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Automorphisms of P_8 Singularities and the Complex Crystallographic Groups

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Abstract—The paper completes the study of symmetries of parabolic function singularities with relation to complex crystallographic groups that was started by the first co-author and his collaborator. We classify smoothable automorphisms of P_8 singularities which split the kernel of the intersection form on the second homology. For such automorphisms, the monodromy groups acting on the duals to the eigenspaces with degenerate intersection form are then identified as some of complex affine reflection groups tabled by V.L. Popov.

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Singularity theory has been maintaining close relations with reflection groups since its early days, starting with the famous works of Arnold and Brieskorn [1, 8]. The first to emerge, as simple function singularities, were the Weyl groups A_k, D_k and E_k. They were followed by the B_k, C_k and F_4 as simple boundary singularities or, equivalently, functions invariant under the involution [2, 22]. Since then appearance of Weyl groups in a singularity problem became a kind of criterion of naturalness of the problem [3–5].

The next to receive a singularity realisation was the list of all Coxeter groups, in the classification of stable Lagrangian maps [9]. Some of the Shephard–Todd groups appeared in [10, 11, 13] in the context of simple functions equivariant with respect to a cyclic group action. And finally, the first examples of complex crystallographic groups came out in [14, 12] in connection with the symmetries of parabolic functions J_{10} and X_9. The affine reflection groups appeared there as monodromy groups, which is similar to the first realisations of other classes of reflection groups. This time it was the equivariant monodromy corresponding to the symmetry eigenspaces H_χ in the vanishing second homology on which the intersection form σ has corank 1.

This paper completes the study of cyclically equivariant parabolic functions started in [14, 12] and considers the P_8 singularities. The approach we are developing here is considerably shorter, with less calculations and more self-contained. This is allowed by a preliminary observation that the modulus parameter may take on only exceptional values if $\operatorname{corank}(\sigma|_{H_\chi}) = 1$: the j-invariant of the underlying elliptic curves must be either 0 or 1728 (see Subsection 2.2).

The structure of the paper is as follows.

Section 1 describes the crystallographic groups which will be involved.

Section 2 gives a classification of smoothable cyclic symmetries of P_8 singularities possessing eigenspaces H_χ with the property as already mentioned.

In Section 3, we obtain distinguished sets of generators in such eigenspaces and calculate the intersection numbers of the elements in the sets. This allows us to describe the Picard–Lefschetz operators generating, as complex reflections, the equivariant monodromy action on the H_χ.

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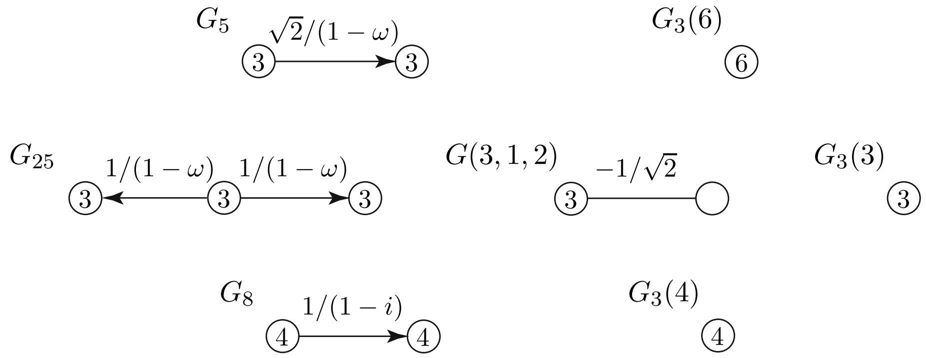

Fig. 1. Linear parts of the crystallographic groups. Each vertex represents a unit root. The order of the corresponding reflection is written inside the vertex (order 2 omitted). An edge $a \to b$ is equipped with the hermitian product $\langle a, b\rangle$ (no orientation is needed when the product is real). There are no edges between orthogonal roots.

Finally, in Section 4, we pass to the vanishing cohomology. In its subspaces dual to the H_χ, we consider hyperplanes formed by all the cocycles taking the same non-zero value on a fixed generator of the kernel of the hermitian intersection form $\sigma|_{H_\chi}$. We show that the equivariant monodromy group acting on each of these hyperplanes is a complex crystallographic group.

1. THE CRYSTALLOGRAPHIC GROUPS

We recall the description of the complex affine reflection groups which will be relevant to our singularity constructions. In the notations of [20], the groups are

$$[K_3(3)], \quad [K_3(4)], \quad [K_3(6)], \quad [G(3,1,2)]_2, \quad [K_5], \quad [K_8], \quad [K_{25}].$$

The linear parts of these groups are Shephard–Todd groups L shown in Fig. 1, in the related notation of [21]. There and in what follows $\omega = \exp(2\pi i/3)$. The rank of L is the number of vertices in its diagram. Each of the groups involved is a semi-direct product of its linear part and translation lattice. The lattice of $[G(3,1,2)]_2$ is spanned over $\mathbb{Z}$ by the $G(3,1,2)$-orbit of any order 2 root, while for any other group it is spanned by the L-orbit of any root. For example, our list contains all three one-dimensional groups, the $[K_3(m)]$, each generated by an order m rotation in $\mathbb{C}$ and the lattice $\mathbb{Z}[1,i]$ if $m = 4$ or $\mathbb{Z}[1,\omega]$ if $m = 3, 6$.

2. THE SYMMETRIES

2.1. Cubic curves. Our first step in the study of linear automorphisms of the surfaces

$$x^3 + y^3 + z^3 + 3\alpha xyz = 0, \qquad \alpha^3 \neq -1,$$

in $\mathbb{C}^3$ is consideration of the projective version of the question. This is a description of automorphisms of projective curves C_α given by the same equations in $\mathbb{C}\mathrm{P}^2$. The question is classical, but, for the sake of the exposition, we indicate a possible elementary approach.

A curve C_α has nine inflection points. There are four triplets of lines in $\mathbb{C}\mathrm{P}^2$ passing through these points, each line containing exactly three inflections and each triplet containing all nine inflections. The lines are zero sets of the linear forms

$$\begin{array}{lll}
x, & y, & z; \\
x+y+z, & x+\omega y+\overline{\omega}z, & x+\overline{\omega}y+\omega z; \\
x+y+\omega z, & x+\omega y+z, & \omega x+y+z; \\
x+y+\overline{\omega}z, & x+\overline{\omega}y+z, & \overline{\omega}x+y+z.
\end{array}$$

Table 1. Automorphisms of projective cubic curves

No.	f	$\kappa_x : \kappa_y : \kappa_z$	Monomial basis of the local ring of f
1	$x^3+y^3+yz^2$	$\omega : 1 : -1$	$1,\ x,\ y,\ z,\ xy,\ xz,\ y^2 \sim z^2,\ xy^2 \sim xz^2$
2	$x^3+y^3+yz^2 \simeq x^3+y^3+z^3$	$\omega : 1 : 1$	$1,\ x,\ y,\ z,\ xy,\ xz,\ y^2 \sim z^2,\ xy^2 \simeq xyz$
3	$x^2y+y^2z+z^2x$	$1 : \omega : \overline{\omega}$	$1,\ x,\ y,\ z,\ x^2 \sim yz,\ y^2 \sim xz,\ z^2 \sim xy,\ xyz$
4	$x^2z+xy^2+z^3$	$1 : i : -1$	$1,\ x,\ y,\ z,\ yz,\ x^2 \sim z^2,\ xz \sim y^2,\ x^3$
5	$x^3+y^3+yz^2+3\alpha xy^2,\ \ \alpha^3 \neq -\frac{1}{4}$	$1 : 1 : -1$	$1,\ x,\ y,\ z,\ xy,\ xz,\ y^2 \sim z^2,\ xy^2 \sim xz^2$
6	$x^3+y^3+z^3+3\alpha xyz,\ \ \alpha^3 \neq -1$	$1 : \omega : \overline{\omega}$	$1,\ x,\ y,\ z,\ xy,\ xz,\ yz,\ xyz$
7	$x^3+y^3+z^3+3\alpha xyz,\ \ \alpha^3 \neq -1$	$1 : 1 : 1$	$1,\ x,\ y,\ z,\ xy,\ xz,\ yz,\ xyz$

A projective automorphism of C_α permutes the triplets and reorders the lines within the triplets. It is sufficient to consider the images of the first triplet only. A routine study of all the possibilities, followed by diagonalising each projective transformation and rewriting the equation of an appropriate curve in the projective eigencoordinates, gives us

Proposition 2.1. *Any projective automorphism of a cubic curve can be reduced, by a choice of projective coordinates, either to the diagonal projective transformation* $(x : y : z) \mapsto (\kappa_x x : \kappa_y y : \kappa_z z)$ *of one of the cubics* $f = 0$ *from Table* 1 *or to the inverse of such a transformation.*

The contents of the last column of Table 1 will be used in Subsection 2.3.

2.2. Kernel character constraints. We now lift a projective automorphism of a cubic curve $f = 0$ to a linear transformation g of $\mathbb{C}^3$. It multiplies f by a non-zero constant and hence acts on the target $\mathbb{C}$ of the function f. We consider deformations of the function-germ $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ as usual: choosing a small ball B in $\mathbb{C}^3$ centred at the origin and deforming f within the ball. However, we restrict our attention to deformations which are g-equivariant with respect to the actions of g on $\mathbb{C}^3$ and $\mathbb{C}$.

Definition 2.2. If the function-germ f admits g-equivariant deformations with non-singular zero levels, we say that the symmetry g of f is *smoothable*.

In this case, the smooth zero level $V \subset B$ of a small g-equivariant deformation of f is a Milnor fibre of f, and g acts on V and on its second homology. Assuming from now on the order r of g finite, we obtain the splitting into a direct sum of character subspaces

$$H_2(V, \mathbb{C}) = \bigoplus_{\chi^r = 1} H_{2,\chi}, \tag{1}$$

so that g acts on each summand as the multiplication by the relevant rth root χ of unity.

We extend the intersection form σ from the lattice $H_2(V, \mathbb{Z})$ to the hermitian form σ_h on $H_2(V, \mathbb{C})$. The symmetric intersection form of a parabolic function singularity in three variables is negative semi-definite, with a two-dimensional kernel.

Definition 2.3. We say that a smoothable symmetry g *splits* the kernel of σ_h if this kernel is shared by two different character subspaces $H_{2,\chi}$. If the restriction of σ_h to a $H_{2,\chi}$ is degenerate, we call χ a *kernel character*.

The aim of our construction, a complex crystallographic group, will be acting on a hyperplane Γ in the cohomology subspace $H^2_\chi = \{\gamma \in H^2(V, \mathbb{C})\colon g^*(\gamma) = \chi\gamma\}$ where χ is a kernel character (cf. [14, 12]). For this, our symmetry g should split the kernel of σ_h, and the hyperplane Γ is defined as the set of all elements of H^2_χ taking the same non-zero value on a fixed generator of the line $\mathrm{Ker}(\sigma_h) \cap H_{2,\chi}$.

Our next step is to show that there is no kernel splitting in any of the modular cases of Proposition 2.1.

Assume g is a lifting of one of the projective automorphisms from Table 1. Take a monomial basis $\varphi_0 = 1, \varphi_1, \dots, \varphi_7$ of the local ring of f, with φ_7 of degree 3. Consider the sub-unfolding of the $\mathcal{R}$-miniversal unfolding of f in which the modulus parameter does not participate:

$$\mathcal{F}\colon \mathbb{C}^3 \times \mathbb{C}^6 \to \mathbb{C} \times \mathbb{C}^6, \qquad (x, y, z, \lambda_1, \dots, \lambda_6) \mapsto \bigl(f + \lambda_1\varphi_1 + \ldots + \lambda_6\varphi_6, \lambda_1, \dots, \lambda_6\bigr).$$

This map is defined globally. Its smooth fibres are Milnor fibres of f. The action of g naturally extends to the source and target of the map, making $\mathcal{F}$ g-equivariant. We will denote the target of $\mathcal{F}$ by $\mathcal{U}$, and points there will be $u = (u_0, \dots, u_6)$.

We simultaneously compactify all fibres V_u, $u \in \mathcal{U}$, of $\mathcal{F}$ adding to each of them a copy $C_u = \overline{V}_u \setminus V_u$ of the same projective curve $f = 0$. Each $\overline{V}_u \subset \mathbb{C}\mathrm{P}^3$ is smooth near C_u.

For non-critical values of u, take the intersection homomorphism $\sigma\colon H_2(V_u; \mathbb{Z}) \to H^2(V_u; \mathbb{Z})$. The Leray map sends $H_1(C_u; \mathbb{Z})$ isomorphically onto the two-dimensional kernel of σ [17].

Consider the 2-form $w = dx \wedge dy \wedge dz \wedge d\lambda_1 \wedge \ldots \wedge d\lambda_6 / du_0 \wedge \ldots \wedge du_6$ on the fibres V_u. According to [17], its residue β along C_u is a non-zero holomorphic 1-form which does not depend on u.

The action of the automorphism g on a g-symmetric fibre V_u extends to $\overline{V}_u$. The residue map $\mathrm{Res}\colon H^2(V_u; \mathbb{C}) \to H^1(C_u; \mathbb{C})$ is g-equivariant: if g^* multiplies a 2-form by χ, then it multiplies the residue of the form by χ too. Therefore, if g splits $\mathrm{Ker}(\sigma_h)$ on V_u, then its action on $H^1(C_u; \mathbb{C})$ must also have two distinct conjugate eigenvalues, with β and $\overline{\beta}$ as eigenvectors. Hence the curve $C_u = \{f = 0\}$ must be either $\mathbb{C}/\mathbb{Z}[1, i]$ or $\mathbb{C}/\mathbb{Z}[1, \omega]$. Respectively, the kernel characters are either $\pm i$ or $(\omega, \overline{\omega})$ or $(-\omega, -\overline{\omega})$.

Thus we have

Proposition 2.4. *A smoothable automorphism g splits the kernel of the intersection form if and only if the section $w = dx \wedge dy \wedge dz/df$ of the vanishing cohomology fibration of f is a g^*-eigenvector with the eigenvalue from the set $\{\pm i, \pm\omega, \pm\overline{\omega}\}$. The eigenvalue and its conjugate are the kernel characters.*

If $g^*(w) = \chi w$, then for the hyperplane Γ mentioned above one can take $\mathrm{Res}^{-1}(\mathrm{Res}(w)) \cap H^2_\chi$.

Since the degree of f is 3, the eigenvalue of g^* on $w = dx \wedge dy \wedge dz/df$ depends only on the projectivisation of g.

Corollary 2.5. *All smoothable liftings of the first four automorphisms from Proposition* 2.1 *split* $\mathrm{Ker}(\sigma_h)$. *On the other hand, none of the liftings of the last three cases does the same.*

Remark. The argument can be easily modified to the symmetries of X_9 and J_{10} function singularities, thus explaining the absence of the moduli in the classification tables in [14, 12].

2.3. Smoothable symmetries. Assume we have two actions of a finite cyclic group, on $(\mathbb{C}^k, 0)$ and on $(\mathbb{C}, 0)$. Consider two function-germs $f_1, f_2\colon (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ equivariant with respect to these actions: $f_i \circ \rho = \rho \circ f_i$ where ρ is a generator of the group. We say that the two functions are *$\mathcal{R}_\rho$-equivalent* if one can be transformed into the other by a ρ-equivariant diffeomorphism-germ of $(\mathbb{C}^k, 0)$.

We apply this notion in the context of our cubic function f, its *diagonal* symmetry g and the induced action of g on $\mathbb{C}$. With a minor abuse of the language, we will still be calling the corresponding equivalence the $\mathcal{R}_g$-equivalence. For example, an $\mathcal{R}_g$-miniversal unfolding of f is the restriction F of the unfolding $\mathcal{F}$ of the previous subsection to $\mathcal{F}^{-1}(U)$, where $U \subset \mathcal{U}$ is the set of fixed points of the natural action of g on $\mathcal{U}$. Equivalently, in this case, for an $\mathcal{R}_g$-miniversal deformation of f one can take $f +$ arbitrary linear combinations of the elements $\psi_1, \dots, \psi_\tau$ of a monomial basis

Table 2. Smoothable symmetries of P_8 singularities splitting $\operatorname{Ker}(\sigma)$

f	$g\colon x,y,z\mapsto$	$\lvert g\rvert$	Versal monomials	Kernel χ	Affine group	Notation
$x^3+y^3+yz^2$	$\omega x, y, -z$	6	$1, y, y^2$	$-\omega, -\overline{\omega}$	$[K_5]$	$C_3^{(3,3)}$
	$x, \overline{\omega}y, -\overline{\omega}z$	6	$1, x$	$-\omega, -\overline{\omega}$	$[K_3(6)]$	$(P_8\vert\mathbb{Z}_6)'$
	$\overline{\omega}x, \omega y, -\omega z$	6	$1, xy$	$-\omega, -\overline{\omega}$	$[K_3(6)]$	$(P_8\vert\mathbb{Z}_6)''$
	$-x, -\overline{\omega}y, \overline{\omega}z$	6	x	$-\omega, -\overline{\omega}$	–	$(P_8/\mathbb{Z}_6)'$
	$i\omega x, iy, -iz$	12	z	$-\omega, -\overline{\omega}$	–	$P_8/\mathbb{Z}_{12}$
	$\omega x, y, z$	3	$1, y, y^2, z$	$\omega, \overline{\omega}$	$[K_{25}]$	$D_4^{(3)}$
	$x, \overline{\omega}y, \overline{\omega}z$	3	$1, x$	$\omega, \overline{\omega}$	$[K_3(6)]$	$(P_8\vert\mathbb{Z}_3)'$
	$\overline{\omega}x, \omega y, \omega z$	3	$1, xy, xz$	$\omega, \overline{\omega}$	$[G(3,1,2)]_2$	$P_8\vert\mathbb{Z}_3$
$x^3+y^3+z^3$	$-x, -\overline{\omega}y, -\overline{\omega}z$	6	x	$\omega, \overline{\omega}$	–	$(P_8/\mathbb{Z}_6)''$
	$-\omega x, -y, -z$	6	y, z	$\omega, \overline{\omega}$	$[K_3(3)]$	$P_8/\mathbb{Z}_6$
$x^2y+y^2z+z^2x$	$\overline{\varepsilon}_9x, \omega\overline{\varepsilon}_9y, \overline{\omega\varepsilon}_9z$	9	1	$\omega, \overline{\omega}$	–	$P_8\vert\mathbb{Z}_9$
$x^2z+xy^2+z^3$	$-x, -iy, z$	4	$1, z, z^2$	$\pm i$	$[K_8]$	$C_3^{(2,4)}$
	$-\omega x,\ -i\omega y,\ \omega z$	12	1	$\pm i$	–	$P_8\vert\mathbb{Z}_{12}$
	$ix,\ -y,\ -iz$	4	x, yz	$\pm i$	$[K_3(4)]$	$P_8/\mathbb{Z}_4$
	$\overline{\varepsilon}_8x, \varepsilon_8y, -\overline{\varepsilon}_8z$	8	y	$\pm i$	–	$P_8/\mathbb{Z}_8$

of the local ring of f that are multiplied by g by the same factor as f. The number τ appearing here will be called the $\mathcal{R}_g$*-codimension* of f.

Proposition 2.6. *Assume a symmetry g is smoothable. Then g multiplies f by the same factor as it multiplies one of the monomials* 1, x, y, *or* z.

Indeed, the conclusion is equivalent to generic fibres of an $\mathcal{R}_g$-miniversal unfolding not being singular at $0\in\mathbb{C}^3$.

With the help of Corollary 2.5 and the last proposition, we obtain after straightforward calculations which we prefer to omit

Theorem 2.7. *The complete list of smoothable symmetries of P_8 function singularities on $\mathbb{C}^3$ which split the kernel of the intersection form is given in Table* 2. *Our classification is up to a choice of generators of the same cyclic group.*

In Table 2, the versal monomials are the monomials participating in $\mathcal{R}_g$-miniversal deformations (these are selected from the last column of Table 1 as those multiplied by g^* by the same factor as f), the affine groups are those which will come out later as monodromy groups acting on the hyperplanes in the character subspaces in the cohomology, and $\varepsilon_k=\exp(2\pi i/k)$.

Remark. During the proof of the theorem, one obtains a few non-smoothable symmetries allowing linear terms in deformations. They are listed in Table 3. It is possible to relate complex crystallographic groups to its two singularities with two-parameter miniversal deformations, but this will be done in another paper.

Of course, one may consider a bit more general problem of finding all possible smoothable symmetry *groups* G of parabolic singularities, which are not necessarily cyclic. In this situation the

Table 3. Non-smoothable symmetries allowing linear terms

f	$g\colon x, y, z \mapsto$	$\lvert g\rvert$	Versal monomials
$x^3 + y^3 + yz^2$	$-\omega x, -y, z$	6	y
$x^2y + y^2z + z^2x$	$\omega x, \overline{\omega} y, z$	3	x, xz
	$-\omega x, -\overline{\omega} y, -z$	6	x
$x^2z + xy^2 + z^3$	$x, iy, -z$	4	z, xz
	$-ix, y, iz$	4	x

second homology splits into irreducible representations of G, and we may be still looking for cases when the kernel of the intersection form is shared by two of them. For P_8, for example, this means that G should contain one of the symmetries g of Table 3, and hence the affine reflection group related to G will be a lower rank subgroup of the one we are relating to g. This does not leave too much dimensional room for further interesting crystallographic groups.

3. DYNKIN DIAGRAMS

The equivariant monodromy of a g-equivariant function singularity f, that is, the one within an $\mathcal{R}_g$-miniversal deformation of f, preserves the direct sum decomposition (1). Its action on an individual summand $H_{2,\chi}$ is generated by the Picard–Lefschetz operators h_e corresponding to vanishing χ-cycles e:

$$h_e(c) = c - (1 - \lambda_e)\langle c, e\rangle e/\langle e, e\rangle.$$

Here $\lambda_e \neq 1$ is the eigenvalue of h_e: $h_e = \lambda_e e$.

In this section we obtain all the necessary information describing the monodromy on the kernel character subspaces of the singularities of Table 2. We choose distinguished sets of generators of the subspaces $H_{2,\chi}$ (cf. [7, 15, 3, 4]), calculate the intersection numbers of the elements of the sets, and find the eigenvalues λ_e. The data will be collected into Dynkin diagrams of the singularities.

Proposition 3.1. *If χ is a kernel character of a symmetric singularity from Table* 2, *then the rank of H^2_χ* (*equivalently, the rank of $H_{2,\chi}$*) *coincides with the $\mathcal{R}_g$-codimension τ of the singularity.*

Proof. If f is g-invariant, then a basis of one of the two H^2_χ is formed, in the notations of Subsection 2.2, by the sections $\varphi_i w$, where the φ_i are all g-invariant monomials within $\{\varphi_0, \dots, \varphi_7\}$. Their number is exactly τ. (Cf. [19, 23].)

If f is g-equivariant rather than invariant, the claim can be verified by a direct case-by-case computation of the eigenvalues of g^* on the sections $\varphi_0 w, \dots, \varphi_7 w$. The observation needs case-free understanding. $\square$

Thus, if $\tau = 1$, then the subspaces $H_{2,\chi}$ in the case of kernel χ are one-dimensional, hence contained in $\mathrm{Ker}(\sigma_h)$, and therefore the monodromy we are interested in is trivial. So, from now on we are considering only the cases of at least two-parameter $\mathcal{R}_g$-miniversal deformations.

We will also forget about the $(P_8|\mathbb{Z}_3)'$ singularity. Indeed, its symmetry g is the inverse of the square of the $(P_8|\mathbb{Z}_6)'$ symmetry, and the two equivariant miniversal deformations coincide. Hence the character subspaces with degenerate intersection form are the same (only the actual character assignments differ: $\chi_3 = \chi_6^{-2}$) and the monodromy is the same.

For each remaining symmetry, we will choose a distinguished set of generators for $H_{2,\chi}$ in the way it has been done in [10, 11, 13]. Namely, let u_* be a point in the complement $\mathbb{C}^\tau \setminus \Delta$ to the discriminant in the base of the $\mathcal{R}_g$-miniversal deformation of our function. Set V_{u_*}/g to be the quotient of the fibre V_{u_*} under the cyclic g-action and $(V_{u_*}/g)'$ its part corresponding to irregular orbits. Denote by π the factorisation map $V_{u_*} \to V_{u_*}/g$. The homology group $H_2(V_{u_*}/g, (V_{u_*}/g)'; \mathbb{Z})$ is spanned by a distinguished set of relative vanishing cycles (such a set is defined in the traditional singularity theory manner). Denote them $c_1, \dots, c_k$. The inverse image $\pi^{-1}(c_j)$ is the cyclic orbit of one of its components, let it be $\widetilde{c}_j$. For a character χ, the chain

$$e_j = \sum_{i=0}^{\operatorname{order}(g)-1} \chi^{-i} g^i(\widetilde{c}_j)$$

is an element of $H_{2,\chi}$. The *vanishing χ-cycles* $e_1, \dots, e_k$ span $H_{2,\chi} = H_{2,\chi}(V_{u_*}, \mathbb{C})$, not necessarily as a basis. They are defined up to multiplication by powers of χ and the sign change.

Examples. (a) The function $x^n + y^2 + z^2$ invariant under the transformation $g\colon (x, y, z) \mapsto (\varepsilon_n x, y, z)$ was denoted $A_1^{(n)}$ in [10]. The factorisation by the group action gives the boundary singularity $A_1\colon w + y^2 + z^2$, $w = x^n$, whose relative vanishing homology is spanned by one semi-cycle $\{w + y^2 + z^2 = 1\colon w, y, z \in \mathbb{R},\ w \geq 0\}$. The corresponding vanishing χ-cycle, $\chi^n = 1$, $\chi \neq 1$, is formed by n hemi-spheres with appropriate coefficients and has self-intersection $-n$ [10], which is consistent with the standard Morse vanishing cycle having self-intersection -2. The Picard–Lefschetz operator h is the classical monodromy of the ordinary A_{n-1} singularity. Its eigenvalue on $H_{2,\chi}$ is $\lambda = \chi$, since the quasi-homogeneous isotopy in the family $x^n + y^2 + z^2 = e^{it}$, $t \in [0, 2\pi]$, finishes with the transformation $(x, y, z) \mapsto (\varepsilon_n x, -y, -z)$ whose action on the homology coincides with that of g.

(b) The A_2-version of the previous singularity is $A_2^{(n)}\colon x^n + y^3 + z^2$, with the same symmetry. For it, each of the $H_{2,\chi}$ is spanned by two similarly defined χ-cycles which may be chosen so that their intersection number is $n/(1 - \chi)$. The two Picard–Lefschetz operators h_j satisfy the standard braiding relation $aba = bab$. Diagrammatically, the $A_2^{(n)}$ singularity is represented by fusing the rectangular $2 \times (n-1)$ Dynkin diagram to the A_2 diagram which may be equipped with the markings indicating the orders of the vertices and the intersection numbers. Cf. the $P_8 \to D_4^{(3)}$ part of Fig. 3.

(c) The singularity $A_m/\mathbb{Z}_m$ is the function $x^{m+1} + yz$ with the equivariant symmetry $g\colon (x, y, z) \mapsto (\varepsilon_m x, \varepsilon_m y, z)$. As shown in [11], its χ-cycle has self-intersection $-m$, χ being any mth root of unity. The quasi-homogeneous argument applied to the $\mathcal{R}_g$-miniversal deformation $x^{m+1} + yz + \alpha x$ demonstrates that the Picard–Lefschetz operator on $H_{2,\chi}$ is the multiplication by χ.

Theorem 3.2. *For the symmetric P_8 singularities and their kernel characters χ, there exist distinguished sets of vanishing χ-cycles described by Dynkin diagrams of Fig.* 2.

Proof. We proceed on the case-by-case basis. We are considering only kernel characters of the P_8 singularities.

$D_4^{(3)}$. Factorisation by the action of $\mathbb{Z}_3$ provides the boundary D_4 singularity. Hence, according to the above examples, a Dynkin diagram of $D_4^{(3)}$ can be obtained by the modification of the standard D_4 diagram: the roots should have squares -3 instead of -2, the Picard–Lefschetz operators h_j become of order 3, and the intersection numbers 1 of pairs of cycles change to $3/(1 - \chi)$. Since $\chi = \omega, \overline{\omega}$, the latter may be reduced to $1 - \omega$ (for both values of χ) using the ambiguities in the choice of the χ-cycles. The diagram may be obtained by the fusion of the cylindrical P_8 diagram as shown in Fig. 3.

$C_3^{(3,3)}$. We extend the previous $\mathbb{Z}_3$-symmetry by $\mathbb{Z}_2$ acting by the sign change on z. This embeds the character subspaces $H_{2,-\omega}$ and $H_{2,-\overline{\omega}}$ of the symmetric singularity $C_3^{(3,3)}$ into $H_{2,\omega}$ and $H_{2,\overline{\omega}}$ of $D_4^{(3)}$ as subspaces anti-invariant under the involution. This is absolutely similar to the relation

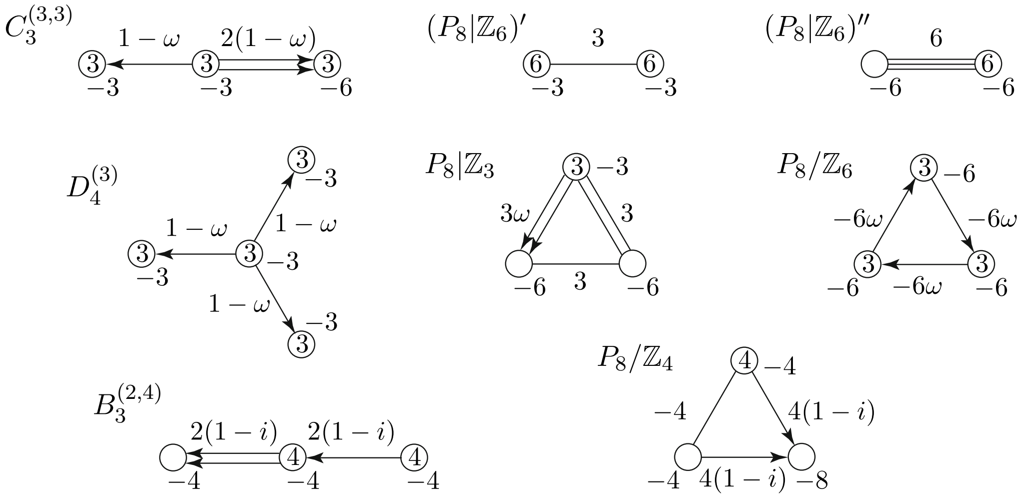

Fig. 2. Dynkin diagrams of the symmetric P_8 singularities. The numbers inside vertices are the orders of the Picard–Lefschetz operators (order 2 omitted). The number next to a vertex is the self-intersection of the χ-cycle. As earlier, edges are marked with the intersection numbers of the cycles. In the diagrams of all *invariant* singularities, the multiplicity of an edge indicates the length of the pair-wise braiding relation inherited from the fundamental group of the complement to the discriminant: $aba = bab$ for a simple edge, $(ab)^2 = (ba)^2$ for a double, and $(ab)^3 = (ba)^3$ for a triple.

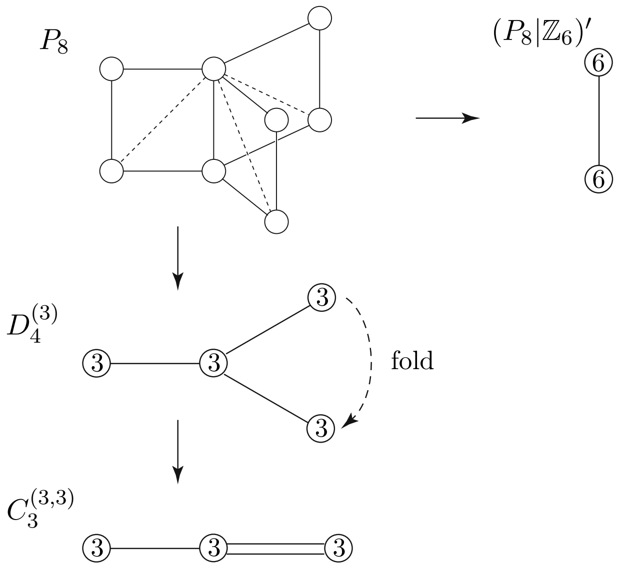

Fig. 3. Obtaining some of the diagrams by fusion and folding.

between the $\mathbb{Z}_2$-symmetric singularity C_3 and the absolute D_4 singularity (cf. [2–4]). Hence the $C_3^{(3,3)}$ diagram of Fig. 2 is provided by the folding of the $D_4^{(3)}$ shown in Fig. 3.

$(P_8|\mathbb{Z}_6)'$. The discriminant in this case is a semi-cubic cusp, which gives the relation $aba = bab$ between the two Picard–Lefschetz operators. A generic point of the discriminant corresponds to the $D_4|\mathbb{Z}_6$ singularity considered in [11], where the self-intersection number of its vanishing χ-cycle was shown to be -3 for $\chi = -\omega, -\overline{\omega}$. Since we are considering the kernel characters, the intersection form on the $H_{2,\chi}$ must be degenerate, which implies that the absolute value of the intersection number of the two χ-cycles is 3. Since this number is in $\mathbb{Z}[1,\omega]$, we can make it 3 using the χ-cycle choice ambiguities once again.

Finally, the Picard–Lefschetz operator $h_{D_4|\mathbb{Z}_6}$ has order 6 as the classical monodromy operator of the absolute D_4 singularity. Its non-trivial eigenvalue on $H_{2,\chi}$ is $\lambda = \overline{\chi}$ since the quasi-homogeneous

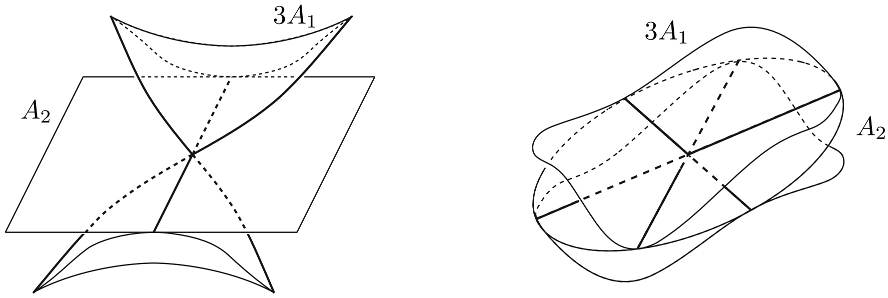

Fig. 4. Two real versions of the $P_8|\mathbb{Z}_3$ discriminant, for $x^3+y^3+yz^2$ and $x^3+y^3-yz^2$.

isotopy $x^2+y^3+yz^2=e^{it}$, $t\in[0,2\pi]$, finishes with the transformation $(x,y,z)\mapsto(-x,\omega y,\omega z)$, which coincides with g^{-1} on the local homology.

Thus we have arrived at the $(P_8|\mathbb{Z}_6)'$ diagram of Fig. 2. Due to the generating reflections coming from the classical monodromy of the D_4 singularities, it is natural to consider it as obtained by the fusion of Fig. 3.

$(P_8|\mathbb{Z}_6)''$. The discriminant of the versal family $x^3+y^3+yz^2+\beta xy+\alpha$ consists of two strata:

$$A_5=\{\alpha=0\}\qquad\text{and}\qquad 3A_1=\{27\alpha+\beta^3=0\}.$$

The cubic tangency of the strata gives the braiding relation between the generators. Of course, the self-intersection of the $3A_1$ vanishing χ-cycle is -6 and the order of the reflection h_{3A_1} is 2. As for the A_5 χ-cycle, it is possible to homotope it to the standard $A_1^{(6)}$ vanishing χ-cycle, with the self-intersection -6 according to Example (a). Now the argument similar to that for the previous singularity gives the intersection number 6 of the two cycles. Quasi-homogeneous considerations of the A_5 local vanishing form $xy+z^6=\varepsilon$ shows that the non-trivial eigenvalue of its monodromy operator on $H_{2,\chi}$ is $\overline{\chi}$ since its local action coincides with g^{-1}. Hence the operator h_{A_5} is of order 6.

$P_8|\mathbb{Z}_3$. The discriminants of two real versions $x^3+y^3\pm yz^2+\alpha+\beta xy+\gamma xz$ of a versal family are the unions of the A_2 stratum $\alpha=0$ and $3A_1$ stratum $(54\alpha+\beta^3\pm 9\beta\gamma^2)^2=(\beta^2\mp 3\gamma^2)^3$. They are shown in Fig. 4. The two strata are simply tangent to each other along their meeting lines.

As in the previous case, it is possible to homotope the A_2 vanishing χ-cycle to the standard $A_1^{(3)}$ vanishing χ-cycle, with the self-intersection -3. An operator h_{A_2} has the eigenvalue χ. The self-intersection of a $3A_1$ χ-cycle is -6, and its monodromy reflection is an involution.

Assume the base point u_* is chosen inside the front lips region of the left diagram of Fig. 4. A generic line through it is vertical. Take on it a path system from u_*: two paths going straight to the $3A_1$ points and the third nearly straight to the A_2 point, bypassing the upper $3A_1$ point on its way (the side is not important). Then the vanishing χ-cycles may be chosen so that

$$\langle e_{3A_1,\mathrm{lower}},e_{3A_1,\mathrm{upper}}\rangle=3\qquad\text{and}\qquad\langle e_{A_2},e_{3A_1,\mathrm{upper}}\rangle=3.$$

Indeed, the absolute values of these intersection numbers must be 3 following the braiding relations between the two pairs of the Picard–Lefschetz operators coming from the singularities of the discriminants. Moreover, both numbers are in $\mathbb{Z}[1,\omega]$ and the χ-cycles may be multiplied by -1 and powers of ω.

Now we see that the last intersection number, $\langle e_{A_2},e_{3A_1,\mathrm{lower}}\rangle$, is either 3ω or $3\overline{\omega}$. This is guaranteed by the degeneracy of the intersection form on $H_{2,\chi}$ and the number being in $\mathbb{Z}[1,\omega]$. However, both options turn out to be equivalent up to a braid transformation.

$C_3^{(2,4)}$. The discriminant coincides with the standard C_3 discriminant. Its smooth stratum is $2A_1$ and the cuspidal is A_3. The latter is nearly $A_1^{(4)}$ of Example (a), but with the order 4 symmetry changing the sign of one of the squared variables; hence the self-intersection of a vanishing χ-cycle e_{A_3} is -4, while its operator h_{A_3} has eigenvalue $-\overline{\chi} = \chi$. Additionally, $\langle e_{2A_1}, e_{2A_1}\rangle$ is also -4 and h_{2A_1} is an involution.

A distinguished set of vanishing χ-cycles consists of $e_{2A_1,1}, e_{2A_1,2}, e_{2A_1}$. The singularities of the discriminant guarantee that the set may be chosen so that the braiding relations between the Picard–Lefschetz operators are exactly those encoded in the diagram of Fig. 2. These imply

$$\langle e_{2A_1,1}, e_{2A_1}\rangle = 0 \qquad \text{and} \qquad |\langle e_{2A_1,1}, e_{2A_1,2}\rangle| = |\langle e_{2A_1,2}, e_{2A_1}\rangle| = 2\sqrt{2}.$$

Since the intersection numbers are in $\mathbb{Z}[1,i]$, we can make $\langle e_{2A_1,1}, e_{2A_1,2}\rangle = \langle e_{2A_1,2}, e_{2A_1}\rangle = 2(1-i)$ by possible multiplication of the cycles by powers of i.

Remark. We prefer calling this singularity C rather than B since it has only one vanishing cycle coming from more than one critical point on one level (cf. [2]).

$P_8/\mathbb{Z}_6$. The discriminant of the versal family $x^3 + y^3 + z^3 + \alpha y + \beta z$ consists of three lines corresponding to the three divisors $\alpha y + \beta z$ of the cubic form $y^3 + z^3$. Each of the lines is a $2A_1^{(3)}$ stratum: each $A_1^{(3)}$ singularity is off the origin and has symmetry $g^2\colon (x,y,z) \mapsto (\overline{\omega}x, y, z)$. Hence all vanishing χ-cycles have self-intersection -6. Any Picard–Lefschetz operator has eigenvalue $\lambda = \overline{\chi}^2 = \chi$.

A distinguished set of χ-cycles consists of three elements. Since the rank of the intersection form on $H_{2,\chi}$ is 1, the absolute value of the intersection number of any pair is 6. Since the number itself is in $\mathbb{Z}[1,\omega]$, we can use the ambiguities in the choice of the cycles and make $\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = -6\omega$. The rank condition implies that $\langle e_3, e_1\rangle$ is also -6ω.

Remark. It is possible to show that the relation between the cycles may be assumed to be $e_1 + e_2 + e_3 = 0$.

$P_8/\mathbb{Z}_4$. The discriminant of the versal family $x^2z + xy^2 + z^3 + \alpha x + \beta yz$ has three strata:

$$A_4\colon\ \alpha = 0, \qquad 2A_2\colon\ \beta = 0, \qquad 4A_1\colon\ 4\alpha + \beta^2 = 0.$$

The A_4 degeneration reduces to the $A_4/\mathbb{Z}_4$ singularity of Example (c). Hence the self-intersection of e_{A_4} is -4. By the quasi-homogeneity, we see that the operator h_{A_4} coincides locally with g and thus has eigenvalue χ on $H_{2,\chi}$. Similarly, each of the two A_2 degenerations is the g^2-symmetric singularity $A_2/\mathbb{Z}_2$. Therefore, the self-intersection of e_{2A_2} is also -4, while h_{2A_2} is an involution. It is clear that a χ-cycle vanishing at $4A_1$ has self-intersection -8, and h_{4A_1} is also an involution.

A distinguished set of vanishing χ-cycles contains one cycle of each of the three kinds. Since the rank of the $H_{2,\chi}$ is 2, the 3×3 intersection matrix must have rank 1. Due to this, after possible multiplication of the cycles by powers of i, we can assume that

$$\langle e_{4A_1}, e_{A_4}\rangle = \langle e_{4A_1}, e_{2A_2}\rangle = 4(1+i).$$

This forces $\langle e_{A_4}, e_{2A_2}\rangle = -4$.

Remark. The relation between the cycles may be assumed to be either $e_{A_4} + e_{4A_1} + ie_{2A_2} = 0$ or $ie_{A_4} + e_{4A_1} + e_{2A_2} = 0$. $\square$

4. AFFINE MONODROMY

We start by recalling how a corank 1 semi-definite hermitian form $\widetilde{q}$ on $\widetilde{V} = \mathbb{C}^{n+1}$ defines an affine reflection group on a hyperplane in the dual space [6, 14, 12].

First of all, choose a basis $e_0', e_1, \dots, e_n$ in $\widetilde{V}$ so that e_0' is in the kernel K of the form. We denote the span of the $e_{j>0}$ by V and write v for the V-component of $\widetilde{v} \in \widetilde{V}$: $\widetilde{v} = v_0 e_0' + v$. Set Q to be the matrix of the restriction $q = \widetilde{q}|_V$: $Q = (\widetilde{q}(e_i, e_j))_{i,j>0}$.

In the dual space $\widetilde{V}^* \simeq K^* \oplus V$, we use coordinates $\alpha_0, \alpha_1, \dots, \alpha_n$ so that a linear functional $\widetilde{\alpha}$ on $\widetilde{V}$ is written as

$$\widetilde{\alpha}(\widetilde{v}) = v_0\alpha_0 + v^{\mathrm{T}}Q\alpha = v_0\alpha_0 + q(v, \overline{\alpha}).$$

Consider a reflection on $\widetilde{V}$ with a root $\widetilde{u} \notin K$ and the eigenvalue λ:

$$A\colon\ \widetilde{v} \mapsto \widetilde{v} - (1-\lambda)\frac{\widetilde{q}(\widetilde{v}, \widetilde{u})\widetilde{u}}{\widetilde{q}(\widetilde{u}, \widetilde{u})}.$$

Then its dual A^* sends each hyperplane $\alpha_0 = \text{const}$ in $\widetilde{V}^*$ into itself, and on such a hyperplane it acts as

$$\alpha \mapsto \alpha - (1-\overline{\lambda})\frac{\alpha_0 u_0 + \overline{q}(\alpha, \overline{u})}{\overline{q}(\overline{u}, \overline{u})}\overline{u},$$

where $\overline{q}$ is the hermitian form on V conjugate to q, with the matrix $\overline{Q} = Q^{\mathrm{T}}$ in the chosen coordinates. For $\alpha_0 \neq 0$, this is an affine reflection on the hyperplane $\Gamma = \{\alpha_0 = \text{const}\} \simeq V$, with the root $\overline{u}$, mirror $\widetilde{a}(\widetilde{u}) = \alpha_0 u_0 + \overline{q}(\alpha, \overline{u}) = 0$ and eigenvalue $\overline{\lambda}$. For $u_0 = 0$, the transformation is linear.

We are now ready to prove our main result.

Theorem 4.1. *Assume a symmetry of a function singularity P_8 on $\mathbb{C}^3$ splits the kernel of the intersection form σ_h and χ is a kernel character. Assume the cohomology character subspace H^2_χ is at least of rank 2. Let $\Gamma \subset H^2_\chi$ be the hyperplane formed by all 2-cocycles taking a fixed non-zero value on a fixed generator of the line $H_{2,\chi} \cap \operatorname{Ker}(\sigma_h)$. Then the equivariant monodromy acts on Γ as a complex crystallographic group. The correspondence between the symmetric singularities and affine groups is given by Table 2.*

Proof. Number the vanishing χ-cycles of each diagram in Fig. 2 from left to right (the numbering of the last two vertices in $D_4^{(3)}$ is not important). Omit the leftmost cycle e_0. In equivariant cases omit also e_2. It is easy to notice that the remaining cycles may be multiplied by appropriate factors so that the encoded hermitian form becomes the negative of the relevant form of Fig. 1 while the orders of the related vertices coincide. Therefore, taking the remaining vanishing cycles as the basic vectors $e_{j>0}$ in $\widetilde{V} = H_{2,\chi}$ in the introduction to this section, we see that their Picard–Lefschetz operators h_j generate the required Shephard–Todd group L on $\Gamma \subset H^2_\chi = \widetilde{V}^*$.

The next task is to obtain the translation lattices of the crystallographic groups. The kernel of any intersection form in Fig. 2 is spanned by the $e_0' = e_0 + \mathbf{a}$, where $\mathbf{a}$ is a linear combination of the $e_{j>0}$. For example, in all $\tau = 2$ cases, $\mathbf{a}$ is a non-zero multiple of e_1 (for equivariant singularities this is due to the two remarks by the end of the previous section).

Assume the order of a root of L which generates the lattice of the required affine group coincides with the order of an operator h_k, $k > 0$. According to what has been said before the theorem, its validity for such a singularity will follow from $\mathbf{a}$ being a multiple of an element of the L-orbit of e_k. Hence only $\tau > 2$ singularities are still to be checked. And we have for them

$$\begin{aligned}
C_3^{(3,3)}\colon &\quad \mathbf{a} = (1-\overline{\omega}, 1) = A_1A_2^2e_1,\\
D_4^{(3)}\colon &\quad \mathbf{a} = (1-\overline{\omega}, 1, 1) = A_1A_3^2A_1e_2,\\
P_8|\mathbb{Z}_3\colon &\quad \mathbf{a} = (\overline{\omega}-\omega, -\omega) = \overline{\omega}A_2A_1^2e_2,\\
C_3^{(2,4)}\colon &\quad \mathbf{a} = (1+i, i) = iA_1^2e_2.
\end{aligned}$$

To make the calculations suitable for any of the two kernel characters, the A_j here are either the Picard–Lefschetz operators or their inverses, but always with the eigenvalues either ω or -1 or i.

The expressions obtained show that the vectors $\mathbf{a}$ are maximal roots of the groups L in the sense of [16]. $\square$

Remark. The multiplicities of vertices and edges in Fig. 2 provide nearly complete presentations of all our rank >1 crystallographic groups as abstract groups. To obtain all defining relations, one should add

(i) $(h_1h_0h_2h_0)^2 = (h_0h_2h_0h_1)^2$ to the $P_8|\mathbb{Z}_3$ diagram of $[G(3,1,2)]_2$ (cf. [18]), which corresponds to one of the tangency lines in Fig. 4, right;

(ii) the condition that the classical monodromy (that is, the product of all the generators) of each singularity has order 3. This is exactly the extra relation from [18].

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