, for example, in [7]. Their number $d(X_\mu)$ is given by the table:

X_μ	A_μ	D_{2k}^+	D_{2k}^-	D_{2k+1}	E_6	E_7	E_8
$d(X_\mu)$	1	$k-1$	k	k	2	4	5

For a fixed simple type X_μ of a real singularity, the R-diagrams have the same number of neutral vertices and the difference of the diagrams is equivalent to the difference of the numbers of the plus- (equivalently, minus-) vertices. Since the numbers of minima, saddles and maxima are the same for M-morsifications from the same M-component, for the numbers $m(X_\mu)$ of M-components of X_μ we have

PROPOSITION A1. $m(X_\mu) \geqslant d(X_\mu)$.

Actually, a stronger statement holds:

THEOREM A1. *The mapping relating an R-diagram of X_μ to an M-component of X_μ is one-to-one.*

COROLLARY A2. $m(X_\mu) = d(X_\mu)$.

PROOF OF THEOREM A1. For $X = A, D$, this is presented in [1, 5]. Thus, we need to study the E case only. In Figure A1 we show all R-diagrams of D_5 and E_μ singularities [7]. The numbers appearing there were explained above. In what follows it is convenient to set $E_5 = D_5$. We also do not distinguish between the '+' and '−' cases.

LEMMA A3. *Each M-component of E_μ contains a connected component of the stratum $E_{\mu-1}$ in its closure.*

PROOF. We consider only $\mu = 6$. The two other cases are similar.

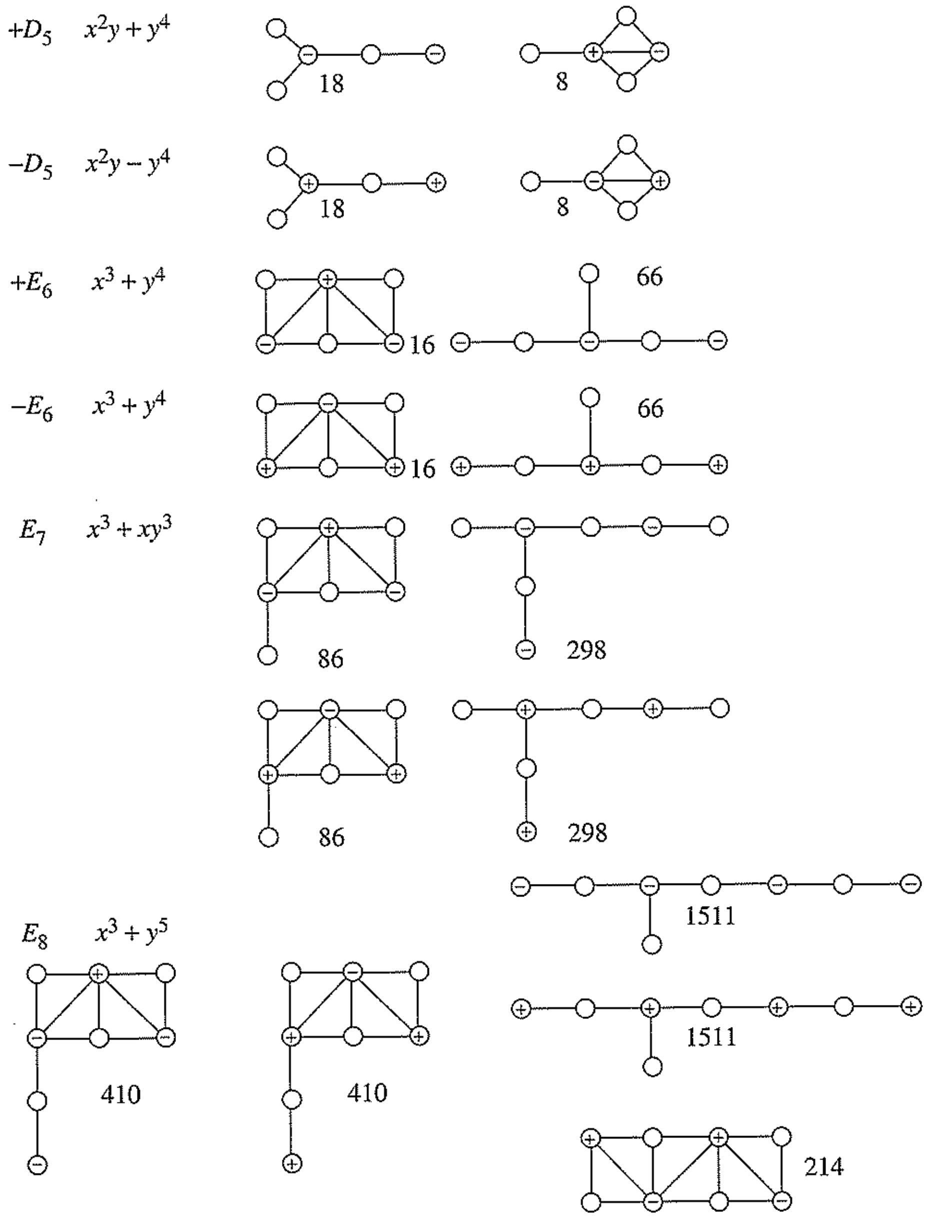


FIGURE A1. Normal forms, R-diagrams and the numbers of very nice components of D_5 and E_μ singularities

Let us take a sabirification of E_6 and consider its R-diagram (one of those given in Figure A1). Note that by omitting the lower right vertex of the diagram, together with all the edges starting at this vertex, we get a D_5 R-diagram. Let us fix the critical value of our sabirification corresponding to the lower right vertex and continuously deform all the other critical values, making them equal. According to [7], this deformation of the critical values induces a continuous deformation of the initial sabirification ending with a function with two critical points, A_1 and D_5, on different levels.

Easy direct computations imply

LEMMA A4. *The stratum $E_{\mu-1}$ in the base of a truncated miniversal deformation of E_μ is homeomorphic to a real line punctured (for E_μ itself omitted) at the origin.*

COROLLARY A5. $m(E_\mu) \leqslant 2m(E_{\mu-1})$.

Now we consider the three cases.

E_6. According to [5], $m(D_5) = 2$. Thus, by the last Corollary, $m(E_6) \leqslant 4$. But there are two different ways to deform a sabirification of E_6 into a function with a D_5 point (take the left vertex in the proof of Lemma A3 instead of the right one). This means that the closure of each of the M-components of E_6 must contain both half-branches of the stratum D_5. Thus $m(E_6) \leqslant 2 = d(E_6)$.

E_7. By Proposition A1 and Corollary A5: $4 = d(E_7) \leqslant m(E_7) \leqslant 2m(E_6) = 4$.

E_8. The E_7 stratum in the truncated versal deformation of E_8 is a one-parameter family $x^3 + y^5 + txy^3$, $t \neq 0$. In addition to the E_7 point, such a function has a Morse point on the other critical level. This point is a local minimum for $t > 0$ and local maximum for $t < 0$. By a slightly more precise argument than in the proof of Lemma A3, it is easy to see that the closures of M-components of E_8 corresponding to the three R-diagrams with both numbers of plus- and minus-vertices positive, contain both half-branches of the stratum E_7. Thus, $m(E_8) \leqslant 2m(E_7) - 3 = 5 = d(E_8)$.

2. **Very nice components.** The base of a truncated miniversal deformation contains a *Maxwell stratum* Σ_m, i.e., a hypersurface corresponding to functions with coinciding values at at least two real critical points. The Maxwell stratum subdivides each M-component into a certain number of open regions each of which consists of morsifications with all the μ critical values different. Such morsifications are called *very nice*. Connected components of the set of very nice morsifications will be called *very nice components*.

THEOREM A2. *The number of very nice components in an M-component of an E_μ function singularity is the number given in Figure* A1 *alongside the R-diagram of the M-component.*

Taking the sum over all R-diagrams of a particular germ, we get

COROLLARY A6. *The numbers of connected components of the set of very nice morsifications of E_6, E_7, and E_8 are respectively* 82, 768, *and* 4056.

PROOF OF THEOREM A2. Consider an M-component $M_\mathcal{D}$ of E_μ corresponding to a certain R-diagram $\mathcal{D}$. Order the vertices of $\mathcal{D}$ in an arbitrary way (in what follows we consider the diagram with this extra ordering). Take a particular sabirification $f \in M_\mathcal{D}$ and deform it inside $M_\mathcal{D}$. This induces a deformation of the ordered set of critical values. By [7], staying in $M_\mathcal{D}$, we can move two critical values independently relatively to each other unless the corresponding vertices are connected by an edge. If there is an edge, the inequality between the critical values remains the same as for the critical values of f. Thus the edges of $\mathcal{D}$ give a system of necessary and sufficient inequalities on the critical values $v_1, \dots, v_\mu$ of a very nice morsification from the M-component $M_\mathcal{D}$.

The same inequalities define a cone $C_\mathcal{D}$ in $\mathbb{R}^{\mu-1} = \{z_1 + \cdots + z_\mu = 0\} \subset \mathbb{R}^\mu$ equipped with the set W of all diagonals $z_i = z_j$. As in [2], shifting the critical