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On Stability of Projections of Lagrangian Varieties*

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Abstract. We show that Lagrangian and Legendre varieties associated with matrix singularities and singularities of composite functions are stable in the sense of a natural modification of Givental's notion of stability of Lagrangian projections.

Key words: Lagrangian variety, stability, matrix singularities, ramified covering, discriminant.

The study of singular Lagrangian and Legendre varieties was initiated by Arnold some twenty-five years ago, when he was investigating singularities in the variational problem of bypassing an obstacle [1]. The first examples of such varieties, open swallowtails, were related to the discriminants of noncrystallographic Coxeter groups [4, 8]. Incorporating these examples into a general context, Givental [4] introduced the notion of stability of Lagrangian and Legendre varieties under perturbations of the symplectic structure and the Lagrangian projection (or, respectively, the contact structure and the Legendre projection) alone, with the diffeomorphic type of the variety being preserved.

Later, it was shown [7] that this notion of stability has an explicit geometric description in terms of generating families, versal deformations of function singularities, and inducing maps.

The interest in the theory of singular Lagrangian and Legendre varieties has been growing recently owing to possible applications to Frobenius structures, D-modules, and other areas.

In the present paper, we extend the results of [7] to a natural modification of Givental's notion of stability and show that the stability condition holds for a wide class of Lagrangian and Legendre varieties associated with matrix singularities (see [2, 3, 6]) and singularities of composite functions [5].

1. 0-Stability

In this section, we recall some standard notions and modify them for later use.

1.1. The Lagrangian setting. A singular *Lagrangian (sub)variety* L of a symplectic space M^{2n} is an n-dimensional analytic subset of M that is Lagrangian in the ordinary sense at all its regular points. A *Lagrangian projection* π is a projection $\pi\colon M \to B^n$ defining a fibration whose fibers are Lagrangian.

The fibers of every Lagrangian fibration possess a well-defined affine structure. Indeed, local coordinates on the base lift to regular functions on the total space, which are pairwise in involution. Hence their Hamiltonian vector fields commute, do not vanish, and are tangent to the fibers.

The restriction $\pi|_L$ of a Lagrangian projection π to a Lagrangian subvariety $L \subset M$ is called a *Lagrangian map*.

Two Lagrangian maps of Lagrangian subvarieties L' and L'' are said to be *equivalent* if there exists a symplectomorphism of the ambient symplectic spaces that maps L' onto L'' and the fibers of one Lagrangian projection onto the fibers of the other. In particular, L' and L'' are symplectomorphic.

The germ of a Lagrangian map $\pi|_L$ of a variety L at a point m is said to be *stable* if the germ of every Lagrangian map $\widetilde{\pi}|_L$ close to $\pi|_L$ at any point $\widetilde{m}$ close to m is equivalent to the germ of

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$\pi|_L$ at some point close to m. Note that only the fibration π is allowed to vary in this definition, while the subvariety L is fixed.

According to Givental [4], this notion of stability is equivalent to the following notion of versality of the map germ $\pi|_L$.

Let $\mathscr{O}_L$ be the algebra of regular functions on L, and let $\mathbf{m}_{B,m}$ be the maximal ideal in the algebra $\mathscr{O}_{B,m}$ of function germs on the base B at the point $\pi(m)$. We define the local algebra of the germ of $\pi|_L$ at m as

$$Q_m = \mathscr{O}_L/((\pi|_L)^*(\mathbf{m}_{B,m}))\mathscr{O}_L.$$

The algebra Q_m is the algebra of restrictions of functions on L to the intersection of L with the fiber $F_{\pi(m)} = \pi^{-1}(\pi(m))$.

We denote the subspace of affine functions on the fiber $F_{\pi(m)}$ (with respect to the corresponding affine structure) by $\mathscr{A}_m$ and the restriction homomorphism that takes each function on the fiber to its restriction to $L \cap F_{\pi(m)}$ by $r\colon \mathscr{A}_m \to Q_m$.

The germ of $\pi|_L$ at $m \in L$ is said to be *versal* if r is surjective and *miniversal* if r is an isomorphism.

Let p, q be local Darboux coordinates on M about m: $p(m) = p_0$, $q(m) = 0$, and $\pi(p,q) = q$. The Weierstrass preparation theorem implies that the versality of $\pi|_L$ at m is equivalent to the representability of every analytic function germ φ on M at $m = (p_0, 0)$ in the form

$$\varphi(p,q) = \psi(p,q) + \sum_{j=1}^{n} a_j(q)p_j + a_0(q), \tag{1}$$

where the a_j, $j \geqslant 0$, are analytic function germs on the base B and the function germ ψ vanishes on L.

Remark. The decomposition means that every function germ on M at m is the sum of a function vanishing on L and a function affine on each fiber. Therefore, each Hamiltonian vector field near m is the sum of a Hamiltonian vector field tangent to L and a Hamiltonian vector field preserving the fibration π. Hence the homotopy method implies that each symplectomorphism germ of M at m close to the identity is the composition of a symplectomorphism preserving L and a symplectomorphism preserving the standard projection π. Since each perturbation of the germ of π in the class of Lagrangian projections is the composition of π with an appropriate symplectomorphism, we see that versality implies stability. See [4] for detailed proofs.

We now proceed to a special case of the theory. Namely, let $M = T^*B$ be the cotangent bundle of a manifold B^n. Let T_0 be the zero section of T^*B, and let $\mathrm{Sym}_0(M)$ be the subgroup of symplectomorphisms of M preserving T_0.

Two Lagrangian maps of Lagrangian subvarieties of the cotangent bundle are said to be 0-*equivalent* if they are equivalent via a symplectomorphism in $\mathrm{Sym}_0(M)$.

Replacing equivalence by 0-equivalence in the definition of stability, we obtain the notion of 0-stability of Lagrangian map germs.

The zero section T_0 determines a linear structure on the fibers of the cotangent bundle. Replacing the space $\mathscr{A}_m$ of affine functions on F_m by the invariantly defined subspace $\mathscr{A}_m^0$ of linear functions, we obtain the definition of 0-*versality*, which is equivalent to the representability of each function germ φ on M at m such that $\varphi|_{T_0} = 0$ in the form

$$\varphi(p,q) = \psi(p,q) + \sum_{j=1}^{n} a_j(q)p_j, \tag{2}$$

where the germs a_j and ψ are the same as in (1) and we assume that the Darboux coordinates p vanish on T_0.

Just as before, 0-versality implies 0-stability.

The *multiplicity* μ of a 0-miniversal germ of a Lagrangian map, that is, the rank of its local algebra treated as a linear space, is equal to n. It is at most n if the germ is 0-versal.

In what follows, we consider only the complex case. Note that the results of this section can also readily be transferred to the real case.

Lemma 1. *The projection* $\pi\colon T^*\mathbb{C}^n \to \mathbb{C}^n$, $(p,q)\mapsto q$, *of a Lagrangian germ* L *at the origin is* 0*-stable if and only if the germs of the products* p_ip_j, $i,j=1,\dots,n$, *admit the decompositions*

$$p_ip_j = \psi_{ij}(p,q) + \sum_{k=1}^{n} c_{ij}^k(q)p_k, \tag{3}$$

where the function germs ψ_{ij} *and* c_{ij}^k *are holomorphic and the* ψ_{ij} *vanish on* L.

Proof. The "only if" part is obvious. To prove the "if" part, we note that the ideal I generated by all quadratic polynomials $P_{ij}(p) = p_ip_j - \sum c_{ij}^k(0)p_k$, $i,j=1,\dots,n$, in the space of all holomorphic function germs on the fiber F_0 is of finite codimension. Modulo I, each function germ on F_0 is an affine function in p. After the projection into the local algebra Q_0, that is, after further reduction modulo functions vanishing on L (more precisely, on $L\cap F_0$), such a function is still affine in p. Hence the 0-versality condition holds. □

An analog of this lemma involving stability (rather than 0-stability) can be found in [4].

We define the *suspension* of a Lagrangian fibration $\pi\colon M\to B$ as the direct product

$$\widehat{\pi} = (\pi,\pi_0)\colon \widehat{M} = M\times T^*\mathbb{C} \to B\times\mathbb{C}$$

of π by the canonical projection $\pi_0\colon T^*\mathbb{C}\to\mathbb{C}$.

The *suspension* of a Lagrangian variety $L\subset M^{2n}$ is the $(n+1)$-dimensional Lagrangian variety $\widehat{L}\subset M\times T^*\mathbb{C}$ defined as the product of L by the line $\ell = \{p_{n+1} = \text{const} \neq 0\}$ in the space $T^*\mathbb{C}$ equipped with the standard Darboux coordinates p_{n+1}, q_{n+1}.

The propositions below readily follow from the definitions.

Proposition 2. *A map germ* $\pi|_L$ *at* $m\in M$ *is* (*mini*)*versal if and only if its suspension* $\widehat{\pi}|_{\widehat{L}}$ *is* 0-(*mini*)*versal at some point* (*and hence at all points*) *of the line* $m\times\ell$ *in* $\widehat{M}$.

Example. The germ of the standard projection π of the Lagrangian submanifold $L\subset T^*\mathbb{C}^n = \{p,q\}$ determined by a generating family $f=f(x,q)$ with parameters $q\in\mathbb{C}^n$ and variables $x\in\mathbb{C}^k$ by the formula

$$L = \{(p,q) \mid \exists x : \partial f/\partial x = 0,\ p = \partial f/\partial q\}$$

is stable if and only if the family germ $f(\,\cdot\,,\,\cdot\,)$ is an $\mathscr{R}^+$-versal deformation of the function germ $f(\,\cdot\,,0)$. The projection is 0-stable if and only if $f(\,\cdot\,,\,\cdot\,)$ is an $\mathscr{R}$-versal deformation of $f(\,\cdot\,,0)$.

Proposition 3. *Consider a Lagrangian subvariety germ* L *in* $\widehat{M} = T^*\mathbb{C}^n\times T^*\mathbb{C}$ *at a point outside the zero section. Suppose that the germ* $\widehat{\pi}|_L$ *is* 0*-versal and* L *belongs to a regular hypersurface in* $\widehat{M}$ *transversal to the* $\partial_{p_{n+1}}$*-direction. Then* $\widehat{\pi}|_L$ *is* 0*-equivalent to the suspension of the versal map germ* $\pi|_{L'}$ *of a Lagrangian subvariety* $L'\subset M = T^*\mathbb{C}^n$.

1.2. The Legendre setting. A singular n-dimensional subvariety of a contact space is said to be *Legendre* if it is Legendre in the ordinary sense at all its regular points. We use the projectivized cotangent bundle $PT^*\mathbb{C}^{n+1}$ and the space $J^1(\mathbb{C}^n,\mathbb{C}) = \{p,q,z\}$ of 1-jets of functions on $\mathbb{C}^n$ equipped with the contact form $\alpha = dz - p\,dq$ as standard (and equivalent) local models of contact $(2n+1)$-dimensional spaces. The definitions of Legendre maps, stability, etc. are similar to those in the Lagrangian case (see also [4]).

The symplectization and contactization functors relate Lagrangian and Legendre germs as follows.

A. The projection $\rho\colon (p,q,z)\mapsto(p,q)$ maps a Legendre variety $\Lambda\subset J^1(\mathbb{C}^n,\mathbb{C})$ onto the Lagrangian variety $\rho(\Lambda)\subset T^*\mathbb{C}^n$.

B. Locally, a Lagrangian fibration and its zero section uniquely determine the Liouville form $\alpha = p\,dq$, which is a primitive of the symplectic form $\omega = d\alpha$. Given a Lagrangian germ $L\subset T^*\mathbb{C}^n$ at a point m, by $L_{0,m}$ we denote the set of points $s\in L$ such that the integral of α along some path γ on L joining m and s is zero.

For simplicity, we assume that the integral is independent of the local path γ, that is, the cohomology class of α on L is zero. (See [4] for examples in which this cohomology class is nonzero.)

If $L_{0,m}$ does not meet the zero section of $T^*\mathbb{C}^n$, then its projectivization is a Legendre (or isotropic) variety in $PT^*\mathbb{C}^n$. The projection $W_0(L,m) = \pi(L_{0,m}) \subset \mathbb{C}^n$ is called the 0-*wave front* of L.

C. For a Lagrangian germ $L \subset T^*\mathbb{C}^n$ at a point m, the set $\Lambda_{L,m} \subset J^1(\mathbb{C}^n, \mathbb{C})$ of points (p,q,z) such that $s = (p,q) \in L$ and the integral of α along a path joining m and s in L is equal to z is a Legendre variety in $J^1(\mathbb{C}^n, \mathbb{C})$.

A symplectomorphism germ $\theta \in \mathrm{Sym}_0(T^*\mathbb{C}^n)$ preserving π preserves α. Hence if $\theta(L') = L''$, then $\theta(L'_{0,m}) = L''_{0,\theta(m)}$. In Darboux coordinates, θ has the form

$$\theta\colon (p,q) \mapsto (P, \check{\theta}(q)),$$

where $\check{\theta}$ is the underlying diffeomorphism of the base and $P = (\check{\theta}^{-1})^* p$ is the value at p of the linear operator on the fibers dual to the inverse of the derivative of θ. In particular, $\check{\theta}(W_0(L', m)) = W_0(L'', \theta(m))$.

These definitions imply the following assertion.

Proposition 4. *Consider a Legendre germ $\Lambda \subset J^1(\mathbb{C}^n, \mathbb{C})$. Suppose that the variety $\widehat{\rho(\Lambda)}$ does not meet the zero section and its standard Lagrangian projection is* 0*-stable. Then the projection of Λ into $J^0(\mathbb{C}^n, \mathbb{C})$ is Legendre stable.*

Conversely, if the projection of Λ into $J^0(\mathbb{C}^n, \mathbb{C})$ is Legendre stable and Λ is quasihomogeneous with positive weights of the variables, then $\widehat{\rho(\Lambda)}$ is 0*-stable.*

2. Stability of Induced Maps

2.1. The critical value theorem. The images of singular points of a Lagrangian variety L under the Lagrangian map $\pi|_L$ and the images of critical points of $\pi|_L$ on the regular part of L form the *caustic* Σ_L of the Lagrangian map.

The caustic of a Lagrangian germ L at a point m of finite multiplicity μ is a proper analytic subset of positive codimension in the base B.

For a point $q \notin \Sigma_L$ close to the base point $\pi(m)$, the inverse image $\pi^{-1}(q) \cap L$ consists of μ distinct points m_i close to m. We can assume that locally π is the standard fibration $T^*B \to B$. This allows us to introduce the *Maxwell set* $M_L \subset B$ as the closure of the set of points $q \notin \Sigma_L$ such that the μ values of z on $\Lambda_{L,m} \cap (\pi \circ \rho)^{-1}(q)$ are not all distinct. If μ is finite, then the Maxwell set is a germ of a proper analytic subset of the base. The union of the caustic and the Maxwell set is called the *bifurcation diagram* $\mathrm{Bif}(\pi, L)$ of the Lagrangian projection.

Consider the Lagrangian projection $\pi\colon T^*\mathbb{C}^n \to \mathbb{C}^n$ of a Lagrangian variety germ L. Let $g\colon \mathbb{C}^k \to \mathbb{C}^n$ be a smooth map germ. If the choice of the base points of the germs is consistent, we define the *induced Lagrangian map* $g^*(\pi|_L)$ as the projection of the subvariety $g^*(L) \subset T^*\mathbb{C}^k$ into $\mathbb{C}^k$.

Theorem 5. *Suppose that the projection germs $\pi|_L$ at m, a point $m \notin T_0$, and $g^*(\pi|_L)$ at a point l such that $\rho(\pi(l)) = \pi(m)$ are* 0*-miniversal and* 0*-stable, respectively. Then the critical value set Ξ_g of g is contained in the union $W_0(L,m) \cup \mathrm{Bif}(\pi, L)$.*

Note that a point of the source space is said to be *critical* if the derivative of the map at the point is not surjective. In particular, all points of the source are critical if its dimension is less than that of the target; in this case, the theorem implies that g maps $\mathbb{C}^k$ into $W_0(L,m) \cup \mathrm{Bif}(\pi, L)$.

The stability analog of Theorem 5 was proved in [7].

Proof. Take a point $q_0 \in \mathbb{C}^n \setminus \Sigma_L$ close to the base point. Its $\pi|_L$-inverse image consists of n distinct points $m_1, \dots, m_n \in F_{q_0}$ different from the origin. The multigerm of $\pi|_L$ at the finite set $\{m_1, \dots, m_n\}$ is 0-versal. (The decomposition (2) holds for multigerms.) Equivalently, the m_i are linearly independent in the fiber F_{q_0}: the restriction of an arbitrary function on the fiber to this set coincides with the restriction of a linear function.

Now let $\lambda_0 \in g^{-1}(q_0)$. Let $I \subset T_{q_0}\mathbb{C}^n$ be the image of the derivative $g_*\colon T_{\lambda_0}\mathbb{C}^k \to T_{q_0}\mathbb{C}^n$. The dual map $g^*\colon F^n_{q_0} \to F^k_{\lambda_0}$ between the fibers of the cotangent bundles is the composition of the projection pr onto the quotient of F_{q_0} by the subspace $I^\vee$ of covectors annihilating I and an embedding. Suppose that the dimension r of $I^\vee$ is positive, that is, λ_0 is a critical point of g. Since $g^*(\pi|_L)$ is 0-stable, it follows that the pr-images of the linearly independent points $m_1, \dots, m_n \in F_{q_0}$ form a linearly independent set in the $(n-r)$-dimensional space $F_{q_0}/I^\vee$. (The images are counted without multiplicities.) Hence the set $\{m_0 = 0, m_1, \dots, m_n\}$ of vertices of an n-simplex $\sigma \subset F_{q_0}$ is taken to the set $\{m'_0 = 0, m'_1, \dots, m'_{n-r}\}$ of vertices of an $(n-r)$-simplex in $F_{q_0}/I^\vee$. In particular, the rank r subspace $I^\vee$ is spanned by all differences $m_i - m_j$ such that $pr(m_i) = pr(m_j)$, that is, by the vectors in all faces of σ contracted by pr to points. (The sum of dimensions of such faces is equal to r.)

Near each of the points m_i, $i = 1, \dots, n$, the Lagrangian variety L is locally the graph of the differential of a function $z = \psi_i(q)$, $\psi(q_0) = 0$. The linearly independent points $m_i \in F_{q_0}$ are the differentials of the ψ_i at q_0.

For each pair $i \neq j$, by $\Delta_{ij} \subset T_{q_0}\mathbb{C}^n$ we denote the hyperplane tangent to the hypersurface $\psi_i(q) - \psi_j(q) = 0$.

For each ℓ, let $\Delta_\ell \subset T_{q_0}\mathbb{C}^n$ be the hyperplane tangent to the hypersurface $\psi_\ell(q) = 0$. The hyperplanes Δ_ℓ and Δ_{ij} are dual to the direction lines of 1-dimensional faces of the simplex $\sigma \subset F_{q_0}$.

The 0-stability of the multigerm $g^*(\pi|_L)$ at the points of F_{λ_0} is equivalent to the condition that the subspace I is the intersection of all subspaces Δ_ℓ such that $pr(m_\ell) = 0$ and all subspaces Δ_{ij} such that $pr(m_i) = pr(m_j) \neq 0$. Hence I is the intersection of subspaces in $T_{q_0}\mathbb{C}^n$ dual to certain faces of the simplex σ. Since I is contained in the tangent cone at q_0 of the critical value set Ξ_g, it follows that the regular strata of Ξ_g near q_0 coincide with the integrable manifolds of the distributions defined, by analogy with I, in the spaces $T_q\mathbb{C}^n$ by some sets of faces of the corresponding n-simplices in the fibers F_q.

It follows from [7, Subsec. 7.1. and 7.2] that such integrable manifolds of maximum dimension containing $\pi(m)$ in their closures are the regular strata of the caustic, the Maxwell set, and, as is easily seen, the wave front $W_0(L, m)$. Hence $\Xi_g \subset W_0(L, m) \cup \mathrm{Bif}(\pi, L)$. □

Theorem 6. *If g is a proper map germ between spaces of the same dimension, then the 0-stability of $g^*(\pi|_L)$ is equivalent to g being a ramified covering with ramification locus contained in $W_0(L, m) \cup \mathrm{Bif}(\pi, L)$.*

Proof. In this case, the regular strata of Ξ_g are $(n-1)$-dimensional. By Theorem 5, 0-stability implies the desired property of the ramification locus. To prove the converse, it suffices to notice that the induced map $g^*(\pi|_L)$ is 0-miniversal outside the ramification locus.

Moreover, it is versal at regular points of the ramification set, as is easily seen from the action of the dual map g^* on the corresponding simplex in the fiber. Hence each holomorphic function germ $\varphi(p, q)$ admits a decomposition (2) with coefficients $a_j(q)$ uniquely determined on the complement of an analytic subset of codimension at least 2. By Hartogs' theorem, the decomposition exists in a full neighborhood of the base point. □

Remark. Suppose that a Lagrangian variety germ L at $m \in T^*\mathbb{C}^n$ is the suspension of a Lagrangian germ L' at a point not contained in the zero section of $T^*\mathbb{C}^{n-1}$. The base $\mathbb{C}^n$ of the suspended Lagrangian fibration contains the distinguished coordinate function q_n corresponding to the second factor in the decomposition $L \simeq L' \times \mathbb{C}$. The caustic and the Maxwell set for L are also isomorphic to the products of the caustic and the Maxwell set for L' by a line (the q_n-axis). On the other hand, the hyperplane tangent to the wave front $W_0(L, m)$ at m is given by the equation $dq_n = 0$.

If, under the assumptions of Theorem 6, the ramification locus Ξ_g contains an $(n-1)$-dimensional component of the caustic or of the Maxwell set, then the direction ∂_{q_n} belongs to the image I of the differential of g at points arbitrary close to m. Hence the composition $q_n \circ g$ is nonsingular at the base point. On the other hand, if the ramification locus is contained in $(n-1)$-dimensional component of the wave front $W_0(L,m)$, then the composition $q_n \circ g$ is necessarily singular at the base point. Otherwise, the hyperplanes tangent to Ξ_g near the base point would not be close to the hyperplane $dq_n = 0$.

2.2. Composite functions. An interesting class of 0-stable Lagrangian projections is provided by versal deformations of composite maps [6].

Given a function germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$, consider the group $\mathscr{K}_f$ (see [6]) that consists of diffeomorphism germs Θ of the product space $(\mathbb{C}^m \times \mathbb{C}^n, (0,0))$ fibered over the projection onto the first factor (i.e., having the form$\Theta\colon (x,y) \mapsto (X(x), Y(x,y))$, $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$, and satisfying $f(Y(x,y)) = f(y)$ for all (x,y)).

The group $\mathscr{K}_f$ naturally acts on the space of map germs $\varphi\colon (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ by taking the graph of each map germ to the graph of another map germ.

Suppose that a map germ φ at the origin has a finite Tyurina number τ with respect to the group $\mathscr{K}_f$. Let $\Phi(x,\lambda) = \varphi(x) + \sum \lambda_s \varphi_s(x)$, $\lambda \in \mathbb{C}^\tau$, be a $\mathscr{K}_f$-miniversal deformation of φ. Consider the composition $F = f \circ \Phi$.

Theorem 7. *The Lagrangian projection defined by the generating family germ $F(x,\lambda)$ is 0-stable.*

Proof. Let $t \in (\mathbb{C},0)$ be an additional parameter. Consider the deformation

$$F_{ij} = f \circ \left(\Phi + t\,\frac{\partial F}{\partial \lambda_j}\,\varphi_i \right)$$

of the composite function $f \circ \varphi$. Since $F_{ij}|_{t=0} = F$ and Φ is K_f-versal, it follows that there exists a family of K_f-equivalences depending on t and inducing F_{ij} from F:

$$F_{ij}(x,\lambda,t) = f \circ \Big(\varphi(X(x,\lambda,t)) + \sum_{s=1}^{\tau} \Lambda_s(\lambda,t)\varphi_s(X(x,\lambda,t)) \Big).$$

Moreover, we can choose the family in such a way that the map $(x,\lambda) \mapsto (X,\Lambda)$ is the identity map for $t = 0$.

Differentiating this equation with respect to t at $t = 0$, we obtain

$$\frac{\partial F}{\partial \lambda_i}\,\frac{\partial F}{\partial \lambda_j} = \sum \frac{\partial F}{\partial x_r}\,\frac{\partial X_r}{\partial t} + \sum \frac{\partial F}{\partial \lambda_k}\,\frac{\partial \Lambda_k}{\partial t}\,.$$

Since $\partial F/\partial \lambda_i = p_i$ and $\partial F/\partial x_r = 0$ on the Lagrangian variety defined by the generating family F, we see that the last relation coincides with the 0-stability criterium of Lemma 1 for this variety. $\square$

Suppose that the germ at the origin of the composite function $h = f \circ \varphi$ has a finite multiplicity μ. The deformation $F = f \circ \Phi$ of h is induced from an $\mathscr{R}$-miniversal deformation H of h by a map germ $g\colon (\mathbb{C}^\tau,0) \to (\mathbb{C}^\mu,0)$ between the deformation bases.

Corollary 8. *If $\tau = \mu$ and the inducing map g is proper, then g is a covering ramified over the 0-wave front of the 0-stable Lagrangian manifold defined by the generating family H.*

Proof. The assertion is trivial if the function germ f is regular (and hence g is a diffeomorphism). Thus we can assume that f has a critical point at the origin. In this case, the composition of g with the projection $\mathbb{C}^\mu \to \mathbb{C}$ along the hyperplane tangent to the discriminant of h at the origin is singular at $0 \in \mathbb{C}^\tau$. Now Theorems 7 and 6 and the remark after Theorem 6 imply the desired assertion.

Remarks. 1. The covering map inducing the determinantal function of a versal matrix deformation of a simple matrix singularity from a versal deformation of the determinantal function of the unperturbed matrix (see [6]) is a special case of the map in Corollary 8.

2. The space of linear functions on a fiber F_q of the cotangent bundle $T^*B \to B$ is the tangent space T_qB. Thus the functions c_{ij}^k defined in (3) for a 0-versal Lagrangian map germ determine a pointwise associative multiplication of vector field germs on the base.

3. Under the assumptions of Corollary 8, the $\mathscr{K}_f$-discriminant of φ is a free divisor. The proof will be published elsewhere.

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