Simple symmetric matrix singularities and the subgroups of Weyl groups A_μ, D_μ, E_μ

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To Vladimir Igorevich from former students

Abstract

We analyse the classification of simple symmetric matrix singularities depending on two parameters which was obtained recently by Bruce and Tari [2]. We show that these singularities are classified by certain reflection subgroups Y of the Weyl groups $X = A_\mu, D_\mu, E_\mu$. The Dynkin diagram of such a subgroup is obtained from the affine diagram of X by deleting vertices of total marking 2: deletion of two 1-vertices corresponds to a 2×2 matrix singularity, and deletion of one 2-vertex gives rise to a 3×3 matrix. The correspondence is based on an isomorphism of the discriminants and on the description of a relevant monodromy group of the determinantal curve. Moreover, the base of a miniversal deformation of a simple matrix singularity turns out to be isomorphic to the quotient of the complex configuration space of the group X by the subgroup Y.

We discuss lattice properties of symmetric matrix families in two variables which, in the case of simple singularities, define the choice of the subgroups.

A wide range of algebro-geometrical problems involve representation of functions as determinants of matrix families. Among those are such classical as finding curves tangent to a given plane curve at all points of their intersection (this goes back to Hesse [6] and Clebsch [4]) and resolutions of Cohen-Macaulay modules over singular hypersurfaces (see [10] and the bibliography there).

However, it seems that a natural question of deforming matrix families (corresponding, for example, to deforming a hypersurface over which a Cohen-Macaulay module is defined) had not been addressed until recently. Only a few years ago, Bruce and Tari studied in detail equivalences of germs of square matrix families, both arbitrary and symmetric ([2, 3]), which in a standard way lead to versal deformations. Bruce and Tari have listed simple singularities of the matrix families, but left a question of relating the lists to anything else open.

In this paper we are establishing one of possible relations, for simple symmetric matrix families in two variables. We are doing this by studying versal deformations of the families and the discriminants in their bases, as well as the vanishing topology of the determinantal curves. The relation we obtain is very similar to that between simple function-germs and the reflection groups A_μ, D_μ, E_μ, which is one of the cornerstones of singularity theory. First of all, since the determinant of a simple matrix family is a simple function, the relevant Weyl group X appears. But representing the function as a determinant introduces certain constraints which give rise to a reflection subgroup $Y \subset X$. It shows up in the following ways.

On one hand, the subgroup Y is the monodromy group of the one-variable stabilisation of the determinantal curve. On the other, the base of a miniversal deformation of a matrix family turns out to be naturally isomorphic to the quotient of the configuration space $\mathbf{C}^\mu$ of the group X by its subgroup Y. The mapping $\Lambda : \mathbf{C}^\mu/Y \to \mathbf{C}^\mu/X$ which completes the factorisation in a sense induces the miniversal matrix deformation from a miniversal deformation of the simple function singularity X.

The reflection subgroups $Y \subset X$ corresponding to the matrix singularities can be conveniently described in terms of the affine Dynkin diagrams (Figure 1). Namely, corank 2 simple symmetric matrix families in two variables are in one-to-one correspondence with those $Y \subset X$ for which a deletion of two 1-vertices from the diagram $\widetilde{X}$ leaves the diagram of Y. Similarly, each corank 3 family corresponds to a deletion of one 2-vertex from $\widetilde{X}$.

Such a diagrammatic way of seeing Y in X follows from lattice properties we are establishing for arbitrary, not necessarily simple, two-variable families of symmetric matrices. However, the matrix realisation of each diagrammatic (or, actually, lattice) possibility is just an experimental result. It would be

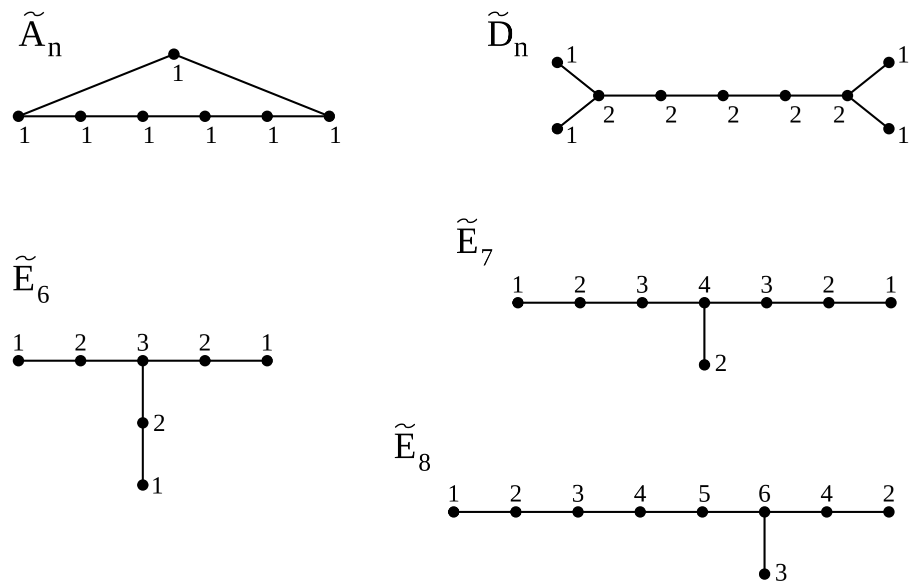

Figure 1: *The affine Dynkin diagrams.*

very interesting to understand if the correspondence observed goes beyond the simple classification.

There arise some other natural questions aiming at generalisations of facts which are true for simple matrix singularities. We formulate them within the exposition.

The paper is organised as follows.

In Section 1, we remind the simple classification from [2] and list all the adjacencies of the simple matrix singularities.

Section 2 introduces the discriminant Σ of a matrix family.

Section 3 deals with the geometry of the discriminant Σ of a simple matrix family. The determinantal function defines a mapping Λ between the bases of miniversal deformations of a matrix singularity and of its determinant. For a simple matrix, the dimensions of the bases are the same. We prove in Subsection 3.1 that in this case Λ is a finite covering.

In Subsection 3.2, this allows us to show that the complement to the discriminant of a simple matrix singularity is a $k(\pi, 1)$ space.

Subsection 3.3 establishes an isomorphism between the base of a miniversal deformation of a simple matrix family and the quotient of the configuration space of the Weyl group by its subgroup.

In Section 4, we study topology of the determinantal curves and of their one-variable stabilisations. The main result there is the characterisation, in terms of the Dynkin diagrams, of the pairs (Weyl group, its subgroup) which classify the simple matrix families. We also establish a number of lattice properties which hold for all 2-parameter families of symmetric matrices, not necessarily simple. These properties restrict very tightly the range of possible group pairs allowed to appear in the simple context. A remarkable feature of the simple classification is that all the group pairs within the range show up.

Section 5, the last one, contains a proof of the fact that the monodromy group of the one-variable stabilisation of the determinantal curve of a simple matrix is exactly the related subgroup of a Weyl group. There we also write out miniversal deformations of all simple matrix singularities. In fact, all the results of this paper have emerged from an observation that the weights of parameters in these deformations match sets of degrees of basic invariants of certain reflection groups, most of which are reducible.

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1 Simple symmetric matrix singularities

Let $Sym_n \simeq \mathbf{C}^{n(n+1)/2}$ be the space of complex symmetric $n \times n$ matrices. A holomorphic map-germ $S : (\mathbf{C}^m, 0) \to Sym_n$ is called a *family of (symmetric order n) matrices*.

Two such families, S_1 and S_2, are said to be *$\mathcal{SG}$-equivalent* ($\mathcal{S}$ for 'symmetric', $\mathcal{G}$ for 'general linear') if there exist a biholomorphism-germ h of $(\mathbf{C}^m, 0)$ and a map-germ $A : (\mathbf{C}^m, 0) \to GL(n, \mathbf{C})$ such that

$$S_1 \circ h = A^T S_2 A \,, \qquad (1)$$

where A^T is the transpose of A.

Allowing in (1) families A of special linear matrices only, we obtain equivalence of matrix families which we shall call the $\mathcal{SS}$-*equivalence* (the second $\mathcal{S}$ stays here for 'special linear').

It is natural to define the *corank of a matrix family* S as the corank of the matrix $S(0)$. Classifying corank k families is equivalent to classifying families S of order k matrices such that $S(0)$ is the zero matrix. For example, in the case of corank 1 matrix families, the $\mathcal{SS}$-classification coincides with the $\mathcal{R}$-classification of function-germs while the $\mathcal{SG}$-classification is the same as the contact classification of functions.

A classification of $\mathcal{SG}$-simple matrix families has been obtained by Bruce and Tari in [2]. Its most interesting part concerns 2-parameter families. If such a family is simple its corank is at most 3. The classification is given in Table 1. Our notation of each singularity there is a pair $(X; Y) =$ (Weyl group; its subgroup generated by reflections). Here X is also the singularity of the determinantal function of the family. The subgroup Y will be shown to be a corresponding monodromy group. The discriminant of the family will be described in terms of the inclusion $Y \subset X$. Relating the subgroups Y to the simple matrix singularities is the main objective of this paper.

In the table, we use the conventions: $A_0 = \{e\}$, $D_2 = A_1 \oplus A_1$, $D_3 = A_3$. In what follows we shall also set $D_1 = \{e\}$ and allow $m = 1$ in the corank 3 series $(D_{k+m}; D_k \oplus D_m)$, thus including the corank 2 series $(D_\mu; D_{\mu-1})$ into it.

A few invariants of the simple matrix families are given in Table 2. The invariants μ_2 and $wt(\Sigma_2)$ are defined in Sections 2 and 3.2 respectively. The values of the invariants strongly suggest certain interplay between the singularities.

Corank 1 matrix singularities will be denoted as the singularities of their determinantal functions.

All adjacencies of corank ≤ 2 simple matrix families can be obtained by composing the standard adjacencies of simple functions and the following:

$(A_{k+m+1}; A_k \oplus A_m) \to A_k, (A_{k+m}; A_{k-1} \oplus A_m), (A_{k+m}; A_k \oplus A_{m-1})$;

$(D_\mu; A_{\mu-1}) \to A_{\mu-1}, (D_{\mu-1}; A_{\mu-2}), (A_{\mu-1}; A_k \oplus A_{\mu-k-2}), k \geq [\frac{\mu-1}{2}]$;

$(D_\mu; D_{\mu-1}) \to D_{\mu-1}, (D_{\mu-1}; D_{\mu-2}), (A_{\mu-1}; A_{\mu-2} \oplus A_0)$;

$(E_6; D_5) \to D_5, (D_5; D_4), (D_5; A_4), (A_5; A_4 \oplus A_0), (A_5; A_3 \oplus A_1)$;

$(E_7; E_6) \to E_6, (E_6; D_5), (D_6; D_5), (D_6; A_5), (A_6; A_5 \oplus A_0), (A_6; A_4 \oplus A_1)$.

Table 1: *Simple corank >1 two-parameter families of symmetric matrices*

$(A_{k+m+1}; A_k \oplus A_m)$ $k \geq m \geq 0$	$\begin{pmatrix} x^{k+1} & y \\ y & x^{m+1} \end{pmatrix}$	$(D_{2k}; A_{2k-1})$ $k \geq 2$	$\begin{pmatrix} y & 0 \\ 0 & xy + x^k \end{pmatrix}$
$(D_\mu; D_{\mu-1})$ $\mu \geq 5$	$\begin{pmatrix} x & 0 \\ 0 & y^2 + x^{\mu-2} \end{pmatrix}$	$(D_{2k+1}; A_{2k})$ $k \geq 2$	$\begin{pmatrix} y & x^k \\ x^k & xy \end{pmatrix}$
$(E_6; D_5)$	$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$	$(E_7; E_6)$	$\begin{pmatrix} x & 0 \\ 0 & x^2 + y^3 \end{pmatrix}$
$(D_{k+m}; D_k \oplus D_m)$ $k \geq m \geq 2$	$\begin{pmatrix} x & 0 & 0 \\ 0 & x^{k-1} & y \\ 0 & y & x^{m-1} \end{pmatrix}$	$(E_6; A_5 \oplus A_1)$	$\begin{pmatrix} x & 0 & y \\ 0 & y^2 & x \\ y & x & 0 \end{pmatrix}$
$(E_7; D_6 \oplus A_1)$	$\begin{pmatrix} x & 0 & y \\ 0 & xy & x \\ y & x & 0 \end{pmatrix}$	$(E_8; E_7 \oplus A_1)$	$\begin{pmatrix} x & 0 & y \\ 0 & y^3 & x \\ y & x & 0 \end{pmatrix}$
$(E_7; A_7)$	$\begin{pmatrix} x & 0 & 0 \\ 0 & y & x \\ 0 & x & y^2 \end{pmatrix}$	$(E_8; D_8)$	$\begin{pmatrix} x & 0 & y^2 \\ 0 & y & x \\ y^2 & x & 0 \end{pmatrix}$

To obtain all adjacencies of all simple matrix singularities, we have to add

$$(D_{k+m}; D_k \oplus D_m) \to (D_{k+m-1}; D_{k-1} \oplus D_m), (D_{k+m-1}; D_k \oplus D_{m-1}),$$
$$(A_{k+m-1}; A_{k-1} \oplus A_{m-1});$$

$$(E_6; A_5 \oplus A_1) \to A_5, (D_5; D_3 \oplus D_2), (D_5; A_4), (A_5; A_3 \oplus A_1), (A_5; A_2 \oplus A_2);$$

$$(E_7; D_6 \oplus A_1) \to D_6, (E_6; A_5 \oplus A_1), (E_6; D_5), (D_6; D_4 \oplus D_2), (D_6; A_5),$$
$$(A_6; A_5 \oplus A_0), (A_6; A_4 \oplus A_1), (A_6; A_3 \oplus A_2);$$

$$(E_7; A_7) \to A_6, (E_6; A_5 \oplus A_1), (D_6; D_3 \oplus D_3), (D_6; A_5), (A_6; A_3 \oplus A_2);$$

$$(E_8; E_7 \oplus A_1) \to E_7, (E_7; D_6 \oplus A_1), (E_7; E_6), (D_7; D_5 \oplus D_2), (D_7; D_6),$$
$$(D_7; A_6), (A_7; A_6 \oplus A_0), (A_7; A_5 \oplus A_1), (A_7; A_4 \oplus A_2);$$

$$(E_8; D_8) \to D_7, A_7, (E_7; A_7), (E_7; D_6 \oplus A_1), (D_7; A_6), (D_7; D_4 \oplus D_3),$$
$$(A_7; A_6 \oplus A_0), (A_7; A_4 \oplus A_2), (A_7; A_3 \oplus A_3).$$

Table 2: *Some numerical invariants of simple matrix singularities* $(X; Y)$

X	A_{k+m+1}	D_μ	D_{k+m}	E_6	E_6	E_7	E_7	E_7	E_8	E_8
Y	$A_k \oplus A_m$	$A_{\mu-1}$	$D_k \oplus D_m$	D_5	$A_5 \oplus A_1$	E_6	$D_6 \oplus A_1$	A_7	$E_7 \oplus A_1$	D_8
$\|X:Y\|$	$\frac{(k+m+2)!}{(k+1)!\,(m+1)!}$	$2^{\mu-1}$	$2\frac{(k+m)!}{k!\,m!}$	27	36	56	63	72	120	135
μ_2	$m+1$	$[\frac{\mu}{2}]$	$2, m=1$ $m+2, m>1$	2	4	3	4	5	4	5
$wt(\Sigma_2)$	$(k+1)(m+1)$	$\frac{\mu^2-\mu}{2}$	$2km$	16	20	27	32	35	56	64

We call a matrix family $S = (s_{ij})$ *quasihomogeneous* if, in addition to the mapping S being quasihomogeneous, we also have

$$weight(s_{ij}s_{k\ell}) = weight(s_{i\ell}s_{kj})$$

for all the entries.

All the families of Table 1 are quasihomogeneous. Moreover, all the $\mathcal{SG}$-fencing (i.e. first non-simple) 2-parameter matrix families contain moduli in their principal parts. Therefore, the table is also the classification list of $\mathcal{SS}$-simple 2-parameter singularities.

Consider the extended tangent space $T_{\mathcal{SS}}S$ to the $\mathcal{SS}$-orbit of a family S in the space of all m-parameter symmetric order n matrix families (here the extension is in the standard sense of allowing diffeomorphisms h in (1) to shift the origin). As usual we define the *Tjurina number* $\tau_{\mathcal{SS}}$ of the family S as the codimension of $T_{\mathcal{SS}}S$ in the space $\mathcal{O}_m^{n(n+1)/2}$ of all variations of the matrix family:

$$\tau_{\mathcal{SS}}(S) = \dim_{\mathbf{C}} \mathcal{O}_m^{n(n+1)/2}/T_{\mathcal{SS}}S\,.$$

The number $\tau_{\mathcal{SG}}(S)$ is defined similarly. It coincides with $\tau_{\mathcal{SS}}(S)$ if S is quasihomogeneous. However, the general behaviour of $\tau_{\mathcal{SS}}$ with respect to other invariants of matrix singularities is much better than that of $\tau_{\mathcal{SG}}$. For example, numerical experiments suggest

Conjecture 1.1 *For any 2-parameter family S of symmetric matrices,*

$$\tau_{\mathcal{SS}}(S) = \mu(\det S)\,.$$

Remark 1.2 The conjecture has been proven in [1] for order 2 matrices, and it is easy to check that it is true for all the singularities of Table 1 too.

In particular, any adjacency of simple matrix families is a composition of those dropping the Tjurina number by 1. Indeed, pairs of singularities in all the adjacencies listed above have neighbouring $\tau_{\mathcal{SS}}$.

Once the quotient $\mathcal{O}_m^{n(n+1)/2}/T_{\mathcal{SS}}S$ is of finite dimension, it can be viewed in the standard way as a base of an $\mathcal{SS}$-miniversal deformation of the family S (in particular, this base is smooth). This follows from the general result due to Damon [5].

Remark 1.3 Apparently one must have $\tau = \mu$ also for 3-parameter families M of arbitrary $n \times n$ matrices and the equivalence $M \sim (AMB) \circ h$, where h is a diffeomorphism-germ of $(\mathbf{C}^3, 0)$, and A and B are 3-parameter families of elements of $SL(n, \mathbf{C})$. According to [1] again, the equality holds for corank < 3 families.

2 Discriminant of a matrix family

From now on we shall be considering only 2-parameter families of symmetric matrices, from the point of view of the $\mathcal{SS}$-equivalence. For short, they will be called just *matrix families* or *matrix singularities*. We always assume $\tau_{\mathcal{SS}}$ finite.

The base $\mathbf{C}^\tau$ of an $\mathcal{SS}$-miniversal deformation of a mapping $S_0 : (\mathbf{C}^2, 0) \to Sym_n$, $\tau = \tau_{\mathcal{SS}}(S_0)$, contains the *discriminant* hypersurface Σ formed by those values of the parameters $\lambda \in \mathbf{C}^\tau$ for which the corresponding perturbation S_λ of S_0 is not transversal to the variety of order n degenerate matrices. The latter condition is equivalent to the function $\varphi_\lambda = \det S_\lambda$ having critical value zero.

The discriminant consists of two components:

Σ_1, which corresponds to the non-transversality to the stratum of corank 1 matrices, and

Σ_2, corresponding to mappings whose image has non-empty intersection with the set of matrices of corank at least 2.

In fact, Σ_1 is the A_1 stratum in the base of a versal deformation, and Σ_2 is the $(A_1; A_0 \oplus A_0)$ stratum.

The *multiplicities*, we denote them μ_1 and μ_2, of the components Σ_1 and Σ_2, are two basic invariants of a matrix family S_0. They can be calculated as follows.

Let us consider, along with a family $S_0 = S_0(x, y)$ of order n symmetric matrices, its generic 1-parameter deformation $\widetilde{S}_0 = \widetilde{S}_0(x, y, t)$. Denote by $\mathrm{Min}_{n-1} \subset \mathcal{O}_{x,y,t}$ the ideal generated by all order $n-1$ minors of the matrix $\widetilde{S}_0$. Then

$$\mu_2 = \dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/\mathrm{Min}_{n-1} \,. \qquad (2)$$

For example, for order 2 matrix families, μ_2 is just the degree of the mapping $\widetilde{S}_0 : \mathbf{C}^3 \to Sym_2$.

Now introduce the ideal $J \subset \mathcal{O}_{x,y,t}$ generated by the function $\det(\widetilde{S}_0)$ and its first order derivatives with respect to x and y. Then

$$\mu_1 + 2\mu_2 = \dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/J\,. \tag{3}$$

Algebraically, the relation is clear since J is of finite codimension in $\mathcal{O}_{x,y,t}$ and has three generators, while the dimension of the quotient for the A_1 and $(A_1; A_0 \oplus A_0)$ singularities is respectively 1 and 2 as it follows from the form of the relevant miniversal deformations. A homological interpretation of formula (3) will be given in Section 4.1.

The values of μ_2 for the simple singularities are given in Table 2.

The multiplicity μ_1 turns out to be well-correlated with the Milnor number of the determinantal curve, at least in the simple cases. Namely, the calculations show that

$$\mu_1(S_0) = \mu(\det S_0) - 1 \tag{4}$$

for any corank 2 simple singularity S_0, and

$$\mu_1(S_0) = \mu(\det S_0) \tag{5}$$

for any corank 3 simple singularity. We comment on the range of validity of these two relations in Section 4.5.

3 The covering mapping between the deformation bases

3.1 The inducing map

The determinantal function on an $\mathcal{SS}$-miniversal family $\{S_\lambda\}$ of a matrix singularity S_0 defines a τ-parameter deformation $\{\varphi_\lambda\}$ of the isolated function singularity φ_0. Like any other deformation, the latter is equivalent to a deformation induced from an $\mathcal{R}$-miniversal deformation of φ_0. Therefore, there exists an inducing holomorphic map-germ

$$\Lambda : (\mathbf{C}^\tau, 0) \to (\mathbf{C}^\mu, 0)\,, \qquad \mu = \mu(\varphi_0)\,, \tag{6}$$

into the base of an $\mathcal{R}$-miniversal deformation of φ_0.

Assume now that S_0 is an $\mathcal{SS}$-simple matrix singularity. Then, due to Remark 1.2, $\tau = \mu$, and Λ is a mapping between spaces of the same dimension.

Theorem 3.1 *For a simple matrix singularity $(X;Y)$, the inducing mapping Λ is a ramified covering of order $|X:Y|$.*

Proof. Take a monomial $\mathcal{SS}$-miniversal deformation of a table singularity $S_0 \in (X;Y)$. Such a deformation is quasihomogeneous with all the parameters λ of positive weights $w_1, \dots, w_\mu$. (All the deformations are given in Section 5.)

Now take a quasihomogeneous $\mathcal{R}$-miniversal deformation of the simple function $\varphi_0 = \det S_0 \in X$. All of its parameters are also of positive weights, $d_1, \dots, d_\mu$.

So, the mapping Λ of (6) is a quasihomogeneous polynomial mapping of type $(w_1, \dots, w_\mu; d_1, \dots, d_\mu)$. For each of the simple matrix singularities, it is easy to write out Λ explicitly and check that $\Lambda^{-1}(0) = 0$ (the cases $X = A_\mu, D_\mu$ follow immediately from the polynomial considerations of Sections 5.1, 5.2 and 5.3 below; the E cases involve not very much illustrative calculations which we prefer to omit). Therefore, $\Lambda : \mathbf{C}^\mu \to \mathbf{C}^\mu$ is indeed a finite covering.

The degree of Λ is $\prod d_i / \prod w_j$. Normalise the weights so that $\max\{d_i\}$ is the Coxeter number of the Weyl group X. A casewise check shows that then the w_j are exactly the degrees of basic invariants of the reflection subgroup $Y \subset X$ (for example, one of these degrees is 1 for the order 2 matrix families). Hence, $\deg \Lambda = |X:Y|$. □

Remark 3.2 As this has been pointed out by Duco van Straten, the covering number 27 for the $(E_6; D_5)$ singularity is actually the number of finite double tangent straight lines of a plane quartic with a degenerate inflection at infinity. Each of the double tangents gives rise to a representation of the quartic's equation in a symmetric determinantal form which is a perturbation of the $(E_6; D_5)$ singularity.

There must exist similar interpretations of the covering numbers in other cases.

3.2 The critical locus of the covering

A miniversal deformation of the $(A_1; A_0 \oplus A_0)$ singularity has determinant

$$\det \begin{pmatrix} x+\lambda_0 & y \\ y & -x+\lambda_0 \end{pmatrix} = -x^2 - y^2 + \lambda_0^2 \,. \tag{7}$$

Thus, at least for simple matrix singularities, the inducing mapping Λ has an order 2 folding along Σ_2. So, Σ_2 is a part of the critical locus $\mathcal{C}$ of Λ.

Proposition 3.3 *For a simple matrix singularity, $\mathcal{C} = \Sigma_2$.*

Proof. The critical locus $\mathcal{C}$ of the quasihomogeneous mapping Λ is given by an equation of weight $\sum d_i - \sum w_j$. It is sufficient to check that the weight of an equation of Σ_2 is the same.

For every simple order n matrix singularity $S_0 = S_0(x, y)$, it is possible to find a quasihomogeneous one-parameter deformation $\widetilde{S}_0 = \widetilde{S}_0(x, y, t)$ (not necessarily generic) for which the number $r = \dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/\mathrm{Min}_{n-1}$ is finite. Then the weight of an equation of Σ_2 is $r \cdot wt(t)$, assuming the weights of x and y normalised as in the proof of Theorem 3.1.

Example 3.4 Consider a 1-parameter quasihomogeneous deformation of the singularity $(E_8; D_8)$:

$$\begin{pmatrix} x & t & y^2 \\ t & y & x \\ y^2 & x & 0 \end{pmatrix} .$$

For it, $r = \dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/\mathrm{Min}_2 = 8$. Since the weight of t here is also 8 (we need the weight of the entire order 3 determinant to be equal to the Coxeter number of E_8, which is 30), the weight of the equation of Σ_2 is 64.

The results of similar calculations for all the singularities are collected in the $wt(\Sigma_2)$ line of Table 2. An immediate check shows that in all the cases $wt(\Sigma_2) = \sum d_i - \sum w_j$. □

Conjecture 3.5 *The simplicity assumption can be omitted in Proposition 3.3.*

The inverse image $\Lambda^{-1}(\Delta)$ of the discriminant $\Delta \subset \mathbf{C}^\mu$ of the function singularity φ_0 is the discriminant Σ of the matrix singularity S_0. Therefore, we have

Corollary 3.6 *For a simple matrix singularity $(X;Y)$, the mapping $\Lambda : \mathbf{C}^\mu \setminus \Sigma \to \mathbf{C}^\mu \setminus \Delta$ is an unramified covering. Hence, $\mathbf{C}^\mu \setminus \Sigma$ is a $k(\pi,1)$ space, where π is a subgroup of index $|X:Y|$ in the Brieskorn braid group B_X of the Weyl group X.*

A better understanding of the group π is provided by the next Subsection.

3.3 Mirror description of the matrix discriminant

Theorem 3.7 *Let $(X;Y)$ be a simple matrix singularity. Consider the mirror arrangement $\mathcal{A}_X \subset \mathbf{C}^\mu$ of the Weyl group X. Let $\mathcal{A}_Y \subset \mathcal{A}_X$ be the mirror arrangement of the reflection subgroup Y, and $\mathcal{A}_{X\setminus Y} \subset \mathcal{A}_X$ the mirrors which are not in $\mathcal{A}_Y$. Then*

$$(\mathbf{C}^\mu, \mathcal{A}_X)/Y \simeq (\mathbf{C}^\mu, \Sigma)\,.$$

Moreover, this biholomorphism provides isomorphisms

$$\mathcal{A}_Y/Y \simeq \Sigma_1 \qquad \textit{and} \qquad \mathcal{A}_{X\setminus Y}/Y \simeq \Sigma_2\,.$$

Proof. We shall base on

Lemma 3.8 *For each simple singularity $(X;Y)$, there exists a surjective mapping*

$$h_{even} : \pi_1(\mathbf{C}^\mu \setminus \Sigma, a) \to Y$$

induced by the mapping Λ from the monodromy representation

$$h : \pi_1(\mathbf{C}^\mu \setminus \Delta, b) \to X\,, \quad b = \Lambda(a)\,,$$

of the suspended function singularity $\widehat{\varphi}_0(x,y,z) = \det\Big(S_0(x,y)\Big) + z^2$.

Details of the induced representation will be given in Section 4.3. A case-by-case proof of the lemma itself is postponed till Section 5.

We shall need the following factorisation mappings:

$$p_Y : (\mathbf{C}^\mu, \mathcal{A}_X) \to (\mathbf{C}^\mu, \mathcal{A}_X)/Y \simeq (\mathbf{C}^\mu, \Sigma')\,,$$

$$p_{X/Y} : (\mathbf{C}^\mu, \Sigma') \to (\mathbf{C}^\mu, \mathcal{A}_X)/X \simeq (\mathbf{C}^\mu, \Delta)$$

and

$$p_X = p_{X/Y} \circ p_Y\,.$$

We identify the target space of $p_{X/Y}$ with the base of an $\mathcal{R}$-miniversal deformation of the function singularity $\widehat{\varphi}_0$.

Lemma 3.9 *The mapping $p_{X/Y}$ induces from h a surjective homomorphism*

$$h' : \pi_1(\mathbf{C}^\mu \setminus \Sigma', a') \to Y', \quad a' \in p_{X/Y}^{-1}(b),$$

where $Y' \subset X$ is a subgroup isomorphic to Y.

Proof. We need to show that the image Y' of h' is isomorphic to Y. Choose $c \in p_Y^{-1}(a')$ and consider a commutative diagram

$$\begin{array}{ccccccccc} & & & & \pi_1(\mathbf{C}^\mu\setminus\Sigma', a') & \xrightarrow{h'} & Y' & & \\ & & & {}^{p_{Y,*}}\nearrow & \downarrow{}^{p_{X/Y,*}} & & \downarrow & \searrow & \\ \{e\} & \longrightarrow & \pi_1(\mathbf{C}^\mu\setminus\mathcal{A}_X, c) & \xrightarrow{p_{X,*}} & \pi_1(\mathbf{C}^\mu\setminus\Delta, b) & \xrightarrow{h} & X & \longrightarrow & \{e\}. \end{array}$$

The horizontal sequence here is exact, and the vertical arrows and $p_{Y,*}$ are embeddings. Hence the five-sequence going from $\{e\}$ to $\{e\}$ via h' is also exact. Therefore, $|X : Y'| = \deg p_{X/Y} = |X : Y|$ and $|Y'| = |Y|$.

On the other hand, the kernel of the monodromy h' contains in particular all loops in $\mathbf{C}^\mu \setminus \Sigma'$ which are contractible in $\mathbf{C}^\mu \setminus (\mathcal{A}_{X\setminus Y}/Y)$. Thus, h' is defined on $\pi_1\big(\mathbf{C}^\mu \setminus (\mathcal{A}_Y/Y), a'\big) \simeq \pi_1(\mathbf{C}^\mu\setminus\Sigma', a')/\pi_1\big(\mathbf{C}^\mu\setminus(\mathcal{A}_{X\setminus Y}/Y), a'\big)$. Therefore, Y' satisfies all the relations which exist in Y and, hence, is isomorphic to a quotient group of Y. But the orders of Y' and Y coincide, so $Y' \simeq Y$. □

For each of our pairs $(X; Y)$, any such subgroup Y' is conjugate to Y in X (in general, a conjugating automorphism of X is allowed to be outer). So, up to a symmetry of $(\mathbf{C}^\mu, \Delta)$ and up to an appropriate choice of a' in $p_{X/Y}^{-1}(b)$, we can assume that $Y' = Y$.

The embedded groups $\Lambda_*\big(\pi_1(\mathbf{C}^\mu \setminus \Sigma, a)\big)$ and $p_{X/Y,*}\big(\pi_1(\mathbf{C}^\mu \setminus \Sigma', a')\big)$ both have the same index $|X : Y|$ in $\pi_1(\mathbf{C}^\mu \setminus \Delta, b)$. The homomorphisms h_{even} and h' are basically restrictions of h to them, and have the same image $Y \subset X$. Hence, $h^{-1}(Y) = \Lambda_*\big(\pi_1(\mathbf{C}^\mu \setminus \Sigma, a)\big) = p_{X/Y,*}\big(\pi_1(\mathbf{C}^\mu \setminus \Sigma', a')\big)$ which implies that the mapping Λ can be lifted via $p_{X/Y}$ to a biholomorphism

$$Bih : (\mathbf{C}^\mu \setminus \Sigma, a) \to (\mathbf{C}^\mu \setminus \Sigma', a').$$

Continuity of Λ and of $p_{X/Y}$ on their entire domains $\mathbf{C}^\mu$ allows to extend Bih to a biholomorphism of the pairs $(\mathbf{C}^\mu, \Sigma) \simeq (\mathbf{C}^\mu, \Sigma')$, $\Lambda = p_{X/Y} \circ Bih$.

Finally, the description of the discriminant components Σ_1 and Σ_2 follows from the decompositions of the inverse images of the discriminant Δ: $\Lambda^{-1}(\Delta) = \Sigma_2 \cup \Sigma_1$ while $p_{X/Y}^{-1}(\Delta) = \mathcal{A}_X/Y = \mathcal{A}_{X\setminus Y}/Y \cup \mathcal{A}_Y/Y$. The first subsets in these unions are the critical loci of Λ and of $p_{X/Y}$ respectively. $\square$

Corollary 3.10 *Identify the bases of miniversal deformations of a simple matrix singularity* $(X;Y)$ *and of the function singularity* X *with the quotients* $\mathbf{C}^\mu/Y$ *and* $\mathbf{C}^\mu/X$. *Then the inducing mapping* Λ *completes a commutative triangle of the factorisation mappings*

$$\begin{array}{ccc} (\mathbf{C}^\mu, \mathcal{A}_X) & \xrightarrow{/Y} & (\mathbf{C}^\mu, \Sigma) \\ {}_{/X}\searrow & & \swarrow_{\Lambda} \\ & (\mathbf{C}^\mu, \Delta) & . \end{array}$$

Our constructions show that the group $\pi = \pi_1(\mathbf{C}^\mu \setminus \Sigma, a)$, when embedded by Λ_* into the Brieskorn group $B_X = \pi_1(\mathbf{C}^\mu \setminus \Delta, b)$, is formed by exactly those loops whose liftings into $\mathbf{C}^\mu \setminus \mathcal{A}_X$ join points from the same Y-orbit (rather than from the same X-orbit as it happens for arbitrary elements of B_X).

Remark 3.11 The numbers of mirrors in $\mathcal{A}_X$ and $\mathcal{A}_Y$ are respectively $\sum(d_i - 1)$ and $\sum(w_j - 1)$. Hence, the weight $\sum d_i - \sum w_j$ of Σ_2 is equal to the number of mirrors in $\mathcal{A}_{X\setminus Y}$.

Theorem 3.7 reduces the adjacency question for simple matrices to analysing the stratification of the mirror arrangements. Namely, we have

Corollary 3.12 *An adjacency* $(X;Y) \to (X';Y')$ *of simple matrix singularities exists if and only if the following holds. Consider the complex configuration space of the Weyl group* X *and any inclusion of* Y *into* X *as a reflection subgroup. Then in the configuration space there must exist a point with a stationary group* X' *such that* $X' \cap Y = Y'$.

The corollary was used to obtain the adjacency lists of Section 1. Of course, for corank 1 matrix families we set $Y = X$. In the $X = E_\mu$ cases the del Pezzo root systems (see, e.g., [8]) make the subgroup search easier. The completeness of the adjacency lists can be checked from the fact that the degree $|X : Y|$ of the covering mapping Λ can be calculated as the sum of the local degrees over a stratum X' in the target of Λ. This is the arithmetic of the $|X:Y|$ line of Table 2. For example,

$$|E_8 : (E_7{\oplus}A_1)| = 120 = 56 + 63 + 1 = |E_7 : E_6| + |E_7 : (D_6{\oplus}A_1)| + |E_7 : E_7|$$

shows that there are indeed just these three matrix singularities $(E_7; Y')$ adjacent to $(E_8; E_7{\oplus}A_1)$. On the other hand, the relation

$$|E_7 : A_7| = 72 = 2 \cdot 36 = 2|E_6 : (A_5{\oplus}A_1)|$$

demonstrates that there is just one singularity with $X' = E_6$ adjacent to $(E_7; A_7)$, and, under the mapping Λ of $(E_7; A_7)$, its stratum covers the E_6 stratum in the $\mathcal{R}$-miniversal deformation of E_7 twice.

Inspection of Table 2 suggests that there must also be certain relation between its $|X:Y|$ entries on one hand and the $wt(\Sigma_2)$ line on the other.

4 Vanishing topology of the determinantal curves

4.1 Distinguished sets of vanishing cycles

A choice of a generic line L in the base $\mathbf{C}^\tau$ of a miniversal deformation of a matrix singularity $S_0 = S_0(x,y)$ provides a family $\widetilde{S}_0 = \widetilde{S}_0(x,y,t)$ as in Section 2 (here t is a coordinate on L). Now let $*$ be a generic point in $\mathbf{C}^\tau$ and S_* the corresponding generic perturbation of S_0. Let L_* be the line parallel to L and passing through $*$. It yields a matrix family $\widetilde{S}_* = \widetilde{S}_*(x,y,t)$. We denote by φ_0, $\widetilde{\varphi}_0$, φ_* and $\widetilde{\varphi}_*$ the corresponding determinantal functions, and by μ and $\widetilde{\mu}$ the Milnor numbers of φ_0 and $\widetilde{\varphi}_0$. We shall also be using curves $V_\varepsilon = \{\varphi_* = \varepsilon\} \subset \mathbf{C}^2_{x,y}$ and surfaces $\widetilde{V}_\varepsilon = \{\widetilde{\varphi}_* = \varepsilon\} \subset \mathbf{C}^3_{x,y,t}$, all localised within appropriate balls. Here ε is a small generic number unless specified, so the varieties V_ε and $\widetilde{V}_\varepsilon$ are smooth while $\widetilde{V}_0$ has Morse singularities.

Consider function t on $\widetilde{V}_0$. Its critical values are intersections of L_* with $\Sigma \subset \mathbf{C}^\tau$: a critical value achieved at a regular point corresponds to an intersection with Σ_1 and the value at a singular point of $\widetilde{V}_0$ gives an intersection with Σ_2. Therefore, function t has μ_1 critical values of the first kind and μ_2 of the second. Also μ_2 is the number of singularities of $\widetilde{V}_0$.

Connect point $*$ on L_* with all the points of $L_* \cap \Sigma$ by a system of paths having no mutual- and self-intersections. As usual, this gives a *distinguished set of vanishing cycles* on V_0 consisting of, what we call them, μ_1 type 1 and μ_2 type 2 cycles. The types are defined as follows.

A local model for a *type 1 cycle* is the real 1-cycle vanishing in the standard A_1 family

$$\det \begin{pmatrix} x^2 + y^2 + \lambda_0 & 0 \\ 0 & I_{n-1} \end{pmatrix} = 0 \tag{8}$$

when the real negative number λ_0 tends to zero. Here I_{n-1} is the order $(n-1)$ identity matrix.

A *type 2 cycle* is modelled by the real 1-cycle which is contracted in the miniversal $(A_1; A_0 \oplus A_0)$ family

$$\det \begin{pmatrix} x + \lambda_0 & y & 0 \\ y & -x + \lambda_0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} = 0\,, \quad \text{that is} \quad x^2 + y^2 - \lambda_0^2 = 0\,,$$

when the real parameter λ_0 tends to zero.

In [1], type 1 and 2 cycles were called respectively long and short.

Proposition 4.1 *A distinguished set of vanishing cycles generates* $H_1(V_0; \mathbf{Z})$.

Proof. Consider the non-trivial part of the exact homological sequence of the pair $(\widetilde{V}_0, V_0)$:

$$0 \to H_2(\widetilde{V}_0) \to H_2(\widetilde{V}_0, V_0) \to H_1(V_0) \to 0\,. \tag{9}$$

The middle term here is freely generated by the thimbles based on a distinguished set of vanishing cycles on V_0. Now the claim follows from the surjectivity of the boundary operator. □

The ranks of the groups in (9) are respectively $\widetilde{\mu} - \mu_2$, $\mu_1 + \mu_2$ and μ. Hence we obtain

Corollary 4.2 $\quad \mu + \widetilde{\mu} = \mu_1 + 2\mu_2\,.$

Since $\mu = \dim_{\mathbf{C}} \mathcal{O}_{x,y}/\langle \varphi_{0,x}, \varphi_{0,y}\rangle$ and $\widetilde{\mu} = \dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/\langle \widetilde{\varphi}_{0,x}, \widetilde{\varphi}_{0,y}, \widetilde{\varphi}_{0,t}\rangle$ (here $\varphi_{0,x} = \partial\varphi_0/\partial x$ etc.), we have

$$\mu_1 + 2\mu_2 = \dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/\langle \widetilde{\varphi}_{0,x}, \widetilde{\varphi}_{0,y}, t\widetilde{\varphi}_{0,t}\rangle \,.$$

The expression (3) for the same quantity from Section 2 is based on counting critical values of function t on a smooth surface $\widetilde{V}_\varepsilon$. The number of these values is

$$\dim_{\mathbf{C}} \mathcal{O}_{x,y,t}/\langle \widetilde{\varphi}_0, \widetilde{\varphi}_{0,x}, \widetilde{\varphi}_{0,y}\rangle = \mu_1 + 2\mu_2 \,.$$

Indeed, each point of $L_* \cap \Sigma_1$ contributes 1 to the dimension of the quotient, and each point of $L_* \cap \Sigma_2$ contributes 2.

4.2 The type 1 sublattice mod2

The Picard-Lefschetz operator corresponding to a type 2 cycle e is the square of the ordinary operator, that is sends any 1-cycle c to $c - 2(c,e)e$ (the brackets denote the intersection number). Therefore, $H_1(V_0, \mathbf{Z}_2)$ contains a well-defined subspace spanned by a distinguished set of type 1 vanishing cycles. We denote this space by $\mathcal{L}_1^{(2)}$.

Proposition 4.3 *For a matrix singularity of corank* > 1,

$$H_1(V_0, \mathbf{Z}_2)/\mathcal{L}_1^{(2)} \simeq \mathbf{Z}_2 \,.$$

Proof. We start with showing that the quotient cannot be bigger.

Lemma 4.4 $\pi_1(\mathbf{C}^\tau \setminus \Sigma_2) = \mathbf{Z}$.

Proof. It is sufficient to show that Σ_2 is irreducible and its only codimension 1 singularities are transversal self-intersections.

Let $S : \mathbf{C}^2_{x,y} \times \mathbf{C}^\tau_\lambda \to Sym_n$ be an $\mathcal{SS}$-versal deformation of a corank n matrix family $S_0 = S_0(x,y)$. Then the rank at $0 \in \mathbf{C}^{2+\tau}$ of the Jacobi matrix of mapping S is maximal, equal to the dimension of Sym_n. Hence, the inverse image under S of the set $\mathcal{D}$ of corank ≥ 2 matrices is diffeomorphic to a cylinder over $\mathcal{D}$, and thus is irreducible. So, Σ_2 is irreducible too since it is the projection of $S^{-1}(\mathcal{D})$ onto the λ-space.

The only codimension 2 local matrix singularity of corank > 1 is $(A_2; A_1 \oplus A_0)$. For it, Σ_2 is smooth. So, the only stratum that contributes to the codimension 1 singularities of the set Σ_2 of an arbitrary matrix family is the bi-germ $2(A_1; A_0 \oplus A_0)$. And its discriminant is a pair of smooth transversal lines in $\mathbf{C}^2$. □

Thus, any two type 2 vanishing cycles coincide modulo $\mathcal{L}_1^{(2)}$.

On the other hand, consider the restriction S' to V_0 of the mapping $S_* : \mathbf{C}^2 \to Sym_n$:

$$S' : V_0 \to W \,,$$

where the target is the space of order n quadratic forms of corank 1. The target is homotopy equivalent to the manifold $U(n)/\big(U(1) \times O(n-1)\big)$ of all $(n-1)$-dimensional isotropic subspaces Π in the $2n$-dimensional symplectic space (see [9], p.178). Here $U(1)$ acts on the kernel K of a quadratic form Q, and $O(n-1)$ acts on the subspace Π hermitian-orthogonal to K and chosen so that Q is positive definite on it. A generator of $H_1(W, \mathbf{Z}_2) \simeq \mathbf{Z}_2$ can be represented by a loop $Q_t(u_1, u_2) + u_3^2 + \ldots + u_n^2$ of corank 1 forms along which a vector in $\mathbf{C}^2$, on which Q_t is positive, homotops to its negative.

A type 2 vanishing cycle

$$\begin{pmatrix} x+1 & y & 0 \\ y & 1-x & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} , \qquad x^2 + y^2 = 1 \,, \qquad x, y \in \mathbf{R} \,, \tag{10}$$

has a parametrisation $x = \cos 2t$, $y = \sin 2t$, $t \in [0, \pi]$. The corresponding quadratic forms are positive on the vectors $(\cos t, \sin t, 0, \ldots, 0)$. Hence, the S'-image of a type 2 cycle generates $H_1(W, \mathbf{Z}_2)$.

Clearly, mapping S' kills all type 1 vanishing cycles. □

In fact, the quotient of $H_1(V_0, \mathbf{Z})$ by the *integer* sublattice spanned by all possible type 1 vanishing cycles (not only those from a distinguished set) is also $\mathbf{Z}_2$ provided the matrix singularity has corank > 1 and $\tau > 1$. This follows from Proposition 4.3 and the validity of the claim for the $(A_2; A_1 \oplus A_0)$ singularity to which any such matrix family is adjacent.

4.3 Stabilisation of the determinantal curve and characterisation of the subgroups Y

Weyl groups arise in the study of function singularities as monodromy groups of simple function-germs on an odd-dimensional manifold. Looking for similar realisations for our reflection subgroups Y we follow this pattern and consider the one-variable stabilisation

$$\widehat{\varphi}_0(x,y,z) = \det(S_0(x,y)) + z^2$$

of the determinantal function φ_0 of a 2-parameter symmetric matrix family S_0. In the same way we stabilise the determinantal function of the entire $\mathcal{SS}$-miniversal family and introduce

$$\widehat{\varphi}_\lambda = \det S_\lambda + z^2\,, \quad \lambda \in \mathbf{C}^\mu\,.$$

Consider now the corresponding one-dimensional suspensions (all of them will be denoted by the hat) of all the topological objects we dealt with in Section 4.1. As before, a distinguished set of suspended vanishing cycles generates $H_2(\widehat{V}_0, \mathbf{Z})$. The Picard-Lefschetz operator on $H_2(\widehat{V}_0, \mathbf{Z})$ associated with a suspended type 1 (type $\widehat{1}$, for short) cycle $\widehat{e}$ is just the standard Picard-Lefschetz operator of a 3-variable function:

$$\sigma \mapsto \sigma + (\sigma, \widehat{e})\widehat{e}\,.$$

For a type $\widehat{2}$ cycle, the monodromy operator is clearly the square of the standard operator, that is the identity.

We shall call the monodromy group Γ of the function family $\{\widehat{\varphi}_\lambda\}$ *the even monodromy group* of the matrix singularity S_0. (The *odd* monodromy group is, of course, that of the non-stabilised family of the determinantal functions.) For a simple matrix singularity $(X;Y)$, this group is clearly a reflection subgroup of X. Lemma 3.8 insists that the subgroup is exactly Y.

Purely diagrammatic observations show that the following is true.

Theorem 4.5 *Let X be one of the Weyl groups A_μ, D_μ, E_μ, and $Y \subset X$ a reflection subgroup whose Dynkin diagram is obtained by deletion of vertices of total marking 2 from the affine diagram of X. There exists a one-to-one correspondence between such group pairs and simple corank > 1 symmetric matrix families in two variables. It relates*

- *deletions of two 1-vertices to corank 2 matrix singularities, and*
- *deletions of a 2-vertex to corank 3 matrices.*

The relevant matrix singularities are exactly those denoted $(X;Y)$.

The next subsection demonstrates that there is absolutely no surprise that only such group pairs arise as pairs (singularity of the determinant of a simple matrix family; monodromy group of the stabilised determinantal curve within the $\mathcal{SS}$-versal deformation). What is indeed surprising in the classification is that all the group pairs singled out are realised.

To proceed, we need to notice that, according to [7, 11],

- the Dynkin diagram Y can be obtained from the affine diagram $\widetilde{X}$ by a deletion of two 1-vertices if and only if the quotient $\mathcal{L}_X/\mathcal{L}_Y$ of the integer lattices is $\mathbf{Z}$ and $\mathcal{L}_Y + \mathbf{Z}\alpha = \mathcal{L}_X$ for any root α of the group X which is not a root of the subgroup Y;
- similar deletion of one 2-vertex is equivalent to the condition $\mathcal{L}_X/\mathcal{L}_Y \simeq \mathbf{Z}_2$.

4.4 The type 1 integer sublattice

The results of this subsection are a sort of 1-dimensional suspension of the results of Section 4.2. Our main object will now be the sublattice $\mathcal{L}_1 \subset H_2(\widehat{V}_0, \mathbf{Z})$ spanned by a distinguished set of type $\widehat{1}$ cycles. Due to the triviality of the Picard-Lefschetz operators corresponding to type $\widehat{2}$ vanishing cycles, $\mathcal{L}_1$ does not depend on the choices and contains all possible type $\widehat{1}$ cycles.

Theorem 4.6 *For any corank 2 matrix singularity,*

$$H_2(\widehat{V}_0, \mathbf{Z})/\mathcal{L}_1 \simeq \mathbf{Z}.$$

Proof. Smooth surface $\widehat{V}_0$ is given by the equation

$$\det \begin{pmatrix} a(x,y) & b(x,y)+z \\ b(x,y)-z & c(x,y) \end{pmatrix} = 0\,.$$

Here the rank of the matrix is never 0. Hence we have a mapping $\beta : \widehat{V}_0 \to \mathbf{C}P^1$ which sends a point to the direction of the image of the matrix.

Consider its action on the second homology:

$$\beta_* : H_2(\widehat{V}_0) \to H_2(\mathbf{C}P^1) = \mathbf{Z}\,.$$

All the type $\widehat{1}$ cycles are in the kernel of β_*. For a standard type $\widehat{2}$ cycle,

$$0 = \det\begin{pmatrix} x+1 & y+z \\ y-z & 1-x \end{pmatrix} = 1 - x^2 - y^2 + z^2\,, \qquad x, y, iz \in \mathbf{R}\,, \tag{11}$$

$\beta(x,y,z) = (y+z)/(1-x)$ is a stereographic projection of the unit 2-sphere. So, β_* sends this cycle to a generator of $H_2(\mathbf{C}P^1)$. □

Theorem 4.7 *For any corank* > 2 *matrix singularity,*

$$H_2(\widehat{V}_0, \mathbf{Z})/\mathcal{L}_1 \simeq \mathbf{Z}_2.$$

Proof. Any such matrix singularity is adjacent to $(D_4; D_2 \oplus D_2)$ for which the lattice quotient is $\mathbf{Z}_2$. Therefore, it is sufficient to prove that $H_2(\widehat{V}_0, \mathbf{Z}_2)/(\mathcal{L}_1 \otimes \mathbf{Z}_2) \simeq \mathbf{Z}_2$.

Let us introduce some notations. We denote by e the type 2 vanishing cycle (10), and by $\widehat{e}$ its suspension

$$0 = \det\begin{pmatrix} x+1 & y & 0 \\ y & 1-x & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} + z^2 = 1 - x^2 - y^2 + z^2\,, \qquad x, y, iz \in \mathbf{R}\,.$$

The subsets $W \subset W' \subset Sym_n$ will be the subsets of respectively all corank 1 and all corank ≤ 1 matrices. We shall denote by e' the 2-disc

$$\det\begin{pmatrix} x+1 & y & 0 \\ y & 1-x & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}, \qquad x, y \in \mathbf{R}\,, \qquad x^2 + y^2 \leq 1\,,$$

which contracts e in W'.

We are going to show that $\widehat{e}$ is a non-trivial element in the homology of the variety

$$\widehat{W} = \{\det(M) + z^2 = 0, M \in W'\} \subset Sym_n \times \mathbf{C}_z\,,$$

into which $\widehat{V}_0$ is naturally mapped. All the homology in our considerations will be with coefficients in $\mathbf{Z}_2$.

The complement to W' in $Sym_n \simeq \mathbf{C}^{n(n+1)/2}$ is of complex codimension 3. Therefore, the Alexander duality implies that $H_j(W') = 0$, $0 < j < 5$. So,

$$H_j(W', W) \simeq H_{j-1}(W)\,, \qquad 1 < j < 5\,. \tag{12}$$

Consider now a fragment of the Smith exact sequence of the double covering $\widehat{W} \to W'$:

$$\ldots \to H_3(W', W) \xrightarrow{\partial_3} H_2(W', W) \oplus H_2(W) \xrightarrow{i_2} H_2(\widehat{W}) \to$$
$$\to H_2(W', W) \xrightarrow{\partial_2} H_1(W', W) \oplus H_1(W) \to \ldots\,.$$

According to (12), the operators ∂_3 and ∂_2 followed by the projections onto the second summands of their targets are isomorphisms. So, the restriction of i_2 onto the first summand of its source is an isomorphism too.

From the proof of Proposition 4.3, e is a non-trivial element in $H_1(W)$. Hence e' is also non-trivial in $H_2(W', W)$. Therefore, $i_2(e')$ is a non-zero element of $H_2(\widehat{W})$. But $i_2(e') = \widehat{e}$. □

We return now to the characterisation of simple matrix singularities. First of all, it is easy to show that their determinants must be simple functions $X = A_\mu, D_\mu, E_\mu$. Let $\Gamma \subset X$ be the $\mathcal{SS}$-monodromy group of the stabilised determinantal curve of such a singularity. We would like to demonstrate, without using Lemma 3.8, that Γ must be one of the subgroups Y which can be obtained from X by the vertex deletion as mentioned in Theorem 4.5.

If our simple singularity has corank 3, Theorem 4.7 guarantees that the quotient $\mathcal{L}_X/\mathcal{L}_\Gamma$ of the integer lattices is $\mathbf{Z}_2$, which immediately restricts us to the choices of the corank 3 case of Theorem 4.5.

For a simple corank 2 matrix, Theorem 4.6 implies $\mathcal{L}_X/\mathcal{L}_\Gamma \simeq \mathbf{Z}$. From Lemma 4.4, $\mathcal{L}_X = \mathcal{L}_\Gamma + \mathbf{Z}\widehat{e}$, where $\widehat{e}$ is any type $\widehat{2}$ vanishing cycle. On the other hand, Theorem 3.1 guarantees that any root of X is either a type $\widehat{1}$ or a type $\widehat{2}$ cycle. But all type $\widehat{1}$ cycles are in $\mathcal{L}_\Gamma$. So, $\mathcal{L}_X = \mathcal{L}_\Gamma + \mathbf{Z}\widehat{e}$, where $\widehat{e}$ is any root of X which is not a root of Γ.

This shows that Theorem 4.5 follows in one direction mainly from the properties of arbitrary (not necessarily simple) matrix singularities, as this has been promised in Section 4.3. The only claim specific for simple matrices we have unfortunately used at this point is Theorem 3.1. However, Theorem 3.1 was used only to guarantee that any vanishing cycle of the determinantal function is either a type 1 or a type 2 cycle. And it is very likely that this much weaker property holds for any matrix singularity, not just for simple.

Remark 4.8 It was noticed in [1] that addition of z^2 to the determinant of an order 2 symmetric matrix family $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a passing to a family $\begin{pmatrix} a & b+z \\ b-z & c \end{pmatrix}$ of arbitrary matrices. For example, all simple (see Remark 1.3) 3-parameter families of arbitrary matrices are related this way to all $\mathcal{SS}$-simple order 2 symmetric families in 2 variables and, hence, a bit more directly to the reflection pairs $(X;Y)$.

4.5 The multiplicity of Σ_1 and the Milnor number

In this subsection we point out cases in which elements of distinguished sets of suspended type 1 vanishing cycles are linearly independent.

Proposition 4.9 *Consider an order 2 matrix singularity which has corank 2 and non-zero 1-jet. For it, $\mu_1 = \mu - 1$.*

Proof. Such a singularity reduces to the form

$$\begin{pmatrix} x & b(y) \\ b(y) & c(x,y) \end{pmatrix}.$$

For its generic 1-parameter deformation one can take

$$\widetilde{S} = \begin{pmatrix} x & b(y)+t \\ b(y)+t & c(x,y)+\gamma t \end{pmatrix},$$

where γ is a generic constant. The codimension of the ideal spanned by the entries of this matrix in $\mathcal{O}_{x,y,t}$ is the minimum r of the orders of b and $c|_{x=0}$. Therefore, $\mu_2 = r$.

On the other hand, the determinant of $\widetilde{S}$ has singularity A_{2r-1}. Now Corollary 4.2 implies the claim. □

Of course, the relation $\mu_1 = \mu - 1$ does not hold in general for corank 2 matrix singularities. For example, a generic order 2 matrix family with the trivial 2-jet has determinant of type X_9, along with $\mu_2 = 4$ and $\widetilde{\mu} = 9$, which implies $\mu_1 = 10$.

For order 3 matrices, numerical experiments suggest

Conjecture 4.10 *Consider a corank 3 matrix singularity whose determinant has a non-zero 3-jet. For it,* $\mu_1 = \mu$.

All simple corank 3 singularities share this property. However, for the family

$$\begin{pmatrix} x^2 & y & 0 \\ y & 0 & x \\ 0 & x & y^2 \end{pmatrix},$$

$\mu = 9$, while $\mu_2 = 4$ and $\widetilde{\mu} > 8$, and thus $\mu_1 > 9$.

5 Even monodromy and discriminants of the simple matrix singularities. Proof of Lemma 3.8

In this section we prove Lemma 3.8 showing, case by case, that the even monodromy group of a simple singularity $(X;Y)$ is Y. This is done mainly by identifying the relevant strata in the base of an $\mathcal{SS}$-miniversal deformation.

In a base of a versal deformation of a corank 2 simple matrix singularity, we point out a stratum of the corank 1 singularities (i.e. of the function singularities) Y. This guarantees that the group $\Gamma = h_{even}\Big(\pi_1(\mathbf{C}^\mu \setminus \Sigma)\Big)$ contains the reflection group Y. On the other hand, Γ cannot be bigger than Y since the multiplicity of Σ_1 is $\mu - 1$ and all of it already comes from the multiplicity of the discriminant of Y.

Majority of corank 3 simple families $(X;Y)$ have subgroups Y reducible: $Y = Y' \oplus Y''$. In a versal deformation of such a singularity, we find the strata $Y', Y'' \subset \Sigma_1$ and check that they are transversal to each other. Since the multiplicities of the discriminants of the function singularities Y' and Y'' now add up to the multiplicity μ of Σ_1, we conclude that $\Gamma = Y' \oplus Y''$.

Finally, for the two remaining corank 3 families, $(E_7; A_7)$ and $(E_8; D_8)$, we calculate relevant Dynkin diagrams which allow us to check the conclusion of Lemma 3.8 for these singularities too.

While looking for specific strata in bases of versal deformations in the $X = A_\mu, D_\mu$ cases, we obtain simple descriptions of the discriminants Σ making Theorem 3.7 obvious.

In what follows, the lower index in the notation of a polynomial is its degree.

5.1 Series $(A_{k+m+1}; A_k \oplus A_m)$

This singularity has a miniversal deformation

$$\begin{pmatrix} p_{k+1}(x) & y \\ y & q_{m+1}(x) \end{pmatrix}, \tag{13}$$

where p and q are monic polynomials, with the sum of the coefficients of respectively x^k and x^m being zero.

The monodromy. The stratum $A_k \subset \Sigma_1$ of the corank 1 matrix singularities is given by the condition of the polynomial p having the only root, of multiplicity $k+1$. Similarly, the stratum A_m corresponds to the polynomial q having a root of multiplicity $m+1$. The two strata meet transversally along the stratum $A_k A_m \subset \Sigma_1$ which is smooth one-dimensional and has a regular parametrisation

$$p = \Big(x - (m+1)t\Big)^{k+1}, \qquad q = \Big(x + (k+1)t\Big)^{m+1}.$$

The multiplicity μ_1 of the discriminant component Σ_1 of $(A_{k+m+1}; A_k \oplus A_m)$ is $\mu - 1 = k + m$. Therefore, a line in the base of deformation (13) passing through a point of the stratum $A_k A_m$ and having a generic direction does not meet Σ_1 anywhere else. Hence, the even monodromy is indeed $A_k \oplus A_m$.

In particular, a distinguished set of vanishing cycles for this singularity can be chosen so that the Dynkin subdiagram for the type $\widehat{1}$ cycles is a disjoint union of the standard A_k and A_m diagrams (cf. Remark 5.1 below).

The discriminant. The discriminant component Σ_1 in the base of deformation (13) corresponds to either p or q having a multiple root. The component Σ_2 is the resultant of p and q. Thus indeed $(\mathbf{C}^{k+m+1}, \Sigma) \simeq \Big(\mathbf{C}^{k+m+1}, W_{A_{k+m+1}}\Big)/A_k \oplus A_m$ as in Theorem 3.7.

The mapping Λ of the base of deformation (13) to the base of the miniversal deformation of the function singularity A_{k+m+1} is given by the coefficients of the polynomial $p(x) \cdot q(x)$. In terms of the root sets of the polynomials, this is a passing from the $A_k \oplus A_m$ orbits on $\mathbf{C}^{k+m+1}$ to the A_{k+m+1} orbits. So, Λ has degree

$$|A_{k+m+1} : (A_k \oplus A_m)| = \frac{(k+m+2)!}{(k+1)!\,(m+1)!}\,. \qquad (14)$$

The fundamental group of the complement $\mathbf{C}^{k+m+1} \setminus \Delta$ to the discriminant of the function singularity A_{k+m+1} is the group B_{k+m+2} of braids on $k+m+2$ strings. The group $\pi_1(\mathbf{C}^{k+m+1} \setminus \Sigma)$ is a subgroup of 2-coloured braids in B_{k+m+2}. The colouring is by two subsets of the endpoints containing $k+1$ and $m+1$ elements: the coloured braids are allowed to realise permutations only within the subsets. The mapping Λ embeds the subgroup into B_{k+m+2}. The index of the subgroup in B_{k+m+2} is given by (14).

5.2 Series $(D_\mu; A_{\mu-1})$

For $\mu = 2k$ and $\mu = 2k+1$, we take for miniversal deformations respectively

$$\begin{pmatrix} y & q_{k-1}(x) \\ q_{k-1}(x) & xy + p_k(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y & q_k(x) \\ q_k(x) & xy + p_k(x) \end{pmatrix}.$$

In the first family p is a monic polynomial and q arbitrary, and in the second vice versa.

The monodromy. One can easily find regular parametrisations of the $A_{\mu-1}$ strata (which are smooth) in the deformations:

$$\begin{pmatrix} y & 0 \\ 0 & xy + (x+t)^k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y & (x+t^2)^k \\ (x+t^2)^k & xy + 2t(x+t^2)^k \end{pmatrix}.$$

Again, a line of a generic direction in the base of the miniversal deformation passing through a point of the $A_{\mu-1}$ stratum cannot meet Σ_1 anywhere else since $\mu_1 = \mu - 1$. So, the even monodromy group Γ is $A_{\mu-1}$.

The discriminant. To understand the discriminant, one can start with eliminating y from the condition that the curve $\det = 0$ in the above deformations is singular. This implies that the discriminant Σ in the both cases can be described as the set of polynomials

$$p^2(x) + 4xq^2(x) \qquad (15)$$

with multiple roots.

Setting $x = z^2$ in (15), we factorise it as

$$\Big(p(z^2) - 2izq(z^2)\Big) \cdot \Big(p(z^2) + 2izq(z^2)\Big) . \tag{16}$$

Notice that the roots of one of the factors here are exactly the negatives of the roots of the other. Thus we can consider the set of roots of (16) as a set of μ dipoles in $\mathbf{C}$ symmetric about the origin: the points in the dipole are marked either $-$ or $+$ depending on which factor they are roots of. Dipoles are allowed to collapse to the origin. For non-discriminantal values of the deformation parameters, at most one of the μ dipoles can collapse and the points in all the other dipoles must all be geometrically distinct. Two dipoles coinciding along with their markings (that is, each of the two factors in (16) having a double root) corresponds to Σ_1. Two dipoles coinciding while the markings being opposite (that is, each of the two factors in (16) has both z_0 and $-z_0$ as its roots) corresponds to Σ_2.

Now, any fixed ordered set of $(z_1, \ldots, z_\mu) \in \mathbf{C}^\mu_z$ is the set of roots of, say the first factor in (16) for a certain choice of the deformation parameters. The factorisation $\mathbf{C}^\mu_z / S_\mu$ by the action of the full symmetric group $S_\mu \simeq A_{\mu-1}$ is a passing to the coefficients of the first factor in (16), that is to the parameters of the $\mathcal{SS}$-miniversal deformation. Therefore, consistently with Theorem 3.7,

$$\Big(\mathbf{C}^\mu_z, W_{D_\mu}\Big)/A_{\mu-1} = (\mathbf{C}^\mu_{p,q}, \Sigma)\,,$$

where W_{D_μ} is the set of the mirrors $z_i = \pm z_j$.

The free term in (15) is a square. Hence, the expansion of (15) provides the mapping Λ of $\mathbf{C}^\mu_{p,q}$ into the space $\mathbf{C}^\mu / D_\mu \simeq \mathbf{C}^\mu_\Lambda$ of the polynomials

$$x^\mu + \Lambda_{\mu-1} x^{\mu-1} + \ldots + \Lambda_1 x + \Lambda_0^2 \tag{17}$$

which also serves as the base of a miniversal deformation of the function singularity D_μ. Passing from (16) to (17) is the consideration of the above dipole sets up to an even number of repolarisations in the μ pairs of points. Therefore, the degree of Λ is $2^{\mu-1}$.

The mapping Λ embeds $\pi_1(\mathbf{C}^\mu_{p,q} \setminus \Sigma)$ into the Brieskorn braid group $B_{D_\mu} = \pi_1(\mathbf{C}^\mu_\Lambda \setminus \Delta_{D_\mu})$ of the Weyl group D_μ.

5.3 Series $(D_{k+m}; D_k \oplus D_m)$

A miniversal deformation now is

$$\begin{pmatrix} x & \alpha & \beta \\ \alpha & p_{k-1}(x) & y \\ \beta & y & q_{m-1}(x) \end{pmatrix}.$$

Here p and q are monic polynomials, $k \geq m \geq 1$, $k+m \geq 4$. Recall that $D_1 = \{e\}$ and $D_2 = A_1 \oplus A_1$.

The monodromy. The stratum $D_k \subset \Sigma_1$ is given by the equations $\alpha = 0$, $p = x^{k-1}$. Similarly to the previous cases, for $m = 1$ (that is for a corank 2 matrix singularity) it provides all the multiplicity $\mu_1 = \mu - 1 = k$ of Σ_1 which guarantees that the even monodromy group is D_k.

For $m > 1$, we notice that the stratum $D_m = \{\beta = 0, q = x^{m-1}\}$ meets the D_k stratum transversally. A line joining generic points of the two strata does not meet Σ_1 elsewhere since $\mu_1 = \mu = k + m$. Therefore, the even monodromy group is indeed $D_k \oplus D_m$, and a distinguished set of vanishing cycles can be chosen so that the Dynkin subdiagram formed by the type $\widehat{1}$ cycles is a disjoint union of the standard D_k and D_μ diagrams.

The discriminant. Eliminating y from the equation $\det = 0$ in assumption that it should define a singular curve, we see that the discriminant Σ of the matrix singularity is the set of polynomials

$$\Big(xp(x) - \alpha^2\Big) \cdot \Big(xq(x) - \beta^2\Big) \qquad (18)$$

with multiple roots: any of the factors in (18) having a multiple root corresponds to Σ_1 and a common root of the two factors corresponds to Σ_2. Expanding (18) and collecting the coefficients of the powers of x, one obtains the mapping Λ into the base of a miniversal deformation (17) of the function singularity D_{k+m}.

The two factors in (18) give rise to the groups B_{D_k} and B_{D_m} when considered just individually. Similar to Section 5.1, the group $\pi_1(\mathbf{C}^{k+m} \setminus \Sigma)$ can be viewed as a subgroup of 2-coloured D-braids (with k 'strings' of one colour and m of the other) in $B_{D_{k+m}}$.

5.4 Exceptional singularities

5.4.1 Corank 2 families

Their miniversal deformations are:

$$(E_6; D_5) : \qquad \begin{pmatrix} x & y^2+a \\ y^2+a & x^2+\varepsilon xy+\delta y^2+\gamma x+\beta y+\alpha \end{pmatrix},$$

$$(E_7; E_6) : \qquad \begin{pmatrix} x & cy^2+by+a \\ cy^2+by+a & x^2+y^3+\delta xy+\gamma x+\beta y+\alpha \end{pmatrix}.$$

One finds the crucial corank 1 strata (both smooth) in the deformations:

$(E_6; D_5) \to D_5$:

$$\det \begin{pmatrix} x & y^2-8t^6 \\ y^2-8t^6 & x^2+8txy+20t^2y^2+20t^4x+96t^5y+112t^8 \end{pmatrix} =$$

$$= X^3 - Y^4 + 2tX^2Y, \text{ where } X = x+2ty+4t^4,\ Y = y+4t^3;$$

$(E_7; E_6) \to E_6$:

$$\det \begin{pmatrix} x & \frac{3}{2}ty^2-9t^5y+13t^9 \\ \frac{3}{2}ty^2-9t^5y+13t^9 & x^2+y^3-6t^2xy+21t^6x-36t^8y+83t^{12} \end{pmatrix} =$$

$$= X^3+XY^3-\tfrac{1}{4}t^2Y^4, \text{ with } X = x-2t^2y+7t^6,\ Y = y-4t^4.$$

Now $\mu_1 = \mu - 1$ implies that the even monodromy groups are respectively D_5 and E_6 as it has been anticipated.

5.4.2 Miniseries $(E_\mu; Y'_{\mu-1}\oplus A_1)$

These have the following miniversal families:

$$(E_6; A_5\oplus A_1) : \qquad \begin{pmatrix} x & \alpha & y \\ \alpha & y^2+cx+by+a & x+\beta \\ y & x+\beta & \gamma \end{pmatrix},$$

$$(E_7; D_6\oplus A_1) : \qquad \begin{pmatrix} x & \alpha & y \\ \alpha & xy+dy^2+cx+by+a & x+\beta \\ y & x+\beta & \gamma \end{pmatrix},$$

$$(E_8; E_7{\oplus}A_1): \qquad \begin{pmatrix} x & \alpha & y \\ \alpha & y^3+exy+dy^2+cx+by+a & x+\beta \\ y & x+\beta & \gamma \end{pmatrix}.$$

In all three cases, the stratum $A_1 \subset \Sigma_1$ contains a smooth component $\gamma = 0$. It meets transversally the complementary smooth one-dimensional strata $Y'_{\mu-1}$:

$(E_6; A_5{\oplus}A_1) \to A_5$:

$$\det \begin{pmatrix} x & 0 & y \\ 0 & y^2+2tx+2t^3 & x+3t^2 \\ y & x+3t^2 & 4t \end{pmatrix} =$$

$$= -\Big(t(x-t^2)-y^2\Big)^2 - (x-t^2)^3;$$

$(E_7; D_6{\oplus}A_1) \to D_6$:

$$\det \begin{pmatrix} x & -\frac{1}{2}t^4 & y \\ -\frac{1}{2}t^4 & xy+ty^2-2t^2x-2t^3y & x+2t^3 \\ y & x+2t^3 & -4t \end{pmatrix} =$$

$$= -X^3 - XY^3 - tX^2Y, \text{ where } X = x+ty-t^3,\ Y = y-t^2;$$

$(E_8; E_7{\oplus}A_1) \to E_7$:

$$\det \begin{pmatrix} x & \frac{37}{2}t^7 & y \\ \frac{37}{2}t^7 & y^3-3txy+21t^3y^2-21t^4x+112t^6y+68t^9 & x+21t^5 \\ y & x+21t^5 & -4t \end{pmatrix} =$$

$$= -X^3 - Y^5 - tXY^3, \text{ with } X = x-4t^2y-14t^5,\ Y = y+5t^3.$$

The transversality and $\mu_1 = \mu$ imply that in each case the even monodromy group is $Y = Y'_{\mu-1}{\oplus}A_1$.

5.4.3 Singularities $(E_\mu; Y_\mu)$ with irreducible subgroups Y_μ

Their miniversal deformations are:

$$(E_7; A_7): \qquad \begin{pmatrix} x & \alpha & by+a \\ \alpha & y & x+\gamma y+\beta \\ by+a & x+\gamma y+\beta & y^2+dy+c \end{pmatrix},$$

$$
(E_8; D_8): \qquad \begin{pmatrix} x & \alpha & y^2+by+a \\ \alpha & y & x+\gamma y+\beta \\ y^2+by+a & x+\gamma y+\beta & ey^2+dy+c \end{pmatrix}.
$$

Setting in the $(E_7; A_7)$ versal deformation all the parameters except for a to be zero gives a $\mathbf{Z}_7$-equivariant determinantal function. We deform it using the parameter c. This provides the complete graph with 7 vertices and (-1)-edges as the Dynkin diagram formed by the type $\widehat{1}$ cycles. Since this is one of possible A_7-diagrams, the even monodromy group Γ is A_7.

Now perturb the $(E_8; D_8)$ singularity by assigning non-zero values to just two of the versal parameters, a and e, $|a| << |e|$. After this, variation of c gives a distinguished set of type $\widehat{1}$ cycles whose Dynkin diagram is the same complete graph as in the $(E_7; A_7)$ case, but with two of its vertices attached to an extra vertex by the edges of weight 1. This is a D_8 graph.

Remark 5.1 The even monodromy groups of all corank 2 simple matrix families could also be obtained from the odd Dynkin diagrams calculated in [1]. These can be easily modified to describe the suspended case.

The modification pattern is illustrated in Figure 2 by the $(A_{k+m+1}; A_k \oplus A_m)$ singularity. There white vertices are type 2 cycles and black are type 1. An edge $e' \to e''$ of multiplicity k means $(e', e'') = k$. Each rhombus gives a relation $a = b - c + d$ between its vertices. A rhombus looses its diagonal when we pass to the surface. The relation becomes $\widehat{a} = \widehat{b} - \widehat{c} - \widehat{d}$. Now clear all the type $\widehat{2}$ cycles off the diagram obtained. This leaves the diagram of $A_k \oplus A_m = \Gamma$.

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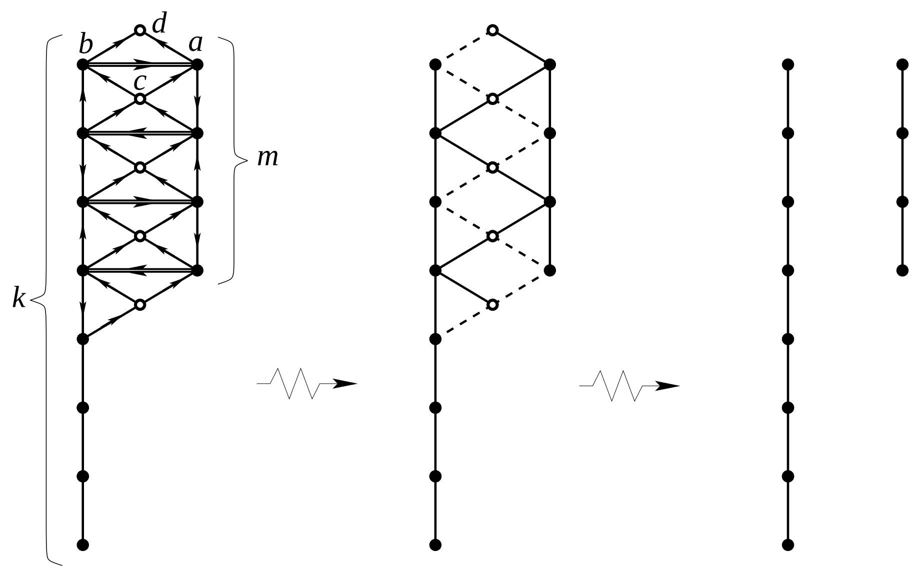

Figure 2: *The* $(A_{k+m+1}; A_k \oplus A_m)$ *singularity: getting from a Dynkin diagram of the curve (left) to the diagrams of the surface (centre) and of the even monodromy group (right).*

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