MONODROMY OF THE IMAGE OF A MAPPING

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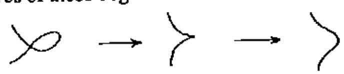
Consider a germ of a mapping $\mathbb{C}^n,0 \longrightarrow \mathbb{C}^{n+1},0$ of a finite left-right ($\mathcal{A}$-) codimension. Let a representative f be defined on a sufficiently small ball $D \subset \mathbb{C}^n$. Let $f_{\lambda_0} : D \longrightarrow \mathbb{C}^{n+1}$ be a stable perturbation of f. D.Mond [10] has proved that the image of f_{λ_0} in $\mathbb{C}^{n+1}$ is homotopy equivalent to a wedge of a finite number of n-dimensional spheres. We shall denote this finite number by σ. It has also been shown by de Jong and van Straten [7] (see also [10]) for $n=2$ that σ is not less than the $\mathcal{A}$-codimension of the germ of f at the origin and σ coincides with this codimension if the germ is quasihomogeneous (for mappings of a line into a plane the corresponding statements are almost trivial, but it is easily seen that if the pair $(n, n+1)$ is out of the nice dimensions [2], i.e. $n>6$, then the same statements are false). Thus, the image V of a stable perturbation of a map-germ $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ is analogous to a Milnor fiber of a function with an isolated critical point. In the present lecture we continue this analogy and define on V (which is a variety with non-isolated singularities) vanishing cycles and an index of intersection with a vanishing cycle and describe the monodromy of the stable image.

0. Mappings of a line into a plane. For better visualization we begin with the simplest case of a map-germ $f\colon \mathbb{C},0 \to \mathbb{C}^2,0$. This case already contains several points useful for the higher dimensions.

When we perturb our germ in a generic way we get an immersion of a smooth curve into a plane. The only singularities of the image of this perturbation are nodes. The image is homotopy equivalent to a wedge of circles the number of which is equal to the number of the nodes.

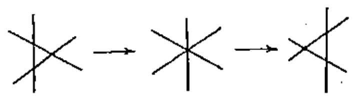
A smooth base Λ of an $\mathcal{A}$-versal deformation f_λ of f contains a *bifurcation diagram* Σ which is the set of the values of the deformation parameters λ corresponding to non-stable perturbations. Σ has three components according to the number of possible local degenerations of the image in generic one-parameter families of maps from $\mathbb{C}$ to $\mathbb{C}^2$. Real representatives of these degenerations are as follows:

i) a cusp



ii) contact of two sheets

iii) a triple point



In each of these cases one can easily see a one-dimensional cycle on the image which vanishes when λ approaches the bifurcation value.

Consider a loop ω in $\Lambda\backslash\Sigma$ with a base point λ_0. The circulation along this loop defines by the lifting homotopy a family of mappings of the image V of f_{λ_0} to the images corresponding to different points of ω and, finally, a mapping to V itself. This last mapping induces an automorphism h_ω of $H_1(V)$ called a *monodromy operator along* ω. When ω runs over the whole fundamental group of $\Lambda\backslash\Sigma$ we get a subgroup $\Gamma\subset \mathrm{Aut}\, H_1(V)$ called the *monodromy group* of the image of the initial map-germ f.

Consider a path γ in Λ from λ_0 to some regular point λ_* of Σ. Suppose only its end to be in Σ. Consider a loop in $\Lambda\backslash\Sigma$ which goes from λ_0 by γ to some point λ' close to its end, then goes round Σ in the positive direction in a line transversal to Σ, and returns to λ_0 by γ again. We call such a loop a *simple loop*. $\pi_1(\Lambda\backslash\Sigma)$ is generated by simple loops. The monodromy operator along a simple loop (called *Picard-Lefschetz operator*) is defined by the events taking place in H_1 when we are going along the small circle around Σ. The lifting homotopy corresponding to this circle gives an automorphism of the image of $f_{\lambda'}$ which is the identity out of a small neighbourhood (in this neighbourhood the image of $f_{\lambda'}$ degenerates if we locally approach Σ). Thus, we have localized the problem of the description of the monodromy of the image. We need now the description of the monodromy in the generic one-parameter families i–iii.

Let us go, for example, round the component of Σ corresponding to the cusp-degeneration. $\mathcal{A}$-versal 1-parameter deformation of the cusp can be written as $f_\lambda: t \mapsto t^2, t^3+\lambda t$. Take a loop $\omega_1: \lambda=-\exp(2\pi i\tau)$, $\tau\in[0,1]$. The map f_{-1} sticks together the points $t=\pm1$. In the complex Fig.1.τ=0 e is the f_{-1}-preimage of the vanishing cycle, Δ is the preimage of a generator of the closed first homology of V. Δ represents how "big" cycles on the "big" stable image come to the small neighbourhood in which the degeneration is localized.

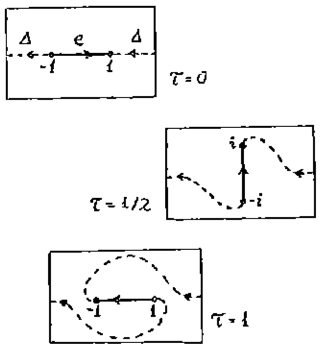


Fig.1

As τ is running from 0 to 1, the points sticking together are going round the origin and, finally, interchange. The events with the cycles are shown in Figs.1.τ=1/2 and τ=1. We see that $h_{\omega_1}: e\mapsto -e,\ \Delta\mapsto\Delta-2e$.

The similar drawings for the case of the contact of two sheets are given in Fig.2. Here it is more convinient to consider the equation $y(y+x^2+\lambda)=0$ for the local family of the images. In this case there are two independent closed 1-cycles represented by Δ_1 and Δ_2 in the preimage. The loop $\omega_2: \lambda=-\exp(2\pi i\tau)$, $\tau\in[0,1]$, provides $h_{\omega_2}: e\mapsto -e$, $\Delta_j\mapsto\Delta_j-e$, $j=1,2$.

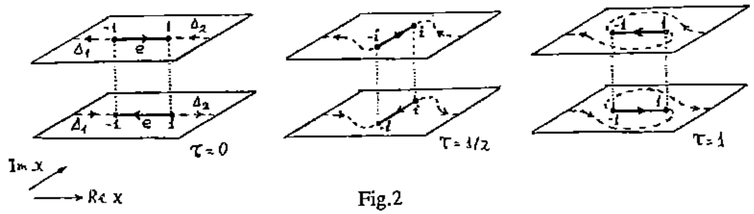
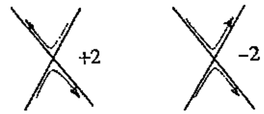


Fig.2

Similarly, for the triple point the local monodromy is the identity on the compact and closed homologies.

Let us introduce an *intersection index* of two 1-dimensional cycles on the image V of a stable perturbation of a map-germ $\mathbb{C}\to\mathbb{C}^2$. This index is calculated by the points of transversal self-intersection of V. Consider only the points in which the both cycles change local branches of the image. If in the given point the change of the sheets for the motion along the cycles is in the same direction, we declare the index to be 2 in this point. If the directions are opposite we declare the index to be -2.



This form is symmetric on $H_1(V)$, not skew-symmetric as it could be expected from an intersection of one-dimensional cycles. The explanation of the symmetry is as follows. Let $V_2\subset\mathbb{C}$ be the preimage of the self-intersection of the stable image. Then our intersection number is the index of intersection in V_2 of the boundaries of the preimages of the cycles under consideration. We shall extend this definition to the case of maps $\mathbb{C}^2\to\mathbb{C}^3$.

The indices of self-intersection (e,e) of the vanishing cycles i–iii are 2, 4 and 6 respectively. The indices of intersection of vanishing and closed cycles in Figs.1 and 2 are 2. Consequently, we get

Theorem 0. The Picard-Lefschetz operator on $H_1(V)$ corresponding to a simple loop going round the cusp or contact component of the bifurcation diagram is the reflection

$$c\mapsto c-2(c,e)e/(e,e)$$

in the plane orthogonal to the vanishing cycle e. The Picard-Lefschetz operator corresponding to the triple point is the identity.

As the intersection index comes from the preimage of the self-intersection, the monodromy of the image is actually defined by the monodromy of this preimage.

Remark. The cycles vanishing at the cusp and at the contact correspond to short and long vanishing cycles appearing in the theory of functions on a manifold with a boundary [9,3].

Example. Consider a map-germ $t \mapsto t^2, t^{2k+1}$, $k \geq 1$. Its bifurcation diagram Σ coincides with the set of irregular orbits of the Weil group B_k, i.e. with the set of polynomials $x^k+\lambda_1 x^{k-1}+...+\lambda_{k-1}x+\lambda_k$ with a zero or multiple root (see Fig.3 for $k=3$). These roots correspond to the cusp and contact degeneration respectively. The monodromy group of the image is generated by the Picard-Lefschetz operators of the simple loops lying on the generic line ℓ shifted from the origin to some base point λ_0. This group coincides with the reflection group B_k. The intersection index on the stable image is given by the Dynkin diagram B_k [1].

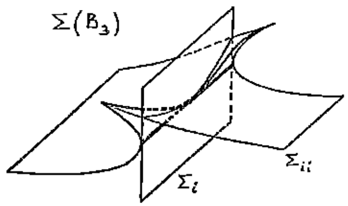


Fig.3

Now we are starting with our main topic of maps from $\mathbb{C}^2$ to $\mathbb{C}^3$. In many points the consideration is similar to the one just carried out. Sometimes we repeat the definitions.

1. Vanishing cycles. Consider an $\mathcal{A}$-versal deformation f_λ, $\lambda \in (\Lambda,0)$, of a map-germ $f: \mathbb{C}^2,0 \longrightarrow \mathbb{C}^3,0$. For a generic value of the deformation parameter the image of f_λ has only stable singularities: Whitney umbrellas and transversal intersections of two or three smooth sheets. There is an *$\mathcal{A}$-bifurcation diagram* Σ in the base Λ of the deformation: Σ is the set of all the values λ for which the corresponding member f_λ of the versal family has unstable multi-singularities. Generally Σ has 5 components which correspond to the following $\mathcal{A}$-codimension 1 degenerations of the image:

1^o. fusion of two Whitney umbrellas;

2^o. tangency of two sheets of the image;

3^o. a smooth sheet passes trough an umbrella;

4^o. tangency of a smooth sheet and of a line of transversal intersection of two other sheets;

5^o. a smooth sheet passes through a point of transversal intersection of three other sheets.

In Fig.4 for special real forms of these degenerations we show the local character of the corresponding real parts of the surfaces $\operatorname{Im} f_\lambda$ for values of λ close to the bifurcation ones. The complement in $\mathbb{R}^3$ of any of the surfaces drawn has just one connected component with compact closure. It is easily seen that the boundary of this component is homeomorphic to the two-dimensional sphere, determines a non-trivial element in $H_2(\operatorname{Im} f_\lambda, \mathbb{Z})$ and collapses to a point when λ tends to the bifurcation value. This boundary we shall call *a vanishing cycle* on $\operatorname{Im} f_\lambda$, $\lambda \notin \Sigma$.

Now we are going to define a distinguished set of vanishing cycles on the stable

image. In order to do this consider a generic line $\ell \simeq \mathbb{C}^1$ passing through $0 \in \Lambda$. Shift it in a generic way from the origin. The new line ℓ' intersects the diagram Σ transversally in a finite number ν of points (ν is the intersection index of ℓ and Σ). Choose a non-bifurcation point λ_0 on ℓ'. Consider on ℓ' a system of ν paths γ_i, without mutual or self intersections, going from λ_0 to the points of the set $\ell' \cap \Sigma$. Enumerate the paths clockwise according to the order in which they leave λ_0. We contract one cycle e_i on the stable image $V = \operatorname{Im} f_{\lambda_0}$ when moving along the path γ_i. The set $e_1, ..., e_\nu$ of cycles is called *distinguished.*

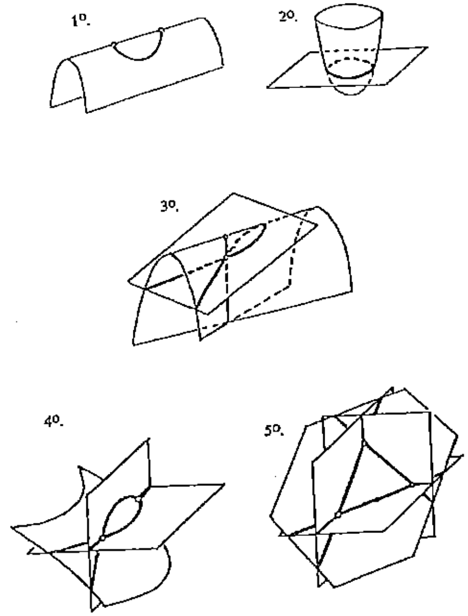


Fig.4

Theorem 1. A distinguished set of cycles generates the group $H_2(V, \mathbb{Z})$.

Proof. The line ℓ' induces a mapping $F': \mathbb{C}^3 \longrightarrow \mathbb{C}^4$, $(x, \lambda) \mapsto (f_\lambda(x), \lambda)$, $\lambda \in \ell'$. This mapping is stable because ℓ' is transverse to Σ. Let $W \simeq \bigvee S^3$ be its image. The exact homology sequence of the pair (W, V) collapses to:

$$0 \longrightarrow H_3(W) \longrightarrow H_3(W,V) \longrightarrow H_2(V) \longrightarrow 0$$

Here the central member is a free group of rank ν, generated by the classes of the thimbles which contract in W the vanishing cycles $e_i \in V$ along the paths γ_i (cf. [6]). Consequently, the cycles e_i generate $H_2(V)$.

The mapping F' we used here is a perturbation of a mapping F induced from the versal deformation by the line ℓ, passing through $0 \in \Lambda$.

Corollary. $\nu = \sigma(f) + \sigma(F)$.

Note that Theorem 1 and its Corollary are also true for mappings $\mathbb{C} \longrightarrow \mathbb{C}^2$ and $\mathbb{C}^3 \longrightarrow \mathbb{C}^4$.

2. Operators of local variation. Consider some path $\gamma(t)$, $t \in [0,1]$, in $\Lambda \setminus \Sigma$ which starts at λ_0. The covering homotopy defines a mapping $V = \operatorname{Im} f_{\gamma(0)} \longrightarrow \operatorname{Im} f_{\gamma(1)}$ (which may be considered to be a diffeomorphism). If the path is a loop we get an automorphism of V.

Definition. The image of the natural homomorphism $\pi_1(\Lambda\setminus\Sigma, \lambda_0) \longrightarrow \operatorname{Aut} H_*(V, \mathbb{Z})$ is called *the group of monodromy of the germ* $f : \mathbb{C}^2,0 \longrightarrow \mathbb{C}^3,0$.

As usual [2 ; 3, ch.4], in order to describe the monodromy group, we consider first the automorphisms of the stable images induced by circulation along small loops going round different components of the bifurcation diagram Σ.

So, let λ_* be a regular point of the diagram Σ and y_* be the point in which the image $\operatorname{Im} f_{\lambda_*}$ has its only unstable singularity. It is easily seen that one can choose a small ball $B \subset \mathbb{C}^3$ centered at y_* and a very small neighbourhood $U \subset \Lambda$ of the point λ_* such that the fibration over U with fiber $\operatorname{Im} f_\lambda \setminus (\operatorname{Im} f_\lambda \cap B)$ is trivial. Consequently, circulation in U along a loop ω based at λ' round Σ defines a homeomorphism of the stable image $\operatorname{Im} f_{\lambda'}$ which may be taken to be the identity out of the set $Y = \operatorname{Im} f_{\lambda'} \cap B$. Thus, this circulation adds to an element $\Delta \in H_*(Y,\partial Y)$ its *variation* $\operatorname{Var}_\omega \Delta \in H_*(Y)$. As the latter homology is non-trivial only in dimension 2, we get $\operatorname{Var}_\omega \Delta = 0$ if $\dim \Delta \neq 2$.

We write out the groups $H_2(Y, \partial Y) \simeq H_2^{cl}(Y)$ (homology with closed supports) for all our five codimension 1 degenerations, pointing out their generators Δ_i. These generators we shall be concerned with during all further considerations. The calculations of the groups are quite elementary and we omit them.

1°. $\mathbb{Z}_2 \oplus \mathbb{Z}$. The generator Δ_1 of $\mathbb{Z}_2$ is the self intersection of Y. The generator Δ_2 of $\mathbb{Z}$ is constructed in the following way: in the case under consideration the $f_{\lambda'}$-preimage of the self intersection is homeomorphic to a cylinder; let α be the generator of this cylinder folded in two by $f_{\lambda'}$; contract α in $\mathbb{C}^2$ by a closed two-dimensional thimble; the $f_{\lambda'}$-image of this thimble is the generator of $\mathbb{Z}$.

2°. $\mathbb{Z}$. Δ_1 is made up by gluing together two closed thimbles which contract in two sheets the generator of the one-dimensional closed homology of the self intersection (the self intersection is homeomorphic to a cylinder).

3°. $\mathbb{Z}_2 \oplus \mathbb{Z}$. Δ_1 is the self intersection. Δ_2 is the real three-faced angle, "vertical" with respect to the three-faced angle of the vanishing cycle (see Fig.4).

4° and 5°. $\mathbb{Z}^2$ and $\mathbb{Z}^4$ respectively. The generators are analogous to the generator Δ_2 of infinite order in the previous case.

As in each case $H_2(Y) = \mathbb{Z}$, the variation operator sends any element of finite order to zero. In order to describe the action of $\operatorname{Var}_\omega$ on the free summands choose orientations for the generators.

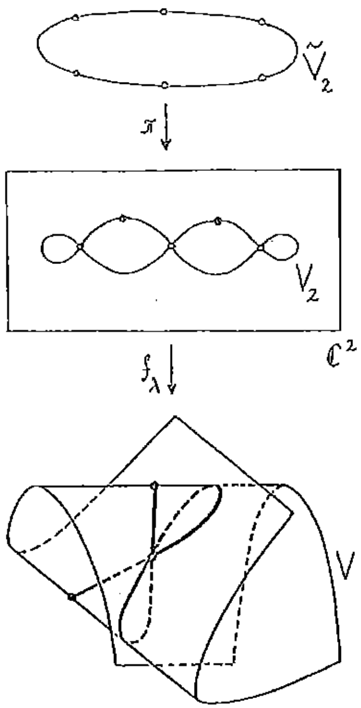


Fig.5

First recall the definitions of some sets, related to the multiple selfintersections of the stable image $V = \operatorname{Im} f_{\lambda_0}$ [8 , 11].

The space $\mathbb{C}^2$ contains a curve V_2 which is the preimage of the self intersection. V_2 has transversal self intersections which are the preimages of the triple points. The normalisation of V_2 is the curve $\tilde V_2 \subset \mathbb{C}^2 \times \mathbb{C}^2$ which is the closure of the set of the pairs (x_1, x_2), $x_1 \neq x_2$, such that $f_{\lambda_0}(x_1) = f_{\lambda_0}(x_2)$. $\tilde V_2$ contains the preimage V_3 of the points of self intersection of the curve V_2 under the projection $\pi : (x_1, x_2) \mapsto x_2$.

Example. In Fig.5 these sets are shown for a stable perturbation of the singularity H_2: $x, y \mapsto x, y^3, xy + y^5$.

For any vanishing cycle e

the dimension of its intersection with a multiple self intersection decreases by 1 when the multiplicity of the self intersection increases by 1. Consider the closure of the f_{λ_o}-preimage of the set $e \cap \operatorname{reg} V$ in $\mathbb{C}^2$. The boundary of this closure is a one-dimensional cycle in V_2. The closure of the π-preimage of $\delta e_2 \cap \operatorname{reg} V_2$ has boundary $\delta_3 e \subset V_3$ (possibly empty).

Let the cycle e vanish for the above mentioned value $\lambda = \lambda_*$. Let Y_2 and Y_3 be analogues of the sets V_2 and V_3 for $Y = \operatorname{Im} f_{\lambda'} \cap B$. Consider some generating cycle $\Delta \in H_2^{cl}(Y)$ of infinite order. It has boundaries $\delta_2\Delta \in H_1^{cl}(Y_2)$ and $\delta_3\Delta \in H_0(Y_3)$. $\delta_3\Delta = 0$ iff $\delta_3 e = \varnothing$. Define the index of intersection of closed and compact cycles on Y:

$$(\Delta, e) = \begin{cases} (\delta_2\Delta, \delta_2 e) & \text{if } \delta_3 e = 0 \\ (\delta_3\Delta, \delta_3 e) & \text{if } \delta_3 e \neq 0 \end{cases}$$

Indices on the right are taken on Y_2 and Y_3 respectively.

Theorem 2. Consider a complex line transversal to the bifurcation diagram Σ at its regular point λ_*. Let ω be a small loop on this line which goes round λ_* once in the positive direction. Then the variations $\operatorname{Var}_\omega \Delta$ of the basic elements of $H_2(Y, \partial Y)$ of infinite order, such that $(\Delta, e) > 0$, are the following:

1^o. $-2e$ 2^o. $-e$ 3^o. $-2e$ 4^o. $-e$ 5^o. 0

The cases 2^o, 4^o and 5^o were investigated in [12], see also [3, §4.1]. The formulae 1^o and 3^o are easily obtained by consideration of the action of $\operatorname{Var}_\omega$ on $H_1^{cl}(Y_2)$ and $H_0(Y_3)$ (cf., e.g. [2, sect.2.1.1.2]).

3. Index of intersection with a vanishing cycle. The description of the operators of local variation suggests that in order to describe the monodromy group of a map-germ $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ we need to define an index of intersection with a vanishing cycle.

Return to the situation of section 1 where we defined a distinguished set of vanishing cycles. Let γ be one of the distinguished paths on a generic line ℓ' which leads from the distinguished point $\lambda_o \in \ell'$ to one of the bifurcation points $\lambda_* \in \ell' \cap \Sigma$. Let e be the cycle vanishing along γ, λ' be a point of γ close to its end and $V' = \operatorname{Im} f_{\lambda'}$. The group $H_2(V')$ is generated by cycles which have one-dimensional intersections with the curve of self intersections of V'. By use of the natural homomorphism $H_2(V') \longrightarrow H_2^{cl}(V')$ we set for a compact cycle c on V':

$$(c, e) = \begin{cases} (\delta_2 c, \delta_2 e) & \text{if } \delta_3 e = 0 \\ (\delta_3 c, \delta_3 e) & \text{if } \delta_3 e \neq 0 \end{cases}$$

The indices on the right are taken on V_2' and V_3'.

We define the intersection index with e on $H_2(V)$ transfering the cycles from V to V' along γ.

The indices of self intersection of the vanishing cycles 1^o–5^o are equal to 0, 0, 6, 12 and 24 respectively.

For two vanishing cycles e and e' on V we have:

if $\delta_3 e = \delta_3 e' = 0$ then $(e, e') = -(e', e)$;
if $\delta_3 e \neq 0 \neq \delta_3 e'$ then $(e, e') = (e', e)$;
if $\delta_3 e \neq 0 = \delta_3 e'$ then $(e', e) = 0$, but not necessarily $(e, e') = 0$.

In the first two cases the intersection index can be computed on V without transfering the cycles onto the corresponding almost-bifurcational images. In the third case the index is zero as the cycle e' can be removed from the cycle e by an isotopy in a neighbourhood of e. In the fourth case the index may be non-zero as in order to calculate it we use only isotopies in a small neighbourhood of the cycle e'.

Thus, the matrix of intersections of the vanishing cycles (when we order them in some way which can differ from the ordering in the distinguished set) has a block structure:

$$\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix},$$

where the matrix A_1 is skew-symmetric and A_2 is symmetric.

4. Picard-Lefschetz operators. The group $\pi_1(\Lambda \backslash \Sigma, \lambda_o)$ is generated by ν classes of simple loops ω_i on $\ell' \backslash \Sigma$ which correspond to a system of distinguished paths γ_i (the simple loop ω_i starts at λ_o, goes along γ_i up to a point close to the end of this path, circulates round the end point counterclockwise and returns to λ_o by γ_i). Consequently, the monodromy group is generated by automorphisms h_{ω_i} of the group $H_2(V)$ induced by circulations along the simple loops. The automorphisms h_{ω_i} are called *Picard-Lefschetz operators.* Their description follows from the Theorem 2.

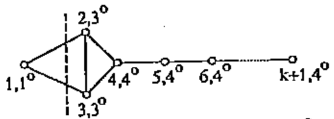
Theorem 3. The Picard-Lefschetz operators corresponding to the vanishing cycles of types 1^o–5^o are the following:

1^o. $c \mapsto c - (c,e)e$ $3^o, 4^o$. $c \mapsto c - 2(c,e)e/(e,e)$
2^o. $c \mapsto c - (c,e)e/2$ 5^o. id

Thus, the operators 3^o and 4^o are reflections into the planes orthogonal to the corresponding cycles. Operators 1^o and 2^o act on the subspace in $H_2(V)$ generated by the cycles of types 1^o and 2^o in the same way as skew-orthogonal reflections in short and long cycles of a function on a manifold with boundary (cf. [1]).

5. Examples. 1. Recall that the list of $\mathcal{A}$-simple map-germs $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ contains as a subset the list of simple function-germs on a half-plane: A_k, B_k, C_k, F_4 [9, 3]. Moreover, contact classes of curves on the x,z-plane with a boundary $z=0$ are in one-to-one correspondence with $\mathcal{A}$-classes of mappings $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ of corank 1 and with stable perturbations without triple points: a curve $g(x,z)=0$ corresponds to a mapping $x,y \mapsto x, y^2, yg(x,y^2)$. We meet here vanishing cycles only of types 1^o and 2^o. A distinguished set of vanishing cycles is a basis in $H_2(V)$. The intersection form, which we have introduced on $H_2(V)$, coincides with the intersection form of a function on a half-plane [1]. In this situation cycles of type 1^o correspond to short cycles of the boundary singularity and cycles of type 2^o correspond to long ones. It is easily seen that this correspondence may be continued up to the correspondence between hypersurfaces in the manifold $\mathbb{C}^n$ with boundary $\mathbb{C}^{n-1}$ and corank 1 mappings $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$ with no triple points on the stable image. This leads to the appearance of Weil groups A_k, B_k, C_k, D_k, E_k and F_4 and their skew-symmetric analogues in the theory of mappings $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$ as the monodromy groups of the corresponding simple singularities.

2. The complement of the list of simple boundary singularities in the list of $\mathcal{A}$-simple mappings $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ consists of one infinite series H_k, $k\geq 2$: $x, y \mapsto x, y^3, xy+y^{3k-1}$. It has $\mathrm{rk}\, H_2(V) = k$ but its distinguished set of cycles contains $k+1$ elements (a stable perturbation of H_2 is shown in Fig.5; two cycles of type 3^o are clearly seen there; the third one, of type 1^o, is equal to the difference of the first two for some choice of orientations). For an appropriate choice of a distinguished point and a system of distinguished paths on a generic line in the base of the versal deformation of the singularity H_k, one gets the following Dynkin diagram:



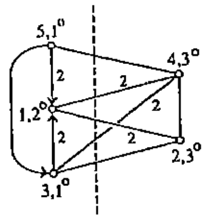
$e_1 = e_2 - e_3$

Here we have indicated the type of each vanishing cycle. The skew-symmetric $(\delta_3 e_i = 0)$ and the symmetric $(\delta_3 e_i \neq 0)$ parts of the diagram are separated by a dashed line. In the symmetric part the weight of an edge is one sixth of the intersection index of the corresponding cycles; in all the other cases the indices and the weights coincide. Weight 1 is omitted. The operator h of classical monodromy (we go along all the simple loops consecutively beginning with the last one) sends:

$e_1 \mapsto e_1$, $\quad e_2 \mapsto e_2 - e_1 - e_4$, $\quad e_3 \mapsto e_3 - e_1 - e_4$, $\quad e_s \mapsto -e_{s+1}$, $4 \leq s \leq k$,
$e_{k+1} \mapsto -e_{k+1} + e_k - e_{k-1} + e_{k-2} - \ldots \pm e_4 \pm 2e_1$

The characteristic polynomial of the operator h is equal to $(\lambda-1)(\lambda^{k-1}-1)$.

3. Every non-simple germ $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ which has triple points on the stabilisation of its image is adjacent to the singularity P_4: $x, y \mapsto x, xy+y^3, xy^2+\alpha y^4$ $(\alpha \neq 0, 1/2, 1, 3/2)$. The Dynkin diagram of this singularity is the following:



$e_1 - e_2 - e_3 + e_4 + e_5 = 0$

The edges in the skew-symmetric left part of the diagram are oriented from the i-th vertex to the j-th one in such a way that the index (e_i, e_j) is equal to the weight of the edge. Here

h: $e_1 \mapsto -e_1 + 2e_5$, $e_2 \mapsto e_1 + e_4 + 3e_5$, $e_3 \mapsto e_1 + e_5$, $e_4 \mapsto e_1 + e_3 + e_4 + e_5$, $e_5 \mapsto -e_3 + e_5$.
$\det(\lambda I - h) = (\lambda-1)(\lambda^3-1)$.

6. Remarks. 1. For mappings $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$, $n>2$, the situation is hypothetically the following. For degenerations of topological $\mathcal{A}$-codimension 1, an n-dimensional sphere must be contracted on the stable image. It is expected that the action of the Picard-Lefschetz operators is described by a matrix of intersections of a distinguished set of vanishing cycles which has block-triangular structure with alternating symmetric and skew-symmetric blocks on the diagonal. See also [5].

2. One more possible generalisation is a description of monodromy of the discriminant of a map-germ $g: \mathbb{C}^n, 0 \longrightarrow \mathbb{C}^p, 0$, $n\geq p$. According to [4] the discriminant of a stable perturbation of an $\mathcal{A}$-finite-codimensional germ g is homotopy equivalent to a wedge of a finite number of $(p-1)$-dimensional spheres. Here the restriction of g to the set of its critical points is the analogue of a mapping $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$.

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