From Theorem 3.2 there follows that for $\gamma > 0$ the function $u_0(x) \equiv 0$ yields a strict local minimum of the functional f. If $\gamma < 0$, then the function $u_0(x) \equiv 0$ is not a point of minimum of this functional. If $\gamma = 0$, then the minimum analysis can be carried out with the aid of Theorem 3.2.

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MONODROMY OF THE IMAGE OF THE MAPPING $C^2 \to C^3$

V. V. Goryunov

UDC 517.3

We consider the germ of the mapping $C^n,0 \to C^{n+1},0$ of finite left-right ($\mathcal{A}$-) codimension. Assume that a representative f is defined on a sufficiently small ball $D \subset C^n$ with center at 0. Let f_{λ_0}: $D \to C^{n+1}$ be a mapping sufficiently close to f with only $\mathcal{A}$ -stable multisingularities (a Whitnification of f). Mond [6] proved that the image of f_{λ_0} in C^{n+1} is homotopically equivalent to the bouquet of a finite number of n-dimensional spheres. We denote this finite number by σ. It was also proved that for $n = 2$, σ is no less than the $\mathcal{A}$-codimension of the initial germ, and for a quasihomogenous mapping it is equal to it (the corresponding claim for a mapping of the line into the plane is almost trivial, but it is easy to see that if the pair (n,n+1) lies outside the range of "nice" dimensions [2], i.e., $n > 6$, the claim is actually false). Thus, the image V of a stable perturbation of the germ of the mapping $C^2 \to C^3$ is analogous to the Milnor fiber of a function with an isolated critical point. In this paper we will extend this analogy, and define vanishing cycles on V (a manifold with nonisolated singularities) and the index of intersection with a vanishing cycle; we will also describe the monodromy of a stable image.

1. Vanishing Cycles. We consdier an $\mathcal{A}$-versal deformation f_λ, $\lambda \in (\Lambda,0)$, of the mapping f: $C^2,0 \to C^3,0$. For a general value of the deformation parameter, the image of f_λ has only stable singularities: the transversal intersection of two or three smooth sheets, and a Whitney hood. The base of the deformation Λ contains the *$\mathcal{A}$-branching diagram* Σ, the hypersurface of values of λ for which the corresponding term of f_λ of a versal family has unstable multisingularities. In the general case — Σ has 5 components, which correspond to the following degeneracies of an image of $\mathcal{A}$-codimension 1:

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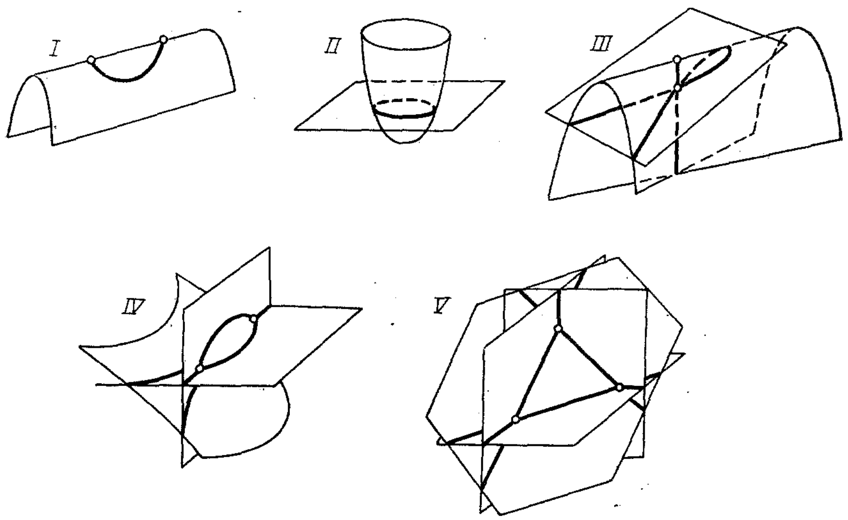


Fig. 1

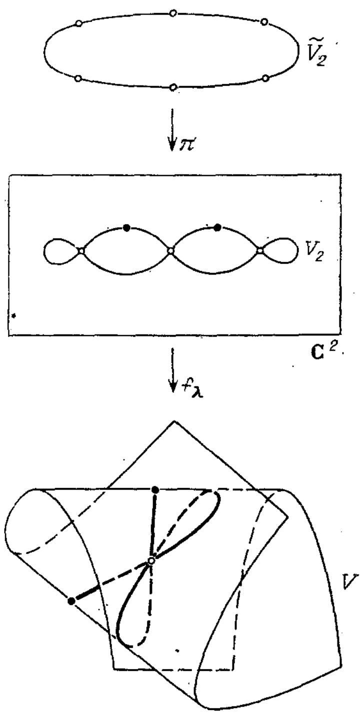


Fig. 2

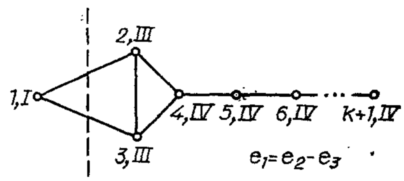
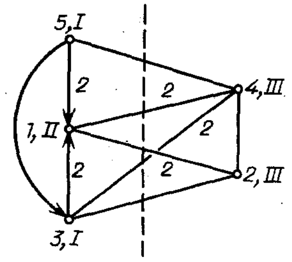


Fig. 3



$e_1 - e_2 - e_3 + e_4 + e_5 = 0$

Fig. 4

Fig. 5

Fig. 6

I. Confluence of two Whitney hoods.

II. Contact between two sheets of the image.

III. Passage of a foreign sheet through a hood.

IV. Contact between the lines of the transversal intersection of two sheets of the image with a third sheet.

V. Passage of a sheet through a point of the transversal intersection of three other sheets.

For specially selected real forms of the degeneracies we have listed, Fig. 1 shows the local form of the corresponding real surfaces of Im f_λ for given values of λ close to the the branching value. The complement (in $\mathbf{R}^3$) of each of the illustrated surfaces has a unique connected component, whose closure is compact. The boundary of the closure of this component is homeomorphic to a two-dimensional sphere, determines a nontrivial element in H_2 (Im f_λ; $\mathbf{Z}$) and contracts to a point as λ approaches a branch point. We call this boundary a vanishing cycle on Im f_λ, $\lambda \notin \Sigma$.

We will now define a distinguished family of vanishing loops on a stable image. To do so, we consider a line $l \simeq \mathbf{C}^1$ in general position that passes through 0. We shift it from zero in the usual way. The line l' thus obtained transversally intersects the diagram Σ in a finite number ν of points (ν is the index of the intersection of l and Σ). We now fix a nonbranching point λ_0 on l', and on l' we consider a system of ν nonintersecting and non-selfintersecting paths γ_i that start at λ_0 and proceed to points of the set $l \cap \Sigma$. We enumerate these paths in the clockwise order that they leave the point λ_0. As motion proceeds along each of these paths on a stable image, $V = \text{Im } f_{\lambda_0} \simeq VS^2$ contracts along a vanishing cycle e_i. We say that the set of cycles $e_1, \ldots, e_\nu$ is distinguished.

THEOREM 1. A distinguished set of cycles generates the group $H_2(V;\mathbf{Z})$.

Proof. The line l' induces the mapping $F'\colon \mathbf{C}^3 \to \mathbf{C}^4$, $(x, \lambda) \mapsto (f_\lambda(x), \lambda)$, $\lambda \in l'$ from a versal deformation. This mapping is stable. Let $W \simeq VS^3$ be its image. Now, consider the segments of the exact homological sequence

$$0 \to H_3(W) \to H_3(W, V) \to H_2(V) \to 0.$$

The central term is a free group of rank ν generated by the classes of frames of vanishing cycles $e_i \subset V$ that contract in W along the paths γ_i (as in [5]). As a result, the cycles e_i generate $H_2(V)$.

The mapping F' we have used is a perturbation of the mapping F induced by a versal deformation by a line l passing through $0 \in \Lambda$.

COROLLARY. $\nu = \sigma(f) + \sigma(F)$.

We should note that analogs of Theorem 1 and its corollary hold for $C^1 \to C^2$ and $C^3 \to C^4$.

2. Local Variational Operators. We now consider any path $\tau(t)$, $t \in [0,1]$ in $\Lambda \setminus \Sigma$ that begins at λ_0. The covering homotopy defines the mapping $V = \operatorname{Im} f_{\gamma(0)} \to \operatorname{Im} f_{\gamma(1)}$ (which can be assumed to be a diffeomorphism). If the path is a loop, we obtain an automorphism V.

Definition. The monodromy group of the germ $f: C^2,0 \to C^3,0$ is the image of the natural homomorphism $\pi_1(\Lambda \setminus \Sigma,\lambda_0) \to \operatorname{Aut} H_*(V;Z)$.

As usual [2; 3, Ch. 4], to describe the mondromy group we first consider automorphisms of the homologies of stable images induced by circuits about small loops around various components of the branching diagram Σ.

Thus, let λ_* be a regular point of the diagram Σ, and let J_* be a point at which the image $\operatorname{Im} f_{\lambda_0}$ has a single unstable singularity. It is not difficult to see that it is possible to choose a sufficiently small ball $B \subset C^3$ with center at J_* and a very small neighborhood $U \subset \Lambda$ of the point λ_* so that the fiber space over U with fiber $\operatorname{Im} f_\lambda \setminus (\operatorname{Im} f_\lambda \cap B)$ is trivial. As a result, a circuit along a loop ω in U about Σ defines a diffeomorphism of a stable image $\operatorname{Im} f_{\lambda'}$ that is the identity outside the set $Y = \operatorname{Im} f_{\lambda'} \cap B$ (λ' is the beginning and end of ω). Thus, this circuit adds its variation $\operatorname{Var}_\omega\Delta$, an element of $H_*(Y)$, to an element Δ of $H_*(Y,\partial Y)$. Since $H_*(Y)$ is homologically nontrivial only in two dimensions, $\operatorname{Var}_\omega \Delta = 0$ if $\dim \Delta \neq 2$.

For our five elementary degeneracies of codimension 1 we will write out the groups $H_2(Y,\partial Y) \approx H_2^{clos}(Y)$, indicating their generating Δ, which we will need later on.

I. $Z_2 \oplus Z$. Δ_1 (order 2) is a selfintersection. Δ_2 is obtained as follows: here the $f_{\lambda'}$-preimage of a selfintersection in C^2 in homeomorphic to a cylinder; let a be the generator of this cyclinder, which the mapping $f_{\lambda'}$ cuts in half; we have a in C^2 contract as a closed two dimensional film; the $f_{\lambda'}$-image of this film also generates the term **Z**.

II. **Z**. Δ_1 is constructed from two closed films that contract in each of the generating sheets of the one-dimensional closed homology of a selfintersection (the selfintersections are homeomorphic to a cylinder).

III. $Z_2 \oplus Z$. Δ_1 is a selfintersection. Δ_2 is a real trihedral angle that is "vertical" relative to the trihedral angle of the vanishing cycle.

IV and V. Z^2 and Z^4. The generators are analogous to the generator Δ_2 of infinite order in the preceding case.

Since $H_2(Y) = Z$ in every case, the variational operator carries any element of finite order into zero. To describe the action of $\operatorname{Var}_\omega$ on the free terms we choose the orientation of generators.

First, we recall the definition of certain sets associated with multiple selfintersections of a stable image $V = \operatorname{Im} f_{\lambda_0}$ [7, 8].

The space C^2 contains a curve V_2 that is the preimage of the selfintersection V. V_2 has transversal selfintersections — the preimages of triple points. We can normalize V_2 by using a curve $\tilde{V}_2 \subset C^2 \times C^2$ that is the closure of the set of pairs (x_1,x_2), $x_1 \neq x_2$, such that $f_{\lambda_0}(x_1) = f_{\lambda_0}(x_2)$. $\tilde{V}_2$ contains the preimage V_3 of the points of selfintersection of the curve V_2 for the projection π: $(x_1, x_2) \mapsto x_1$.

Example. In Fig. 2 these sets are shown for the Whitnification of the singularity H_2: $x, y \mapsto x, y^3, xy + y^5$.

On any vanishing cycle e the dimension of an intersection with a multiple selfintersection of the image V drops by 1 when the multiplicity of the selfintersection rises by 1. We will consider the closure of the f_{λ_0}-preimage of the set $e \cap \operatorname{reg} V$ in C^2. The boundary $\delta_2 e$ of this closure is a one-dimensional cycle in V_2. The closure of the π-preimage of $\delta_2 e \cap \operatorname{reg} V_2$ has the boundary $\delta_3 e \subset V_3$ (which may be empty).

Assume that the cycle e vanishes on the values of $\lambda = \lambda_*$ considered above. Let Y_2 and Y_3 be the analogs of the sets V_2 and V_3 for $Y = \operatorname{Im} f_{\lambda'} \cap B$. Consider some basic cycle of infinite order $\Delta \in H_2^{clos}(Y)$. It has boundary $\delta_2\Delta \in H_1^{clos}(Y_2)$ and $\delta_3\Delta \in H_0(Y_3)$. Here $\delta_3\Delta = 0$ if and only if $\delta_3 e = \varnothing$. The indices of intersection of closed and compact cycles on Y are as follows:

$$(\Delta, e) = \begin{cases} (\delta_2\Delta, \delta_2 e), & \text{if} \quad \delta_3 e = 0, \\ (\delta_3\Delta, \delta_3 e), & \text{if} \quad \delta_3 e \neq 0. \end{cases}$$

The indices on the right respectively refer to Y_2 and Y_3.

THEOREM 2. Consider a complex line transverse to the branching diagram Σ at a regular point λ_*. Let ω be a small loop on this line that circles λ_* once in the positive direction. Then the variations $\operatorname{Var}_\omega\Delta$ of the basis elements of infinite order in $H_2(Y,\partial Y)$ such that $(\Delta,e) > 0$ are:

I. 2e.

II. e.

III. 2e.

IV. e.

V. 0.

Cases II, IV, and V were examined in [4] (see also [3, §4.1]). Formulas for I and III follow easily from consideration of the effect of Var_ω on $H_1^{clos}(Y_2)$ and $H_0(Y_3)$ (cf., for example, [2, para. 2.1.1.2]).

3. Index of Intersection with a Vanishing Loop. The description we have given for local variational operators assumes that to describe the monodromy group for the germ of $C^2 \to C^3$ we must introduce the index of intersection with a vanishing cycle into the homology of a stable image. For this we refer to the situation used to define a distinguished set of vanishing cycles in Para. 1: Let γ be one of the distinguished paths on a line l' in general position that leads from a distinguished point $\lambda_0 \in l'$ to one of the branching points $\lambda_* \in l' \cap \Sigma$, and let e be a loop that vanishes along γ. Let λ' be a point near the end of γ, $V' = \mathrm{Im}\, f_{\lambda'}$. The group $H_2(V')$ is generated by cycles that have one-dimensional intersections with selfintersecting lines of V′. Using the natural homomorphism $H_2(V') \to H_2^{clos}(V')$, we set, for a compact cycle c on V′,

$$(c, e) = \begin{cases} (\delta_2 c, \delta_2 e), & \text{if} \quad \delta_3 e = 0, \\ (\delta_3 c, \delta_3 e), & \text{if} \quad \delta_3 e \neq 0. \end{cases}$$

The indices on the right refer to V_2' and V_3'.

The movement of cycles from the manifold V to the manifold V′ along the path γ determines the index of intersection with e on $H_2(V)$.

The indices of intersection for vanishing cycles I-V are 0, 0, 6, 12, and 24, respectively.

If e and e′ are two vanishing cycles on V, then

$$\begin{aligned} &\delta_3 e = \delta_3 e' = 0 \Rightarrow (e, e') = -(e', e), \\ &\delta_3 e \neq 0 \neq \delta_3 e' \Rightarrow (e, e') = (e', e), \\ &\delta_3 e \neq 0 = \delta_3 e' \Rightarrow (e', e) = 0, \\ &\delta_3 e \neq 0 = \delta_3 e' \not\Rightarrow (e, e') = 0. \end{aligned}$$

In the first two cases the indices can be immediately calculated for V, without moving the cycles to the "almost branching" image. In the third case the index is zero, since the cycle e′ can be removed from the cycle e by an isotopy in the neighborhood of e. In the fourth case the index need not vanish, since, in computing it, we used only an isotopy in a small neighborhood of e′.

Thus, the matrix for the intersections of vanishing cycles (with the cycles ordered in some way that need not be the same as in the distinguished set) has the block form

$$\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix},$$

where the matrix A_1 is skew symmetric and A_2 is symmetric.

4. Picard–Lefshets Operators. The group $\pi_1(\Lambda \setminus \Sigma, \lambda_0)$ is generated by ν classes of simple loops ω_i on $l' \setminus \Sigma$ that correspond to a system of distinguished paths γ_i (a simple loop ω_i leaves λ_0, proceeds along γ_i almost to the end of the path, then loops clockwise about the endpoint and returns to λ_0 along γ_i). As a result, the monodromy group is generated by the automorphisms of the group $H_2(V)$ that are induced by circuits in simple loops. The automorphisms h_{ω_i} are called Picard–Lefshets operators. Their description is a consequence of Theorem 2.

THEOREM 3. The Picard–Lefshets operators corresponding to vanishing loops e of the forms I-V are of the form

$$\begin{aligned} &\mathrm{I} - c \mapsto c - (c, e)\, e; \\ &\mathrm{II} - c \mapsto c - 1/2\, (c, e)\, e; \\ &\mathrm{III},\ \mathrm{IV} - c \mapsto c - 2\, (c, e)\, e/(e, e); \\ &\mathrm{V} - \mathrm{id}. \end{aligned}$$

Thus, operators III and IV are mappings into planes orthogonal to the corresponding cycles. Operators I and II on a space in $H_2(V)$ generated by cycles of types I and II act as skew-orthogonal mappings into shorter and longer cycles of functions on bounded manifolds.

5. Examples. 1°. Recall that the list of $\rightharpoondown$ -simple germs of mappings $C^2 \to C^3$ contains, as a subset, the list of simple functions functions on the halfplane: A_k, B_k, C_k, and F_4 (see [3]). By the degree of a germ we mean the maximum number of different preimages that a point can have with a relatively small shift of the germ (the shift need not be general). Then, on the whole, the contact classes of the curves on the (x,z) plane with boundary z = 0 have a one-to-one relationship to the $\mathcal{A}$-classes of the mappings $C^2 \to C^3$ of degree 2 and rank 1: the mapping $x, y \mapsto x, y^2, yg(x, y^2)$. corresponds to the curve g(x,z) = 0. The Whitnification of such mappings has no triple points, so here we encounter only cycles of types I and II. A distinguished set of vanishing cycles forms a basis for $H_2(V)$. The form we introduced for intersections on $H_2(V)$ coincides with the form for intersections of the function g on the halfplane [1]. Here short cycles of boundary singularities correspond to cycles of type I, and long cycles correspond to cycles of type II. This correspondence clearly extends to a correspondence of hypersurfaces in the manifolds C^n with boundary C^{n-1} and the mappings $C^n \to C^{n+1}$ of degree 2 and rank 1. This leads to the appearance of the Weyl groups A_k, B_k, C_k, D_k, E_k, and F_4 as well as their skew-symmetric analogs in the theory of mappings $C^n \to C^{n+1}$ as monodromy groups of the corresponding simple singularities.

2°. The $\mathcal{A}$-simple mappings $C^2 \to C^3$ that do not reduce to boundary singularities are exhausted by the single infinite series H_k, $k \geqslant 2$: $x, y \mapsto x, y^3, xy + y^{3k+1}$. In it rk $H_2(V)$ = k, but a distinguished set contains k + 1 members (in Fig. 2, which shows the Whitnification of H_2, we can clearly see two vanishing cycles of type III; a third, of type I, is equal, when the orientation is chosen properly, to the difference of the first two). It is possible to choose a distinguished point and a system of distinguished paths on a line in general position to the base of a versal deformation for the singularities of H_k so that the Dynkin diagram takes the form shown in Fig. 3.

We will state the type of each vanishing cycle that appears here. The skew-symmetric ($\delta_3 e_i = 0$) and symmetric ($\delta_3 e_i \neq 0$) parts of the diagram are separated by a dashed line. In the symmetric part the weight of the arcs is one sixth of the index of intersection of the corresponding cycles; in the remaining cases the weights and indices are the same. The weight 1 is omitted. The classical monodromy operator h (successive circuit of simple loops, beginning with the last) yields

$$e_1 \mapsto e_1, \quad e_2 \mapsto e_2 - e_1 - e_4, \quad e_3 \mapsto e_3 - e_1 - e_4,$$
$$e_s \mapsto -e_{s+1}, \quad 4 \leqslant s \leqslant k,$$
$$e_{k+1} \mapsto -e_{k+1} + e_k - e_{k-1} + e_{k-2} - \ldots \pm e_4 \pm 2e_1.$$

The characteristic polynomial of the operator h is $(\lambda - 1)(\lambda^{k-1} - 1)$.

3°. Any nonsimple germ of $C^2 \to C^3$ with degree greater than 2 is associated with the singularity P_4: $x, y \mapsto x, xy + y^3, xy^2 + \alpha y^4$ $(\alpha \neq 0, 1/2, 1, 3/2)$. The Dynkin diagram of P_4 is shown in Fig. 4.

The arcs in the skew-symmetric (left) part of the diagram are directed from i-th to j-th vertices so that the indices (e_i,e_j) are the same as the multiplicities of the corresponding arcs. Here

$$h: e_1 \mapsto -e_1 + 2e_5, \quad e_2 \mapsto e_1 + e_4 + 3e_5,$$
$$e_3 \mapsto -e_1 + e_5,$$
$$e_4 \mapsto e_1 + e_3 + e_4 + e_5, \quad e_5 \mapsto -e_3 + e_5,$$
$$\det(\lambda E - h) = (\lambda - 1)(\lambda^3 - 1).$$

6. Remarks. 1°. Consider the image V of the Whitnification of a mapping $C^1 \to C^2$. The intersection index of two one-dimensional cycles on V that is analogous to that introduced in Para. 3 is taken at points of transversal selfintersection of V. Here we must consider only the points at which the cycles under consideration pass from one sheet of the image to another. If, at a given point, the change in sheets upon movement along both cycles occurs in the same order, the index is taken equal to two. If the orders are opposite, the index is taken equal to –2. Such a form on $H_1(V)$ is symmetric.

With the above noted analogy with functions on bounded manifolds taken into account, we show a short cycle in Fig. 5 (Cartesian sheet), and a long cycle is illustrated in Fig. 6. The series $t \to t^2, t^{2k+1}$ of simple singularities has the Dynkin diagram of B_k. The Picard—Lefshets operators corresponding to the long and short cycles are mappings in orthogonal planes. The identity operator corresponds to the cycle that vanishes into the point of triple selfintersection with square of 6.

2°. For mappings $C^n \to C^{n+1}$, n > 2, the situation is hypothetically as follows. In the case of degeneracies of topological $\mathcal{A}$-codimension 1, an n-dimensional sphere must contract to a topologically stable image. It is to be expected that the action of Picard—Lefshets operators is described by intersection matrices for vanishing cycles with block-triangular form and alternating symmetric and skew-symmetric blocks on the diagonal.

3°. Still another possible generalization is description of the monodromy for the discriminant of the mapping g: $C^n,0 \to C^p,0$, $n \geq p$. It was announced in [6] that a common small shift in the germ g of finite $\mathcal{A}$-codimension is homotopically equivalent to the bouquet of a finite number of (p – 1)-dimensional spheres. Here the analog of the mapping $C^n \to C^{n+1}$ is provided by the restriction of g to the set of critical points.

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SIMPLE PROOF OF MACDONALD'S IDENTITIES FOR THE SERIES A

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1. Macdonald [1] associated a quite remarkable identity in the ring of formal power series in several variables with each root system R. A variety of different interpretations and proofs (see [2-7]) were subsequently associated with these identities. The goal of the present article is to provide a simple proof of the series of Macdonald identities for root systems of type A_l, using only combinatorial and elementary algebraic considerations.

For the system A_1 the Macdonald identity reduces to the classical Gauss–Jacobi identity. It is of the form

$$J(y,z)=\prod_{m\geq 1}(1+yz^{m-1})(1+y^{-1}z^m)(1-z^m)=\sum_{r\in Z}y^r z^{h(r)} \tag{1}$$

(here and below we use the notation $h(r) = r(r - 1)/2$). The author presented a simple combinatorial proof of (1) in [8]. It provides the foundation for the following discussion.

In order to construct the Macdonald identities we will need the following notation: We set $M_n = \{a = (a_1, \ldots, a_n) \in \mathbf{Z}^n \mid a_1 + \ldots + a_n = 1 + \ldots + n\}$. For $a \in \mathbf{Z}$ and natural n we denote the residue of a modulo n by $\mathrm{res}_n\, a \in \mathbf{Z}/n\mathbf{Z}$. For each sequence $(\nu_1,\ldots,\nu_n)$ of residues modulo n we define the number $\varepsilon(\nu_1,\ldots,\nu_n)$ to be equal to 0 or ± 1 according to the following rules: if all of the ν_i are different, i.e., $(\nu_1,\ldots,\nu_n)$ is a permutation of the sequence $(\mathrm{res}_n\, 1, \mathrm{res}_n\, 2,\ldots,\mathrm{res}_n\, n)$, then $\varepsilon(\nu_1,\ldots,\nu_n)$ is the sign of the permutation; otherwise we set $\varepsilon(\nu_1,\ldots,\nu_n) = 0$. For each $a = (a_1,\ldots,a_n) \in \mathbf{Z}^n$ we set $\varepsilon(a) = \varepsilon(\mathrm{res}_n\, a_1,\ldots,\mathrm{res}_n\, a_n)$ and $\varphi(a) = (a_1^2 + \ldots + a_n^2 - 1^2 - \ldots - n^2)/2n$. We will write the Macdonald identities in terms of this notation.

THEOREM 1. For every $n \geq 2$ the identity

$$\Big(\prod_{m\geq 1}(1-x^m)^{n-1}\Big)\prod_{1\leq j<i\leq n}\prod_{m\geq 1}\Big(1-\frac{x_i}{x_j}x^{m-1}\Big)\Big(1-\frac{x_j}{x_i}x^m\Big)=$$
$$=\sum_{a\in M_n}\varepsilon(a)\,x^{\varphi(a)}x_1^{1-a_1}\cdot\ldots\cdot x_n^{n-a_n} \tag{2}$$

holds in the ring of formal power series in x with coefficients from the ring of Laurent polynomials in $x_1,\ldots,x_n$.

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