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Introduction

Associated to each unstable map-germ \( f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0 \), where \((n, p)\) are in Mather’s range of “nice dimensions”, there is a “stabilisation”; that is, a locally stable mapping \( f : U \to \mathbb{C}^p \), where \( U \) is some contractible neighbourhood of 0 in \( \mathbb{C}^n \). When \( n < p \), the image \( Y \) of \( f \) plays the same rôle in the theory of singularities of mappings, as does the Milnor fibre in the theory of isolated complete intersection singularities. Our aim in this paper is to describe the topology of \( Y \), and, in the case where \( f_0 \) is quasihomogeneous, to make a start in the study of its canonical mixed Hodge structure. Our main results concern germs of corank 1, since in this case the spectral sequence used to calculate the vanishing cohomology degenerates at \( E_1 \), making explicit calculation very easy.

The key to our description of the topology of the image \( Y \) is provided by the multiple point spaces \( D_k(f) \) (the \( k \)-th multiple point space \( D_k(f) \) is in this context the closure, in \( U^k \), of the set of \( k \)-tuples of pairwise distinct points having the same image under \( f \) (see Section 2 below)). When \( f_0 \) is a finitely determined corank 1 map-germ, each germ \( D_k(f_0), 0 \), for \( 2 \leq k \leq p/(p - n) \), is an isolated complete intersection singularity (\([19]\)). Replacing \( f_0 \) by its stable perturbation \( f \), we smooth each space \( D_k(f_0) \); \( D_k(f) \) is thus a Milnor fibre of \( D_k(f_0) \), and is therefore a Stein manifold with the homotopy type of a wedge of spheres (\([9]\)). Moreover, the natural projections \( D_k(f) \to D_k^{-1}(f) \) all turn out to be stable mappings (see Section 2).

Section 1 makes precise the notion of a stable perturbation of a map-germ.

In Section 2, we construct an alternating semi-simplicial resolution \( \text{Alt} \mathbb{Z}_p \) of the constant sheaf \( \mathbb{Z}_Y \), which relates the topology of \( Y \) to that of the multiple point spaces of \( f \). When the original map-germ \( f_0 \) has corank 1, that is, when \( \dim \ker d f_0(0) = 1 \), \( D_k(f) \) has the homotopy of a wedge of spheres, and in consequence the spectral sequence for the hypercohomology of the corresponding rational complex \( \text{Alt} \mathbb{Q}_p \) degenerates at \( E_1 \) and we obtain a rather succinct relation between the rational cohomology of \( Y \) and the \( S_k \)-alternating part of the rational cohomology of the \( D_k \) (Theorem 2.6). This is most interesting when \( p = n + 1 \). In this case, by a theorem of Lê, \( Y \) itself has the homotopy of a wedge...
of spheres in dimension $n$ ([24]); it turns out that the filtration on $H^n(Y; \mathbb{Q})$ coming from the spectral sequence, has successive quotients isomorphic to the $S_k$-alternating part of $H^{n-k+1}(D^k; \mathbb{Q})$.

By an extension of this method, we also calculate the rational cohomology of the image multiple point schemes $M_k \subseteq Y$; it turns out that when $p = n + 1$, they have rational cohomology only in the middle dimension (Theorem 2.8).

In Section 3 we use the results of Section 2 to reprove and extend some numerical formulae due to W. L. Marar, relating the ranks of homology groups of $Y$ to those of the $D^k$ and their intersections with the multi-diagonals.

In Section 4 we concentrate on the case where $f_0$ is quasihomogeneous, adapting results of Greuel and Hamm on the dimension of spaces of forms, to the alternating case, and prove numerical formulae which express the Betti numbers of $Y$, in terms of the quasihomogeneous type of $f_0$. In the process we obtain a description of the space of alternating holomorphic forms on a space with the action of a finite group generated by reflections.

In Section 5, we continue the study of quasihomogeneous corank 1 map-germs, with the aim of calculating the invariants of Deligne's mixed Hodge structure on the image of a stabilisation. In this case the stabilisation may be taken to have domain $\mathbb{C}^n$, and its image $Y$ and multiple point schemes $D^k$ can be embedded in appropriate weighted projective spaces. We define mixed Hodge sheaves on $Y$ by means of alternating mixed Hodge sheaves on the $D^k$, using the resolution of $C_Y$ from Section 2, and then use an alternating version of Hamm's calculation of the Hodge numbers of quasihomogeneous isolated complete intersection singularities, to obtain formulae for the Hodge numbers of $Y$, in terms of the quasihomogeneous type of $f$.

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1. Good representatives

Let $f_0: \mathbb{C}^n, 0 \to \mathbb{C}^p, 0 (n < p)$ be a finitely $\mathcal{A}$-determined map-germ of discrete stable type (i.e., in a versal unfolding of $f_0$ there only appear a finite number of right-left equivalence classes of stable germs; this is guaranteed, for example, by the hypothesis that $f_0$ be of corank 1). We are interested in studying a particular class of stable mappings associated with $f_0$, the so called stable perturbations, which we define as follows, following [17, 18]: let $F: \mathbb{C}^n \times \mathbb{C}^d, 0 \to \mathbb{C}^p \times \mathbb{C}^d, 0$ be an unfolding of $f_0$, with $F(x, t) = (f(x), t)$. Choose a proper representative
$F: \mathcal{U} \to \mathcal{W} \times \mathcal{Z}$, where $\mathcal{U}$ and $\mathcal{W}$ and $\mathcal{Z}$ are open neighbourhoods of 0 in $\mathbb{C}^n$, $\mathbb{C}^p$ and $\mathbb{C}^d$ respectively, such that

(i) $F^{-1}(0) = 0$

(ii) $F: \mathcal{U} \to \mathcal{W} \times \mathcal{Z}$ is a finite map (i.e. proper with finite fibres).

Now let $I_{\text{rel}}(F) = \{(y, t) \in \mathcal{W} \times \mathcal{Z}|$ the germ of $f_t$ at $f_t^{-1}(y) \cap \mathcal{U}_t$ is not $\mathcal{A}$-stable\}$, where $\mathcal{U}_t = \{x \in \mathbb{C}^n|(x, t) \in \mathcal{U}\}$. As $F$ is finite, $I_{\text{rel}}(F)$ is an analytic subset of $\mathcal{W} \times \mathcal{Z}$. Since $f_0$ is finitely determined, $0 \in \mathbb{C}^n$ is an isolated point of the fibre over $0 \in \mathbb{C}^d$ of the projection $\pi: I_{\text{rel}}(F) \to \mathbb{C}^d$ (this is in fact equivalent to finite determinacy, see e.g. [30]); thus by shrinking $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{Z}$, we may suppose that $\pi: I_{\text{rel}}(F) \to \mathcal{W}$ is finite. Choose $\varepsilon > 0$ such that $F(\mathcal{U}) \cap \mathbb{C}^p \times \{0\}$ is stratified transverse (with respect to some Whitney stratification of $F(\mathcal{U})$) to the sphere $S_{\varepsilon}$ of centre 0 and radius $\varepsilon'$, for every $\varepsilon'$ with $0 < \varepsilon' \leq \varepsilon$ (i.e. such that $B_\varepsilon(0)$ is a Milnor ball for $f_0(U_0)$). Then by the properness of $\pi$, there exists a neighbourhood $\mathcal{W}_0$ of 0 in $\mathcal{W}$, such that

(i) $I_{\text{rel}}(F) \cap (\mathbb{C}^p \times \mathcal{W}_0)$ is contained in $\text{int}(B_\varepsilon(0)) \times \mathcal{W}_0$.

(ii) The stable type stratification of $F(\mathcal{W}) \setminus I_{\text{rel}}(F)$ is transverse to $S_{\varepsilon} \times \mathcal{W}_0$. (Note that off $I_{\text{rel}}(F)$, the stable type stratification of $F(\mathcal{W})$ is the minimal Whitney stratification).

Now restrict $F$ to $U = F^{-1}(B_\varepsilon(0) \times \mathcal{W}_0)$. We call the new map $F: U \to B_\varepsilon(0) \times \mathcal{W}_0$ a good representative of $F$. Let $\mathcal{B} = \pi(I_{\text{rel}}(F))$, and suppose that it is a proper subset of $\mathcal{W}_0$; this is guaranteed if $(n, p)$ are nice dimensions, cf. [20], or if $f_0$ is of corank 1. As a consequence of Thom’s Second Isotopy Lemma, the family of mappings

$$U \cap \pi^{-1}(\mathcal{W}_0 \setminus \mathcal{B}) \xrightarrow{F} B_\varepsilon(0) \times (\mathcal{W}_0 \setminus \mathcal{B}) \xrightarrow{} \mathcal{W}_0 \setminus \mathcal{B}$$

is locally topologically trivial. Details of the proof are given in [16, 17]. In particular, setting $U_t = \{x \in \mathbb{C}^n|(x, t) \in U\}$, then up to $C^0$-$\mathcal{A}$-equivalence, the map $f_t: U_t \to B_\varepsilon(0)$ is independent of the choice of $t \in W_0 \setminus \mathcal{B}$ (for $W_0 \setminus \mathcal{B}$ is connected). We will call such a map a stable perturbation of $f_0$. In fact, up to $C^0$-$\mathcal{A}$-equivalence, there is a unique stable perturbation of $f_0$; for any stable perturbation is bianalytically equivalent to one contained in a versal unfolding, and the stable perturbations of $f_0$ contained in any two versal unfoldings are easily shown to be $C^0$-$\mathcal{A}$-equivalent.

If $\mathcal{B}$ is not a proper analytic subset of the base of a versal unfolding $F$ of $f_0$, as may occur if $(n, p)$ lie outside the range of nice dimensions, then by replacing $I_{\text{rel}}(F)$ in the previous construction by the set of points where $j^k f_t$ fails to be
multitransverse to the canonical stratification of the jet bundle, one can show that, again up to $C^0$-$\mathcal{A}$-equivalence, $f_0$ has a well defined topologically stable perturbation. In the nice dimensions, the notions of stability and topological stability coincide.

Suppose that $f_i: U_i \to B_i(0) \subseteq \mathbb{C}^p$ is a stable or topologically stable perturbation of $f_0$ as described above. Then the image $Y_i$ of $f_i$ is a Stein space, as are all of the multiple point spaces $M_k(f_i)$ in the image. For each is the intersection of an analytic subspace of an open set in $\mathbb{C}^p$, with the closed ball $B_i(0)$, and, since the distance squared function on Euclidean space is strictly plurisubharmonic, the affirmation is a consequence of e.g. Corollary 10 of Chapter IX of [8]. By results of H. Hamm, [10], it follows that $Y_i$ and the $M_k$ have the homotopy type of CW complexes of half of their real dimension, and in particular have no cohomology above the middle dimension.

When $p = n + 1$, it is possible to show that $Y_i$ is in fact homotopy equivalent to a wedge of spheres of dimension $n$ (see e.g. [24]); however, when $p \geq n + 2$, it turns out that no such simple description of $Y_i$ is possible: as we shall see in the next section, $Y_i$ may have cohomology in dimensions $p - (p - n - 1)k - 1$ for all integers $k$ for which $p - (p - n)k \geq 0$.

2. Multiple point spaces and alternating semi-simplicial resolutions

In this section we use alternating semisimplicial resolutions to compute the rational cohomology of the image of a finite mapping, with special emphasis on the case of a stable perturbation of a map-germ $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0, (n < p)$. The reader may find it helpful to refer to the example on page 52 while reading it.

First we define a collection of spaces associated with any continuous mapping $f: X \to Y$ of topological spaces:

$D^k(f)$ (or $D^k$ where there is no danger of confusion), is the $k$-fold multiple point space of $f$:

$$D^k = \text{closure}\{(x_1, \ldots, x_k) \in X^k: f(x_1) = \cdots = f(x_k), \ x_i \neq x_j \ \text{if} \ i \neq j\}$$

for $1 \leq k < \infty$.

There are continuous mappings $e^{i,k}: D^k \to D^{k-1}$, defined by

$$e^{i,k}(x_1, \ldots, x_k) = (x_1, \ldots, \hat{x}_i, \ldots, x_k), \ \text{for} \ 1 \leq i \leq k.$$ 

The spaces $D^k$, together with the maps $e^{i,k}$, constitute a semisimplicial object in the category of topological spaces. Observe that $f$ induces well defined maps $D^k \to Y$, which we call $e^k$, with $e^{k+1} \circ e^{i,k+1} = e^{k+1}$ (thus $(D \to Y)$ is a semisimplicial object over $Y$). However, we will make no use of this notion, although our
calculation of the cohomology of the image is a modification of well known
techniques of semisimplicial resolution.

We now suppose that \( f \) is a finite, proper map (and we will continue to do so
throughout this section).

**The alternating complex**

Consider the complex of sheaves

\[
\begin{array}{ccccccc}
\varepsilon^1_* (Z_X) & \delta_1 & \varepsilon^2_* (Z_D) & \delta_2 & \varepsilon^3_* (Z_D) & \delta_3 & \ldots \\
\end{array}
\]

where \( \delta_k : \varepsilon^k_* (Z_{D^k}) \to \varepsilon^{k+1}_* (Z_{D^{k+1}}) \) is equal to \( \sum_{j=1}^{k+1} (-1)^{k+j} (\varepsilon^j_* (Z_{D^j})) \) (here \( D^1 \) is
just \( X \)).

In general the complex \( Z_{D^*} \) is not exact: exactness fails at points of \( Y \) lying
under points where some \( D^k \) meets one of the diagonals. If, for example, \( y = f(x) \), \( f^{-1}(y) = \{ x \} \) and \( (x, x) \in D^2 \), but \( (x, x, x) \notin D^3 \), then each of \( \varepsilon_* (Z_X) \) and \( \varepsilon^2_* (Z_{D^2}) \) has stalk at \( y \) isomorphic to \( \mathbb{Z} \), while all of the other sheaves in the
complex have stalk at \( y \) equal to \( 0 \). It follows by counting the rank of the stalks
that exactness is not possible at \( y \). Note that precisely this configuration arises
(at \( y = 0 \)) if we take \( f \) to be the stable map \( \mathbb{C}^2 \to \mathbb{C}^3 \) defined by
\[
f(x_1, x_2) = (x_1, x_2, x_1 x_2),
\]
whose image \( Y \) has a pinch point singularity at \( 0 \).

In order to obtain exactness, we restrict to the alternating subcomplex, which
we now define. There is a natural continuous action of the symmetric group \( S_k 
\)
on \( \varepsilon_* (Z_{D^k}) \), defined as follows: \( S_k \) acts on \( D^k \) by permuting the factors; as \( \varepsilon_* \) is \( S_k \)-invariant, for any open set \( U \subseteq Y \), \( (\varepsilon_* )^{-1} (U) \) is mapped to itself by the
permutation action. Thus, \( S_k \) acts on \( \Gamma((\varepsilon^k_* )^{-1} (U), Z_{D^k}) \) by the permutation
representation coming from the permutation action on the set of connected
components of \( (\varepsilon^k_* )^{-1} (U) \), and hence acts (continuously) also on \( \varepsilon^k_* (Z_{D^k}) \). We
denote the action of \( \sigma \in S_k \) by \( \sigma^* \). We let \( \text{Alt}(\varepsilon^k_* (Z_{D^k})) \) be the subsheaf of \( \varepsilon^k_* (Z_{D^k}) \)
on which \( S_k \) acts by the alternating representation, i.e.

\[
\text{Alt}(\varepsilon^k_* (Z_{D^k}))_y = \{ s \in \varepsilon^k_* (Z_{D^k})_y \mid \text{ for all } \sigma \in S_k, \sigma^* s = \text{sign}(\sigma) s \}
\]

and we denote the complex \( \{ \text{Alt} \varepsilon^k_* (Z_{D^k}), \delta \} \) by \( \text{Alt} Z_{D^*} \).

We remark that this construction is alluded to by Deligne in [0, pp. 31–32],
but is not described in detail.

2.1. PROPOSITION. The complex \( \text{Alt} Z_{D^*} \) is exact.

**Proof.** Let \( y \in Y \) be a point with exactly \( m+1 \) preimages, \( x_0, \ldots, x_m \). We
prove exactness of the stalk complex \( \text{Alt} Z_{D^*} \), by showing that it is isomorphic
to the simplicial cochain complex \( \{ C(\Delta^m, Z), d_* \} \), where \( \Delta^m \) is an \( m \)-simplex. It
will be helpful in what follows to write \( X = D^1 \), and \( f = \varepsilon^1_* \).
Recall that since $f$, and hence $\varepsilon^k$, is finite,

$$\varepsilon^k_* (\mathbb{Z}_{D^k})_y \cong \bigoplus_{x \in (\mathbb{Z})^{-1}(y)} \mathbb{Z}_{D^k, x}.$$ 

Let $0 \leq i_1, \ldots, i_k \leq m$, and if $(x_{i_1}, \ldots, x_{i_k}) \in D^k$, let $\chi(x_{i_1}, \ldots, x_{i_k})$ be the member of $\bigoplus_{x \in (\mathbb{Z})^{-1}(y)} \mathbb{Z}_{D^k, x}$ which is 1 at $(x_{i_1}, \ldots, x_{i_k})$ and 0 elsewhere. Denote by $\text{Alt} \chi(x_{i_1}, \ldots, x_{i_k})$ the element

$$\sum_{\sigma \in S_k} \text{sign}(\sigma) \chi(x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(k)}});$$

if $i_j = i_l$ for any $j \neq l$, then $\text{Alt} \chi(x_{i_1}, \ldots, x_{i_k}) = 0$. Hence, $\varepsilon^k_* (\mathbb{Z}_{D^k})_y$ has as free basis the elements $\text{Alt} \chi(x_{i_1}, \ldots, x_{i_k})$ with $0 \leq i_1 < \cdots < i_k \leq m$.

Now let $\Delta^m = (v_0, \ldots, v_m)$ be the standard $m$-simplex, and for $0 \leq i_1 < \cdots < i_k \leq m$, let $(v_{i_1}, \ldots, v_{i_k})$ be the $(k-1)$-face with vertices $v_{i_1}, \ldots, v_{i_k}$, oriented in some standard way. Let $\xi_{(v_{i_1}, \ldots, v_{i_k})}$ be the simplicial $(k-1)$-cochain on the simplicial complex generated by $\Delta^m$, which takes the value 1 on $(v_{i_1}, \ldots, v_{i_k})$, and 0 on the other $(k-1)$-faces of $\Delta^m$. Then for any

$$\sigma \in S_k, \xi_{(v_{i_1}, \ldots, v_{i_k})}(\binom{v_{i_{\sigma(1)}}, \ldots, v_{i_{\sigma(k)}}})$$

is equal to $\text{sign}(\sigma)$.

It follows that the map of complexes $\{\varphi_k\}: \text{Alt} \mathbb{Z}_{D^m, y} \rightarrow \{C^{-1}(\Delta^m, \mathbb{Z}), d_{-1}\}$ determined by

$$\varphi_k(\text{Alt} \chi(x_{i_1}, \ldots, x_{i_k})) = \xi_{(v_{i_1}, \ldots, v_{i_k})}$$

is well-defined, and bijective. One checks easily that $\varphi_{k+1} \circ \delta_k = d_k \circ \varphi_k$; thus, the two chain complexes are isomorphic, and since $\{C^{(-1)}(\Delta^m, \mathbb{Z}), d_{-1}\}$ is exact, so is $\{\text{Alt} \mathbb{Z}_{D^m, y}, \delta\}$.

Exactly the same proof shows that the complex $\text{Alt} \mathbb{Q}_{D^m}$, obtained by replacing $\mathbb{Z}$ by $\mathbb{Q}$ in the previous construction, is also exact. We will make use of this complex rather than the integer complex, because while, as we shall shortly see, $H^*(Y, \text{Alt} \varepsilon^k_* (\mathbb{Q}_{D^k}))$ is equal to the alternating part $H^*_{\text{Alt} \mathbb{Q}}(D^k, \mathbb{Q})$ of $H^*(D^k, \mathbb{Q})$, the relation is not so simple over $\mathbb{Z}$.

Now let $0 \rightarrow \mathbb{Q}_{D^k} \rightarrow I_k$ be an injective resolution of the constant sheaf $\mathbb{Q}_{D^k}$; pushing it down to $Y$ we obtain an injective resolution of $\varepsilon^k_* (\mathbb{Q}_{D^k})$. Since $S_k$ acts on $\varepsilon^k_* (\mathbb{Q}_{D^k})$, we can choose $I_k$ so that $S_k$ acts on $\varepsilon^k_*(I_k)$ too, by taking $I_k$ to be the canonical resolution of Godement ([3] II.4.3). Then we have
2.2. LEMMA. Under these circumstances, the complex $\text{Alt}_k e^k_{\ast} I_{k}$ is an injective resolution of $\text{Alt}_k e^k_{\ast} (QD_k)$.

Proof. Define an indempotent operator $\text{Alt}_k$ on each $I^k_k$, by

$$\text{Alt}_k = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) \sigma^\ast.$$ 

Each of the differentials in the complex $e^k_{\ast} I_{k}$ commutes with the action of $S_k$, and thus with the operator $\text{Alt}_k$. As this operator is indempotent, it is an easy exercise to show that $\text{Alt}_k e^k_{\ast} I_{k}^k$ is injective, and moreover to deduce, from the exactness of $e^k_{\ast} I_{k}$, that the complex $\text{Alt}_k e^k_{\ast} I_{k}$ is exact. \square

Now by lifting the differentials $\delta^k: e^k_{\ast} (QD_k) \to e^{k+1}_{\ast} (QD_{k+1})$ to sheaf homomorphisms

$$\delta^k_k: \text{Alt}_k e^k_{\ast} (I^k_k) \to \text{Alt}_{k+1} e^{k+1}_{\ast} (I^k_{k+1})$$

we obtain a double complex $\{\text{Alt}_k e^k_{\ast} (I^k_k), \delta^k_k, d^k_k\}$. By a standard argument, the total complex $K'$, with

$$K'^q = \bigoplus_{i+k=q+1} \text{Alt}_k (e^k_{\ast} (I^k_i)),$$

is exact, and is thus an injective resolution of $Q_Y$. Therefore the complex $\Gamma(Y, K')$, obtained by taking global sections, computes the cohomology of $Y$. Now

$$\Gamma(Y, \text{Alt}_k (e^k_{\ast} (I^k_k)) = \text{Alt}_k (\Gamma(D^k, I^k_k));$$

since the differential of the complex $\Gamma(D^k, I^k_k)$ commutes with the idempotent operator $\text{Alt}$, which is defined on this complex in the obvious way, we have

2.3. PROPOSITION. The spectral sequence associated to the filtration

$$F^p \Gamma(Y, K'^q) = \bigoplus_{k \geq p+1} \Gamma(Y, \text{Alt}(e^k_{\ast} (I^k_k)))$$

has $E_1^{p,q}$ term equal to $H^p_{\text{Alt}_{q+1}} (D^{q+1}, Q)$ (where $H^p_{\text{Alt}_{q+1}} (D^{q+1}, Q)$ is the alternating part of $H^q(D^{q+1}, Q)$).

Proof. The $E_1^{p,q}$ term of the spectral sequence is equal to $H^q(\text{Alt}_{p+1} (\Gamma(D^{p+1}, I^k_k), d^p_{p+1})$. As just described, this is equal to $H^q_{\text{Alt}_{q+1}} (D^{q+1}, Q))$. \square

2.4. REMARK. (ii) The above construction breaks down if $Z$ is replaced by $Q$; the idempotence of the operator $\text{Alt}_k$ is an essential ingredient in the proof of 2.2, and over $Z$ it is not possible to construct such an idempotent operator, since one
cannot divide by \( k! \). Indeed 2.3 is false if we replace \( \mathbb{Q} \) by \( \mathbb{Z} \). This is shown by the following example. Let \( X \) be the closed northern hemisphere of \( S^2 \), and \( f: X \to Y = \mathbb{RP}^2 \) the restriction to \( X \) of the usual quotient mapping. This map has double points but no triple points. Projection onto the first factor induces a homeomorphism of \( D^2(f) \) onto the equator of \( S^2 \), where the involution is the antipodal map. Since this map is orientation-preserving, \( H^*_{\text{Alt}^2}(D^2(f); \mathbb{Z}) = 0 \); but \( \mathbb{RP}^2 \) is not an integer-homology disc.

(ii) The injective resolutions \( I_k \) above can be replaced by any fine resolution (to whose push-forward to \( Y \) the \( S_k \)-action on \( c_k^*(\Omega_{D^k}) \) lifts), if the aim is simply that of calculating \( H^*(Y) \). Thus, in particular, if all of the \( D^k \) are smooth Stein spaces then one may calculate \( H^*(Y; \mathbb{C}) \) by using the resolutions \( 0 \to C_{D^k} \to \Omega_{D^k} \), where \( \Omega_{D^k} \) is the (exact) complex of sheaves of germs of holomorphic differential forms (c.f. [6], §4).

**An exercise**

To end this paragraph we give a simple example. Among the five codimension 1 singularities of maps from surfaces to 3-space we find the "birth of two triple points"—the multi-germ consisting of three immersions which are pairwise transverse, but in which the curve of intersection of each pair of immersed sheets is (first order) tangent to the third. Over \( \mathbb{R} \) there are two inequivalent stable perturbations of this configuration, shown in Fig. 1(a) and (b). Figure 1(a) does indeed have two triple points, which are imaginary in 1(b). In each case \( X \) consists of the disjoint union of three 2-cells, \( X_1, X_2 \) and \( X_3 \), and so \( X \times X \) has nine connected components. As \( f \) is an immersion, \( D^2(f) \) has no component in any \( X_i \times X_i \), but for each \( i \neq j \), \( D^2(f) \cap (X_i \times X_j) \) is a line. The \( \mathbb{Z}_2 \) action on \( D^2(f) \) interchanges \( D^2(f) \cap (X_i \times X_j) \) and \( D^2(f) \cap (X_j \times X_i) \), and it follows that \( H^0_{\text{Alt}^2}(D^2; \mathbb{C}) \) has rank 3. In 1(a), \( D^3 \) consists of two faithful \( S_3 \) orbits, each of

![Fig. 1.](image-url)
which contributes one dimension to $H^0_{\text{AH}_3}(D^3; \mathbb{Q})$. We urge the reader to compute the spectral sequence of 2.3 for each of them. The outcome should be clear from the drawings, and the computation is particularly easy since the $D^k$ have cohomology only in dimension 0.

**Stable perturbations of corank 1 map-germs and simplicial stable mappings**

We can now use 2.2 to compute explicitly the rational cohomology of the image $Y$ of a stable perturbation of a map-germ $f_0: \mathbb{C}^n, 0 \to \mathbb{C}^m, 0 (n < m)$ of corank 1. Let $f_t: U_t \to \mathbb{C}^m$ be such a perturbation, with image $Y_t$, and let the spaces $D^k$ be constructed as above (with $X = U$). In [19], the space $D^k$ (constructed for the map $f_t$) was denoted $\overline{D}^k(f_t)$. Here we abandon the tilde. We recall the principal result of [19]:

> **2.5. THEOREM.** ([19], 2.14). (i) The map-germ $f: \mathbb{C}^n, x \to \mathbb{C}^m, y$ is stable if and only if for all $k$ with $2 \leq k$, the germ of $D^k(f)$ at $(x, x, \ldots, x) \in (\mathbb{C}^n)^k$ is smooth of dimension $m - (m - n)k$, or empty;

(ii) $f$ is finitely determined (for $\mathcal{A}$-equivalence) if and only if for all $k$ with $2 \leq k \leq m/(m - n)$, $D^k(f)$ is a complete intersection of dimension $m - (m - n)k$, with (at most) isolated singularity at $(x, x, \ldots, x) \in (\mathbb{C}^n)^k$. □

Note that [19] defines multiple point schemes $D^k(f)$ for corank 1 map-germs $f$, by means of explicit equations, rather than the multiple point spaces defined here. In fact for finitely determined corank 1 germs $\mathbb{C}^n, 0 \to \mathbb{C}^m, 0$, the two definitions coincide provided $k < m/(m - n)$ (in other words, for those $k$ such that $\dim D^k(f) > 0$). This is because for such germs genuine $k$-tuple points are dense in the scheme $D^k(f)$ (essentially by 2.5—see [19], page 563). When $k = m/(m - n)$, the scheme $D^k(f)$ may contain $k$-tuples $(x_1, \ldots, x_k)$ where not all the $x_i$ are distinct, which are clearly not in the space $D^k(f)$ defined here. However, the $S_k$ orbit of such a point does not support any alternating 0-th cohomology; so we conclude that one may use the scheme-theoretic $D^k(f)$ defined in [19], in place of the space $D^k(f)$ defined here, and obtain the same spectral sequence converging to the cohomology of the image $Y$, from $E_1$ onwards.

If $F: U \to V \times T \subseteq \mathbb{C}^m \times \mathbb{C}^d$ is a good representative of a stable parametrised unfolding of $f_0$, then the spaces $D^k(F)$, which by 2.5(i) are all smooth, fibre over $T$, with fibre over $t \in T$ equal to $D^k(f_t)$. If $t$ lies in the complement of the bifurcation set $\mathcal{B}$, by 2.5(i) $D^k(f_t)$ is smooth, and is in fact a Milnor fibre for the isolated complete intersection singularity $D^k(f_0), 0$. It follows, by results of Hamm [9] (see also [15], Chapter 5) that for each $k$, $D^k(f_t)$ has the homotopy type of a wedge of spheres of middle dimension, so that its reduced cohomology is concentrated in this dimension. Hence,
2.6. THEOREM. In these circumstances, the spectral sequence described above collapses at the $E_1$ term, and thus we have

(i) if $m = n + 1$, then $H^n(Y, \mathbb{Q})$ is isomorphic to

$$\bigoplus_{k=2}^{n+1} H^{n-k+1}_{\text{Alt}_k}(D^k(f_i), \mathbb{Q}),$$

and $H^p(Y, \mathbb{Q}) = 0$ for $1 \leq p < n$ and for $p > n$.

(ii) if $m - n \geq 2$, then for each integer $k$ with $2 \leq k \leq m/(m - n)$,

$$H^{m-(m-n-1)k+1}(Y, \mathbb{Q}) \simeq H^{n-(m-n)k}_{\text{Alt}_k}(D^k, \mathbb{Q})$$

and $H^p(Y, \mathbb{Q})$ vanishes for all other positive values of $p$.

Proof. We have $E_1^{p,q} = H^q_{\text{Alt}_{p+1}}(D^{p+1}, \mathbb{Q})$, and so the first differential runs from $H^q_{\text{Alt}_{p+1}}(D^{p+1}, \mathbb{Q})$ to $H^{q+1}_{\text{Alt}_{p+2}}(D^{p+2}, \mathbb{Q})$. For each value of $q$, there is at most one value of $p$ for which $H^q_{\text{Alt}_{p+1}}(D^{p+1}, \mathbb{Q})$ is non-vanishing; this follows from the fact that for each $p$, there is at most one value of $q = q(p) > 0$ such that $H^q(D^{p+1}, \mathbb{Q}) \neq 0$, and the sequence $q(p)$ is strictly decreasing, while for $q = 0$, it holds because if $\dim(D^{p+1}(f_i)) > 0$, then $D^{p+1}$ is connected and so $H^0_{\text{Alt}_{p+1}}(D^{p+1}, \mathbb{Q}) = 0$, and there is at most one value of $p$ for which $\dim(D^{p+1}) = 0$. It follows that the spectral sequence collapses at the $E_1$ term, as claimed. Since the spectral sequence converges to the cohomology of $Y$, (i) and (ii) follows. \qed

Note that the spectral sequence in the example on page 52 does not collapse at $E_1$, even if we replace $\mathbb{R}$ by $\mathbb{C}$; however, this does not contradict 2.6, since in the example we are dealing with a stable perturbation of a multi-germ.

2.7. REMARK. (i) The hypothesis that $f_0$ be of corank 1 is not necessary in order to guarantee that the spaces $D^p(f_i)$ be smooth. The only requirement here is that all of the singularities of the stable perturbation $f_i$ should be of corank 1. This is also guaranteed if $n < 2(m - n + 2)$.

(ii) The conclusion of 2.6 continues to hold for stable perturbations of map-germs $C^2, 0 \to C^3, 0$ of corank 2. In this case, since $D^2(f_i)$ is a smooth, non-compact complex curve, it has cohomology only in dimensions 0 and 1, and so the spectral sequence collapses at the $E_1$ term as in the proof of 2.6.

(iii) 2.6 is valid also in a slightly wider context; for example, where the domain of $f_0$ is an isolated complete intersection singularity and the domain $U_i$ of $f_i$ is a smoothing. In this case, provided the multiple point schemes $D^k(f_i)$ still have cohomology concentrated in the middle dimension, the only change to the calculation is the addition of one further summand in the cohomology of $Y_i$, coming from the cohomology of $U_i$. This arises in the study of projections of
complete intersection singularities to smooth complex spaces (cf [4]), and also, in the context that principally concerns us, if we are interested in the images of the maps \( \epsilon^{i,k} \): \( D^k(f_i) \to D^{k-1}(f_i) \). For here, the domain of \( \epsilon^{i,k} \) is the smoothing \( D^k(f_i) \) of the isolated complete intersection singularity \( D^k(f_0) \). Now \( D^j(\epsilon^{i,k}) \cong D^{k+j-1}(f_i) \); for (taking \( i = k \) to simplify notation) an ordered \( j \)-tuple of points in \( D^k(f_i) \) having the same image under \( \epsilon^{k,k} \) must be of the form

\[
((x_1, \ldots, x_{k-1}, x_k), (x_1, \ldots, x_{k-1}, x_{k+1}), \ldots, (x_1, \ldots, x_{k-1}, x_{k+j-1})),
\]

and we define an isomorphism \( D^j(\epsilon^{i,k}) \to D^{k+j-1}(f_i) \) by sending this point to \((x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+j-1})\). This is the “method of iteration” used by Kleiman in [12]. Incidentally, this shows (by 2.5(i)) that the maps \( \epsilon^{i,k} \) are themselves locally stable, so that the family of spaces \( D^k(f_i) \) and mappings \( \epsilon^{i,k} \) form a simplicial stable mapping. It also explains the observation in [25] (in the case of generic maps of 3-folds into \( \mathbb{P}^4 \)) that the source double point space \( \epsilon^{1,2}(D^2(f)) \) has the same singularities as the image of a stable map from 2-space to 3-space. In a similar vein, the source triple point set \( D^3(\epsilon^{1,2}) \) (for a stable map \( f \) from \( n \)-space to \( p \)-space) has the singularities of the image double point set of a stable mapping from \( p - 2(p - n) \)-space to \( n \)-space.

(iv) It is known that when \( m = n + 1 \), \( Y_t \) actually has the homotopy type of a wedge of spheres of dimension \( n \). This follows from a theorem of Lê [12, 13]; see [24]. Thus \( H^n(Y_t, \mathbb{Z}) \) is a free abelian group, as are the groups \( H^{m-(m-n)k}(D^k, \mathbb{Z}) \).

However, as we have seen in 2.4(i), 2.3 does not hold over \( \mathbb{Z} \), and in order to relate the integer cohomology of the \( D^k \) to that of \( Y_t \), some more work is required (see [5]).

**Rational cohomology of the image multiple point sets**

We now use the same technique to compute the cohomology of the spaces \( M_p(f_i) = \epsilon^p(D^p) \subseteq Y_t \), which can also be described as the locus of zeros of the \( (p - 1) \)st Fitting ideal sheaf of \( f_{r_1}(\mathcal{O}_Y) \). To lighten the notation, we abandon the subscript \( t \) on \( Y_t, f_t \) etc.

Let \( D^j_p \) be the (reduced) image of \( D^j \) in \( D^j \) under any one of the Cartesian projections. We calculate the cohomology of \( M_p \) by means of the following exact complex of sheaves on \( M_p \):

\[
0 \to \mathcal{Q}_{M_p} \to \mathcal{Q}_{D^j} \to \text{Alt}_2(\mathcal{Q}_{D^j}) \to \cdots \to \text{Alt}_{p-1}(\mathcal{Q}_{D^j,p-1}) \to \text{Alt}_p(\mathcal{Q}_{D^j,p})
\]

\[
\to \text{Alt}_{p+1}(\mathcal{Q}_{D^j,p+1}) \to \cdots
\]

(here we have omitted the symbols \( \epsilon^j \), etc.). Exactness is a consequence of 2.1, for
denoting by $f_p$ the restriction of $f$ to $D^p$, we have $D^j(f_p) = D^p$ for $j < p$ and $D^j(f_p) = D^j$ for $j \geq p$.

2.8. THEOREM. Suppose that $f: U \rightarrow Y$ is a stable perturbation of a finitely determined corank 1 map-germ $\mathbb{C}^n, 0 \rightarrow \mathbb{C}^{n+1}, 0$. Then all of the spaces $M_k$, for $2 \leq k \leq n + 1$, have rational cohomology only in dimension $n - k + 1$.

Proof. This is proved by induction on $k$. The possibility of carrying out an induction is based on the principle of iteration: namely, that

$$D^p_\ell(f) = M_p(\varepsilon^{\ell+1}; D^2(f) \rightarrow U)$$

and, more generally, since $D^k(\varepsilon^{i,j}) \simeq D^{k+j-1}(f)$,

$$D^p_q(f) = M_{p-q}(\varepsilon^{i,j+1}; D^{q+1}(f) \rightarrow D^q(f)) \quad (\text{for } q < p).$$

The induction hypothesis is in fact slightly stronger than the theorem itself. It is

Hyp($p - 1$): Let $g: U \rightarrow V$ be a proper, stable map of affine Stein manifolds, with $\dim(U) = \dim(V) - 1 = n$, and suppose that $g$ has only corank 1 singularities; suppose, moreover that $U$, and all of the spaces $D^k(g)$, have reduced rational cohomology only in dimension $n - k + 1$, and that all of the spaces $M_p(g)$ are Stein spaces. Then for $1 \leq q \leq p - 1$, $M_q(g)$ has reduced rational cohomology only in dimension $n - q + 1$.

Note that Theorem 2.6 (together with 2.7(iii)) establishes that Hyp(1) holds.

The induction step is proved as follows. By 2.4, the cohomology of $M_p(g)$ is computed by a spectral sequence with $E_r^{s,t} = H^{r+s,t,s+1}(D^{p+1}(g_p), \mathbb{Q})$ (where $g_p$ is the restriction of $g$ to $D^p(g)$). Now

$$D^j(g_p) = D^j_f(g) = M_{p-j}(\varepsilon^{i,j+1}; D^{j+1}(g) \rightarrow D^j(g)) \quad \text{for } j < p$$

and

$$D^j(g_p) = D^j(g) \quad \text{for } j \geq p.$$
In particular, $E_1^{i,k} = 0$ for all $j,k$ with $j + k < n - p + 1$; since the spectral sequence converges to $H^*(M_p(g), \mathbb{Q})$, it follows that $H^k(M_p(g), \mathbb{Q}) = 0$ for $k < n - p + 1$. As $M_p(g)$ is a Stein space of dimension $n - p + 1$ it has cohomology only in dimension less than or equal to $n - p + 1$. Thus, we have proved Hyp($p$).

The filtration on the cohomology of the image

We give now an alternative description of the filtration on the cohomology of the image $Y_i$ obtained from the spectral sequence with which we calculated the cohomology of the double complex in 2.6. The previous description of $H^k(M_p(f), \mathbb{Q})$, in the case $p = 2$, sheds light on this filtration, as follows. It is clear from Fig. 1 (in which the first index is the vertical one) that in the spectral sequence with which we calculated $H^{n-1}(M_2; \mathbb{Q})$, we have

$$E_1^{1,n-1} = E_2^{1,n-1} = H_{Alt_1}^{n-1}(D^2(f), \mathbb{Q})/d_1(H^{n-1}(D_1^2, \mathbb{Q})) ;$$

as $M_2(f)$ is a Stein space of dimension $n - 1$ and thus $H^n(M_2(f), \mathbb{Q}) = 0$, it follows that $E_{\infty}^{1,n-1} = 0$, and so

$$d_1: H^{n-1}(D_1^2, \mathbb{Q}) \rightarrow H_{Alt_2}^{n-1}(D^2(f), \mathbb{Q})$$

is onto.

Similarly, $d_2: E_2^{0,n-1} = \text{Ker} d_1 \rightarrow E_2^{2,n-2} = H_{Alt_3}^{n-2}(D^3(f), \mathbb{Q})$ is onto; in fact the succession of differentials $d_i$, each defined on the kernel of its predecessor, has $d_i$ mapping onto $E_i^{n-i} = H_{Alt_i+1}^{n-i}(D^{i+1}(f), \mathbb{Q})$, because $E_{i+1}^{n-i} = E_{\infty}^{n-i} = 0$. The successive kernels $E_2^{0,n-1}$ form a decreasing filtration on $E_1^{0,n-1} = H^{n-1}(D_1^2, \mathbb{Q})$, with $H^{n-1}(M_2, \mathbb{Q}) = E_0^{0,n-1} = E_{n+1}^{0,n-1}$ as the smallest term. Now there is an exact sequence

$$0 \rightarrow H^{n-1}(M_2; \mathbb{Q}) \rightarrow H^{n-1}(D_1^2; \mathbb{Q}) \rightarrow H^n(Y_i; \mathbb{Q}) \rightarrow H^n(U_i; \mathbb{Q}) \rightarrow 0$$

(coming from the short exact sequence (3) below).
The descending filtration \( \{E_r^{0,n-1}\}_{1 \leq r \leq n+1} \) on \( H^{n-1}(D^2_i; \mathbb{Q}) \) gives a descending filtration \( \{F_r\}_{0 \leq r \leq n+1} \) on \( H^n(Y; \mathbb{Q}) \), with \( F_r = E_r^{0,n-1}/H^{n-1}(M_2^i; \mathbb{Q}) \); adding \( F^0 = H^n(Y; \mathbb{Q}) \), we obtain a descending filtration \( \{F_r\}_{0 \leq r \leq n+1} \) with \( F_r/F_{r+1} \cong H^{n-r}_{\text{alg}}(D^{r+1}; \mathbb{Q}) \) (here \( D^1 = U_t \)). This filtration coincides with the one coming from the spectral sequence with which we calculated \( H^n(Y; \mathbb{Q}) \).

Wheels turning at different speeds

As above, let \( f: U \to Y \) be a stabilisation of a map-germ of corank 1 from \( \mathbb{C}^n, 0 \) to \( \mathbb{C}^m, 0 \); write \( U = X \), \( D^k(f) = D^k \). Comparison of the exact complexes

\[
0 \to \mathbb{Q}_Y \to f_*(\mathbb{Q}_X) \to \text{Alt}_2^e(Q_{D^2}) \to \text{Alt}_3^e(Q_{D^3}) \to \cdots \quad (1)
\]

and

\[
0 \to \mathbb{Q}_{M_2} \to f_*(\mathbb{Q}_{D^2}) \to \text{Alt}_2^e(Q_{D^3}) \to \text{Alt}_3^e(Q_{D^4}) \to \cdots \quad (2)
\]

shows that there is a short exact sequence

\[
0 \to \mathbb{Q}_{M_2} \to f_*(\mathbb{Q}_{D^2}) \to f_*(\mathbb{Q}_X)/\mathbb{Q}_Y \to 0. \quad (3)
\]

Now to calculate the cohomology of \( D^2_i \) by the method of the beginning of this section, one makes use of the exact complex

\[
0 \to \mathbb{Q}_{D^2_i} \to \mathbb{Q}_{D^2} \to \text{Alt}_2^e(Q_{D^2}) \to \text{Alt}_3^e(Q_{D^3}) \to \text{Alt}_4^e(Q_{D^4}) \to \cdots. \quad (4)
\]

Here \( \tilde{e}^k: D^k \to X \) is induced by the Cartesian projection \( X^k \to X \) which forgets all but the first component, and \( S_{k-1} \) acts on \( D^k \) by permuting the last \( k - 1 \) components.

Exactness of this complex is a consequence of 2.1, taking \( D^2 \) and \( D^2 \) as \( X \) and \( Y \); for by the principle of iteration, \( D^k=1(\tilde{e}^k: D^k \to X) = D^k(f: X \to Y) \), the \( S_{k-1} \) action on \( D^k \) being the one just described.

If we shorten the complex (1) by replacing the first three terms by

\[0 \to f_*(\mathbb{Q}_X)/\mathbb{Q}_Y \to \cdots \]

and call the resulting complex \( (1') \), we find that there is a morphism of complexes \( \theta: f_*(4) \to (1') \) extending the morphism \( f_*(\mathbb{Q}_{D^2_i}) \to f_*(\mathbb{Q}_X)/\mathbb{Q}_Y \), defined as follows: for \( y \in M_2 \), we have

\[
f_*(\text{Alt}_{k-1}\tilde{e}_*^k(Q_{D^k}))(y) = \bigoplus_{x \in f^{-1}(y)} \text{Alt}_{k-1}\tilde{e}_*^k(Q_{D^k})_x.
\]
Now $\text{Alt}_{k-1}^k(\mathbb{Q}_{D^k})$ is generated as $\mathbb{Q}$-vector space by elements $\text{Alt}_{k-1}^k(x, x_2, \ldots, x_k)$, where

$$(x, x_2, \ldots, x_k) \in (\mathbf{e}^k)^{-1}(y) \subseteq D^k.$$

Define $\theta_{k,y} : f_*(\text{Alt}_{k-1}^k(\mathbb{Q}_{D^k}))_y \to \text{Alt}_{k}^k(\mathbb{Q}_{D^k})_y$ by sending $\text{Alt}_{k-1}^k(x, x_2, \ldots, x_k)$ to $\text{Alt}_{k}^k(x, x_2, \ldots, x_k)$. Observe that this definition does not depend on the choice of order of $x_2, \ldots, x_k$, since any permutation induces the same sign change in $\text{Alt}_{k-1}^k(x, x_2, \ldots, x_k)$ and in $\text{Alt}_{k}^k(x, x_2, \ldots, x_k)$. It follows that the $\theta_{k,y}$ fit together to give a morphism of sheaves, $\theta_k$.

It is straightforward to check that the $\theta_k$ commute with the differentials in the complexes $f_*(\mathbf{4})$ and $(1')$.

The morphism of cohomology groups $H^p_{\text{Alt}_{k-1}}(D^k, \mathbb{Q}) \to H^p_{\text{Alt}_k}(D^k, \mathbb{Q})$ induced by $\theta_k$ is formally the same as $\theta_k$ itself; it is thus in fact equal to $\text{Alt}_k$ (defined here without $k$ in the denominator).

Let $K_k = \text{Ker}(\theta_k)$, and denote the (exact) complex

$$0 \to \mathbb{Q}_{M^2} \to K_2 \to K_3 \to K_4 \to \cdots$$

by $\text{Ker}(\theta)$. We now have a short exact sequence of exact complexes

$$0 \to \text{Ker}(\theta) \to f_*(\mathbf{4}) \xrightarrow{\theta} (1') \to 0. \quad (5)$$

Now by taking injective resolutions of each of the sheaves $K_i$, and forming a double complex in the usual way, we obtain as total complex an injective resolution of $\mathbb{Q}_{M^2}$; by taking global sections, we may thus calculate the cohomology of $M^2$. In fact the spectral sequence coming from the first filtration of the double complex collapses at $E_1$, just as in 2.6; for from the short exact sequence

$$0 \to K_k \to f_*(\text{Alt}_{k-1}^k(\mathbb{Q}_{D^k})) \xrightarrow{\theta_k} \text{Alt}_{k}^k(\mathbb{Q}_{D^k}) \to 0,$$

we obtain a long exact sequence in which the map $H^p_{\text{Alt}_{k-1}}(D^k, \mathbb{Q}) \to H^p_{\text{Alt}_k}(D^k, \mathbb{Q})$ is simply the epimorphism $\text{Alt}_k$; since $H^p(D^k, \mathbb{Q})$ vanishes except when $p = 0$ and $m - (m - n)k$, it follows that $H^p(K_k)$ also vanishes except when $p = m - (m - n)k$, and the long exact sequence collapses to the short exact sequence

$$0 \to H^{m - (m - n)k}(K_k) \to H^{m - (m - n)k}_{\text{Alt}_{k-1}}(D^k, \mathbb{Q}) \to H^{m - (m - n)k}_{\text{Alt}_k}(D^k, \mathbb{Q}) \to 0.$$
2.9. **Proposition.** (i) When \( m = n + 1 \), the spectral sequence just described induces a filtration on \( H^{n-1}(M_2, \mathbb{Q}) \) with successive quotients naturally isomorphic to

\[
\ker[\text{Alt}_k : H^{n-k+1}_{\text{Alt}_k-1}(D^k, \mathbb{Q}) \to H^{n-k+1}_{\text{Alt}_k}(D^k, \mathbb{Q})],
\]

so that

\[
H^{n-1}(M_2, \mathbb{Q}) \cong \bigoplus_{k=2}^{n+1} \ker[\text{Alt}_k : H^{n-k+1}_{\text{Alt}_k-1}(D^k, \mathbb{Q}) \to H^{n-k+1}_{\text{Alt}_k}(D^k, \mathbb{Q})]
\]

(ii) If \( m > n + 1 \), then \( H^p(M_2, \mathbb{Q}) \) is isomorphic to

\[
\ker[\text{Alt}_k : H^{m-(m-n)k}_{\text{Alt}_k-1}(D^k, \mathbb{Q}) \to H^{m-(m-n)k}_{\text{Alt}_k}(D^k, \mathbb{Q})]
\]

when \( p = m - k(m - n - 1) - 2 \), and is equal to 0 for other positive values of \( p \).

\[\square\]

2.10. **Remark.** The proofs given here show that we have the short exact sequence of complexes (5) over \( \mathbb{Z} \) as well as over \( \mathbb{Q} \).

3. **Marar’s formulae**

In [16], Washington Luiz Marar obtains formulae for the Euler characteristic of the image \( V \) of a good representative of a stable perturbation of a finitely determined corank 1 map-germ \( \mathbb{C}^n, 0 \to \mathbb{C}^m, 0 \), in terms of the Milnor numbers of the (singularities at 0 of the) associated multiple points schemes \( D^k(f) \), 0 and their intersections \( D^k(f, \mathcal{P}) \), 0 with the various multi-diagonals. Here, using the results of the previous section, we reprove and strengthen these formulae, by showing how the ranks of the cohomology groups \( \text{Alt}_k(H^{m-(m-n)k}(D^k, \mathbb{Q})) \) may be expressed in terms of the Milnor numbers \( \mu(D^k(f, \mathcal{P}), 0) \).

Let \( G \) be a finite group, and let \( \mathcal{R} \) be the ring of all \( \mathbb{Q} \) linear representations of \( G \). Denote elements of \( \mathcal{R} \) by \([V]\). Recall that for a topological space \( X \) on which \( G \) acts, one has the **equivariant Euler characteristic** \( \chi_G(X) \) as an element of \( \mathcal{R} \):

\[
\chi_G(X) = \sum_q (-1)^q[H_q(X, \mathbb{Q})]
\]

where \( G \) acts on \( H_q(X, \mathbb{Q}) \) in the natural way (see [29]). If \( X \) has the structure of a cell complex which is respected by the \( G \)-action, then for \( g \in G \), \( \chi_G(X)(g) \) is
equal to the topological Euler characteristic of the fixed point set $X^g$ of $g$ ([29]).

Let $G$ act on $\mathbb{C}^m$ and let $X \subset \mathbb{C}^m$ be a $G$-invariant Milnor fibre of the germ at 0 of a $G$-invariant isolated complete intersection singularity $X_0$ of codimension $c$. Then

$$\chi_G(X) = [Q] + (-1)^{m-c}[H]$$

where $[Q]$ is the trivial 1-dimensional representational representation of $G$ and $H = H^{m-c}(X, Q)$. Let $g \in G$ and suppose that $X_0^g$ is an isolated complete intersection singularity, which is also of codimension $c$ in $(\mathbb{C}^m)^g$; if $X^g$ is smooth, then it is a Milnor fibre of $X_0^g$, and setting $d_g = \dim(\mathbb{C}^m)^g$, and letting $\mu_g$ be the Milnor number of $X_0^g$, we have

$$X_G(X)(g) = 1 + (-1)^{d_g} \mu_g$$

and so

$$[H](g) = (-1)^{m+d_g} \mu_g.$$  \hspace{1cm} (3.1)

Now consider a map-germ $f: \mathbb{C}^n, 0 \to \mathbb{C}^{n+p}, 0$ of corank 1, and let $f_t: U_t \to V$ be a good representative of a stable perturbation of $f$. Let $X^0 = D^k(f), 0$, and let $X = D^k(f_t)$; then indeed $X$ is a Milnor fibre of $X_0$. By a suitable choice of coordinates on $\mathbb{C}^n$ and $\mathbb{C}^{n+p}$, $X_0$ and $X$ may be embedded in $\mathbb{C}^{n-1+k}$, in such a way that the natural action of the group $G = S_k$ on $X_0$ and $X$ is induced by permutation of the last $k$ coordinates ([19]). Let $H = H_{m-k(m-n)}(D^k(f), Q)$. In order to calculate the dimension of $\text{Alt}_k(H)$, we proceed as follows: $\text{Alt}_k(H)$ is the maximal subspace of $H$ on which $S_k$ acts via its sign representation. As the character of the 1-dimensional sign representation is exactly the sign $\sigma$, $\text{dim}_Q \text{Alt}_k(H)$, which is just the multiplicity of this representation in $H$, is given by the inner product of characters:

$$\text{dim}_Q \text{Alt}_k(H) = \frac{1}{k!} \sum_{g \in S_k} \sigma(g)[H](g).$$

Now in order to calculate the right hand side, suppose that $1 = k_1 < k_2 < \cdots < k_r$, and that in the cycle decomposition of $g$ there are $a_i$ cycles of length $k_i$ (so that $\sum a_i k_i = k$). Then $\sigma(g) = (-1)^{\sum_a(k_i - 1)} = (-1)^{k - \sum_a}$. We calculate $[H](g)$ by using 2.1; for $X^0$ is the intersection of $X = D^k(f_t)$ with the multi-diagonal in $(\mathbb{C}^{n-1+k})^g$ consisting of all points $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_k)$ where $y_i = y_j$ if $i$ and $j$ appear in the same cycle in $g$. Thus $X_g$ is isomorphic to the multiple point scheme $D^k(f, \mathcal{P})$ where $\mathcal{P}$ is the partition $(k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_r, \ldots, k_r)$ ($k_i$ appearing $a_i$ times) of $k$. Now $D^k(f, \mathcal{P})$ is a Milnor fibre of the
isolated complete intersection singularity $D^k(f_0, \mathcal{P}_g, 0)$, whose codimension in $(\mathbb{C}^{n-1+k})^g$ is equal to that of $D^k(f_0)$ in $\mathbb{C}^{n-1+k}$, and thus, as

$$d_g = \dim(\mathbb{C}^{n-1+k})^g = n - 1 + \sum \alpha_i,$$

we have

$$[H](g) = (-1)^{n-k+1+d_g} \mu(D^k(f_0, \mathcal{P}_g, 0), 0) = (-1)^{k+\sum \alpha_i} \mu(D^k(f_0, \mathcal{P}_g, 0), 0).$$

Therefore

$$\dim_{\mathcal{Q}} \text{Alt}_k(H) = \frac{1}{k!} \sum_{\mathcal{P} \in S_k} \mu(D^k(f_0, \mathcal{P}_g, 0), 0).$$

Now combinatorial arguments show that for each fixed partition $\mathcal{P}$ as above, there are $k!/(\Pi i \alpha_i k_i^{\alpha_i})$ elements in $S_k$ with $\mathcal{P}_g = \mathcal{P}$, and hence

$$\dim_{\mathcal{Q}} \text{Alt}_k(H) = \sum_{\mathcal{P}} \left\{ \mu(D^k(f_0, \mathcal{P}, 0) \left( \prod_i \alpha_i k_i^{\alpha_i} \right) \right\}$$

(3.2)

where the sum is taken over all partitions

$$\mathcal{P} = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_r, \ldots, k_r)$$

(where $k_i$ appears $\alpha_i$ times), in which $1 \leq k_1 < k_2 < \cdots < k_r$.

By using 2.6 one obtains formulae for the rank of the cohomology groups of the image of $f_i$, which imply Marar's formulae. The only difference is that Marar incorporates an expression for $\mu(D^k(f_0)/S_k)$ into his formulae.

4. Quasihomogeneous mappings

The main aim of this section is to obtain expressions for the Betti numbers of the image of a stable perturbation $f_i$ of a quasihomogeneous corank 1 map-germ $f_0: \mathbb{C}^n, 0 \to \mathbb{C}^s, 0$, in terms of the weights and degrees. We shall do this by using the results of Section 2, and so as the first step we calculate the rank of the alternating part of the cohomology of the multiple point spaces $D^k(f_i)$.

Let us first recall from [6] some information about isolated complete intersection singularities (ICIS). Let $\varphi = (\varphi_1, \ldots, \varphi_s): \mathbb{C}^n, 0 \to \mathbb{C}^s, 0$ be an analytic germ, such that $\varphi^{-1}(0)$ is an ICIS. Let $\Omega^r = \{\Omega^p, d\}_{0 \leq p}$ be the complex of
germs of holomorphic differential forms on $\mathbb{C}^m, 0$, and consider the complex of relative forms on $\mathbb{C}^m, 0$:

$$\Omega_p^\varphi = \Omega^p \setminus \sum_{i=1}^{s} d\varphi_i \wedge \Omega^{p-1}.$$

Let $\mu$ be the Milnor number of $(\varphi^{-1}(0), 0)$.

4.1. **PROPOSITION [6].** $\Omega^{m-s}_{\varphi}/d\Omega^{m-s-1}_{\varphi}$ is a free $\varphi^{-1}(0,\mathbb{C},0)$ module of rank $\mu$ if $m > s$ and of rank $\mu + 1$ if $m = s$. $\blacksquare$

In the next proposition, we do not assume that $\varphi^{-1}(0)$ is an ICIS, but we do assume that it is a complete intersection.

4.2. **PROPOSITION [6].** Denote by $\Sigma(\varphi)$ the set of critical points of $\varphi$, and by $\varphi'$ the map-germ $(\varphi_1, \ldots, \varphi_{s-1})$. Provided $p \leq m - \dim \Sigma(\varphi)$, the sequence

$$0 \rightarrow \Omega_{\varphi'}^0 \xrightarrow{d\varphi'} \Omega_{\varphi'}^1 \xrightarrow{d\varphi'} \cdots \xrightarrow{d\varphi'} \Omega_{\varphi'}^p \rightarrow 0$$

is exact. $\blacksquare$

As a consequence of 4.2, we have

4.3. **PROPOSITION [6].** If $p \leq m - \dim \Sigma(\varphi)$, the following sequence is exact:

$$0 \rightarrow \varphi^{-1}(0,\mathbb{C},0) \rightarrow \Omega_{\varphi}^0 \xrightarrow{d} \Omega_{\varphi}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\varphi}^{p-1}$$

$$\xrightarrow{d} \Omega_{\varphi}^p \rightarrow \Omega_{\varphi}^p/d\Omega_{\varphi}^{p-1} \rightarrow 0$$

$\blacksquare$

As a consequence, if $\varphi^{-1}(0)$ is an ICIS, then the relative de Rham complex of holomorphic forms calculates the cohomology of the Milnor fibre.

Now let us consider the alternated versions of 4.1–4.3. For any linear space $V$ equipped with a linear $S_k$-action, we denote by $V^{alt}$ the maximal subspace on which $S_k$ acts via its sign representation: $V^{alt} = \{v \in V | \sigma(v) = \text{sign}(\sigma)v \text{ for all } \sigma \in S_k\}$.

Choose some coordinate system on $(\mathbb{C}^m, 0)$ and let $S_k$ act on $\mathbb{C}^m$ by permuting the last $k$ coordinates (we assume $m \geq k$). Suppose that every coordinate function $\varphi_i$ of the map-germ $\varphi:(\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^s, 0)$ (where $m \geq s$) is $S_k$-invariant. Then $\varphi^{-1}(0,\mathbb{C},0)$ is a subring of the ring of $S_k$-invariant functions on $(\mathbb{C}^m, 0)$. The space of relative alternating forms $\Omega_{\varphi}^{p, alt}/\Sigma d\varphi_i \wedge \Omega^{p-1, alt}$, which is equal to
(\Omega^p/\Sigma d\varphi_1 \wedge \Omega^{p-1})^{\text{alt}}$, and which we denote by \(\Omega^p_{\varphi, \text{alt}}\), is a module over the ring of \(S_k\)-invariant functions, and thus over \(\varphi^{-1}(\mathcal{O}_{C_0})\). Suppose that \(\varphi^{-1}(0)\) is an ICIS. The Milnor fibre \(\{\varphi = \varepsilon\}\) is \(S_k\)-invariant, and so we get an \(S_k\) action on its cohomology. Let \(\mu^{\text{alt}}\) be the rank of the \(S_k\)-alternating part of its middle dimensional cohomology.

4.4. PROPOSITION. \(\Omega^{m-s}_{\varphi, \text{alt}}/d\Omega^{m-s-1}_{\varphi, \text{alt}}\) is a free \(\varphi^{-1}(\mathcal{O}_{C_0})\)-module of rank \(\mu^{\text{alt}}\).

Proof. Since \(\Omega^{m-s}_{\varphi, \text{alt}}/d\Omega^{m-s-1}_{\varphi, \text{alt}} = (\Omega^{m-s}_{\varphi}/d\Omega^{m-s-1}_{\varphi})^{\text{alt}}\), it is a direct summand of the free \(\varphi^{-1}(\mathcal{O}_{C_0})\)-module \(\Omega^{m-s}_{\varphi}/d\Omega^{m-s-1}_{\varphi}\) (for every representation of \(S_k\) is completely reducible). It follows that it is free. Moreover, if \(y\) is a regular value of \(\varphi\),

\[
(\Omega^{m-s}_{\varphi}/d\Omega^{m-s-1}_{\varphi})_y \otimes (\mathcal{O}_{C_0,y}/\mathcal{M}_y) = H^{m-s}(\varphi^{-1}(y); \mathbb{C})
\]

and so its alternating part is just \(H^{m-s}(\varphi^{-1}(y); \mathbb{C})^{\text{alt}}\). This proves that the rank of \(\Omega^{m-s}_{\varphi, \text{alt}}/d\Omega^{m-s-1}_{\varphi, \text{alt}}\) is \(\mu^{\text{alt}}\). \(\Box\)

Now consider the alternating parts of the sequences in Propositions 4.2 and 4.3. Notice that for \(S_k\)-invariant \(\varphi\), the differentials in these sequences commute with the \(S_k\)-action. Thus, we get

4.5. PROPOSITION. Let \(p \leq m - \dim(\Sigma(\varphi))\). Then the sequence

\[
0 \rightarrow \Omega^{0, \text{alt}}_{\varphi} \xrightarrow{d\varphi_1 \wedge} \Omega^{1, \text{alt}}_{\varphi} \xrightarrow{d\varphi_1 \wedge} \cdots \xrightarrow{d\varphi_1 \wedge} \Omega^{s, \text{alt}}_{\varphi} \rightarrow \Omega^{s, \text{alt}}_{\varphi} \rightarrow 0
\]

is exact. \(\Box\)

As a consequence,

4.6. PROPOSITION. If \(\varphi^{-1}(0)\) is an ICIS, there is an exact sequence

\[
0 \rightarrow \Omega^{0, \text{alt}}_{\varphi} \xrightarrow{d} \Omega^{1, \text{alt}}_{\varphi} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{m-s, \text{alt}}_{\varphi} \rightarrow \Omega^{m-s, \text{alt}}_{\varphi} \rightarrow 0.
\]

\(\Box\)

\(S_k\)-alternating forms

As above, let \(S_k\) act on \(\mathbb{C}^m\) by permuting the last \(k\) coordinates. In order to describe the spaces \(\Omega^p_{\varphi, \text{alt}}\) of relative \(S_k\)-alternating forms for a germ at \(0 \in \mathbb{C}^m\) of \(S_k\)-invariant mapping \(\varphi\), we start with a description of the absolute \(S_k\)-alternating forms in the case \(m = k\).

We will consider the problem in a more general setting – for a finite group \(G\) of
linear automorphisms of $\mathbb{C}^k$, generated by reflections (recall that a reflection is a linear automorphism which leaves fixed all the points of a hyperplane; one can treat the real case as well). In this subsection we will denote by $\Omega^{p, \text{alt}}$ the space of germs at 0 of holomorphic $p$-forms $\omega$ such that $g^* \omega = \det(g) \omega$ for all $g \in G$; we will call these forms "$G$-alternating". What is $\Omega^{p, \text{alt}}$?

It is well known that the ring $\mathcal{O}^{\text{sym}}_k$ of germs at 0 of $G$-invariant functions is generated by exactly $k$ independent functions, say $h_1, \ldots, h_k$. Let $I$ be the $p$-tuple $(i_1, \ldots, i_p)$, with $i_1 < i_2 < \cdots < i_p$, and let $J = (j_1, \ldots, j_{k-p})$ be its complement in the set $\{1, \ldots, k\}$, with $j_1 < j_2 < \cdots < j_{k-p}$. Consider the $(k - p)$-gradient vector $\nabla h_I = \nabla h_{j_1} \wedge \nabla h_{j_2} \wedge \cdots \wedge \nabla h_{j_{k-p}}$. Let $\omega_I$ be the contraction of the form $dy_1 \wedge dy_2 \wedge \cdots \wedge dy_k$ along $\nabla h_I$ (where the $y_i$ are linear coordinates on $\mathbb{C}^k$). Then $\omega_I$ is a $G$-alternating $p$-form.

4.7. PROPOSITION. The space $\Omega^{p, \text{alt}}$ of germs at 0 of $G$-alternating holomorphic $p$-forms on $\mathbb{C}^k$ is a free $\mathcal{O}^{\text{sym}}_k$-module, with free basis consisting of all $\omega_I$ such that $|I| = p$.

Proof (cf. [26]).

STEP 1. Let $L$ be the field of germs at 0 of meromorphic functions. Then the forms $\omega_I$ are $L$-linearly independent in $\Omega^{p, \text{alt}}$. For suppose that $|I| = p \omega_I = 0$ is a relation, with $\alpha_I \in L$. Evaluating this relation on $\nabla h_I$ for some fixed $I$, we get $0 = \alpha_I \omega_I(\nabla h_I) = \pm \alpha_I$, where

$$\Delta = \det(\partial(h_1, \ldots, h_k)/\partial(y_1, \ldots, y_k));$$

as $\Delta$ is not identically zero, $\alpha_I = 0$.

STEP 2. We show that any alternating $p$-form belongs to $\Sigma_{|I| = p} \mathcal{O}^{\text{sym}}_k \omega_I$. As the number of different $\omega_I$, with $|I| = p$, is $C_p^k$, they generate $\Omega^{p, \text{alt}} \otimes_{\mathcal{O}^{\text{sym}}_k} L$ linearly over $L$. So for any holomorphic $p$-form $\omega$, we can write $\omega = \Sigma_{|I| = p} \beta_I \omega_I$, for some $\beta_I \in L$. Now if $\omega$ is $G$-alternating, then alternation of this expression over $G$ gives

$$\omega = \frac{1}{|G|} \sum_{g \in G} \det(g)^{-1} g^* \omega = \frac{1}{|G|} \sum_{g \in G} \sum_{|I| = p} g^* (\beta_I) \omega_I = \sum_{|I| = p} \gamma_I \omega_I,$$

where the $\gamma_I \in L$ are $G$-invariant. Again by evaluating such an expression on $\nabla h_I$ for a fixed $I$, we get a holomorphic function $\varphi_I = \gamma_I \Delta$. This is a $G$-alternating function, and so by the lemma from [26], $\varphi_I = \Delta \varphi_I$ for some $\varphi_I \in \mathcal{O}^{\text{sym}}$. Thus, we are done. \qed

Now let us consider the bigrading on the space $\Omega^{\cdot, \text{alt}}$ by the weight of the form, (with weight $y_i = \text{weight } dy_i = 1$) and by the degree of the form. Let $P(\Omega^{\cdot, \text{alt}}, t, \tau) = \sum c_{l,m} t^l \tau^m$ be the corresponding Poincaré series (i.e. $c_{l,m}$ is the
dimension of the $\mathbb{C}$-vector space of $m$-forms of weight $l$. Let $d_1, \ldots, d_k$ be the weights of the basic invariants $h_1, \ldots, h_k$ (which of course can be chosen to be homogeneous). We have

4.8. COROLLARY.

$$P(\Omega^{alt}, t, \tau) = \prod_{i=1}^{k} \frac{t^{d_i} - 1 + \tau t}{1 - t^{d_i}}$$

Proof. This is immediate from the previous discussion. The numerator provides the Poincaré polynomial for the set of generators $\omega_I$, and the denominator provides the Poincaré series for $\mathcal{O}_k^{sym}$. \hfill \Box

Since the Poincaré series for the space of forms of fixed degree $p$, is the coefficient of $\tau^p$ in this series, we have

4.9. COROLLARY.

$$P(\Omega^{p, alt}, t) = \text{res}_{\tau = 0} t^{-p - 1} \prod_{i=1}^{k} \frac{t^{d_i} - 1 + \tau t}{1 - t^{d_i}}$$

EXAMPLE. When $S_k$ acts on $\mathbb{C}^k$ by permutations of the coordinates, then $d_i = i$ for $i = 1, \ldots, k$, and we get

$$P(\Omega^{alt}, t, \tau) = \prod_{i=1}^{k} \frac{t^{i-1} + \tau t}{1 - t^i} \quad \text{and} \quad P(\Omega^{p, alt}, t) = \text{res}_{\tau = 0} t^{-p - 1} \prod_{i=1}^{k} \frac{t^{i-1} + \tau t}{1 - t^i}$$

We shall use these expressions in what follows. Note that there is another expression for the latter series:

4.10. PROPOSITION. When $G = S_k$ acts by permuting the coordinates on $\mathbb{C}^k$,

$$P(\Omega^{p, alt}, t) = t^{p+1/2(k-p)(k-p-1)} \prod_{i=1}^{p} (1 - t^i)^{-1} \prod_{j=1}^{k-p} (1 - t^j)^{-1}$$

Proof (independent of 4.7). Consider a $p$-form

$$\omega = \text{Ady}_1 \wedge \cdots \wedge dy_p + \text{terms with other } p\text{-tuples of } dy_j\text{s}.$$ 

It is easily seen that if $\omega$ is an $S_k$-alternating form, the holomorphic function $A$ must be symmetric with regard to permutations of $y_1, \ldots, y_p$, and alternating with respect to permutations of $y_{p+1}, \ldots, y_k$. So

$$A \in \mathcal{O}_k^{sym} \otimes \mathcal{O}_k^{alt} \otimes \mathcal{O}_k^{sym} \{V.d.m.(y_{p+1}, \ldots, y_k)\}$$
where $V.d.m.(y_{p+1}, \ldots, y_k)$ is the Vandermonde determinant. On the other hand, any such $A$ uniquely determines an $S_k$-alternating $p$-form $\omega$. As the Poincaré series of $\partial_y^{\text{sym}, y_p} \cdots \partial_y^{\text{sym}, y_p}$ is $\prod_{i=1}^{p} (1 - t^i)^{-1}$, and $\deg(V.d.m.(y_{p+1}, \ldots, y_k)) = \frac{1}{2}(k-p)(k-p-1)$, the statement is proved. \hfill \Box

$S_k$-alternating forms on a symmetric quasihomogeneous ICIS

Let $x_1, \ldots, x_n, y_1, \ldots, y_k$ be coordinates in $(\mathbb{C}^{n+k}, 0)$ and let the group $S_k$ act by permutation of the $y$ coordinates. Let $w_0, w_1, \ldots, w_n; d_1, \ldots, d_s$ be positive integers. Let $\varphi: (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^s, 0)$, with $n + k \geq s$, be an $S_k$ invariant mapping, quasihomogeneous of type $(w_1, \ldots, w_n, w_0, \ldots, w_0; d_1, \ldots, d_s)$.

Suppose that $\varphi^{-1}(0)$ is an ICIS, and let

$$H^{\text{alt}} = \Omega^{n+k-s, \text{alt}}_\varphi / (d\Omega^{n+k-s-1, \text{alt}}_\varphi + \sum p_i \Omega^{n+k-s, \text{alt}}_\varphi),$$

be the top cohomology of the complex of relative $S_k$-alternating holomorphic forms on $\varphi^{-1}(0)$. It inherits a grading, with $wt(x_i) = wt(dx_i) = w_i$, $wt(y_i) = wt(dy_i) = w_0$. Let $P(H^{\text{alt}}; t)$ be the Poincaré series of $H^{\text{alt}}$ with respect to this grading.

4.11. THEOREM.

$$P(H^{\text{alt}}; t) = \text{res}_{t=0} \frac{\tau^{-n-k+s-1}}{1 + \tau} \prod_{i=1}^{k} \frac{\tau^{i(l-1)w_0} + \tau^{w_0}n}{1 - t^{jw_0}} \prod_{j=1}^{n} \frac{1 + \tau^{w_j}}{1 - t^{w_j}} \prod_{l=1}^{s} \frac{1 - t^{d_l}}{1 + \tau^{d_l}}$$

We give the proof below.

By Proposition 4.4, $\dim(H^{\text{alt}}) = \mu^{\text{alt}}$. So we have

4.12. COROLLARY. $\mu^{\text{alt}} = P(H^{\text{alt}}; 1)$.

Proof of 4.11 1. Consider the space of $S_k$-alternating forms on $\mathbb{C}^{n+k}; \Omega^{*, \text{alt}} = \Omega^{*, \text{alt}}_k \otimes \mathbb{C} \Omega_n$. It has a natural bigrading, by the quasihomogeneous weight $l$ and by the degree $p$ of the form. Let $P(\Omega^{*, \text{alt}}; t, \tau) = \Sigma c_{l, p} t^l \tau^p$ be the corresponding Poincaré series. Then $P(\Omega^{*, \text{alt}}) = P(\Omega^{*, \text{alt}}_k) \cdot P(\Omega_n)$ and by Corollary 4.8 we get

$$P(\Omega^{*, \text{alt}}; t, \tau) = \prod_{l=1}^{k} \frac{\tau^{i(l-1)w_0} + \tau^{w_0}n}{1 - t^{jw_0}} \prod_{j=1}^{n} \frac{1 + \tau^{w_j}}{1 - t^{w_j}}$$

2. By induction we obtain from Proposition 4.5 that

$$P(\Omega^{*, \text{alt}}_\varphi; t, \tau) = \text{res}_{t=0} \frac{\tau^{-p-1}}{1 + \tau} P(\Omega^{*, \text{alt}}; t, \tau) \prod_{l=1}^{s} \frac{1 - t^{d_l}}{1 + \tau^{d_l}}$$
By Proposition 4.6,

\[ P(\Omega^{n+k-s, \text{alt}}_\varphi / d\Omega^{n+k-s-1, \text{alt}}_\varphi, t) = \sum_{p=0}^{n+k-s} (-1)^{n+k-s-p} P(\Omega^{\text{alt}}_\varphi, t) \]

\[ = \res_{\tau=0} \frac{\tau^{-n-k+s-1}}{1 + \tau} P(\Omega^{\text{alt}}_\varphi, t) \prod_{l=1}^{s} (1 + \tau t^{d_l})^{-1} \]

4. By Proposition 4.4, the space \( \Omega^{n+k-s, \text{alt}}_\varphi / d\Omega^{n+k-s-1, \text{alt}}_\varphi \) is a free \( \phi^{-1}(\mathcal{C}_z, 0) \)-module. The generators of this module are \( \mathbb{C} \)-linear generators of the space \( H^{\text{alt}}_\varphi \).

Thus,

\[ P(H^{\text{alt}}, t) = P(\Omega^{n+k-s, \text{alt}}_\varphi / d\Omega^{n+k-s-1, \text{alt}}_\varphi, t) / P(\phi^{-1}(\mathcal{C}_z, 0), t) \]

As \( P(\phi^{-1}(\mathcal{C}_z, 0); t) = \prod_{l=1}^{s} (1 - t^{d_l})^{-1} \), we are done.

Our quasihomogeneous mapping \( \varphi \) is equivariant with respect to the following \( U(1) \)-action:

\[ \lambda \cdot (x_1, \ldots, x_n, y_1, \ldots, y_k) = (\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n, \lambda^{w_0} y_1, \ldots, \lambda^{w_0} y_k) \]

in the source \( \mathbb{C}^{n+k} \), and \( \lambda \cdot (z_1, \ldots, z_s) = (\lambda^{d_1} z_1, \ldots, \lambda^{d_s} z_s) \) in the target \( \mathbb{C}^s \).

Consider the non-critical level \( \{ \varphi = \varepsilon \} \). Let \( d \) be the greatest common factor of all of the \( d_i \) such that \( \varepsilon_i \neq 0 \). Consider the loop \( \varepsilon \cdot \exp(2\pi i \rho / d), \rho \in [0, 1] \), in \( \mathbb{C}^s \). It induces an endomorphism \( h^{\text{alt}} \) of the \( S_k \)-alternating cohomology of the Milnor fibre \( \{ \varphi = \varepsilon \} \). This endomorphism \( h^{\text{alt}} \) is called the alternating quasihomogeneous monodromy. Let \( D \in \mathbb{Z}[\mathbb{C}^*] \) be the divisor of the characteristic polynomial of \( h^{\text{alt}} \).

Then we have

4.13. COROLLARY. \( D = P(H^{\text{alt}}; \langle \exp(2\pi i / d) \rangle) \).

Indeed, quasihomogeneous forms which represent a \( \mathbb{C} \)-basis of \( H^{\text{alt}}_\varphi \) also give a basis for the alternating cohomology of the Milnor fibre, and \( h^{\text{alt}} \) simply multiplies each such form of weight \( l \) by \( \exp(2\pi i l / d) \).

The cohomology of the image of a quasihomogeneous mapping

We now apply the results of this section and Section 2 to calculate the Betti numbers of the image of a stable perturbation of a quasihomogeneous corank 1 map-germ \( f_0: \mathbb{C}^{n+1}, 0 \to \mathbb{C}^{n+r}, 0 \), where \( r > 1 \) (for notational reasons, it is convenient to consider mappings with domain \( \mathbb{C}^{n+1} \) rather than \( \mathbb{C}^n \)).
We can choose coordinates \((x_1, \ldots, x_n, y)\) in the source and coordinates in the target, so that \(f_0\) takes the form

\[
f_0(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n, f_{0,1}(x, y), \ldots, f_{0,r}(x, y)).
\]

Then the multiple point scheme \(D^k(f_0)\) embeds into \(\mathbb{C}^n \times \mathbb{C}^k\), where it is defined by the equations

\[
F_{i,j}(x_1, \ldots, x_n, y_1, \ldots, y_k) = \text{V.d.m.}(y, f_{0,j})/\text{V.d.m.}(y)
\]

where \(\text{V.d.m.}(y)\) is the Vandermonde determinant \(\det[y^{\alpha_i-1}]_{1 \leq i, \alpha \leq k}\), and \(\text{V.d.m.}(y, f_{0,j})\) is the determinant obtained from \(\text{V.d.m.}(y)\) by replacing \(y_i\) by \(f_{0,j}(x, y_i)\) for \(1 \leq i \leq k\) (see [19], §2). Each point \((x, y_1, \ldots, y_k)\) satisfying these equations corresponds to a \(k\)-tuple \((x, y_1), \ldots, (x, y_k)\) of points of \(\mathbb{C}^{n+1}\) having the same image under \(f_0\). Each \(F_{i,j}\) is invariant with respect to the \(S_k\)-action on \(\mathbb{C}^{n+k}\) in which the last \(k\) coordinates are permuted. We shall denote by \(F_k\) the mapping \(\mathbb{C}^{n+k} \rightarrow \mathbb{C}^{(k-1)r}\) with components \(F_{i,j}\), \(1 \leq j \leq r\), \(1 \leq i \leq k - 1\). Then if \(f\) is a stable perturbation of \(f_0\), \(D^k(f)\) is a Milnor fibre of the ICIS \((F_k^{-1}(0), 0)\) ([19]).

Now suppose in addition that \(f_0\) is quasihomogeneous, with respect to weights \(w_i\) for the variables \(x_i\), and \(w_0\) for \(y\), with weight \((f_{0,j}) = d_j\). Then weight \((F_{i,j}) = d_j - lw_0\), and so by Theorem 4.11 the Poincaré series of the top alternating cohomology of the multiple point space \(D^k(f)\) is given by the polynomial

\[
R_k(t) = \text{res}_{t=0} \frac{\tau^{n-k+r(k-1)-1}}{1 + \tau} \prod_{a=1}^{k} \frac{t^{(a-1)w_0} + \tau t^{w_0}}{1 - t^{aw_0}}
\]

\[
\prod_{i=1}^{n} \frac{1 + \tau t^{w_i}}{1 - t^{w_i}} \prod_{j=1,\ldots,r}^{1} \frac{1 - t^{d_j - lw_0}}{1 + \tau t^{d_j - lw_0}}
\]

\[
= \text{res}_{t=0} \tau^{n-k+r(k-1)-1} t^{1/2k(k-1)w_0} \prod_{a=2}^{k} (1 - t^{aw_0})^{-1} \prod_{a=3}^{k} (1 + \tau t^{2-2a} w_0)
\]

\[
\prod_{i=0}^{n} \frac{1 + \tau t^{w_i}}{1 - t^{w_i}} \prod_{j=1,\ldots,r}^{1} \frac{1 - t^{d_j - lw_0}}{1 + \tau t^{d_j - lw_0}}
\]

The decomposition of the cohomology of the image of a stable perturbation into the direct sum of the alternating cohomology of the multiple point spaces \(D^k\), given by Theorem 2.6, then leads to

4.14. THEOREM. If \(r = 2\), then the \(n+1\)-st Betti number \(\beta_{n+1}(Y_t)\) of the image \(Y_t\) of a stable perturbation of \(f_0\) is equal to \(\sum_{k=2}^{n+\frac{2}{3}} R_k(1)\).
If \( r > 2 \), then if \( 2 \leq k \leq (n + r)/(r - 1) \), we have \( \beta_{n+k-(k-1)(r-1)}(Y_i) = R_k(1) \).

In both cases, the remaining Betti numbers \( \beta_i \), for \( i > 0 \), vanish. \( \square \)

4.15. REMARK. Corollary 4.13 determines the eigenvalues of the quasihomogeneous monodromy of the image of a stable perturbation of \( f_0 \).

4.16. EXAMPLES. We list only the non-zero Betti numbers:

1. \( \mathbb{C}^{n+1} \to \mathbb{C}^{2n+2} \):

\[
\beta_1 = \prod_{j=1}^{n+2} (d_j - w_0)/2w_0^2 \prod_{i=1}^{n} w_i.
\]

Note that \( \beta_1 \) here is just the number of points of self-intersection of the image.

2. \( \mathbb{C}^{n+1} \to \mathbb{C}^{2n+1}, n > 1: \)

\[
\beta_2 = \frac{\prod_{j=1}^{n+1} (d_j - w_0)}{2w_0^2w_1 \cdots w_n} \left[ \sum_{j=1}^{n+1} d_j - (n+1)w_0 - \sum_{i=0}^{n} w_i \right]
\]

3. \( \mathbb{C}^2 \to \mathbb{C}^3 \) (c.f. [23])

\[
\beta_2 = \frac{(d_1 - w_0)(d_2 - w_0)}{6w_0^3w_1} [d_1 + w_0](d_2 + w_0) - 6w_0^2 - 3w_0w_1]
\]

4. \( \mathbb{C}^4 \to \mathbb{C}^6: \)

\[
\beta_2 = \frac{\prod_{j=1}^{3} (d_j - w_0)(d_j - 2w_0)}{6w_0^3w_1w_2w_3}
\]

\[
\beta_3 = \frac{\prod_{j=1}^{3} (d_j - w_0)}{2w_0^3w_1w_2w_3} \left[ 4 \sum_{j=1}^{3} (d_j - w_0)^2 + \sum_{1 \leq \alpha < \beta \leq 3} (d_\alpha + d_\beta - 2w_0)^2 - \sum_{i=0,1,2,3} (d_j + w_i - w_0)^2 + \sum_{0 \leq \alpha < \beta \leq 3} (w_\alpha + w_\beta)^2 \right].
\]

5. Hodge numbers of the image of a stable perturbation of a quasihomogeneous map-germ

Let \( f_0: \mathbb{C}^{n+1}, 0 \to \mathbb{C}^{n+r}, 0 \) with

\[
f_0(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n, f_{0,1}(x, y), \ldots, f_{0,r}(x, y)),
\]
be as in Section 4 a quasihomogeneous map-germ of corank 1 and of finite codimension, with $r \geq 2$. It is easy to see from the characterisation of stability in terms of multiple point schemes given as Theorem 2.5 above, and from the construction of a versal deformation of $f_0$ that $f_0$ has a stable perturbation in “negative weight”; that is, it is possible to find a stable $f$ such that for each $j$, $f_j - f_0,j = \Sigma_s m_{j,s}$ with each monomial $m_{j,s}$ of weight $d_{j,s}$ less than the weight $d_j$ of $f_0,j$.

Now the mapping $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+r}$ is the affine part of the mapping $\tilde{f} : \mathbb{P}(w_1, \ldots, w_n, w_0, 1) \to \mathbb{P}(w_1, \ldots, w_n, d_1, \ldots, d_r, 1)$ of weighted projective spaces given by $\tilde{f}(x, y, z) = (x, \tilde{f}_1, \ldots, \tilde{f}_r, z)$ where

$\tilde{f}_j(x, y, z) = f_0,j(x, y) + \sum_s z^{d_j - d_{j,s}} m_{j,s}(x, y)$.

The multiple point spaces $D^k(f)$ are compact algebraic varieties which can be embedded in $\mathbb{P}(w_1, \ldots, w_n, w_0, \ldots, w_0, 1)$ (where $w_0$ appears $k$ times). If $f_\infty = \tilde{f}|z=0$, then $D^k(f_\infty)$ embeds in $\mathbb{P}(w_1, \ldots, w_n, w_0, \ldots, w_0)$, which is just the subspace $\{z = 0\}$ of $\mathbb{P}(w_1, \ldots, w_n, w_0, \ldots, w_0, 1)$. Now $D^k(f_\infty)$ is the weighted projectivisation of $D^k(f_0)$, which is smooth outside the origin; thus $D^k(f_\infty)$ is quasismooth, and its only singularities are cyclic quotient singularities. The affine part $\{z \neq 0\}$ of $D^k(\tilde{f})$ is just $D^k(f)$, and hence is smooth; since moreover $D^k(\tilde{f}) \cap \{z = 0\} = D^k(f_\infty)$ is itself quasismooth, the affine cone over $D^k(\tilde{f})$ is thus smooth outside 0, and hence $D^k(\tilde{f})$ is quasismooth also. Since both $D^k(\tilde{f})$ and $D^k(f_\infty)$ are compact algebraic varieties, they have a canonical mixed Hodge structure (\cite{[0]}, \cite{[1]}, \cite{[28]}). In fact the structures are pure, for both $D^k(\tilde{f})$ and $D^k(f_\infty)$ are projective and are $V$-manifolds, (cf. \cite{[28]}, Chapter I, §5); that is, they are locally the quotient of a smooth space by the action of an finite group of holomorphic automorphisms.

In this section we use the results of Sections 2 and 4 to express the Hodge numbers of the mixed Hodge structure on the cohomology of the image of $f$, in terms of the cohomology of the multiple point spaces of $f$ and $f_\infty$.

**Complete intersections**

As in Section 4, let $\varphi : \mathbb{C}^{m}, 0 \to \mathbb{C}^{s}, 0$, with $m > s$, be quasihomogeneous of type $(w_1, \ldots, w_m; d_1, \ldots, d_s)$, and suppose that $\varphi^{-1}(0)$ is an ICIS. Then the same equations $\varphi = 0$ define an $(m - s - 1)$-dimensional quasismooth complete intersection $X$ in the $(m - 1)$-dimensional weighted projective space.
\( P = \mathbb{P}(w_1, \ldots, w_m) \) (see [28], [2]). \( X \) has a pure Hodge structure ([28]). Let \( F^p \) be the Hodge filtration on \( H^*(X, \mathbb{C}) \).

We now recall from [28] the description (due to R. O. Buchweitz) of the primitive parts \( P^{p,q} \) of the spaces \( \text{Gr}^p H^{p+q}(X; \mathbb{C}) \), which are non-trivial for \( p + q > 0 \) only if \( p + q = m - s - 1 \):

Let \( S \) be the graded ring \( S = \mathbb{C}[z_1, \ldots, z_m] \), with \( wt(z_i) = w_i \), and consider its graded quotient \( R = S/\langle \varphi_1, \ldots, \varphi_s \rangle \). We shall use the following graded \( R \)-modules:

- \( \omega_R \), the graded module corresponding to the sheaf of holomorphic forms of top degree \( m - s - 1 \) on \( X \); it is a free \( R \)-module of rank 1, with generator \( \omega_0 \) of weight \( \Sigma w_i - \Sigma d_j \) (in fact \( \omega_0 \) is given by contraction of the form \( dz_1 \wedge \cdots \wedge dz_m / \varphi_1 \wedge \cdots \wedge \varphi_s \) with the Euler vector field \( e = \Sigma w_i z_i \partial / \partial z_i \) on \( \mathbb{C}^n \));

- \( \mathcal{N} \), corresponding to the normal bundle of \( X \) in \( \mathbb{P} \); it is a free \( R \)-module of rank \( s \), with generators \( u_j \) of weight \( -d_j \);

- \( \theta \), corresponding to the restriction of the tangent bundle of \( \mathbb{P} \) to \( X \); it is a free \( R \)-module of rank \( m \), with generators \( v_i \) of weight \( -w_i \);

- \( S^\alpha \mathcal{N} \), the \( \alpha \)-th symmetric power of \( \mathcal{N} \);

- \( \wedge^\beta \theta \), the \( \beta \)-th exterior power of \( \theta \);

- \( M_{\alpha,\beta} = S^\alpha \mathcal{N} \otimes_R \wedge^\beta \theta \otimes_R \omega_R \), which is a free \( R \)-module on generators \( u_1^i \cdots u_s^i \otimes v_i \wedge \cdots \wedge v_i \otimes w_0 \), where \( c_j \geq 0 \) and \( \Sigma c_j = \alpha \), and \( 0 \leq i_1 < \cdots < i_\beta \leq m \).

We shall also consider the mappings \( d\varphi: \theta \to \mathcal{N} \) (differentiation of \( \varphi \) along a vector field) given by

\[
\sum r_i v_i \to \sum J \left( \sum r_i \partial \varphi_j / \partial z_i \right) u_j
\]

and

\[
e \wedge: \wedge^\beta \theta \to \wedge^{\beta+1} \theta \) (wedging with the Euler field).

These mappings induce mappings

\[
d\varphi: M_{\alpha,\beta} \to M_{\alpha+1,\beta-1} \quad \text{and} \quad e \wedge: M_{\alpha,\beta} \to M_{\alpha,\beta+1}.
\]

Consider the following diagram, for a fixed value of \( q \):
Construct a complex $\Lambda(q)^*$ by taking the direct sums of the weight zero parts of the modules (i.e. global sections of the corresponding sheaves) on the same dashed line in the diagram:

$$\Lambda(q)^i = \bigoplus_{\alpha, \beta \geq 0, \alpha \geq i, 2\alpha + \beta = q + i} (M_{\alpha, \beta})_0 \quad |i| \leq q$$

$$\Lambda(q)^i = 0 \quad |i| > q$$

The differential in this complex is a certain combination of $d\varphi$ and $e$ (neither of these mappings change the weight).

Then $P_{m-s-1-q,q}$ is the only non-trivial cohomology of the complex $\Lambda(q)^*$ (see [28]). Its dimension $h^{m-s-1-q,q}$ may be calculated as follows:

5.1. PROPOSITION ([11]).

$$h^{m-s-1-q,q} = (-1)^q \left\{ t^{-1} \frac{1 + ut^{-\beta_{\ast}}}{1 + u} \prod \frac{1}{1 + t^{w_i}} \prod \frac{1 - t^d_j}{1 + ut^{-d}_j} \right\}$$
(here the limits for $i$ and $j$ are the obvious ones; we shall omit them in what follows).

Proof. We have

$$h^{m-s-1-q,q} = \sum_{|i| \leq q} (-1)^{q-i} \dim \Lambda(q)^i = \sum_{|i| \leq q} (-1)^{q-i} \sum_{x, \beta \geq 0, x \geq i} \dim(M_{x, \beta})_0$$

$$= \sum_{x, \beta \geq 0, 2x + \beta = q + i} (-1)^{\beta} \dim(M_{x, \beta})_0.$$

Let us evaluate this sum.

Consider the $R$-module

$$M = \bigoplus_{x, \beta \geq 0} M_{x, \beta} = S^\cdot \mathcal{N} \otimes_R \mathcal{A} \otimes_R \omega_R.$$

It has three gradings: by the weight $r$ and the degrees $x$ and $\beta$. The Poincaré series for its free $R$-generators is

$$A(t, u, v) = t^{\Sigma w_i - \Sigma d_j} \prod (1 + vt^{-w_i}) \prod (1 - ut^{-d_j}).$$

(so that the Poincaré series for $M$ is the product of $A(t, u, v)$ with the Poincaré series for $R$). Here the coefficient of $t^x u^r v^\beta$ is the number of generators of $M_{x, \beta}$ of weight $r$. Also $\dim(M_{x, \beta})_0$ is the coefficient of $t^x u^r v^\beta$ in the Poincaré series $P(M; t, u, v) = A(t, u, v) \cdot P(R; t)$, where $P(R; t) = \prod (1 - t^{d_j}) / \prod (1 - t^{w_i})$. Consequently, $\sum_{x, \beta = \gamma} (-1)^{\beta} \dim(M_{x, \beta})_0$ is the coefficient of $t^x u^r$ in $B(t, u) = P(M; t, u, -u)$. In order to obtain the sum of such expressions for $0 \leq j \leq q$, we have to take the coefficient of $t^x u^r$ in the series $C(t, u) = B(t, u)/(1 - u)$. In the statement of the proposition we simply point out that one can obtain the same number from the series $C(t, -u)$, by means of residues. \qed

The symmetric case

Now let $\varphi: \mathbb{C}^{n+k}, 0 \to \mathbb{C}^s, 0$ with $n + k > s$, be quasihomogeneous of type $(w_1, \ldots, w_n, w_0, \ldots, w_0; d_1, \ldots, d_s)$ (where $w_0$ appears $k$ times), and such that each of its coordinate functions $\varphi_j$ is invariant under the $S_k$ action on $\mathbb{C}^{n+k}$ in which the last $k$ coordinate are permuted. We want to express the rank of the $S_k$-alternating part of the primitive cohomology of the quasismooth complete intersection $\{\varphi = 0\}$ in the corresponding weighted projective space, in terms of the weights $w_i$ and $d_j$. 
5.2. PROPOSITION.

\[ h_0^{n+k-s-1-q,q;\text{alt}} = (-1)^q \operatorname{res}_{t=0} \operatorname{res}_{u=0} \left\{ t^{-1} \frac{u^{-q-1}}{1+u} \prod_{i \geq 1} \frac{1+ut^{-w_i}}{1-t^{w_i}} \prod_{l=1}^{k} \frac{1+ut(l-2)w_0}{1-t^{l}w_0} \times \prod_{j \geq 1} \frac{1-ut^{d_j}}{1+ut^{-d_j}} t^{kw_0 + \Sigma_{i \geq 1} w_i - \Sigma_{j \geq 1} d_j} \right\} \]

Proof. Let us take the alternating part of the complex \( \Lambda(q') \) of the previous section. Under our assumptions, the mappings \( d_\phi \) and \( e \wedge \) are \( S_k \)-equivariant. Hence, we need to take the \( S_k \)-alternating parts of the modules

\[ M_{\alpha,\beta} = \bigoplus (u^1 \wedge \ldots \wedge u^s) \otimes_{C} (\hat{\theta} \otimes_{C} \omega_0) \]

(where the direct sum is taken over all \( c_j \geq 0 \) such that \( \Sigma c_j = d \)). As \( u_1, \ldots, u_s \) are \( S_k \)-invariant and \( \omega_0 \) is \( S_k \)-alternating, taking the \( S_k \)-alternating part of \( M_{\alpha,\beta} \) is the same as taking the \( S_k \)-invariant part \( (\Lambda(\hat{\theta})_{\text{sym}} \otimes \hat{\theta}) \). Following [26], we have \( (\Lambda(\hat{\theta})_{\text{sym}} \otimes \hat{\theta}) = \Lambda(\theta_{\text{sym}}) \), and \( \theta_{\text{sym}} \) is \( R_{\text{sym}} \)-freely generated by the gradients of the \( n+k \) basic \( S_k \)-invariant functions. Thus, for the bigrading of \( (\Lambda(\hat{\theta})_{\text{sym}} \otimes \hat{\theta}) \) by the weight and the degree of the form, we get

\[ P(\Lambda(\theta_{\text{sym}}); t, v) = P(R_{\text{sym}}; t) \cdot D(t, v) \]

where \( D(t, v) \) is the Poincaré series of the exterior algebra on the basic equivariant vector fields,

\[ D(t, v) = \prod_{i \geq 1} (1 + vt^{-w_i}) \cdot \prod_{l=1}^{k} (1 + ut(l-2)w_0). \]

Now \( R_{\text{sym}} \) is a quotient of the ring of \( S_k \)-invariant polynomials on \( \mathbb{C}^{n+k} \) by the ideal generated by \( \varphi_1, \ldots, \varphi_s \). So,

\[ P(R_{\text{sym}}; t) = \prod_{i \geq 1} (1 - t^{w_i})^{-1} \cdot \prod_{l=1}^{k} (1 - t^{l}w_0)^{-1} \cdot \prod_{j \geq 1} (1 - t^{d_j}). \]

The proposition is now proved by repeating the arguments from the proof of proposition 5.1.

5.3. REMARK. The multiplicities of the eigenvalues of the quasi-homogeneous monodromy on the alternating cohomology can be expressed in the same way as in the non-symmetric case (cf. [11]).
Hodge numbers of the stable image

In order to obtain the canonical mixed Hodge structure (MHS) on the image of a lower stable perturbation $f$ of a quasihomogeneous corank 1 mapping $f_0 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+r}$, we begin with consideration of its $k$-point space $D^k = D^k(f)$ which is also a lower stable perturbation of the quasihomogeneous ICIS $D^k(f_0)$. Now $\bar{D}^k = D^k(\tilde{f})$ (see the beginning of this section) is a compactification of the smooth complete intersection $D^k$. Set $D^k = D^k(f_\infty) = \bar{D}^k \setminus D^k$. Then $(\bar{D}^k, D^k)$ is a pair of V-manifolds, i.e. it is locally a quotient of an action of a finite subgroup of $GL(n; \mathbb{C})$ on the pair $(\mathbb{C}^{\rho_k}, \mathbb{C}^{\rho_k-1})$, where $\rho_k = \dim D^k$ (here $\mathbb{C}^{\rho_k-1}$ is mapped into itself by every element of the subgroup). For this fact see [2, subsection 3.1]. As $\bar{D}^k$ is a V-manifold, its singular set $\Sigma$ has codimension $\geq 2$. Let $j : \bar{D}^k \setminus \Sigma \to \bar{D}^k$ be the inclusion map. Following [27] we define

$$\tilde{\Omega}_{\bar{D}^k}(\log D^k) = j_* \Omega_{\bar{D}^k \setminus \Sigma}(\log(D^k \setminus \Sigma))$$

As in [28, §10], this is a resolution of $\mathbb{C}_{D^k}$ and, thus,

$$H^*(D^k; \mathbb{C}) \cong \mathbb{H}^*(\bar{D}^k, \tilde{\Omega}_{\bar{D}^k}(\log D^k)).$$

In order to obtain the canonical mixed Hodge structure (MHS) on this cohomology we define the Hodge and weight filtrations on the logarithmic sheaves as follows:

- Hodge (decreasing): $F^s \tilde{\Omega}_{\bar{D}^k}(\log D^k) = \begin{cases} \tilde{\Omega}_{\bar{D}^k}(\log D^k) & \text{if } p \geq s \\ 0 & \text{if } p < s \end{cases}$

- weight (increasing): $W^s \tilde{\Omega}_{\bar{D}^k}(\log D^k) = \tilde{\Omega}_{\bar{D}^k}(\log D^k) \setminus \tilde{\Omega}_{\bar{D}^k}^{p-s}$

(later term is $j_* \Omega_{\bar{D}^k \setminus \Sigma}$).

We have $Gr^w_s \tilde{\Omega}_{\bar{D}^k}(\log D^k) \neq 0$ only for $s = 0, 1$ [27, page 532], and in fact

- $Gr^w_0 \tilde{\Omega}_{\bar{D}^k}(\log D^k) = \tilde{\Omega}_{\bar{D}^k}$

- $Gr^w_1 \tilde{\Omega}_{\bar{D}^k}(\log D^k) \simeq i_* \tilde{\Omega}_{\bar{D}^k} [-1]$ (by the residue map).

Here $i : D^k \subset \bar{D}^k$ and $j' : D^k \setminus \Sigma \subset D^k_\infty$ are inclusions (where $\Sigma'$ is the singular locus of $D^k_\infty$) and $\tilde{\Omega}_{\bar{D}^k} = j_* \tilde{\Omega}_{\bar{D}^k \setminus \Sigma'}$.

Define

- $F^p H^p(D^k; \mathbb{C}) = \text{image of } \mathbb{H}^p(\bar{D}^k; F^p \tilde{\Omega}_{\bar{D}^k}(\log D^k)) \text{ in } H^p(D^k; \mathbb{C})$

- $W_s H^p(D^k; \mathbb{C}) = \text{image of } \mathbb{H}^p(\bar{D}^k; W_s \tilde{\Omega}_{\bar{D}^k}(\log D^k)) \text{ in } H^p(D^k; \mathbb{C})$. 

Here $s$ is a non-negative integer.
As in [28, Theorem 10.3] one can see that $W$ is already defined over $\mathbb{Q}$ and $W$ and $F$ define a mixed Hodge structure on $H^*(D^k)$.

We find

\[ \text{Gr}_\rho^W H^p(D^k; \mathbb{C}) = P^p(\overline{D}^k; \mathbb{C}) \quad \text{(the primitive part of } H^p(\overline{D}^k; \mathbb{C})), \]
\[ \text{Gr}_{\rho+1}^W H^p(D^k; \mathbb{C}) \simeq P^{p-1}(D_{\infty}; \mathbb{C}) \]

(here $\rho = \rho_k$). The non-primitive parts of $H^*(\overline{D}^k; \mathbb{C})$ and $H^*(D_{\infty}; \mathbb{C})$ which come from the cohomology of the weighted projective spaces are cancelled by the $W$-spectral sequence, which degenerates at $E_2$.

Now we go to the stable image $Y$ of $f$.

For each $k = 0, 1, \ldots$, consider the sheaf complex

\[ K^k = \text{Alt}_{k+1} c_{k+1} \overline{D}^k \cdot (\log D_{k+1}^{k+1}) \text{ on } \overline{Y}. \]

The mappings $e^{i,k+1}: \overline{D}^{k+1} \to \overline{D}^k$, $i = 1, \ldots, k+1$, induce operators

\[ \delta^k = \sum_{i=1}^{k+1} (-1)^{i+k}(e^{i,k+1})^*: K^{k-1} \to K^k. \]

For each $k$, the complex $K^k$ is a resolution of $\text{Alt}_{k+1} c_{k+1} (C_{D_{k+1}})$. The complex $\{\text{Alt}_{k+1} c_{k+1} (C_{D_{k+1}}), \delta^k\}_k$ is a resolution of $C_Y$, by Section 2. Thus, the double complex $K^{k}$ also resolves $C_Y$. Let $K^*$ be the associated total complex: $K^m = \bigoplus_{p+q=m} K^{p,q}$. Its hypercohomology is the cohomology of $Y$.

Following Steenbrink [28, §13] define filtrations

\[ F^s K^m = \bigoplus_{p+q=m} F^s K^{p,q}, \quad W^s K^m = \bigoplus_{p+q=m} W^{s+p} K^{p,q}. \]

In the same way as for $D^k$, these filtrations give rise to Hodge and weight filtrations on $H^*(Y)$.

For our situation we get:

\[ \text{Gr}_s^W K^m = \text{Gr}_s^W K^{-s,m+s} \oplus \text{Gr}_1^W K^{1-s,m+s-1} \quad \text{for } s \leq 0, \]
\[ \text{Gr}_1^W K^m = \text{Gr}_1^W K^{0,m}, \quad \text{Gr}_{>1}^W = 0. \]

Thus, with $Y$ the image of the mapping $f: \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ we obtain, on $H^{n+1}(Y)$,

\[ \text{Gr}_{n+1}^W \simeq H^n_{\text{Alt}_{2}}(D_{\infty}; \mathbb{Q}), \]
\[ \text{Gr}_{n+1-k}^W \simeq H^{n-k} \text{Alt}_{k+1}(D_{k+1}; \mathbb{Q}) \oplus H^{n-k} \text{Alt}_{k+2}(D_{\infty}; \mathbb{Q}), \quad k = 1, \ldots, n+1 \]

(note that $D_{n+2}^{n+2} = D_{\infty}^{n+3} = \emptyset$)
(here we use the whole cohomology, not only the primitive part, as alternation kills the non-primitive forms).

Adding the Hodge filtration we get

\[ h^{p,q} = h^{p,q; \text{alt}}(D^{n+2-(p+q)}(f)) + h^{p,q; \text{alt}}(D^{n+3-(p+q)}(f)). \]

For a mapping \( \mathbb{C}^{n+1} \to \mathbb{C}^{n+r} \), with \( r > 2 \), all of the groups except \( H^{n+k-(r-1)(k-1)}(Y; \mathbb{C}), \) for \( 2 \leq k \leq (n + r)/(r - 1) \), are trivial by Proposition 2.6, and the MHS on each of these exceptional groups coincides with the MHS on \( H^{n+k-r(k-1)}; \text{alt}(D^k(f); \mathbb{C}) \).

We now calculate the numbers \( h^{p,q} \) in the case of a map \( f_0: \mathbb{C}^{n+1} \to \mathbb{C}^{n+r}, \) 
\((x_1, \ldots, x_n, y) \to (x_1, \ldots, x_n, f_0, (x, y), \ldots, f_0, (x, y))\), where each \( f_{0,j} \) is quasi-homogeneous of type \((w_1, \ldots, w_n, w_0, d_j)\), in terms of the weights. The compactification \( \bar{D}^k = \bar{D}^k(f) \) of the \( k \)-point space \( D^k = D^k(f) \) is a quasismooth complete intersection in \((n+k)\)-dimensional weighted projective space \( \mathbb{P}(w_1, \ldots, w_n, w_0, \ldots, w_0, 1) \). Now \( D^k = D^k(f_\infty) = \bar{D}^k \backslash D^k \) lies in the hyperplane on which the weight \( 1 \) coordinate vanishes; \( \bar{D}^k \) and \( D^k_\infty \) are defined by equations of weights \( d_j - lw_0, j = 1, \ldots, r, l = 1, \ldots, k - 1 \), invariant under the permutations of the weight \( w_0 \) coordinates (see Section 4). Let us denote by \( \rho_k \) the dimension \( n + k - r(k - 1) \) of \( D^k(f) \) and introduce the series

\[ Q_k(t, u) = \frac{1}{1 + u} \prod_{i \geq 1} \left( \frac{1 + ut^{-w_i}}{1 - t^{w_i}} \right) \prod_{i=1}^{k} \left( \frac{1 + ut^{(l-2)^w_0}}{1 - t^{l^w_0}} \right) \cdot \prod_{j=1}^{r} \prod_{l=1}^{k-1} \left( \frac{1 - t^d_j - lw_0}{1 + ut^{-d_j + lw_0}} \right) t^{k(1 + \frac{1}{2}(k-1)(k-1))w_0 + \sum_{i>1} w_i - (k-1)\sum d_j} \]

The preceding discussion, together with Proposition 5.2, now yields

5.4. PROPOSITION. The non-zero Hodge numbers of the unique non-trivial alternating cohomology of \( D^k(f) \) are given by the following formulae:

\[ h^{\rho_k-q, q; \text{alt}}(D^k(f)) = (-1)^q \res_{t=0} \res_{u=0} u^{-q-1} \frac{1 + ut^{-1}}{1 - t} Q_k(t, u), \quad q = 0, \ldots, \rho_k \]

\[ h^{\rho_k+1-q, q; \text{alt}}(D^k(f)) = (-1)^{q-1} \res_{t=0} \res_{u=0} t^{-1} u^{-q} Q_k(t, u) \quad q = 1, \ldots, \rho_k. \]

On the right in these formulae are the numbers \( h^{\rho_k, q; \text{alt}} \) and \( h^{\rho_k-1, q-1; \text{alt}} \) for (the primitive parts of) \( H^{\rho_k}(\bar{D}^k(f)) \) and \( H^{\rho_k-1}(D^k(f_\infty)) \) respectively.

5.5. EXAMPLES. The following table shows the Hodge numbers \( h^{p,q} \) for the stable images of the simple singularities of mappings \( \mathbb{C}^2 \to \mathbb{C}^3 \) and of the first
non-simple one [22] (all of them are quasihomogeneous of corank 1):

<table>
<thead>
<tr>
<th>$S_{2k+1}$, $B_{2k+1}$, $C_{2k+1}$</th>
<th>$h^{1,1}$</th>
<th>$h^{1,0} = h^{0,1}$</th>
<th>$h^{0,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{2k}$, $B_{2k}$, $C_{2k}$</td>
<td>1</td>
<td>$k$</td>
<td>0</td>
</tr>
<tr>
<td>$F_4$</td>
<td>0</td>
<td>$k$</td>
<td>0</td>
</tr>
<tr>
<td>$H_k$</td>
<td>1</td>
<td>0</td>
<td>$k - 1$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that $h^{0,0}$ is always the number of triple points of the stable image.

References


