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Vanishing cohomology of singularities of mappings

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Introduction

Associated to each unstable map-germ $f_0\colon \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, where (n, p) are in Mather's range of "nice dimensions", there is a "stabilisation"; that is, a locally stable mapping $f\colon U \to \mathbb{C}^p$, where U is some contractible neighbourhood of 0 in $\mathbb{C}^n$. When $n < p$, the image Y of f plays the same rôle in the theory of singularities of mappings, as does the Milnor fibre in the theory of isolated complete intersection singularities. Our aim in this paper is to describe the topology of Y, and, in the case where f_0 is quasihomogeneous, to make a start in the study of its canonical mixed Hodge structure. Our main results concern germs of corank 1, since in this case the spectral sequence used to calculate the vanishing cohomology degenerates at E_1, making explicit calculation very easy.

The key to our description of the topology of the image Y is provided by the multiple point spaces $D^k(f)$ (the k-th multiple point space $D^k(f)$ is in this context the closure, in U^k, of the set of k-tuples of pairwise distinct points having the same image under f (see Section 2 below)). When f_0 is a finitely determined corank 1 map-germ, each germ $D^k(f_0), 0$, for $2 \leqslant k \leqslant p/(p-n)$, is an isolated complete intersection singularity ([19]). Replacing f_0 by its stable perturbation f, we smooth each space $D^k(f_0)$; $D^k(f)$ is thus a Milnor fibre of $D^k(f_0)$, and is therefore a Stein manifold with the homotopy type of a wedge of spheres ([9]). Moreover, the natural projections $D^k(f) \to D^{k-1}(f)$ all turn out to be stable mappings (see Section 2).

Section 1 makes precise the notion of a stable perturbation of a map-germ.

In Section 2, we construct an *alternating semi-simplicial resolution* $\mathbf{Alt}\, \mathbb{Z}_{D^\cdot}$ of the constant sheaf $\mathbb{Z}_Y$, which relates the topology of Y to that of the multiple point spaces of f. When the original map-germ f_0 has corank 1, that is, when $\dim \operatorname{Ker} df_0(0) = 1$, $D^k(f)$ has the homotopy of a wedge of spheres, and in consequence the spectral sequence for the hypercohomology of the corresponding rational complex $\mathbf{Alt}\, \mathbb{Q}_{D^\cdot}$ degenerates at E_1 and we obtain a rather succinct relation between the rational cohomology of Y and the S_k-alternating part of the rational cohomology of the D^k (Theorem 2.6). This is most interesting when $p = n + 1$. In this case, by a theorem of Lê, Y itself has the homotopy of a wedge of spheres in dimension n ([24]); it turns out that the filtration on $H^n(Y; \mathbb{Q})$ coming from the spectral sequence, has successive quotients isomorphic to the S_k-alternating part of $H^{n-k+1}(D^k; \mathbb{Q})$.

By an extension of this method, we also calculate the rational cohomology of the image multiple point schemes $M_k \subseteq Y$; it turns out that when $p = n + 1$, they have rational cohomology only in the middle dimension (Theorem 2.8).

In Section 3 we use the results of Section 2 to reprove and extend some numerical formulae due to W. L. Marar, relating the ranks of homology groups of Y to those of the D^k and their intersections with the multi-diagonals.

In Section 4 we concentrate on the case where f_0 is quasihomogeneous, adapting results of Greuel and Hamm on the dimension of spaces of forms, to the alternating case, and prove numerical formulae which express the Betti numbers of Y, in terms of the quasihomogeneous type of f_0. In the process we obtain a description of the space of alternating holomorphic forms on a space with the action of a finite group generated by reflections.

In Section 5, we continue the study of quasihomogeneous corank 1 map-germs, with the aim of calculating the invariants of Deligne's mixed Hodge structure on the image of a stabilisation. In this case the stabilisation may be taken to have domain $\mathbb{C}^n$, and its image Y and multiple point schemes D^k can be embedded in appropriate weighted projective spaces. We define mixed Hodge sheaves on Y by means of alternating mixed Hodge sheaves on the D^k, using the resolution of $\mathbb{C}_Y$ from Section 2, and then use an alternating version of Hamm's calculation of the Hodge numbers of quasihomogeneous isolated complete intersection singularities, to obtain formulae for the Hodge numbers of Y, in terms of the quasihomogeneous type of f.

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1. Good representatives

Let $f_0: \mathbb{C}^n, 0 \to \mathbb{C}^p, 0\, (n < p)$ be a finitely $\mathscr{A}$-determined map-germ of discrete stable type (i.e., in a versal unfolding of f_0 there only appear a finite number of right-left equivalence classes of stable germs; this is guaranteed, for example, by the hypothesis that f_0 be of corank 1). We are interested in studying a particular class of stable mappings associated with f_0, the so called *stable perturbations*, which we define as follows, following [17, 18]: let $F: \mathbb{C}^n \times \mathbb{C}^d, 0 \to \mathbb{C}^p \times \mathbb{C}^d, 0$ be an unfolding of f_0, with $F(x, t) = (f_t(x), t)$. Choose a proper representative

$F: \mathscr{U} \to \mathscr{W} \times \mathscr{X}$, where $\mathscr{U}$ and $\mathscr{W}$ and $\mathscr{X}$ are open neighbourhoods of 0 in $\mathbb{C}^n$, $\mathbb{C}^p$ and $\mathbb{C}^d$ respectively, such that

(i) $F^{-1}(0) = 0$

(ii) $F: \mathscr{U} \to \mathscr{W} \times \mathscr{X}$ is a finite map (i.e. proper with finite fibres).

Now let $I_{\mathrm{rel}}(F) = \{(y, t) \in \mathscr{W} \times \mathscr{X} \mid$ the germ of f_t at $f_t^{-1}(y) \cap \mathscr{U}_t$ is not $\mathscr{A}$-stable$\}$, where $\mathscr{U}_t = \{x \in \mathbb{C}^n \mid (x, t) \in \mathscr{U}\}$. As F is finite, $I_{\mathrm{rel}}(F)$ is an analytic subset of $\mathscr{W} \times \mathscr{X}$. Since f_0 is finitely determined, $0 \in \mathbb{C}^n$ is an isolated point of the fibre over $0 \in \mathbb{C}^d$ of the projection $\pi: I_{\mathrm{rel}}(F) \to \mathbb{C}^d$ (this is in fact equivalent to finite determinacy, see e.g. [30]); thus by shrinking $\mathscr{U}$, $\mathscr{W}$ and $\mathscr{X}$, we may suppose that $\pi: I_{\mathrm{rel}}(F) \to \mathscr{W}$ is finite. Choose $\varepsilon > 0$ such that $F(\mathscr{U}) \cap \mathbb{C}^p \times \{0\}$ is stratified transverse (with respect to some Whitney stratification of $F(\mathscr{U})$) to the sphere $S_{\varepsilon'}$ of centre 0 and radius ε', for every ε' with $0 < \varepsilon' \leqslant \varepsilon$ (i.e. such that $B_\varepsilon(0)$ is a Milnor ball for $f_0(U_0)$). Then by the properness of π, there exists a neighbourhood $\mathscr{W}_0$ of 0 in $\mathscr{W}$, such that

(i) $I_{\mathrm{rel}}(F) \cap (\mathbb{C}^p \times \mathscr{W}_0)$ is contained in $\mathrm{int}(B_\varepsilon(0)) \times \mathscr{W}_0$.

(ii) The stable type stratification of $F(\mathscr{U}) \backslash I_{\mathrm{rel}}(F)$ is transverse to $S_\varepsilon \times \mathscr{W}_0$. (Note that off $I_{\mathrm{rel}}(F)$, the stable type stratification of $F(\mathscr{U})$ is the minimal Whitney stratification).

Now restrict F to $U = F^{-1}(B_\varepsilon(0) \times \mathscr{W}_0)$. We call the new map $F: U \to B_\varepsilon(0) \times \mathscr{W}_0$ a *good representative* of F. Let $\mathscr{B} = \pi(I_{\mathrm{rel}}(F))$, and suppose that it is a proper subset of $\mathscr{W}_0$; this is guaranteed if (n, p) are nice dimensions, cf. [20], or if f_0 is of corank 1. As a consequence of Thom's Second Isotopy Lemma, the family of mappings

$$\begin{tikzcd} U \cap \pi^{-1}(\mathscr{W}_0 \backslash \mathscr{B}) \arrow[rr, "F"] \arrow[dr] & & B_\varepsilon(0) \times (\mathscr{W}_0 \backslash \mathscr{B}) \arrow[dl] \\ & \mathscr{W}_0 \backslash \mathscr{B} & \end{tikzcd}$$

is locally topologically trivial. Details of the proof are given in [16, 17]. In particular, setting $U_t = \{x \in \mathbb{C}^n \mid (x, t) \in U\}$, then up to C^0-$\mathscr{A}$-equivalence, the map $f_t: U_t \to B_\varepsilon(0)$ is independent of the choice of $t \in W_0 \backslash \mathscr{B}$ (for $W_0 \backslash \mathscr{B}$ is connected). We will call such a map a *stable perturbation* of f_0. In fact, up to C^0-$\mathscr{A}$-equivalence, there is a unique stable perturbation of f_0; for any stable perturbation is bianalytically equivalent to one contained in a versal unfolding, and the stable perturbations of f_0 contained in any two versal unfoldings are easily shown to be C^0-$\mathscr{A}$-equivalent.

If $\mathscr{B}$ is not a proper analytic subset of the base of a versal unfolding F of f_0, as may occur if (n, p) lie outside the range of nice dimensions, then by replacing $I_{\mathrm{rel}}(F)$ in the previous construction by the set of points where $j^k f_t$ fails to be multitransverse to the canonical stratification of the jet bundle, one can show that, again up to C^0-$\mathscr{A}$-equivalence, f_0 has a well defined *topologically stable* perturbation. In the nice dimensions, the notions of stability and topological stability coincide.

Suppose that $f_t\colon U_t \to B_\varepsilon(0) \subseteq \mathbb{C}^p$ is a stable or topologically stable perturbation of f_0 as described above. Then the image Y_t of f_t is a Stein space, as are all of the multiple point spaces $M_k(f_t)$ in the image. For each is the intersection of an analytic subspace of an open set in $\mathbb{C}^p$, with the closed ball $B_\varepsilon(0)$, and, since the distance squared function on Euclidean space is strictly plurisubharmonic, the affirmation is a consequence of e.g. Corollary 10 of Chapter IX of [8]. By results of H. Hamm, [10], it follows that Y_t and the M_k have the homotopy type of CW complexes of half of their real dimension, and in particular have no cohomology above the middle dimension.

When $p = n + 1$, it is possible to show that Y_t is in fact homotopy equivalent to a wedge of spheres of dimension n (see e.g. [24]); however, when $p \geqslant n + 2$, it turns out that no such simple description of Y_t is possible: as we shall see in the next section, Y_t may have cohomology in dimensions $p - (p - n - 1)k - 1$ for all integers k for which $p - (p - n)k \geqslant 0$.

2. Multiple point spaces and alternating semi-simplicial resolutions

In this section we use *alternating semisimplical resolutions* to compute the rational cohomology of the image of a finite mapping, with special emphasis on the case of a stable perturbation of a map-germ $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $(n < p)$. The reader may find it helpful to refer to the example on page 52 while reading it.

First we define a collection of spaces associated with any continuous mapping $f\colon X \to Y$ of topological spaces:

$D^k(f)$ (or D^k where there is no danger of confusion), is the *k-fold multiple point space of f*:

$$D^k = \text{closure}\{(x_1,\dots,x_k) \in X^k\colon f(x_1) = \cdots = f(x_k),\ x_i \neq x_j \quad \text{if } i \neq j\}$$

for $1 \leqslant k < \infty$.

There are continuous mappings $\varepsilon^{i,k}\colon D^k \to D^{k-1}$, defined by

$$\varepsilon^{i,k}((x_1,\dots,x_k)) = (x_1,\dots,\hat{x}_i,\dots,x_k), \quad \text{for} \quad 1 \leqslant i \leqslant k.$$

The spaces D^k, together with the maps $\varepsilon^{i,k}$, constitute a *semisimplicial object* in the category of topological spaces. Observe that f induces well defined maps $D^k \to Y$, which we call ε^k, with $\varepsilon^k \circ \varepsilon^{i,k+1} = \varepsilon^{k+1}$ (thus $(D^\cdot \to Y)$ is a *semisimplicial object over* Y). However, we will make no use of this notion, although our calculation of the cohomology of the image is a modification of well known techniques of semisimplicial resolution.

We now suppose that f is a finite, proper map (and we will continue to do so throughout this section).

The alternating complex

Consider the complex of sheaves

$$\mathbb{Z}_{D^{\cdot}}\; 0 \longrightarrow \mathbb{Z}_Y \longrightarrow \varepsilon^1_*(\mathbb{Z}_X) \xrightarrow{\delta_1} \varepsilon^2_*(\mathbb{Z}_{D^2}) \xrightarrow{\delta_2} \varepsilon^3_*(\mathbb{Z}_{D^3}) \xrightarrow{\delta_3} \cdots$$

where $\delta_k\colon \varepsilon^k_* \mathbb{Z}_{D^k} \to \varepsilon^{k+1}_* \mathbb{Z}_{D^{k+1}}$ is equal to $\sum_{j=1}^{k+1} (-1)^{k+j} (\varepsilon^{j,k+1})^*$ (here D^1 is just X).

In general the complex $\mathbb{Z}_{\cdot D^{\cdot}}$ is not exact: exactness fails at points of Y lying under points where some D^k meets one of the diagonals. If, for example, $y = f(x)$, $f^{-1}(y) = \{x\}$ and $(x, x) \in D^2$, but $(x, x, x) \notin D^3$, then each of $\mathbb{Z}_Y, \varepsilon^1_*(\mathbb{Z}_X)$ and $\varepsilon^2_*(\mathbb{Z}_{D^2})$ has stalk at y isomorphic to $\mathbb{Z}$, while all of the other sheaves in the complex have stalk at y equal to 0. It follows by counting the rank of the stalks that exactness is not possible at y. Note that precisely this configuration arises (at $y = 0$) if we take f to be the stable map $\mathbb{C}^2 \to \mathbb{C}^3$ defined by $f(x_1, x_2) = (x_1, x_2^2, x_1 x_2)$, whose image Y has a pinch point singularity at 0.

In order to obtain exactness, we restrict to the *alternating subcomplex*, which we now define. There is a natural continuous action of the symmetric group S_k on $\varepsilon^k_*(\mathbb{Z}_{D^k}))$, defined as follows: S_k acts on D^k by permuting the factors; as ε^k is S_k-invariant, for any open set $U \subseteq Y$, $(\varepsilon^k)^{-1}(U)$ is mapped to itself by the permutation action. Thus, S_k acts on $\Gamma((\varepsilon^{k)^{-1}}(U), \mathbb{Z}_{D^k})$ by the permutation representation coming from the permutation action on the set of connected components of $(\varepsilon^k)^{-1}(U)$, and hence acts (continuously) also on $\varepsilon^k_*(\mathbb{Z}_{D^k})$. We denote the action of $\sigma \in S_k$ by σ^*. We let $\mathrm{Alt}(\varepsilon^k_*(\mathbb{Z}_{D^k}))$ be the subsheaf of $\varepsilon^k_*(\mathbb{Z}_{D^k})$ on which S_k acts by the alternating representation, i.e.

$$\mathrm{Alt}(\varepsilon^k_*(\mathbb{Z}_{D^k}))_y = \{s \in \varepsilon^k_*(\mathbb{Z}_{D^k})_y | \quad \text{for all } \sigma \in S_k,\ \sigma^* s = \mathrm{sign}(\sigma) s\}$$

and we denote the complex $\{\mathrm{Alt}\, \varepsilon^k_*(\mathbb{Z}_{D^k}), \delta\}$ by $\mathbf{Alt}\, \mathbb{Z}_{D^{\cdot}}$.

We remark that this construction is alluded to by Deligne in [0, pp. 31–32], but is not described in detail.

2.1. PROPOSITION. *The complex* $\mathbf{Alt}\, \mathbb{Z}_{D^{\cdot}}$ *is exact.*

Proof. Let $y \in Y$ be a point with exactly $m + 1$ preimages, $x_0, \ldots, x_m$. We prove exactness of the stalk complex $\mathbf{Alt}\, \mathbb{Z}_{D^{\cdot}, y}$ by showing that it is isomorphic to the simplicial cochain complex $\{C^{\cdot}(\Delta^m, \mathbb{Z}), d_{\cdot}\}$, where Δ^m is an m-simplex. It will be helpful in what follows to write $X = D^1$, and $f = \varepsilon^1$.

Recall that since f, and hence ε^k, is finite,

$$\varepsilon_*^k(\mathbb{Z}_{D^k})_y \simeq \bigoplus_{x\in(\varepsilon^k)^{-1}(y)} \mathbb{Z}_{D^k,x}.$$

Let $0 \leqslant i_1,\ldots,i_k \leqslant m$, and if $(x_{i_1},\ldots,x_{i_k})\in D^k$, let $\chi_{(x_{i_1},\ldots,x_{i_k})}$ be the member of $\bigoplus_{x\in(\varepsilon^k)^{-1}(y)}\mathbb{Z}_{D^k,x}$ which is 1 at $(x_{i_1},\ldots,x_{i_k})$ and 0 elsewhere. Denote by $\operatorname{Alt}\chi_{(x_{i_1},\ldots,x_{i_k})}$ the element

$$\sum_{\sigma\in S_k} \operatorname{sign}(\sigma)\chi_{(x_{i_{\sigma(1)}},\ldots,x_{i_{\sigma(k)}})};$$

if $i_j = i_l$ for any $j \neq l$, then $\operatorname{Alt}\chi_{(x_{i_1},\ldots,x_{i_k})} = 0$. Hence, $\operatorname{Alt}\varepsilon_*^k(\mathbb{Z}_{D^k})_y$ has as free basis the elements

$$\operatorname{Alt}\chi_{(x_{i_1},\ldots,x_{i_k})} \text{ with } 0 \leqslant i_1 < \cdots < i_k \leqslant m.$$

Now let $\Delta^m = (v_0,\ldots,v_m)$ be the standard m-simplex, and for $0 \leqslant i_1 < \cdots < i_k \leqslant m$, let $(v_{i_1},\ldots,v_{i_k})$ be the $(k-1)$-face with vertices $v_{i_1},\ldots,v_{i_k}$, oriented in some standard way. Let $\xi_{(v_{i_1},\ldots,v_{i_k})}$ be the simplicial $(k-1)$-cochain on the simplicial complex generated by Δ^m, which takes the value 1 on $(v_{i_1},\ldots,v_{i_k})$, and 0 on the other $(k-1)$-faces of Δ^m. Then for any

$$\sigma\in S_k,\ \xi_{(v_{i_1},\ldots,v_{i_k})}((v_{i_{\sigma(1)}},\ldots,v_{i_{\sigma(k)}}))$$

is equal to $\operatorname{sign}(\sigma)$.

It follows that the map of complexes $\{\varphi_k\}$: $\mathbf{Alt}\,\mathbb{Z}_{D^\cdot,y}\to\{C^{\cdot-1}(\Delta^m,\mathbb{Z}),\ d_{\cdot-1}\}$ determined by

$$\varphi_k(\operatorname{Alt}\chi_{(x_{i_1},\ldots,x_{i_k})}) = \xi_{(v_{i_1},\ldots,v_{i_k})}$$

is well-defined, and bijective. One checks easily that $\varphi_{k+1}\circ\delta_k = d_k\circ\varphi_k$; thus, the two chain complexes are isomorphic, and since $\{C^\cdot(\Delta^m,\mathbb{Z}),d_\cdot\}$ is exact, so is $\{\operatorname{Alt}\,\mathbb{Z}_{D^\cdot y},\delta_\cdot\}$. □

Exactly the same proof shows that the complex $\operatorname{Alt}\mathbb{Q}_{D^\cdot}$, obtained by replacing $\mathbb{Z}$ by $\mathbb{Q}$ in the previous construction, is also exact. We will make use of this complex rather than the integer complex, because while, as we shall shortly see, $H^*(Y, \operatorname{Alt}\varepsilon_*^k(\mathbb{Q}_{D^k}))$ is equal to the alternating part $H^*_{\mathrm{Alt}_k}(D^k,\mathbb{Q})$ of $H^*(D^k,\mathbb{Q})$, the relation is not so simple over $\mathbb{Z}$.

Now let $0\to\mathbb{Q}_{D^k}\to\mathbf{I}_k^\cdot$ be an injective resolution of the constant sheaf $\mathbb{Q}_{D^k}$; pushing it down to Y we obtain an injective resolution of $\varepsilon_*^k(\mathbb{Q}_{D^k})$. Since S_k acts on $\varepsilon_*^k(\mathbb{Q}_{D_k})$, we can choose $I_k^\cdot$ so that S_k acts on $\varepsilon_*^k(I_k^\cdot)$ too, by taking $\mathbf{I}_k^\cdot$ to be the canonical resolution of Godement ([3] II.4.3). Then we have

2.2. LEMMA. *Under these circumstances, the complex* $\mathrm{Alt}_k\varepsilon^k_*\mathbf{I}^{\cdot}_k$ *is an injective resolution of* $\mathrm{Alt}_k\varepsilon^k_*(\mathbb{Q}_{D^k})$.

Proof. Define an indempotent operator Alt_k on each I^j_k, by

$$\mathrm{Alt}_k = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sign}(\sigma)\sigma^*.$$

Each of the differentials in the complex $\varepsilon^k_*\mathbf{I}^{\cdot}_k$ commutes with the action of S_k, and thus with the operator Alt_k. As this operator is indempotent, it is an easy exercise to show that $\mathrm{Alt}_k\varepsilon^k_* I^j_k$ is injective, and moreover to deduce, from the exactness of $\varepsilon^k_*\mathbf{I}^{\cdot}_k$, that the complex $\mathrm{Alt}_k\varepsilon^k_*\mathbf{I}^{\cdot}_k$ is exact. □

Now by lifting the differentials $\delta_k\colon \varepsilon^k_*(\mathbb{Q}_{D^k}) \to \varepsilon^{k+1}_*(\mathbb{Q}_{D^{k+1}})$ to sheaf homomorphisms

$$\delta^i_k\colon \mathrm{Alt}_k\varepsilon^k_*(I^i_k) \to \mathrm{Alt}_{k+1}\varepsilon^{k+1}_*(I^i_{k+1})$$

we obtain a double complex $\{\mathrm{Alt}_k\varepsilon^k_*(I^i_k), \delta^i_k, d^i_k\}$. By a standard argument, the total complex $K^{\cdot}$, with

$$K^q = \bigoplus_{i+k=q+1}\mathrm{Alt}_k(\varepsilon^k_*(I^i_k)),$$

is exact, and is thus an injective resolution of $\mathbb{Q}_Y$. Therefore the complex $\Gamma(Y, K^{\cdot})$ obtained by taking global sections, computes the cohomology of Y. Now

$$\Gamma(Y, \mathrm{Alt}_k(\varepsilon^k_*(I^i_k)) = \mathrm{Alt}_k(\Gamma(D^k, I^i_k));$$

since the differential of the complex $\Gamma(D^k, I^{\cdot}_k)$ commutes with the idempotent operator Alt, which is defined on this complex in the obvious way, we have

2.3. PROPOSITION. *The spectral sequence associated to the filtration*

$$F^p\Gamma(Y, K^q) = \bigoplus_{k\geqslant p+1}\Gamma(Y, \mathrm{Alt}(\varepsilon^k_*(I^i_k)))$$

has $E_1^{p,q}$ *term equal to* $H^q_{\mathrm{Alt}_{p+1}}(D^{p+1}, \mathbb{Q}))$ (*where* $H^q_{\mathrm{Alt}_{p+1}}(D^{p+1}, \mathbb{Q}))$ *is the alternating part of* $H^q(D^{p+1}; \mathbb{Q})$).

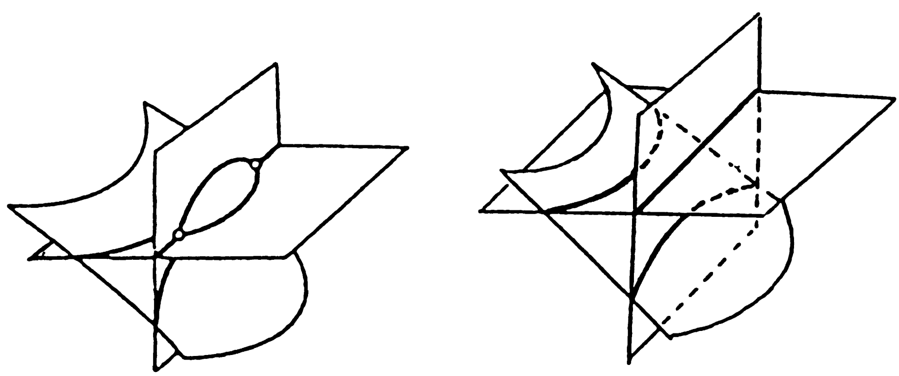
Proof. The $E_1^{p,q}$ term of the spectral sequence is equal to $H^q(\mathrm{Alt}_{p+1}(\Gamma(D^{p+1}, I^{\cdot}_k), d^{\cdot}_{p+1})$. As just described, this is equal to $H^q_{\mathrm{Alt}_{p+1}}(D^{p+1}, \mathbb{Q}))$. □

2.4. REMARK. (ii) The above construction breaks down if $\mathbb{Z}$ is replaced by $\mathbb{Q}$; the idempotence of the operator Alt_k is an essential ingredient in the proof of 2.2, and over $\mathbb{Z}$ it is not possible to construct such an idempotent operator, since one cannot divide by $k!$. Indeed 2.3 is false if we replace $\mathbb{Q}$ by $\mathbb{Z}$. This is shown by the following example. Let X be the closed northern hemisphere of S^2, and $f: X \to Y = \mathbb{RP}^2$ the restriction to X of the usual quotient mapping. This map has double points but no triple points. Projection onto the first factor induces a homeomorphism of $D^2(f)$ onto the equator of S^2, where the involution is the antipodal map. Since this map is orientation-preserving, $H^*_{\mathrm{Alt}_2}(D^2(f); \mathbb{Z}) = 0$; but $\mathbb{RP}^2$ is not an integer-homology disc.

(ii) The injective resolutions $\mathbf{I}^{\cdot}_k$ above can be replaced by any fine resolution (to whose push-forward to Y the S_k-action on $\varepsilon^k_*(\mathbb{Q}_{D^k})$ lifts), if the aim is simply that of calculating $H^*(Y)$. Thus, in particular, if all of the D^k are smooth Stein spaces then one may calculate $H^*(Y; \mathbb{C})$ by using the resolutions $0 \to \mathbb{C}_{D^k} \to \Omega^{\cdot}_{D^k}$, where $\Omega^{\cdot}_{D^k}$ is the (exact) complex of sheaves of germs of holomorphic differential forms (c.f. [6], §4).

An exercise

To end this paragraph we give a simple example. Among the five codimension 1 singularities of maps from surfaces to 3-space we find the "birth of two triple points" – the multi-germ consisting of three immersions which are pairwise transverse, but in which the curve of intersection of each pair of immersed sheets is (first order) tangent to the third. Over $\mathbb{R}$ there are two inequivalent stable perturbations of this configuration, shown in Fig. 1(a) and (b). Figure 1(a) does indeed have two triple points, which are imaginary in 1(b). In each case X consists of the disjoint union of three 2-cells, X_1, X_2 and X_3, and so $X \times X$ has nine connected components. As f is an immersion, $D^2(f)$ has no component in any $X_i \times X_i$, but for each $i \neq j$, $D^2(f) \cap (X_i \times X_j)$ is a line. The $\mathbb{Z}_2$ action on $D^2(f)$ interchanges $D^2(f) \cap (X_i \times X_j)$ and $D^2(f) \cap (X_j \times X_i)$, and it follows that $H^0_{\mathrm{Alt}_2}(D^2; \mathbb{Q})$ has rank 3. In 1(a), D^3 consists of two faithful S_3 orbits, each of



(a) (b)

Fig. 1.

which contributes one dimension to $H^0_{\mathrm{Alt}_3}(D^3; \mathbb{Q})$. We urge the reader to compute the spectral sequence of 2.3 for each of them. The outcome should be clear from the drawings, and the computation is particularly easy since the D^k have cohomology only in dimension 0.

Stable perturbations of corank 1 *map-germs and simplicial stable mappings*

We can now use 2.2 to compute explicitly the rational cohomology of the image Y_t of a stable perturbation of a map-germ $f_0\colon \mathbb{C}^n, 0 \to \mathbb{C}^m, 0$ $(n < m)$ of corank 1. Let $f_t\colon U_t \to \mathbb{C}^m$ be such a perturbation, with image Y_t, and let the spaces D^k be constructed as above (with $X = U$). In [19], the space D^k (constructed for the map f_t) was denoted $\tilde{D}^k(f_t)$. Here we abandon the tilde. We recall the principal result of [19]:

2.5. THEOREM. ([19], 2.14). (i) *The map-germ $f\colon \mathbb{C}^n, x \to \mathbb{C}^m, y$ is stable if and only if for all k with $2 \leqslant k$, the germ of $D^k(f)$ at $(x, x, \ldots, x) \in (\mathbb{C}^n)^k$ is smooth of dimension $m - (m - n)k$, or empty;*

(ii) *f is finitely determined (for $\mathscr{A}$-equivalence) if and only if for all k with $2 \leqslant k \leqslant m/(m - n)$, $D^k(f)$ is a complete intersection of dimension $m - (m - n)k$, with (at most) isolated singularity at $(x, x, \ldots, x) \in (\mathbb{C}^n)^k$.* □

Note that [19] defines multiple point *schemes* $D^k(f)$ for corank 1 map-germs f, by means of explicit equations, rather than the multiple point *spaces* defined here. In fact for finitely determined corank 1 germs $\mathbb{C}^n, 0 \to \mathbb{C}^m, 0$, the two definitions coincide provided $k < m/(m - n)$ (in other words, for those k such that $\dim D^k(f) > 0$). This is because for such germs genuine k-tuple points are dense in the scheme $D^k(f)$ (essentially by 2.5 – see [19], page 563). When $k = m/(m - n)$, the scheme $D^k(f)$ may contain k-tuples $(x_1, \ldots, x_k)$ where not all the x_i are distinct, which are clearly not in the space $D^k(f)$ defined here. However, the S_k orbit of such a point does not support any alternating 0-th cohomology; so we conclude that one may use the scheme-theoretic $D^k(f)$ defined in [19], in place of the space $D^k(f)$ defined here, and obtain the same spectral sequence converging to the cohomology of the image Y, from E_1 onwards.

If $F\colon U \to V \times T \subseteq \mathbb{C}^m \times \mathbb{C}^d$ is a good representative of a stable parametrised unfolding of f_0, then the spaces $D^k(F)$, which by 2.5(i) are all smooth, fibre over T, with fibre over $t \in T$ equal to $D^k(f_t)$. If t lies in the complement of the bifurcation set $\mathscr{B}$, by 2.5(i) $D^k(f_t)$ is smooth, and is in fact a Milnor fibre for the isolated complete intersection singularity $D^k(f_0), 0$. It follows, by results of Hamm [9] (see also [15], Chapter 5) that for each k, $D^k(f_t)$ has the homotopy type of a wedge of spheres of middle dimension, so that its reduced cohomology is concentrated in this dimension. Hence, 2.6. THEOREM. *In these circumstances, the spectral sequence described above collapses at the E_1 term, and thus we have*

(i) *if $m = n + 1$, then $H^n(Y_t, \mathbb{Q})$ is isomorphic to*

$$\bigoplus_{k=2}^{n+1} H^{n-k+1}_{\mathrm{Alt}_k}(D^k(f_t), \mathbb{Q})),$$

and $H^p(Y_t, \mathbb{Q}) = 0$ for $1 \leqslant p < n$ and for $p > n$.

(ii) *if $m - n \geqslant 2$, then for each integer k with $2 \leqslant k \leqslant m/(m-n)$,*

$$H^{m-(m-n-1)k-1}(Y_t, \mathbb{Q}) \simeq H^{m-(m-n)k}_{\mathrm{Alt}_k}(D^k, \mathbb{Q})$$

and $H^p(Y_t, \mathbb{Q})$ vanishes for all other positive values of p.

Proof. We have $E_1^{p,q} = H^q_{\mathrm{Alt}_{p+1}}(D^{p+1}, \mathbb{Q}))$, and so the first differential runs from $H^q_{\mathrm{Alt}_{p+1}}(D^{p+1}, \mathbb{Q}))$ to $H^q_{\mathrm{Alt}_{p+2}}(D^{p+2}, \mathbb{Q}))$. For each value of q, there is at most one value of p for which $H^q_{Alt_{p+1}}(D^{p+1}, \mathbb{Q}))$ is non-vanishing; this follows from the fact that for each p, there is at most one value of $q = q(p) > 0$ such that $H^q(D^{p+1}, \mathbb{Q}) \neq 0$, and the sequence $q(p)$ is strictly decreasing, while for $q = 0$, it holds because if $\dim(D^{p+1}(f_t)) > 0$, then D^{p+1} is connected and so $H^0_{\mathrm{Alt}_{p+1}}(D^{p+1}, \mathbb{Q})) = 0$, and there is at most one value of p for which $\dim(D^{p+1}) = 0$. It follows that the spectral sequence collapses at the E_1 term, as claimed. Since the spectral sequence converges to the cohomology of Y_t, (i) and (ii) follows. □

Note that the spectral sequence in the example on page 52 does not collapse at E_1, even if we replace $\mathbb{R}$ by $\mathbb{C}$; however, this does not contradict 2.6, since in the example we are dealing with a stable perturbation of a *multi*-germ.

2.7. REMARK. (i) The hypothesis that f_0 be of corank 1 is not necessary in order to guarantee that the spaces $D^p(f_t)$ be smooth. The only requirement here is that all of the singularities of the stable perturbation f_t should be of corank 1. This is also guaranteed if $n < 2(m - n + 2)$.

(ii) The conclusion of 2.6 continues to hold for stable perturbations of map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ of corank 2. In this case, since $D^2(f_t)$ is a smooth, non-compact complex curve, it has cohomology only in dimensions 0 and 1, and so the spectral sequence collapses at the E_1 term as in the proof of 2.6.

(iii) 2.6 is valid also in a slightly wider context; for example, where the domain of f_0 is an isolated complete intersection singularity and the domain U_t of f_t is a smoothing. In this case, provided the multiple point schemes $D^k(f_t)$ still have cohomology concentrated in the middle dimension, the only change to the calculation is the addition of one further summand in the cohomology of Y_t, coming from the cohomology of U_t. This arises in the study of projections of complete intersection singularities to smooth complex spaces (cf [4]), and also, in the context that principally concerns us, if we are interested in the images of the maps $\varepsilon^{i,k}: D^k(f_t) \to D^{k-1}(f_t)$. For here, the domain of $\varepsilon^{i,k}$ is the smoothing $D^k(f_t)$ of the isolated complete intersection singularity $D^k(f_0)$. Now $D^j(\varepsilon^{i,k}) \simeq D^{k+j-1}(f_t)$; for (taking $i=k$ to simplify notation) an ordered j-tuple of points in $D^k(f_t)$ having the same image under $\varepsilon^{k,k}$ must be of the form

$$((x_1, \ldots, x_{k-1}, x_k), (x_1, \ldots, x_{k-1}, x_{k+1}), \ldots, (x_1, \ldots, x_{k-1}, x_{k+j-1})),$$

and we define an isomorphism $D^j(\varepsilon^{i,k}) \to D^{k+j-1}(f_t)$ by sending this point to $(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+j-1})$. This is the "method of iteration" used by Kleiman in [12]. Incidentally, this shows (by 2.5(i)) that the maps $\varepsilon^{i,k}$ are themselves locally stable, so that the family of spaces $D^k(f_t)$ and mappings $\varepsilon^{i,k}$ form a simplicial stable mapping. It also explains the observation in [25] (in the case of generic maps of 3-folds into $\mathbb{P}^4$) that the source double point space $\varepsilon^{i,2}(D^2(f))$ has the same singularities as the image of a stable map from 2-space to 3-space. In a similar vein, the source triple point set $D_1^3(f)$ (for a stable map f from n-space to p-space) has the singularities of the image double point set of a stable mapping from $p - 2(p - n)$-space to n-space.

(iv) It is known that when $m = n + 1$, Y_t actually has the homotopy type of a wedge of spheres of dimension n. This follows from a theorem of Lê [12, 13]; see [24]. Thus $H^n(Y_t, \mathbb{Z})$ is a free abelian group, as are the groups $H^{m-(m-n)k}_{\mathrm{Alt}_k}(D^k, \mathbb{Z})$. However, as we have seen in 2.4(i), 2.3 does not hold over $\mathbb{Z}$, and in order to relate the integer cohomology of the D^k to that of Y_t, some more work is required (see [5]).

Rational cohomology of the image multiple point sets

We now use the same technique to compute the cohomology of the spaces $M_p(f_t) = \varepsilon^p(D^p) \subseteq Y_t$, which can also be described as the locus of zeros of the $(p - 1)$'st Fitting ideal sheaf of $f_{t*}(\mathcal{O}_U)$. To lighten the notation, we abandon the subscript t on Y_t, f_t etc.

Let D_j^p be the (reduced) image of D^p in D^j under any one of the Cartesian projections. We calculate the cohomology of M_p by means of the following exact complex of sheaves on M_p:

$$0 \to \mathbb{Q}_{M_p} \to \mathbb{Q}_{D_1^p} \to \mathrm{Alt}_2(\mathbb{Q}_{D_2^p}) \to \cdots \to \mathrm{Alt}_{p-1}(\mathbb{Q}_{D_{p-1}^p}) \to \mathrm{Alt}_p(\mathbb{Q}_{D^p})$$
$$\to \mathrm{Alt}_{p+1}(\mathbb{Q}_{D^{p+1}}) \to \cdots$$

(here we have omitted the symbols ε_*^j etc.). Exactness is a consequence of 2.1, for denoting by f_p the restriction of f to D_1^p, we have $D^j(f_p) = D_j^p$ for $j < p$ and $D^j(f_p) = D^j$ for $j \geqslant p$.

2.8. THEOREM. *Suppose that $f: U \to Y$ is a stable perturbation of a finitely determined corank 1 map-germ $\mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$. Then all of the spaces M_k, for $2 \leqslant k \leqslant n+1$, have rational cohomology only in dimension $n-k+1$.*

Proof. This is proved by induction on k. The possibility of carrying out an induction is based on the principle of iteration: namely, that

$$D_1^p(f) = M_p(\varepsilon^{i,2}: D^2(f) \to U)$$

and, more generally, since $D^k(\varepsilon^{i,j}) \simeq D^{k+j-1}(f)$,

$$D_q^p(f) = M_{p-q}(\varepsilon^{i,q+1}: D^{q+1}(f) \to D^q(f)) \quad (\text{for } q < p).$$

The induction hypothesis is in fact slightly stronger than the theorem itself. It is

Hyp($p-1$): Let $g: U \to V$ be a proper, stable map of affine Stein manifolds, with $\dim(U) = \dim(V) - 1 = n$, and suppose that g has only corank 1 singularities; suppose, moreover that U, and all of the spaces $D^k(g)$, have reduced rational cohomology only in dimension $n-k+1$, and that all of the spaces $M_p(g)$ are Stein spaces. Then for $1 \leqslant q \leqslant p-1$, $M_q(g)$ has reduced rational cohomology only in dimension $n-q+1$.

Note that Theorem 2.6 (together with 2.7(iii)) establishes that Hyp(1) holds.

The induction step is proved as follows. By 2.4, the cohomology of $M_p(g)$ is computed by a spectral sequence with $E_1^{r,s} = H^s_{\mathrm{Alt}_{r+1}}(D^{r+1}(g_p), \mathbb{Q}))$ (where g_p is the restriction of g to $D_1^p(g)$). Now

$$D^j(g_p) = D_j^p(g) = M_{p-j}(\varepsilon^{i,j+1}: D^{j+1}(g) \to D^j(g)) \quad \text{for } j < p$$

and

$$D^j(g_p) = D^j(g) \quad \text{for } j \geqslant p.$$

By Hyp($p-1$), $D^j(g_p)$ has cohomology only in dimension $n-p+1$ for $1 \leqslant j < p$, and in dimension $n-j+1$ for $p \leqslant j$. It follows that

$$\begin{aligned} E_1^{j,n-p+1} &= H^{n-p+1}_{\mathrm{Alt}_{j+1}}(D^{j+1}(g_p), \mathbb{Q}) = H^{n-p+1}_{\mathrm{Alt}_{j+1}}(M_{p-j+1}(\varepsilon^{i,j-1}), \mathbb{Q}) \\ &\quad \text{for } 1 \leqslant j < p-1, \\ E_1^{j,n-j} &= H^{n-p+1}_{\mathrm{Alt}_{j+1}}(D^{j+1}(g), \mathbb{Q}) \quad \text{for } p-1 \leqslant j, \quad \text{and} \\ E_1^{j,k} &= 0 \quad \text{otherwise.} \end{aligned}$$

In particular, $E_1^{j,k} = 0$ for all j, k with $j + k < n - p + 1$; since the spectral sequence converges to $H^*(M_p(g), \mathbb{Q})$, it follows that $H^k(M_p(g), \mathbb{Q}) = 0$ for $k < n - p + 1$. As $M_p(g)$ is a Stein space of dimension $n - p + 1$ it has cohomology only in dimension less than or equal to $n - p + 1$. Thus, we have proved Hyp(p). $\square$

The filtration on the cohomology of the image

We give now an alternative description of the filtration on the cohomology of the image Y_t obtained from the spectral sequence with which we calculated the cohomology of the double complex in 2.6. The previous description of $H^k(M_p(f), \mathbb{Q})$, in the case $p = 2$, sheds light on this filtration, as follows. It is clear from Fig. 1 (in which the first index is the vertical one) that in the spectral sequence with which we calculated $H^{n-1}(M_2; \mathbb{Q})$, we have

$$E_\infty^{1,n-1} = E_2^{1,n-1} = H^{n-1}_{\mathrm{Alt}_2}(D^2(f), \mathbb{Q})/d_1(H^{n-1}(D_1^2, \mathbb{Q}));$$

$$\begin{array}{cccccccc}
H^0_{\mathrm{Alt}_{n+1}}(D^{n+1}) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & H^1_{\mathrm{Alt}_n}(D^n) & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & H^{n-2}_{\mathrm{Alt}_3}(D^3) & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & H^{n-1}_{\mathrm{Alt}_2}(D^2) & 0 \\
 & & & & & & \uparrow d_1 & \\
0 & 0 & 0 & \cdots & 0 & 0 & H^{n-1}(D_1^2) & 0
\end{array}$$

Fig. 1

as $M_2(f)$ is a Stein space of dimension $n - 1$ and thus $H^n(M_2(f), \mathbb{Q}) = 0$, it follows that $E_\infty^{1,n-1} = 0$, and so

$$d_1 \colon H^{n-1}(D_1^2, \mathbb{Q}) \to H^{n-1}_{\mathrm{Alt}_2}(D^2(f), \mathbb{Q})$$

is onto.

Similarly, $d_2 \colon E_2^{0,n-1} = \operatorname{Ker} d_1 \to E_2^{2,n-2} = H^{n-2}_{\mathrm{Alt}_3}(D^3(f), \mathbb{Q})$ is onto; in fact the succession of differentials d_i, each defined on the kernel of its predecessor, has d_i mapping onto $E_i^{i,n-i} = H^{n-i}_{\mathrm{Alt}_{i+1}}(D^{i+1}(f), \mathbb{Q})$, because $E_{i+1}^{i,n-i} = E_\infty^{i,n-i} = 0$. The successive kernels $E_r^{0,n-1}$ form a decreasing filtration on $E_1^{0,n-1} = H^{n-1}(D_1^2, \mathbb{Q})$, with $H^{n-1}(M_2, \mathbb{Q}) = E_\infty^{0,n-1} = E_{n+1}^{0,n-1}$ as the smallest term. Now there is an exact sequence

$$0 \to H^{n-1}(M_2; \mathbb{Q}) \to H^{n-1}(D_1^2; \mathbb{Q}) \to H^n(Y_t; \mathbb{Q}) \to H^n(U_t; \mathbb{Q}) \to 0$$

(coming from the short exact sequence (3) below).

The descending filtration $\{E_r^{0,n-1}\}_{1\leqslant r\leqslant n+1}$ on $H^{n-1}(D_1^2;\mathbb{Q})$ gives a descending filtration $\{F^r\}_{1\leqslant r\leqslant n+1}$ on $H^n(Y_t;\mathbb{Q})$, with $F^r = E_r^{0,n-1}/H^{n-1}(M_2;\mathbb{Q})$; adding $F^0 = H^n(Y_t;\mathbb{Q})$, we obtain a descending filtration $\{F^r\}_{0\leqslant r\leqslant n+1}$ with $F^r/F^{r+1} \simeq H^{n-r}_{\mathrm{Alt}_{r+1}}(D^{r+1};\mathbb{Q})$ (here $D^1 = U_t$). This filtration coincides with the one coming from the spectral sequence with which we calculated $H^n(Y;\mathbb{Q})$.

Wheels turning at different speeds

As above, let $f\colon U\to Y$ be a stabilisation of a map-germ of corank 1 from $\mathbb{C}^n, 0$ to $\mathbb{C}^m, 0$; write $U = X$, $D^k(f) = D^k$. Comparison of the exact complexes

$$0\longrightarrow \mathbb{Q}_Y\longrightarrow f_*(\mathbb{Q}_X)\xrightarrow{\delta_1}\mathrm{Alt}_2\varepsilon^2_*(\mathbb{Q}_{D^2})\xrightarrow{\delta_2}\mathrm{Alt}_3\varepsilon^3_*(\mathbb{Q}_{D^3})\xrightarrow{\delta_3}\cdots \tag{1}$$

and

$$0\longrightarrow \mathbb{Q}_{M_2}\longrightarrow f_*(\mathbb{Q}_{D_1^2})\xrightarrow{\delta_1}\mathrm{Alt}_2\varepsilon^2_*(\mathbb{Q}_{D^2})\xrightarrow{\delta_2}\mathrm{Alt}_3\varepsilon^3_*(\mathbb{Q}_{D^3})\xrightarrow{\delta_3}\cdots \tag{2}$$

shows that there is a short exact sequence

$$0\to\mathbb{Q}_{M_2}\to f_*(\mathbb{Q}_{D_1^2})\to f_*(\mathbb{Q}_X)/\mathbb{Q}_Y\to 0. \tag{3}$$

Now to calculate the cohomology of D_1^2 by the method of the beginning of this section, one makes use of the exact complex

$$0\to\mathbb{Q}_{D_1^2}\to\mathbb{Q}_{D^2}\to\mathrm{Alt}_2\tilde\varepsilon^3_*(\mathbb{Q}_{D^3})\to\mathrm{Alt}_3\tilde\varepsilon^4_*(\mathbb{Q}_{D^4})\to\mathrm{Alt}_4\tilde\varepsilon^5_*(\mathbb{Q}_{D^5})\to\cdots. \tag{4}$$

Here $\tilde\varepsilon^k\colon D^k\to X$ is induced by the Cartesian projection $X^k\to X$ which forgets all but the first component, and S_{k-1} acts on D^k by permuting the last $k-1$ components.

Exactness of this complex is a consequence of 2.1, taking D^2 and D_1^2 as X and Y; for by the principle of iteration, $D^{k-1}(\tilde\varepsilon^k\colon D^k\to X) = D^k(f\colon X\to Y)$, the S_{k-1} action on D^k being the one just described.

If we shorten the complex **(1)** by replacing the first three terms by $0\to f_*(\mathbb{Q}_X)/\mathbb{Q}_Y\to$, and call the resulting complex **(1′)**, we find that there is a morphism of complexes $\theta\colon f_*\mathbf{(4)}\to\mathbf{(1')}$ extending the morphism $f_*(\mathbb{Q}_{D_1^2})\to f_*(\mathbb{Q}_X)/\mathbb{Q}_Y$, defined as follows: for $y\in M_2$, we have

$$f_*(\mathrm{Alt}_{k-1}\tilde\varepsilon^k_*(\mathbb{Q}_{D^k}))_y = \bigoplus_{x\in f^{-1}(y)}\mathrm{Alt}_{k-1}\tilde\varepsilon^k_*(\mathbb{Q}_{D^k})_x.$$

Now $\mathrm{Alt}_{k-1}\tilde{\varepsilon}^k_*(\mathbb{Q}_{D^k})_x$ is generated as $\mathbb{Q}$-vector space by elements $\mathrm{Alt}_{k-1}\chi_{(x,x_2,\ldots,x_k)}$, where

$$(x, x_2, \ldots, x_k) \in (\varepsilon^k)^{-1}(y) \subseteq D^k.$$

Define $\theta_{k,y}: f_*(\mathrm{Alt}_{k-1}\tilde{\varepsilon}^k_*(\mathbb{Q}_{D^k}))_y \to \mathrm{Alt}_k \varepsilon^k_*(\mathbb{Q}_{D^k})_y$ by sending $\mathrm{Alt}_{k-1}\chi_{(x,x_2,\ldots,x_k)}$ to $\mathrm{Alt}_k \chi_{(x,x_2,\ldots,x_k)}$. Observe that this definition does not depend on the choice of order of $x_2, \ldots, x_k$, since any permutation induces the same sign change in $\mathrm{Alt}_{k-1}\chi_{(x,x_2,\ldots,x_k)}$ and in $\mathrm{Alt}_k \chi_{(x,x_2,\ldots,x_k)}$. It follows that the $\theta_{k,y}$ fit together to give a morphism of sheaves, θ_k.

It is straightforward to check that the θ_k commute with the differentials in the complexes f_***(4)** and **(1′)**.

The morphism of cohomology groups $H^p_{\mathrm{Alt}_{k-1}}(D^k, \mathbb{Q}) \to H^p_{\mathrm{Alt}_k}(D^k, \mathbb{Q})$ induced by θ_k is formally the same as θ_k itself; it is thus in fact equal to Alt_k (defined here without $k!$ in the denominator).

Let $K_k = \mathrm{Ker}(\theta_k)$, and denote the (exact) complex

$$0 \to \mathbb{Q}_{M_2} \to K_2 \to K_3 \to K_4 \to \cdots$$

by $\mathrm{Ker}(\theta)$. We now have a short exact sequence of exact complexes

$$0 \longrightarrow \mathrm{Ker}(\theta) \longrightarrow f_*(4) \xrightarrow{\theta} (1') \to 0. \tag{5}$$

Now by taking injective resolutions of each of the sheaves K_i, and forming a double complex in the usual way, we obtain as total complex an injective resolution of $\mathbb{Q}_{M_2}$; by taking global sections, we may thus calculate the cohomology of M_2. In fact the spectral sequence coming from the first filtration of the double complex collapses at E_1, just as in 2.6; for from the short exact sequence

$$0 \longrightarrow K_k \longrightarrow f_*\mathrm{Alt}_{k-1}(\tilde{\varepsilon}^k_*(\mathbb{Q}_{D^k})) \xrightarrow{\theta_k} \mathrm{Alt}_k \varepsilon^k_*(\mathbb{Q}_{D^k}) \longrightarrow 0,$$

we obtain a long exact sequence in which the map $H^p_{\mathrm{Alt}_{k-1}}(D^k, \mathbb{Q}) \to H^p_{\mathrm{Alt}_k}(D^k, \mathbb{Q})$ is simply the epimorphism Alt_k; since $H^p(D^k, \mathbb{Q}))$ vanishes except when $p = 0$ and $m - (m - n)k$, it follows that $H^p(K_k)$ also vanishes except when $p = m - (m - n)k$, and the long exact sequence collapses to the short exact sequence

$$0 \to H^{m-(m-n)k}(K_k) \to H^{m-(m-n)k}_{\mathrm{Alt}k-1}(D^k, \mathbb{Q}) \to H^{m-(m-n)k}_{\mathrm{Alt}_k}(D^k, \mathbb{Q}) \to 0.$$

We conclude

2.9. PROPOSITION. (i) *When $m = n + 1$, the spectral sequence just described induces a filtration on $H^{n-1}(M_2, \mathbb{Q})$ with successive quotients naturally isomorphic to*

$$\mathrm{Ker}[\mathrm{Alt}_k\colon H^{n-k+1}_{\mathrm{Alt}k-1}(D^k, \mathbb{Q}) \to H^{n-k+1}_{\mathrm{Alt}_k}(D^k, \mathbb{Q})],$$

so that

$$H^{n-1}(M_2, \mathbb{Q}) \simeq \bigoplus_{k=2}^{n+1} \mathrm{Ker}[\mathrm{Alt}_k\colon H^{n-k+1}_{\mathrm{Alt}k-1}(D^k, \mathbb{Q}) \to H^{n-k+1}_{\mathrm{Alt}_k}(D^k, \mathbb{Q})]$$

(ii) *If $m > n + 1$, then $H^p(M_2, \mathbb{Q})$ is isomorphic to*

$$\mathrm{Ker}[\mathrm{Alt}_k\colon H^{m-(m-n)k}_{\mathrm{Alt}k-1}(D^k, \mathbb{Q}) \to H^{m-(m-n)k}_{\mathrm{Alt}_k}(D^k, \mathbb{Q})]$$

when $p = m - k(m - n - 1) - 2$, and is equal to 0 for other positive values of p. □

2.10. REMARK. The proofs given here show that we have the short exact sequence of complexes **(5)** over $\mathbb{Z}$ as well as over $\mathbb{Q}$.

3. Marar's formulae

In [16], Washington Luiz Marar obtains formulae for the Euler characteristic of the image V of a good representative of a stable perturbation of a finitely determined corank 1 map-germ $\mathbb{C}^n, 0 \to \mathbb{C}^m, 0$, in terms of the Milnor numbers of the (singularities at 0 of the) associated multiple points schemes $D^k(f), 0$ and their intersections $D^k(f, \mathscr{P}), 0$ with the various multi-diagonals. Here, using the results of the previous section, we reprove and strengthen these formulae, by showing how the ranks of the cohomology groups $\mathrm{Alt}_k(H^{m-(m-n)k}(D^k, \mathbb{Q}))$ may be expressed in terms of the Milnor numbers $\mu(D^k(f, \mathscr{P}), 0)$.

Let G be a finite group, and let $\mathscr{R}$ be the ring of all $\mathbb{Q}$ linear representations of G. Denote elements of $\mathscr{R}$ by $[V]$. Recall that for a topological space X on which G acts, one has the *equivariant Euler characteristic* $\chi_G(X)$ as an element of $\mathscr{R}$:

$$\chi_G(X) = \sum_q (-1)^q [H_q(X, \mathbb{Q})]$$

where G acts on $H_q(X, \mathbb{Q})$ in the natural way (see [29]). If X has the structure of a cell complex which is respected by the G-action, then for $g \in G$, $\chi_G(X)(g)$ is equal to the topological Euler characteristic of the fixed point set X^g of g ([29]).

Let G act on $\mathbb{C}^m$ and let $X \subseteq \mathbb{C}^m$ be a G-invariant Milnor fibre of the germ at 0 of a G-invariant isolated complete intersection singularity X_0 of codimension c. Then

$$\chi_G(X) = [\mathbb{Q}] + (-1)^{m-c}[H]$$

where $[\mathbb{Q}]$ is the trivial 1-dimensional representational representation of G and $H = H^{m-c}(X, \mathbb{Q})$. Let $g \in G$ and suppose that X_0^g is an isolated complete intersection singularity, which is also of codimension c in $(\mathbb{C}^m)^g$; if X^g is smooth, then it is a Milnor fibre of X_0^g, and setting $d_g = \dim(\mathbb{C}^m)^g$, and letting μ_g be the Milnor number of X_0^g, we have

$$X_G(X)(g) = 1 + (-1)^{d_g - c}\mu_g$$

and so

$$[H](g) = (-1)^{m+d_g}\mu_g. \tag{3.1}$$

Now consider a map-germ $f: \mathbb{C}^n, 0 \to \mathbb{C}^{n+p}, 0$ of corank 1, and let $f_t: U_t \to V$ be a good representative of a stable perturbation of f. Let $X^0 = D^k(f), 0$, and let $X = D^k(f_t)$; then indeed X is a Milnor fibre of X_0. By a suitable choice of coordinates on $\mathbb{C}^n$ and $\mathbb{C}^{n+p}$, X_0 and X may be embedded in $\mathbb{C}^{n-1+k}$, in such a way that the natural action of the group $G = S_k$ on X_0 and X is induced by permutation of the last k coordinates ([19]). Let $H = H_{m-k(m-n)}(D^k(f), \mathbb{Q})$. In order to calculate the dimension of $\mathrm{Alt}_k(H)$, we proceed as follows: $\mathrm{Alt}_k(H)$ is the maximal subspace of H on which S_k acts via its sign representation. As the character of the 1-dimensional sign representation is exactly the sign σ, $\dim_{\mathbb{Q}} \mathrm{Alt}_k(H)$, which is just the multiplicity of this representation in H, is given by the inner product of characters:

$$\dim_{\mathbb{Q}} \mathrm{Alt}_k(H) = \frac{1}{k!} \sum_{g \in S_k} \sigma(g)[H](g).$$

Now in order to calculate the right hand side, suppose that $1 \leqslant k_1 < k_2 < \cdots < k_r$, and that in the cycle decomposition of g there are α_i cycles of length k_i, (so that $\Sigma\, \alpha_i k_i = k$). Then $\sigma(g) = (-1)^{\Sigma \alpha_i (k_i - 1)} = (-1)^{k - \Sigma \alpha_i}$. We calculate $[H](g)$ by using 2.1; for X^g is the intersection of $X = D^k(f_t)$ with the multi-diagonal in $(\mathbb{C}^{n-1+k})^g$ consisting of all points $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_k)$ where $y_i = y_j$ if i and j appear in the same cycle in g. Thus X_g is isomorphic to the multiple point scheme $D^k(f, \mathscr{P}_g)$ where $\mathscr{P}_g$ is the partition $(k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_r, \ldots, k_r)$ (k_i appearing α_i times) of k. Now $D^k(f, \mathscr{P}_g)$ is a Milnor fibre of the isolated complete intersection singularity $D^k(f_0, \mathscr{P}_g), 0$, (see [19]), whose codimension in $(\mathbb{C}^{n-1+k})^g$ is equal to that of $D^k(f_0)$ in $\mathbb{C}^{n-1+k}$, and thus, as

$$d_g = \dim(\mathbb{C}^{n-1+k})^g = n - 1 + \Sigma\alpha_i,$$

we have

$$[H](g) = (-1)^{n-k+1+d_g}\mu(D^k(f_0, \mathscr{P}), 0) = (-1)^{k+\Sigma\alpha_i}\mu(D^k(f_0, \mathscr{P}_g), 0).$$

Therefore

$$\dim_{\mathbb{Q}} \mathrm{Alt}_k(H) = \frac{1}{k!}\sum_{g\in S_k} \mu(D^k(f_0, \mathscr{P}_g), 0).$$

Now combinatorial arguments show that for each fixed partition $\mathscr{P}$ as above, there are $k!/(\Pi_i\, \alpha_i!\, k_i^{\alpha_i})$ elements in S_k with $\mathscr{P}_g = \mathscr{P}$, and hence

$$\dim_{\mathbb{Q}} \mathrm{Alt}_k(H) = \sum_{\mathscr{P}} \left\{\mu(D^k(f_0, \mathscr{P}), 0)\Big/\left(\prod_i \alpha_i!\, k_i^{\alpha_i}\right)\right\} \tag{3.2}$$

where the sum is taken over all partitions

$$\mathscr{P} = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_r, \ldots, k_r)$$

(where k_i appears α_i times), in which $1 \leqslant k_1 < k_2 < \cdots < k_r$.

By using 2.6 one obtains formulae for the rank of the cohomology groups of the image of f_t, which imply Marar's formulae. The only difference is that Marar incorporates an expression for $\mu(D^k(f_0)/S_k)$ into his formulae.

4. Quasihomogeneous mappings

The main aim of this section is to obtain expressions for the Betti numbers of the image of a stable perturbation f_t of a quasihomogeneous corank 1 map-germ $f_0\colon \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, in terms of the weights and degrees. We shall do this by using the results of Section 2, and so as the first step we calculate the rank of the alternating part of the cohomology of the multiple point spaces $D^k(f_t)$.

Let us first recall from [6] some information about isolated complete intersection singularities (ICIS). Let $\varphi = (\varphi_1, \ldots, \varphi_s)\colon \mathbb{C}^m, 0 \to \mathbb{C}^s, 0$ be an analytic germ, such that $\varphi^{-1}(0)$ is an ICIS. Let $\Omega^\cdot = \{\Omega^p, d\}_{0 \leqslant p}$ be the complex of germs of holomorphic differential forms on $\mathbb{C}^m, 0$, and consider the complex of relative forms on $\mathbb{C}^m, 0$:

$$\Omega_\varphi^p = \Omega^p \Big/ \sum_{i=1}^{s} d\varphi_i \wedge \Omega^{p-1}.$$

Let μ be the Milnor number of $(\varphi^{-1}(0), 0)$.

4.1. PROPOSITION [6]. *$\Omega_\varphi^{m-s}/d\Omega_\varphi^{m-s-1}$ is a free $\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0})$ module of rank μ* if *$m > s$ and of rank $\mu + 1$ if $m = s$.* □

In the next proposition, we do not assume that $\varphi^{-1}(0)$ is an ICIS, but we do assume that it is a complete intersection.

4.2. PROPOSITION [6]. *Denote by $\Sigma(\varphi)$ the set of critical points of φ, and by φ' the map-germ $(\varphi_1, \ldots, \varphi_{s-1})$. Provided $p \leqslant m - \dim \Sigma(\varphi)$, the sequence*

$$0 \longrightarrow \Omega^0_{\varphi'} \xrightarrow{d\varphi_s \wedge} \Omega^1_{\varphi'} \xrightarrow{d\varphi_s \wedge} \cdots \xrightarrow{d\varphi_s \wedge} \Omega^p_{\varphi'} \longrightarrow \Omega^p_\varphi \longrightarrow 0$$

is exact. □

As a consequence of 4.2, we have

4.3. PROPOSITION [6]. *If $p \leqslant m - \dim \Sigma(\varphi)$, the following sequence is exact:*

$$0 \longrightarrow \varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0}) \longrightarrow \Omega^0_\varphi \xrightarrow{d} \Omega^1_\varphi \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_\varphi$$

$$\xrightarrow{d} \Omega^p_\varphi \longrightarrow \Omega^p_\varphi/d\Omega^{p-1}_\varphi \longrightarrow 0$$

□

As a consequence, if $\varphi^{-1}(0)$ is an ICIS, then the relative de Rham complex of holomorphic forms calculates the cohomology of the Milnor fibre.

Now let us consider the alternated versions of 4.1–4.3. For any linear space V equipped with a linear S_k-action, we denote by V^{alt} the maximal subspace on which S_k acts via its sign representation: $V^{\mathrm{alt}} = \{v \in V | \sigma(v) = \mathrm{sign}(\sigma)v$ for all $\sigma \in S_k\}$.

Choose some coordinate system on $(\mathbb{C}^m, 0)$ and let S_k act on $\mathbb{C}^m$ by permuting the last k coordinates (we assume $m \geqslant k$). Suppose that every coordinate function φ_i of the map-germ $\varphi: (\mathbb{C}^m, 0) \to (\mathbb{C}^s, 0)$ (where $m \geqslant s$) is S_k-invariant. Then $\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0})$ is a subring of the ring of S_k-invariant functions on $(\mathbb{C}^m, 0)$. The space of relative alternating forms $\Omega^{p,\mathrm{alt}}/\Sigma\, d\varphi_i \wedge \Omega^{p-1,\mathrm{alt}}$, which is equal to

$(\Omega^p/\Sigma\, d\varphi_i \wedge \Omega^{p-1})^{\text{alt}}$, and which we denote by $\Omega_\varphi^{p,\text{alt}}$, is a module over the ring of S_k-invariant functions, and thus over $\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0})$. Suppose that $\varphi^{-1}(0)$ is an ICIS. The Milnor fibre $\{\varphi = \varepsilon\}$ is S_k-invariant, and so we get an S_k action on its cohomology. Let μ^{alt} be the rank of the S_k-alternating part of its middle dimensional cohomology.

4.4. PROPOSITION. *$\Omega_\varphi^{m-s,\text{alt}}/d\Omega_\varphi^{m-s-1,\text{alt}}$ is a free $\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0})$-module of rank μ^{alt}.*

Proof. Since $\Omega_\varphi^{m-s,\text{alt}}/d\Omega_\varphi^{m-s-1,\text{alt}} = (\Omega_\varphi^{m-s}/d\Omega_\varphi^{m-s-1})^{\text{alt}}$, it is a direct summand of the free $\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0})$-module $\Omega_\varphi^{m-s}/d\Omega_\varphi^{m-s-1}$ (for every representation of S_k is completely reducible). It follows that it is free. Moreover, if y is a regular value of φ,

$$(\Omega_\varphi^{m-s}/d\Omega_\varphi^{m-s-1})_y \underset{\mathcal{O}_{\mathbb{C}^s,y}}{\otimes} (\mathcal{O}_{\mathbb{C}^s,y}/\mathcal{M}_y) = H^{m-s}(\varphi^{-1}(y);\mathbb{C})$$

and so its alternating part is just $H^{m-s}(\varphi^{-1}(y);\mathbb{C})^{\text{alt}}$. This proves that the rank of $\Omega_\varphi^{m-s,\text{alt}}/d\Omega_\varphi^{m-s-1,\text{alt}}$ is μ^{alt}. □

Now consider the alternating parts of the sequences in Propositions 4.2 and 4.3. Notice that for S_k-invariant φ, the differentials in these sequences commute with the S_k-action. Thus, we get

4.5. PROPOSITION. *Let $p \leqslant m - \dim(\Sigma(\varphi))$. Then the sequence*

$$0 \longrightarrow \Omega_{\varphi'}^{0,\text{alt}} \xrightarrow{d\varphi_s\wedge} \Omega_{\varphi'}^{1,\text{alt}} \xrightarrow{d\varphi_s\wedge} \cdots \xrightarrow{d\varphi_s\wedge} \Omega_{\varphi'}^{p,\text{alt}} \longrightarrow \Omega_\varphi^{p,\text{alt}} \longrightarrow 0$$

is exact. □

As a consequence,

4.6. PROPOSITION. *If $\varphi^{-1}(0)$ is an ICIS, there is an exact sequence*

$$0 \longrightarrow \Omega_\varphi^{0,\text{alt}} \xrightarrow{d} \Omega_\varphi^{1,\text{alt}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_\varphi^{m-s,\text{alt}}$$

$$\longrightarrow \Omega_\varphi^{m-s,\text{alt}}/d\Omega_\varphi^{m-s-1,\text{alt}} \longrightarrow 0.$$ □

S_k-alternating forms

As above, let S_k act on $\mathbb{C}^m$ by permuting the last k coordinates. In order to describe the spaces $\Omega_\varphi^{p,\text{alt}}$ of relative S_k-alternating forms for a germ at $0 \in \mathbb{C}^m$ of S_k-invariant mapping φ, we start with a description of the absolute S_k-alternating forms in the case $m = k$.

We will consider the problem in a more general setting – for a finite group G of linear automorphisms of $\mathbb{C}^k$, generated by reflections (recall that a reflection is a linear automorphism which leaves fixed all the points of a hyperplane; one can treat the real case as well). In this subsection we will denote by $\Omega^{p,\mathrm{alt}}$ the space of germs at 0 of holomorphic p-forms ω such that $g^*\omega = \det(g)\omega$ for all $g \in G$; we will call these forms "G-alternating". What is $\Omega^{p,\mathrm{alt}}$?

It is well known that the ring $\mathcal{O}_k^{\mathrm{sym}}$ of germs at 0 of G-invariant functions is generated by exactly k independent functions, say $h_1,\ldots,h_k$. Let I be the p-tuple $(i_1,\ldots,i_p)$, with $i_1 < i_2 < \cdots < i_p$, and let $J = (j_1,\ldots,j_{k-p})$ be its complement in the set $\{1,\ldots,k\}$, with $j_1 < j_2 < \cdots < j_{k-p}$. Consider the $(k-p)$-gradient vector $\nabla h_J = \nabla h_{j_1} \wedge \nabla h_{j_2} \wedge \cdots \wedge \nabla h_{j_{k-p}}$. Let ω_I be the contraction of the form $dy_1 \wedge dy_2 \wedge \cdots \wedge dy_k$ along ∇h_J (where the y_i are linear coordinates on $\mathbb{C}^k$). Then ω_I is a G-alternating p-form.

4.7. PROPOSITION. *The space $\Omega^{p,\mathrm{alt}}$ of germs at $0 \in \mathbb{C}^k$ of G-alternating holomorphic p-forms on $\mathbb{C}^k$ is a free $\mathcal{O}_k^{\mathrm{sym}}$-module, with free basis consisting of all ω_I such that $|I| = p$.*

Proof (cf. [26]).

STEP 1. Let L be the field of germs at $0 \in \mathbb{C}^k$ of meromorphic functions. Then the forms ω_I are L-linearly independent in $\Omega^p \otimes_{\mathcal{O}_{\mathbb{C}^k,0}} L$. For suppose that $\Sigma_{|I|=p}\alpha_1\omega_I = 0$ is a relation, with $\alpha_I \in L$. Evaluating this relation on ∇h_I for some fixed I, we get $0 = \alpha_I\omega_I(\nabla h_I) = \pm\alpha_I\Delta$, where

$$\Delta = \det(\partial(h_1,\ldots,h_k)/\partial(y_1,\ldots,y_k));$$

as Δ is not identically zero, $\alpha_I = 0$.

STEP 2. We show that any alternating p-form belongs to $\Sigma_{|I|=p}\mathcal{O}_k^{\mathrm{sym}}\omega_I$. As the number of different ω_I, with $|I| = p$, is C_k^p, they generate $\Omega^p \otimes_{\mathcal{O}_{\mathbb{C}^m,0}} L$, linearly over L. So for any holomorphic p-form ω, we can write $\omega = \Sigma_{|I|=p}\beta_I\omega_I$, for some $\beta_I \in L$. Now if ω is G-alternating, then alternation of this expression over G gives

$$\omega = \frac{1}{|G|}\sum_{g\in G}\det(g)^{-1}g^*\omega = \frac{1}{|G|}\sum_{g\in G}\sum_{|I|=p} g^*(\beta_I)\omega_I = \sum_{|I|=p}\gamma_I\omega_I,$$

where the $\gamma_I \in L$ are G-invariant. Again by evaluating such an expression on ∇h_I for a fixed I, we get a holomorphic function $\varepsilon_I = \gamma_I\Delta$. This is a G-alternating function, and so by the lemma from [26], $\varepsilon_I = \Delta\varphi_I$ for some $\varphi_I \in \mathcal{O}^{\mathrm{sym}}$. Thus, we are done. □

Now let us consider the bigrading on the space $\Omega^{\cdot,\mathrm{alt}}$ by the weight of the form, (with weight $y_i = \text{weight}\, dy_i = 1$) and by the degree of the form. Let $P(\Omega^{\cdot,\mathrm{alt}}, t, \tau) = \Sigma c_{l,m}t^l\tau^m$ be the corresponding Poincaré series (i.e. $c_{l,m}$ is the dimension of the $\mathbb{C}$-vector space of m-forms of weight l. Let $d_1,\ldots,d_k$ be the weights of the basic invariants $h_1,\ldots,h_k$ (which of course can be chosen to be homogeneous). We have

4.8. COROLLARY.

$$P(\Omega^{\cdot,\mathrm{alt}}; t, \tau) = \prod_{i=1}^{k} \frac{t^{d_i - 1} + \tau t}{1 - t^{d_i}}$$

Proof. This is immediate from the previous discussion. The numerator provides the Poincaré polynomial for the set of generators ω_I, and the denominator provides the Poincaré series for $\mathcal{O}_k^{\mathrm{sym}}$. □

Since the Poincaré series for the space of forms of fixed degree p, is the coefficient of τ^p in this series, we have

4.9. COROLLARY.

$$P(\Omega^{p,\mathrm{alt}}; t) = \mathrm{res}_{\tau=0}\tau^{-p-1} \prod_{i=1}^{k} \frac{t^{d_i - 1} + \tau t}{1 - t^{d_i}} \qquad \square$$

EXAMPLE. When S_k acts on $\mathbb{C}^k$ by permutations of the coordinates, then $d_i = i$ for $i = 1,\ldots,k$, and we get

$$P(\Omega^{\cdot,\mathrm{alt}}; t, \tau) = \prod_{i=1}^{k} \frac{t^{i-1} + \tau t}{1 - t^{i}} \quad \text{and} \quad P(\Omega^{p,\mathrm{alt}}; t) = \mathrm{res}_{\tau=0}\tau^{-p-1} \prod_{i=1}^{k} \frac{t^{i-1} + \tau t}{1 - t^{i}}$$

We shall use these expressions in what follows. Note that there is another expression for the latter series:

4.10. PROPOSITION. *When $G = S_k$ acts by permuting the coordinates on $\mathbb{C}^k$,*

$$P(\Omega^{p,\mathrm{alt}}; t) = t^{p + 1/2(k-p)(k-p-1)} \prod_{i=1}^{p} (1 - t^i)^{-1} \prod_{j=1}^{k-p} (1 - t^j)^{-1}$$

Proof (independent of 4.7). Consider a p-form

$$\omega = A dy_1 \wedge \cdots \wedge dy_p + \text{terms with other } p\text{-tuples of } dy_j'\text{s}.$$

It is easily seen that if ω is an S_k-alternating form, the holomorphic function A must be symmetric with regard to permutations of $y_1,\ldots, y_p$, and alternating with respect to permutations of $y_{p+1},\ldots, y_k$. So

$$A \in \mathcal{O}^{\mathrm{sym}}_{y_1,\ldots,y_p} \otimes_{\mathbb{C}} \mathcal{O}^{\mathrm{alt}}_{y_{p+1},\ldots,y_k} = \mathcal{O}^{\mathrm{sym}}_{y_1,\ldots,y_p} \otimes_{\mathbb{C}} \mathcal{O}^{\mathrm{sym}}_{y_{p+1},\ldots,y_k} \{\mathrm{V.d.m.}(y_{p+1},\ldots, y_k)\}$$

where V.d.m.$(y_{p+1},\dots,y_k)$ is the Vandermonde determinant. On the other hand, any such A uniquely determines an S_k-alternating p-form ω. As the Poincaré series of $\mathcal{O}^{\mathrm{sym}}_{y_1,\dots,y_p}$ is $\Pi_{i=1}^p(1-t^i)^{-1}$, and $\deg(\mathrm{V.d.m.}(y_{p+1},\dots,y_k)) = \frac{1}{2}(k-p)(k-p-1)$, the statement is proved. □

S_k-alternating forms on a symmetric quasihomogeneous ICIS

Let $x_1,\dots,x_n, y_1,\dots,y_k$ be coordinates in $(\mathbb{C}^{n+k},0)$ and let the group S_k act by permutation of the y coordinates. Let $w_0, w_1,\dots,w_n; d_1,\dots,d_s$ be positive integers. Let $\varphi\colon(\mathbb{C}^{n+k},0)\to(\mathbb{C}^s,0)$, with $n+k\geqslant s$, be an S_k invariant mapping, quasihomogeneous of type $(w_1,\dots,w_n,w_0,\dots,w_0;d_1,\dots,d_s)$.

Suppose that $\varphi^{-1}(0)$ is an ICIS, and let

$$H^{\mathrm{alt}} = \Omega_\varphi^{n+k-s,\mathrm{alt}}/(d\Omega_\varphi^{n+k-s-1,\mathrm{alt}} + \textstyle\sum \varphi_i\Omega_\varphi^{n+k-s,\mathrm{alt}}),$$

be the top cohomology of the complex of relative S_k-alternating holomorphic forms on $\varphi^{-1}(0)$. It inherits a grading, with $wt(x_i)=wt(dx_i)=w_i$, $wt(y_i)=wt(dy_i)=w_0$. Let $P(H^{\mathrm{alt}};t)$ be the Poincaré series of H^{alt} with respect to this grading.

4.11. THEOREM.

$$P(H^{\mathrm{alt}};t) = \mathrm{res}_{\tau=0}\frac{\tau^{-n-k+s-1}}{1+\tau}\prod_{i=1}^k\frac{t^{(i-1)w_0}+\tau t^{w_0}}{1-t^{iw_0}}\prod_{j=1}^n\frac{1+\tau t^{w_j}}{1-t^{w_j}}\prod_{l=1}^s\frac{1-t^{d_l}}{1+\tau t^{d_l}}$$

We give the proof below.

By Proposition 4.4, $\dim(H^{\mathrm{alt}})=\mu^{\mathrm{alt}}$. So we have

4.12. COROLLARY. $\mu^{\mathrm{alt}} = P(H^{\mathrm{alt}};1)$.

Proof of 4.11 1. Consider the space of S_k-alternating forms on $\mathbb{C}^{n+k}\colon \Omega^{\cdot,\mathrm{alt}} = \Omega_k^{\cdot,\mathrm{alt}}\otimes_{\mathbb{C}}\Omega_n$. It has a natural bigrading, by the quasihomogeneous weight l and by the degree p of the form. Let $P(\Omega^{\cdot,\mathrm{alt}};t,\tau)=\Sigma\, c_{l,p}t^l\tau^p$ be the corresponding Poincaré series. Then $P(\Omega^{\cdot,\mathrm{alt}}) = P(\Omega_k^{\cdot,\mathrm{alt}})\cdot P(\Omega_n)$ and by Corollary 4.8 we get

$$P(\Omega^{\cdot,\mathrm{alt}};t,\tau) = \prod_{i=1}^k\frac{t^{(i-1)w_0}+\tau t^{w_0}}{1-t^{iw_0}}\cdot\prod_{j=1}^n\frac{1+\tau t^{w_j}}{1-t^{w_j}}$$

2. By induction we obtain from Proposition 4.5 that

$$P(\Omega_\varphi^{p,\mathrm{alt}};t) = \mathrm{res}_{\tau=0}\tau^{-p-1}P(\Omega^{\cdot,\mathrm{alt}};t,\tau)\prod_{l=1}^s(1+\tau t^{d_l})^{-1}$$

3. By Proposition 4.6,

$$P(\Omega_\varphi^{n+k-s,\mathrm{alt}}/d\Omega_\varphi^{n+k-s-1,\mathrm{alt}};t) = \sum_{p=0}^{n+k-s} (-1)^{n+k-s-p} P(\Omega_\varphi^{p,\mathrm{alt}};t)$$

$$= \mathrm{res}_{\tau=0} \frac{\tau^{-n-k+s-1}}{1+\tau} P(\Omega^{\cdot,\mathrm{alt}};t,\tau) \prod_{l=1}^{s} (1+\tau t^{d_l})^{-1}$$

4. By Proposition 4.4, the space $\Omega_\varphi^{n+k-s,\mathrm{alt}}/d\Omega_\varphi^{n+k-s-1,\mathrm{alt}}$ is a free $\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0})$-module. The generators of this module are $\mathbb{C}$-linear generators of the space H^{alt}. Thus,

$$P(H^{\mathrm{alt}};t) = P(\Omega_\varphi^{n+k-s,\mathrm{alt}}/d\Omega_\varphi^{n+k-s-1,\mathrm{alt}};t)/P(\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0});t).$$

As $P(\varphi^{-1}(\mathcal{O}_{\mathbb{C}^s,0});t) = \Pi_{l=1}^s (1-t^{d_l})^{-1}$, we are done. □

Our quasihomogeneous mapping φ is equivariant with respect to the following $U(1)$-action:

$$\lambda\cdot(x_1,\dots,x_n,y_1,\dots,y_k) = (\lambda^{w_1}x_1,\dots,\lambda^{w_n}x_n,\lambda^{w_0}y_1,\dots,\lambda^{w_0}y_k)$$

in the source $\mathbb{C}^{n+k}$, and $\lambda\cdot(z_1,\dots,z_s) = (\lambda^{d_1}z_1,\dots,\lambda^{d_s}z_s)$ in the target $\mathbb{C}^s$.

Consider the non-critical level $\{\varphi = \varepsilon\}$. Let d be the greatest common factor of all of the d_i such that $\varepsilon_i \neq 0$. Consider the loop $\varepsilon\cdot\exp(2\pi i\rho/d)$, $\rho\in[0,1]$, in $\mathbb{C}^s$. It induces an endomorphism h^{alt} of the S_k-alternating cohomology of the Milnor fibre $\{\varphi = \varepsilon\}$. This endomorphism h^{alt} is called the *alternating quasihomogeneous monodromy*. Let $D\in\mathbb{Z}[\mathbb{C}^*]$ be the divisor of the characteristic polynomial of h^{alt}. Then we have

4.13. COROLLARY. $D = P(H^{\mathrm{alt}};\langle\exp(2\pi i/d)\rangle)$. □

Indeed, quasihomogeneous forms which represent a $\mathbb{C}$-basis of H^{alt} also give a basis for the alternating cohomology of the Milnor fibre, and h^{alt} simply multiplies each such form of weight l by $\exp(2\pi il/d)$.

The cohomology of the image of a quasihomogeneous mapping

We now apply the results of this section and Section 2 to calculate the Betti numbers of the image of a stable perturbation of a quasihomogeneous corank 1 map-germ $f_0\colon\mathbb{C}^{n+1}, 0\to\mathbb{C}^{n+r}, 0$, where $r>1$ (for notational reasons, it is convenient to consider mappings with domain $\mathbb{C}^{n+1}$ rather than $\mathbb{C}^n$).

We can choose coordinates $(x_1,\dots,x_n, y)$ in the source and coordinates in the target, so that f_0 takes the form

$$f_0(x_1,\dots,x_n, y) = (x_1,\dots,x_n, f_{0,1}(x, y),\dots,f_{0,r}(x, y)).$$

Then the multiple point scheme $D^k(f_0)$ embeds into $\mathbb{C}^n \times \mathbb{C}^k$, where it is defined by the equations

$$F_{l,j}(x_1,\dots,x_n, y_1,\dots, y_k) = \text{V.d.m.}_l(y, f_{0,j})/\text{V.d.m.}(y)$$

where V.d.m.(y) is the Vandermonde determinant $\det[y_i^{\alpha-1}]_{1\leqslant i,\alpha\leqslant k}$, and V.d.m.$_l(y, f_{0,j})$ is the determinant obtained from V.d.m.(y) by replacing y_i^l by $f_{0,j}(x, y_i)$ for $1 \leqslant i \leqslant k$ (see [19], §2). Each point $(x, y_1,\dots, y_k)$ satisfying these equations corresponds to a k-tuple $(x, y_1),\dots,(x, y_k)$ of points of $\mathbb{C}^{n+1}$ having the same image under f_0. Each $F_{l,j}$ is invariant with respect to the S_k-action on $\mathbb{C}^{n+k}$ in which the last k coordinates are permuted. We shall denote by F_k the mapping $\mathbb{C}^{n+k} \to \mathbb{C}^{(k-1)r}$ with components $F_{l,j}$, $1 \leqslant j \leqslant r$, $1 \leqslant l \leqslant k-1$. Then if f is a stable perturbation of f_0, $D^k(f)$ is a Milnor fibre of the ICIS $(F_k^{-1}(0), 0)$ ([19]).

Now suppose in addition that f_0 is quasihomogeneous, with respect to weights w_i for the variables x_i, and w_0 for y, with weight $(f_{0,j}) = d_j$. Then weight $(F_{l,j}) = d_j - lw_0$, and so by Theorem 4.11 the Poincaré series of the top alternating cohomology of the multiple point space $D^k(f)$ is given by the polynomial

$$R_k(t) = \text{res}_{\tau=0} \frac{\tau^{-n-k+r(k-1)-1}}{1+\tau} \prod_{\alpha=1}^{k} \frac{t^{(\alpha-1)w_0} + \tau t^{w_0}}{1 - t^{\alpha w_0}}$$
$$\prod_{i=1}^{n} \frac{1+\tau t^{w_i}}{1-t^{w_i}} \prod_{\substack{j=1,\dots,r \\ l=1,\dots,k-1}} \frac{1-t^{d_j-lw_0}}{1+\tau t^{d_j-lw_0}}$$
$$= \text{res}_{\tau=0} \tau^{-n-k+r(k-1)-1} \cdot t^{1/2k(k-1)w_0} \prod_{\alpha=2}^{k} (1 - t^{\alpha w_0})^{-1} \prod_{\alpha=3}^{k} (1 + \tau t^{(2-\alpha)w_0})$$
$$\prod_{i=0}^{n} \frac{1+\tau t^{w_i}}{1-t^{w_i}} \prod_{\substack{j=1,\dots,r \\ l=1,\dots,k-1}} \frac{1-t^{d_j-lw_0}}{1+\tau t^{d_j-lw_0}}$$

The decomposition of the cohomology of the image of a stable perturbation into the direct sum of the alternating cohomology of the multiple point spaces D^k, given by Theorem 2.6, then leads to

4.14. THEOREM. *If $r = 2$, then the $n+1-$st Betti number $\beta_{n+1}(Y_t)$ of the image Y_t of a stable perturbation of f_0 is equal to $\Sigma_{k=2}^{n+2} R_k(1)$.*

If $r > 2$, then if $2 \leqslant k \leqslant (n+r)/(r-1)$, we have $\beta_{n+k-(k-1)(r-1)}(Y_t) = R_k(1)$.

In both cases, the remaining Betti numbers β_i, for $i > 0$, vanish. □

4.15. REMARK. Corollary 4.13 determines the eigenvalues of the quasihomogeneous monodromy of the image of a stable perturbation of f_0.

4.16. EXAMPLES. We list only the non-zero Betti numbers:

1. $\mathbb{C}^{n+1} \to \mathbb{C}^{2n+2}$.

$$\beta_1 = \prod_{j=1}^{n+2} (d_j - w_0)/2w_0^2 \prod_{i=1}^{n} w_i.$$

Note that β_1 here is just the number of points of self-intersection of the image.

2. $\mathbb{C}^{n+1} \to \mathbb{C}^{2n+1}$, $n > 1$:

$$\beta_2 = \frac{\prod_{j=1}^{n+1} (d_j - w_0)}{2w_0^2 w_1 \cdots w_n} \left[\sum_{j=1}^{n+1} d_j - (n+1)w_0 - \sum_{i=0}^{n} w_i \right]$$

3. $\mathbb{C}^2 \to \mathbb{C}^3$ (c.f. [23])

$$\beta_2 = \frac{(d_1 - w_0)(d_2 - w_0)}{6w_0^3 w_1} [d_1 + w_0)(d_2 + w_0) - 6w_0^2 - 3w_0 w_1]$$

4. $\mathbb{C}^4 \to \mathbb{C}^6$:

$$\beta_2 = \frac{\prod_{j=1}^{3} (d_j - w_0)(d_j - 2w_0)}{6w_0^3 w_1 w_2 w_3}$$

$$\beta_3 = \frac{\prod_{j=1}^{3} (d_j - w_0)}{2w_0^2 w_1 w_2 w_3} \left[4 \sum_{j=1}^{3} (d_j - w_0)^2 + \sum_{1 \leqslant \alpha < \beta \leqslant 3} (d_\alpha + d_\beta - 2w_0)^2 - \right.$$
$$\left. - \sum_{\substack{i=0,1,2,3 \\ j=1,2,3}} (d_j + w_i - w_0)^2 + \sum_{0 \leqslant \alpha < \beta \leqslant 3} (w_\alpha + w_\beta)^2 \right].$$

5. Hodge numbers of the image of a stable perturbation of a quasihomogeneous map-germ

Let $f_0: \mathbb{C}^{n+1}, 0 \to \mathbb{C}^{n+r}, 0$ with

$f_0(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n, f_{0,1}(x, y), \ldots, f_{0,r}(x, y)),$

be as in Section 4 a quasihomogeneous map-germ of corank 1 and of finite $\mathscr{A}$-codimension, with $r \geqslant 2$. It is easy to see from the characterisation of stability in terms of multiple point schemes given as Theorem 2.5 above, and from the construction of a versal deformation of f_0 that f_0 has a stable perturbation in "negative weight"; that is, it is possible to find a stable f such that for each j, $f_j - f_{0,j} = \Sigma_s m_{j,s}$ with each monomial $m_{j,s}$ of weight $d_{j,s}$ less than the weight d_j of $f_{0,j}$.

Now the mapping $f\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+r}$ is the affine part of the mapping

$$\bar{f}\colon \mathbb{P}(w_1,\ldots,w_n,\, w_0,\, 1) \to \mathbb{P}(w_1,\ldots,w_n,\, d_1,\ldots,d_r,\, 1)$$

of weighted projective spaces given by $\bar{f}(x, y, z) = (x, \bar{f}_1,\ldots,\bar{f}_r, z)$ where

$$\bar{f}_j(x,\, y,\, z) = f_{0,j}(x,\, y) + \sum_s z^{d_j - d_{js}} m_{j,s}(x,\, y).$$

The multiple point spaces $D^k(f)$ are compact algebraic varieties which can be embedded in $\mathbb{P}(w_1,\ldots,w_n, w_0,\ldots,w_0, 1)$ (where w_0 appears k times). If $f_\infty = \bar{f}|_{z=0}$, then $D^k(f_\infty)$ embeds in $\mathbb{P}(w_1,\ldots,w_n, w_0,\ldots,w_0)$, which is just the subspace $\{z = 0\}$ of $\mathbb{P}(w_1,\ldots,w_n,\, w_0,\ldots,w_0, 1)$. Now $D^k(f_\infty)$ is the weighted projectivisation of $D^k(f_0)$, which is smooth outside the origin; thus $D^k(f_\infty)$ is quasismooth, and its only singularities are cyclic quotient singularities. The affine part $\{z \neq 0\}$ of $D^k(\bar{f})$ is just $D^k(f)$, and hence is smooth; since moreover $D^k(\bar{f}) \cap \{z = 0\} = D^k(f_\infty)$ is itself quasismooth, the affine cone over $D^k(\bar{f})$ is thus smooth outside 0, and hence $D^k(\bar{f})$ is quasismooth also. Since both $D^k(\bar{f})$ and $D^k(f_\infty)$ are compact algebraic varieties, they have a canonical mixed Hodge structure ([0], [1], [28]). In fact the structures are pure, for both $D^k(\bar{f})$ and $D^k(f_\infty)$ are projective and are *V-manifolds*, (cf. [28], Chapter I, §5); that is, they are locally the quotient of a smooth space by the action of an finite group of holomorphic automorphisms.

In this section we use the results of Sections 2 and 4 to express the Hodge numbers of the mixed Hodge structure on the cohomology of the image of f, in terms of the cohomology of the multiple point spaces of f and f_∞.

Complete intersections

As in Section 4, let $\varphi\colon \mathbb{C}^m, 0 \to \mathbb{C}^s, 0$, with $m > s$, be quasihomogeneous of type $(w_1,\ldots,w_m; d_1,\ldots,d_s)$, and suppose that $\varphi^{-1}(0)$ is an ICIS. Then the same equations $\varphi = 0$ define an $(m - s - 1)$-dimensional quasismooth complete intersection X in the $(m - 1)$-dimensional weighted projective space

$\mathbb{P} = \mathbb{P}(w_1, \ldots, w_m)$ (see [28], [2]). X has a pure Hodge structure ([28]). Let F^p be the Hodge filtration on $H^*(X, \mathbb{C})$.

We now recall from [28] the description (due to R. O. Buchweitz) of the primitive parts $P^{p,q}$ of the spaces $\mathrm{Gr}^p_F H^{p+q}(X; \mathbb{C})$, which are non-trivial for $p + q > 0$ only if $p + q = m - s - 1$:

Let S be the graded ring $S = \mathbb{C}[z_1, \ldots, z_m]$, with $wt(z_i) = w_i$, and consider its graded quotient $R = S/(\varphi_1, \ldots, \varphi_s)$. We shall use the following graded R-modules:

ω_R, the graded module corresponding to the sheaf of holomorphic forms of top degree $m - s - 1$ on X; it is a free R-module of rank 1, with generator ω_0 of weight $\Sigma w_i - \Sigma d_j$ (in fact ω_0 is given by contraction of the form $dz_1 \wedge \cdots \wedge dz_m/d\varphi_1 \wedge \cdots \wedge d\varphi_s$ with the Euler vector field $e = \Sigma w_i z_i \partial/\partial z_i$ on $\mathbb{C}^m$);

$\mathcal{N}$, corresponding to the normal bundle of X in $\mathbb{P}$; it is a free R-module of rank s, with generators u_j of weight $-d_j$;

θ, corresponding to the restriction of the tangent bundle of $\mathbb{P}$ to X; it is a free R-module of rank m, with generators v_i of weight $-w_i$;

$S^\alpha \mathcal{N}$, the α-th symmetric power of $\mathcal{N}$;

$\bigwedge^\beta \theta$, the β-th exterior power of θ;

$M_{\alpha,\beta} = S^\alpha \mathcal{N} \otimes_R \bigwedge^\beta \theta \otimes_R \omega_R$, which is a free R-module on generators $u_1^{c_1} \cdots u_s^{c_s} \otimes v_{i_1} \wedge \cdots \wedge v_{i_\beta} \otimes w_0$, where $c_j \geqslant 0$ and $\Sigma c_j = \alpha$, and $0 \leqslant i_1 < \cdots < i_\beta \leqslant m$.

We shall also consider the mappings $d\varphi\colon \theta \to \mathcal{N}$ (differentiation of φ along a vector field) given by

$$\sum r_i v_i \to \sum_j \left(\sum_i r_i \partial\varphi_j/\partial z_i \right) u_j$$

and

$$e \wedge : \textstyle\bigwedge^\beta \theta \to \bigwedge^{\beta+1} \theta \text{ (wedging with the Euler field).}$$

These mappings induce mappings

$$d\varphi\colon M_{\alpha,\beta} \to M_{\alpha+1,\beta-1} \quad \text{and} \quad e\wedge : M_{\alpha,\beta} \to M_{\alpha,\beta+1}.$$

Consider the following diagram, for a fixed value of q:

$$
\begin{tikzcd}
M_{0,0} \arrow[d] \\
M_{0,1} \arrow[r] \arrow[d] & M_{1,0} \arrow[d] \\
M_{0,2} \arrow[r] \arrow[d] & M_{1,1} \arrow[r] \arrow[d] & M_{2,0} \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
M_{0,q-2} \arrow[r] \arrow[d] & M_{1,q-3} \arrow[r] \arrow[d] & M_{2,q-4} \arrow[r] \arrow[d] & \cdots \arrow[r] & M_{q-2,0} \arrow[d] \\
M_{0,q-1} \arrow[r] \arrow[d] & M_{1,q-2} \arrow[r] \arrow[d] & M_{2,q-3} \arrow[r] \arrow[d] & \cdots \arrow[r] & M_{q-2,1} \arrow[r] \arrow[d] & M_{q-1,0} \arrow[d] \\
M_{0,q} \arrow[r] & M_{1,q-1} \arrow[r] & M_{2,q-2} \arrow[r] & \cdots \arrow[r] & M_{q-2,2} \arrow[r] & M_{q-1,2} \arrow[r] & M_{q,0}
\end{tikzcd}
\qquad
\xrightarrow{d\varphi} \qquad \big\downarrow e\wedge
$$

Construct a complex $\Lambda(q)^{\cdot}$ by taking the direct sums of the weight zero parts of the modules (i.e. global sections of the corresponding sheaves) on the same dashed line in the diagram:

$$
\begin{aligned}
\Lambda(q)^i &= \bigoplus_{\substack{\alpha,\beta\geqslant 0,\ \alpha\geqslant i\\ 2\alpha+\beta=q+i}} (M_{\alpha,\beta})_0 \qquad |i|\leqslant q\\
\Lambda(q)^i &= 0 \qquad\qquad\qquad\qquad |i|>q
\end{aligned}
$$

The differential in this complex is a certain combination of $d\varphi$ and $e\wedge$ (neither of these mappings change the weight).

Then $P^{m-s-1-q,q}$ is the only non-trivial cohomology of the complex $\Lambda(q)^{\cdot}$ (see [28]). Its dimension $h^{m-s-1-q,q}$ may be calculated as follows:

5.1. PROPOSITION ([11]).

$$
h^{m-s-1-q,q}=(-1)^q \operatorname{res}_{t=0}\operatorname{res}_{u=0}\left\{t^{-1}\frac{u^{-q-1}}{1+u}\prod\frac{1+ut^{-w_i}}{1-t^{w_i}}\prod\frac{1-t^{d_j}}{1+ut^{-d_j}}\,t^{\Sigma w_i-\Sigma d_j}\right\}
$$

(here the limits for i and j are the obvious ones; we shall omit them in what follows).

Proof. We have

$$h^{m-s-1-q,q} = \sum_{|i|\leqslant q} (-1)^{q-i} \dim \Lambda(q)^i = \sum_{|i|\leqslant q} (-1)^{q-i} \sum_{\substack{\alpha,\beta\geqslant 0, \alpha\geqslant i\\ 2\alpha+\beta=q+i}} \dim(M_{\alpha,\beta})_0$$

$$= \sum_{\substack{\alpha,\beta\geqslant 0\\ \alpha+\beta\leqslant q}} (-1)^{\beta} \dim(M_{\alpha,\beta})_0.$$

Let us evaluate this sum.

Consider the R-module

$$M = \bigoplus_{\alpha,\beta\geqslant 0} M_{\alpha,\beta} = S^{\cdot}\mathcal{N} \otimes_R {\textstyle\bigwedge}^{\cdot}\theta \otimes_R \omega_R.$$

It has three gradings: by the weight r and the degrees α and β. The Poincaré series for its free R-generators is

$$A(t, u, v) = t^{\Sigma w_i - \Sigma d_j} \prod (1 + vt^{-w_i}) / \prod (1 - ut^{-d_j})$$

(so that the Poincaré series for M is the product of $A(t, u, v)$ with the Poincaré series for R). Here the coefficient of $t^r u^\alpha v^\beta$ is the number of generators of $M_{\alpha,\beta}$ of weight r. Also $\dim(M_{\alpha,\beta})_0$ is the coefficient of $t^0 u^\alpha v^\beta$ in the Poincaré series $P(M; t, u, v) = A(t, u, v) \cdot P(R; t)$, where $P(R; t) = \Pi(1 - t^{d_j})/\Pi(1 - t^{w_i})$. Consequently, $\Sigma_{\alpha+\beta=\gamma}(-1)^\beta \dim(M_{\alpha,\beta})_0$ is the coefficient of $t^0 u^\gamma$ in $B(t, u) = P(M; t, u, -u)$. In order to obtain the sum of such expressions for $0 \leqslant j \leqslant q$, we have to take the coefficient of $t^0 u^q$ in the series $C(t, u) = B(t, u)/(1 - u)$. In the statement of the proposition we simply point out that one can obtain the same number from the series $C(t, -u)$, by means of residues. □

The symmetric case

Now let $\varphi: \mathbb{C}^{n+k}, 0 \to \mathbb{C}^s, 0$ with $n + k > s$, be quasihomogeneous of type $(w_1, \ldots, w_n, w_0, \ldots, w_0; d_1, \ldots, d_s)$ (where w_0 appears k times), and such that each of its coordinate functions φ_j is invariant under the S_k action on $\mathbb{C}^{n+k}$ in which the last k coordinate are permuted. We want to express the rank of the S_k-alternating part of the primitive cohomology of the quasismooth complete intersection $\{\varphi = 0\}$ in the corresponding weighted projective space, in terms of the weights w_i and d_j.

5.2. PROPOSITION.

$$h_0^{n+k-s-1-q,q;\mathrm{alt}}$$
$$= (-1)^q \operatorname{res}_{t=0} \operatorname{res}_{u=0} \left\{ t^{-1} \frac{u^{-q-1}}{1+u} \prod_{i\geq 1} \frac{1+ut^{-w_i}}{1-t^{w_i}} \prod_{l=1}^{k} \frac{1+ut^{(l-2)w_0}}{1-t^{lw_0}} \times \right.$$
$$\left. \times \prod_{j\geq 1} \frac{1-t^{d_j}}{1+ut^{-d_j}} t^{kw_0+\Sigma_{i\geq 1} w_i - \Sigma_{j\geq 1} d_j} \right\}$$

Proof. Let us take the alternating part of the complex $\Lambda(q)^\cdot$ of the previous section. Under our assumptions, the mappings $d\varphi$ and $e\wedge$ are S_k-equivariant. Hence, we need to take the S_k-alternating parts of the modules

$$M_{\alpha,\beta} = \oplus (u_1^{c_1} \dots u_s^{c_s} \otimes_{\mathbb{C}} \textstyle\bigwedge^\beta \theta \otimes_{\mathbb{C}} \omega_0)$$

(where the direct sum is taken over all $c_j \geq 0$ such that $\Sigma\, c_j = d$). As $u_1, \dots, u_s$ are S_k-invariant and ω_0 is S_k-alternating, taking the S_k-alternating part of $M_{\alpha,\beta}$ is the same as taking the S_k-invariant part $(\bigwedge^\beta \theta)^{\mathrm{sym}}$ of $\bigwedge^\beta \theta$. Following [26], we have$(\bigwedge^\cdot \theta)^{\mathrm{sym}} = \bigwedge^\cdot (\theta^{\mathrm{sym}})$, and θ^{sym} is R^{sym}-freely generated by the gradients of the $n+k$ basic S_k-invariant functions. Thus, for the bigrading of $(\bigwedge^\cdot \theta)^{\mathrm{sym}}$ by the weight and the degree of the form, we get

$$P(\textstyle\bigwedge^\cdot \theta^{\mathrm{sym}}; t, v) = P(R^{\mathrm{sym}}; t) \cdot D(t, v)$$

where $D(t, v)$ is the Poincaré series of the exterior algebra on the basic equivariant vector fields,

$$D(t, v) = \prod_{i\geq 1} (1 + vt^{-w_i}) \cdot \prod_{l=1}^{k} (1 + vt^{(l-2)w_0}).$$

Now R^{sym} is a quotient of the ring of S_k-invariant polynomials on $\mathbb{C}^{n+k}$ by the ideal generated by $\varphi_1, \dots, \varphi_s$. So,

$$P(R^{\mathrm{sym}}; t) = \prod_{i\geq 1} (1 - t^{w_i})^{-1} \cdot \prod_{l=1}^{k} (1 - t^{lw_0})^{-1} \cdot \prod_{j\geq 1} (1 - t^{d_j}).$$

The proposition is now proved by repeating the arguments from the proof of proposition 5.1. □

5.3. REMARK. The multiplicities of the eigenvalues of the quasihomogeneous monodromy on the alternating cohomology can be expressed in the same way as in the non-symmetric case (cf. [11]).

Hodge numbers of the stable image

In order to obtain the canonical mixed Hodge structure (MHS) on the image of a lower stable perturbation f of a quasihomogeneous corank 1 mapping $f_0: \mathbb{C}^{n+1} \to \mathbb{C}^{n+r}$, we begin with consideration of its k-point space $D^k = D^k(f)$ which is also a lower stable perturbation of the quasihomogeneous ICIS $D^k(f_0)$. Now $\bar{D}^k = D^k(\bar{f})$ (see the beginning of this section) is a compactification of the smooth complete intersection D^k. Set $D^k_\infty = D^k(f_\infty) = \bar{D}^k \backslash D^k$. Then $(\bar{D}^k, D^k_\infty)$ is a pair of V-manifolds, i.e. it is locally a quotient of an action of a finite subgroup of $\mathrm{GL}(n; \mathbb{C})$ on the pair $(\mathbb{C}^{\rho_k}, \mathbb{C}^{\rho_k - 1})$, where $\rho_k = \dim D^k$ (here $\mathbb{C}^{\rho_k - 1}$ is mapped into itself by every element of the subgroup). For this fact see [2, subsection 3.1]. As $\bar{D}^k$ is a V-manifold, its singular set Σ has codimension $\geqslant 2$. Let $j: \bar{D}^k \backslash \Sigma \to \bar{D}^k$ be the inclusion map. Following [27] we define

$$\tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty) = j_* \Omega^{\cdot}_{\bar{D}^k \backslash \Sigma}(\log(D^k_\infty \backslash \Sigma))$$

As in [28, §10], this is a resolution of $\mathbb{C}_{D^k}$ and, thus,

$$H^*(D^k, \mathbb{C}) = \mathbb{H}^*(\bar{D}^k, \tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty)).$$

In order to obtain the canonical mixed Hodge structure (MHS) on this cohomology we define the Hodge and weight filtrations on the logarithmic sheaves as follows:

$$\text{Hodge (decreasing): } F^s \tilde{\Omega}^p_{\bar{D}^k}(\log D^k_\infty) = \begin{cases} \tilde{\Omega}^p_{\bar{D}^k}(\log D^k_\infty) & \text{if } p \geqslant s \\ 0 & \text{if } p < s \end{cases}$$

$$\text{weight (increasing): } W_s \tilde{\Omega}^p_{\bar{D}^k}(\log D^k_\infty) = \tilde{\Omega}^s_{\bar{D}^k}(\log D^k_\infty) \wedge \tilde{\Omega}^{p-s}_{\bar{D}^k}$$

(the last term is $j_* \Omega^{p-s}_{\bar{D}^k \backslash \Sigma}$).

We have $\mathrm{Gr}^W_s \tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty) \neq 0$ only for $s = 0, 1$ [27, page 532], and in fact

$$\mathrm{Gr}^W_0 \tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty) = \tilde{\Omega}^{\cdot}_{\bar{D}^k}$$

$$\mathrm{Gr}^W_1 \tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty) \simeq i_* \tilde{\Omega}^{\cdot}_{D^k_\infty}[-1] \quad \text{(by the residue map).}$$

Here $i: D^k_\infty \hookrightarrow \bar{D}^k$ and $j': D^k_\infty \backslash \Sigma' \hookrightarrow D^k_\infty$ are inclusions (where Σ' is the singular locus of D^k_∞) and $\tilde{\Omega}^{\cdot}_{D^k_\infty} = j'_* \Omega^{\cdot}_{D^k_\infty - \Sigma'}$.

Define

$$F^s H^p(D^k; \mathbb{C}) = \text{image of } \mathbb{H}^p(\bar{D}^k; F^s \tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty)) \text{ in } H^p(D^k; \mathbb{C})$$

$$W_s H^p(D^k; \mathbb{C}) = \text{image of } \mathbb{H}^p(\bar{D}^k; W_{s-p} \tilde{\Omega}^{\cdot}_{\bar{D}^k}(\log D^k_\infty)) \text{ in } H^p(D^k; \mathbb{C}).$$

As in [28, Theorem 10.3] one can see that W is already defined over $\mathbb{Q}$ and W and F define a mixed Hodge structure on $H^*(D^k)$.

We find

$$\mathrm{Gr}_\rho^W H^\rho(D^k; \mathbb{C}) = P^\rho(\bar{D}^k; \mathbb{C}) \quad \text{(the primitive part of } H^\rho(\bar{D}^k; \mathbb{C})),$$

$$\mathrm{Gr}_{\rho+1}^W H^\rho(D^k; \mathbb{C}) \simeq P^{\rho-1}(D_\infty^k; \mathbb{C})$$

(here $\rho = \rho_k$). The non-primitive parts of $H^*(\bar{D}^k; \mathbb{C})$ and $H^*(D_\infty^k; \mathbb{C})$ which come from the cohomology of the weighted projective spaces are cancelled by the W-spectral sequence, which degenerates at E_2.

Now we go to the stable image Y of f.

For each $k = 0, 1, \ldots,$ consider the sheaf complex

$$K^{k,\cdot} = \mathrm{Alt}_{k+1}\bar{\varepsilon}_*^{k+1}\tilde{\Omega}^{\cdot}_{\bar{D}^{k+1}}(\log D_\infty^{k+1}) \text{ on } \bar{Y}.$$

The mappings $\varepsilon^{i,k+1}\colon \bar{D}^{k+1} \to \bar{D}^k$, $i = 1, \ldots, k+1$, induce operators

$$\delta^{k-1} = \sum_{i=1}^{k+1} (-1)^{i+k}(\varepsilon^{i,k+1})^*\colon K^{k-1,\cdot} \to K^{k,\cdot}.$$

For each k, the complex $K^{k,\cdot}$ is a resolution of $\mathrm{Alt}_{k+1}\varepsilon_*^{k+1}(\mathbb{C}_{D^{k+1}})$. The complex $\{\mathrm{Alt}_{k+1}\varepsilon_*^{k+1}(\mathbb{C}_{D^{k+1}}), \delta^k\}_k$ is a resolution of $\mathbb{C}_Y$, by Section 2. Thus, the double complex $K^{\cdot\cdot}$ also resolves $\mathbb{C}_Y$. Let $K^\cdot$ be the associated total complex: $K^m = \bigoplus_{p+q=m} K^{p,q}$. Its hypercohomology is the cohomology of Y.

Following Steenbrink [28, §13] define filtrations

$$F^s K^m = \bigoplus_{p+q=m} F^s K^{p,q}, \quad W_s K^m = \bigoplus_{p+q=m} W^{s+p} K^{p,q}.$$

In the same way as for D^k, these filtrations give rise to Hodge and weight filtrations on $H^*(Y)$.

For our situation we get:

$$\mathrm{Gr}_s^W K^m = \mathrm{Gr}_0^W K^{-s,m+s} \oplus \mathrm{Gr}_1^W K^{1-s,m+s-1} \quad \text{for } s \leqslant 0,$$

$$\mathrm{Gr}_1^W K^m = \mathrm{Gr}_1^W K^{0,m}, \quad \mathrm{Gr}_{>1}^W = 0.$$

Thus, with Y the image of the mapping $f\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ we obtain, on $H^{n+1}(Y)$,

$$\mathrm{Gr}_{n+1}^W \simeq H^n_{\mathrm{Alt}_2}(D_\infty^2; \mathbb{Q}),$$

$$\mathrm{Gr}_{n+1-k}^W \simeq H^{n-k+1}_{\mathrm{Alt}_{k+1}}(D^{k+1}; \mathbb{Q}) \oplus H^{n-k}_{\mathrm{Alt}_{k+2}}(D_\infty^{k+2}; \mathbb{Q}),\ k = 1, \ldots, n+1$$

(note that $D_\infty^{n+2} = D_\infty^{n+3} = \emptyset$)

(here we use the whole cohomology, not only the primitive part, as alternation kills the non-primitive forms).

Adding the Hodge filtration we get

$$h^{p,q} = h^{p,q;\text{alt}}(D^{n+2-(p+q)}(f)) + h^{p,q;\text{alt}}(D^{n+3-(p+q)}(f)).$$

For a mapping $\mathbb{C}^{n+1} \to \mathbb{C}^{n+r}$, with $r > 2$, all of the groups except $H^{n+k-(r-1)(k-1)}(Y; \mathbb{C})$, for $2 \leqslant k \leqslant (n+r)/(r-1)$, are trivial by Proposition 2.6, and the MHS on each of these exceptional groups coinsides with the MHS on $H^{n+k-r(k-1);\text{alt}}(D^k(f); \mathbb{C})$.

We now calculate the numbers $h^{p,q}$ in the case of a map f_0: $\mathbb{C}^{n+1} \to \mathbb{C}^{n+r}$, $(x_1, \ldots, x_n, y) \to (x_1, \ldots, x_n, f_{0,1}(x, y), \ldots, f_{0,r}(x, y))$, where each $f_{0,j}$ is quasi-homogeneous of type $(w_1, \ldots, w_n, w_0; d_j)$, in terms of the weights. The compactification $\bar{D}^k = \bar{D}^k(\bar{f})$ of the k-point space $D^k = D^k(f)$ is a quasismooth complete intersection in $(n+k)$-dimensional weighted projective space $\mathbb{P}(w_1, \ldots, w_n, w_0, \ldots, w_0, 1)$. Now $D^k_\infty = D^k(f_\infty) = \bar{D}^k \backslash D^k$ lies in the hyperplane on which the weight 1 coordinate vanishes; $\bar{D}^k$ and D^k_∞ are defined by equations of weights $d_j - lw_0, j = 1, \ldots, r, l = 1, \ldots, k-1$, invariant under the permutations of the weight w_0 coordinates (see Section 4). Let us denote by ρ_k the dimension $n + k - r(k-1)$ of $D^k(f)$ and introduce the series

$$Q_k(t, u) = \frac{1}{1+u} \prod_{i \geqslant 1} \frac{1 + ut^{-w_i}}{1 - t^{w_i}} \prod_{l=1}^{k} \frac{1 + ut^{(l-2)w_0}}{1 - t^{lw_0}}$$
$$\cdot \prod_{\substack{j=1,\ldots,r \\ l=1,\ldots,k-1}} \frac{1 - t^{d_j - lw_0}}{1 + ut^{-d_j + lw_0}} t^{k(1 + \frac{1}{2}r(k-1))w_0 + \Sigma_{i \geqslant 1} w_i - (k-1)\Sigma d_j}$$

The preceding discussion, together with Proposition 5.2, now yields

5.4. PROPOSITION. *The non-zero Hodge numbers of the unique non-trivial alternating cohomology of* $D^k(f)$ *are given by the following formulae:*

$$h^{\rho_k - q, q; \text{alt}}(D^k(f)) = (-1)^q \operatorname{res}_{t=0} \operatorname{res}_{u=0} u^{-q-1} \frac{1 + ut^{-1}}{1 - t} Q_k(t, u), \quad q = 0, \ldots, \rho_k$$

$$h^{\rho_k + 1 - q, q; \text{alt}}(D^k(f)) = (-1)^{q-1} \operatorname{res}_{t=0} \operatorname{res}_{u=0} t^{-1} u^{-q} Q_k(t, u) \quad q = 1, \ldots, \rho_k.$$

On the right in these formulae are the numbers $h^{\rho_k - q, q; \text{alt}}_{[0]}$ *and* $h^{\rho_k - q, q-1; \text{alt}}_{[0]}$ *for (the primitive parts of)* $H^{\rho_k}(\bar{D}^k(f))$ *and* $H^{\rho_k - 1}(D^k(f_\infty))$ *respectively.* □

5.5. EXAMPLES. The following table shows the Hodge numbers $h^{p,q}$ for the stable images of the simple singularities of mappings $\mathbb{C}^2 \to \mathbb{C}^3$ and of the first non-simple one [22] (all of them are quasihomogeneous of corank 1):

	$h^{1,1}$	$h^{1,0}=h^{0,1}$	$h^{0,0}$
S_{2k+1}, B_{2k+1}, C_{2k+1}	1	k	0
S_{2k}, B_{2k}, C_{2k}	0	k	0
F_4	0	2	0
H_k	1	0	$k-1$
P_4	1	1	1

Note that $h^{0,0}$ is always the number of triple points of the stable image.

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