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BRIEF COMMUNICATIONS

Simple Functions on Space Curves

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We classify simple singularities of functions on space curves. We show that their bifurcation sets have the same properties as those of functions on smooth manifolds and complete intersections [3, 4]: the $k(\pi, 1)$-theorem for the bifurcation diagram of functions is true, and both this diagram and the discriminant are Saito's free divisors.

1. Space curves and functions on them. A germ of any reduced space curve is a determinantal variety: it is the zero set of all order n minors of a germ of some $n \times (n+1)$-matrix M on $\mathbb{C}^3$. We treat a space curve as such a matrix.

Definition 1.1. Two curve-germs at the origin, given by matrices M and M', are said to be *equivalent* if there exist two invertible matrix-germs A and B and a biholomorphism-germ h of $(\mathbb{C}^3, 0)$ such that $AMB = M' \circ h$.

Simple space curves were recently classified in [8].

We say that the *corank* of a space curve-germ is c if the rank at the origin of its defining matrix is $n - c$. Such a curve can be given by a $c \times (c+1)$-matrix that is just the zero matrix at the origin.

Definition 1.2. *A function on a space curve* is a pair of germs (M, f) on $(\mathbb{C}^3, 0)$, where M is an $n \times (n+1)$-matrix and f a function.

Definition 1.3. Two germs of functions on curves at the origin, (M, f) and (M', f'), are $\mathscr{R}_c$*-equivalent* if there exist two invertible matrix-germs A and B, a biholomorphism-germ h of $(\mathbb{C}^3, 0)$ and a function g belonging to the ideal generated by the maximal minors of M such that

$$(AMB, f + g) = (M' \circ h, f' \circ h).$$

The notion of $\mathscr{R}_c$-equivalence satisfies all the conditions of Damon's good geometrical equivalence [6]. Thus all the standard theorems like those of versality and finite determinacy are valid in our case.

2. Simple singularities.

Theorem 2.1. *A simple function on a space curve is a function on either a plane curve or on a corank 2 curve. Accordingly, the complete list of simple functions consists of two parts given in Table* 1.

The first half of the list was obtained in [7]. The meaning of the last column of the table is explained in Sect. 4.

All the adjacencies of the listed functions on plane curves are compositions of adjacencies within the series obtained by reducing the indices (we additionally set $B_2 = C_{1,1}$ and $F_3 = B_3$) and the adjacencies

$$F_{p+q+1} \to C_{p,q} \to A_{p+q-1}.$$

Likewise, one obtains obvious adjacencies of the singularities within the second half of the table by reducing the indices in the series. Some other adjacencies showing the relations between the series are as follows:

$$C_{p,q,r} \to C_{p,q}, \quad \dot{F}_k \to F_{k-2}, \quad \dot{F}_{p+q+3} \to C_{p,q,1}, \quad \check{E}_6 \to \dot{F}_5.$$

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Table 1

notation	curve equations	function	restrictions	index
A_k	y	x^{k+1}	$k \geqslant 0$	$(k+1)^{k-1}$
$C_{p,q}$	xy	$x^p + y^q$	$p \geqslant q \geqslant 1$	$\dfrac{(p+q-1)!\,p^p q^q}{(p-1)!\,(q-1)!}$
B_k	$x^2 + y^k$	y	$k \geqslant 3$	1
F_k	$x^2 + y^3$	$\begin{cases} y^r \\ xy^r \end{cases}$	$k = 2r+1 \geqslant 5$ $k = 2r+4 \geqslant 4$	$\dfrac{(k-2)(k-1)^k k}{24}$
$C_{p,q,r}$	$\begin{vmatrix} x & y & 0 \\ 0 & y & z \end{vmatrix}$	$x^p + y^q + z^r$	$p \geqslant q \geqslant r \geqslant 1$	$\dfrac{(p+q+r+1)!\,p^p q^q r^r}{(p-1)!\,(q-1)!\,(r-1)!}$
$\dot{F}_k$	$\begin{vmatrix} x & y & 0 \\ y^2 & x & z \end{vmatrix}$	$z + \begin{cases} y^r \\ xy^r \end{cases}$	$k = 2r+3 \geqslant 5$ $k = 2r+6 \geqslant 6$	$\dfrac{(k-3)^k(k-2)(k-1)k}{24}$
$\check{E}_6$	$\begin{vmatrix} x & y & z \\ z^2 & x & y \end{vmatrix}$	z	—	3^5

Any nonsimple function is either a function on a corank > 2 curve or is adjacent to one of the following two nonsimple functions on plane curves:

$$X_9^*: \quad x + \alpha y^2 \quad \text{on } x^2 + y^4 = 0.$$

$$J_{10}^*: \quad x + \alpha y \quad \text{on } x^3 + y^3 = 0.$$

Here α is a generic complex number (modulus). Both singularities are adjacent to B_4 and F_4. Additionally, X_9^* is adjacent to $C_{3,1}$.

3. Bifurcation varieties as free divisors. Let (x_1, x_2, x_3) be coordinates in $(\mathbb{C}^3, 0)$ and $\mathscr{O}_3$ the space of holomorphic function-germs on it. We identify the space of germ-pairs consisting of an $n \times (n+1)$-matrix and a function with the module $\mathscr{O}_3^{n(n+1)+1}$. The tangent space $T(M, f)$ to the (extended) $\mathscr{R}_c$-equivalence class of a germ (M, f) in this module is the $\mathscr{O}_3$-submodule generated by the elements

$$\begin{aligned} &(E_{ij}^n M, 0), && i, j = 1, \dots, n, \\ &(M E_{kl}^{n+1}, 0), && k, l = 1, \dots, n+1, \\ &(0, \varphi_r), && r = 1, \dots, n+1, \\ &(\partial M/\partial x_s, \partial f/\partial x_s), && s = 1, 2, 3. \end{aligned}$$

Here E_{ij}^n is the $n \times n$-matrix with unit entry at the intersection of the ith row with the jth column and with zero entries in all other positions.

We set $\tau(M, f) = \dim_{\mathbb{C}} \mathscr{O}_3^{n(n+1)+1}/T(M, f)$ and call τ the *Tyurina number* of the function singularity. This is the dimension of the base of an $\mathscr{R}_c$-miniversal deformation of (M, f). Such a deformation can be taken in the form $(M, f) + \lambda_1 e_1 + \dots + \lambda_\tau e_\tau$, where the λ_i are parameters and the e_i are elements of $\mathscr{O}_3^{n(n+1)+1}$ that are projected to a linear basis of the quotient $\mathscr{O}_3^{n(n+1)+1}/T(M, f)$.

For the singularities $C_{p,q}$ and $C_{p,q,r}$, we have $\tau = p + q$ and $\tau = p + q + r + 1$, respectively. For all other simple singularities in our list, τ is a subscript in the notation. For X_9^* and J_{10}^*, τ is 6.

Now let $\mu(M, f)$ be the number of Morse critical points that a generic small perturbation of f has on a generic smoothing of M. We refer to $\mu(M, f)$ as the *Milnor number* of the singularity.

Conjecture 3.1. $\tau(M,f)=\mu(M,f)$.

The conjecture is true for functions on complete intersections [7], for $\mathscr{R}_c$-simple singularities, and in some other special cases.

Definition 3.2. Consider the base $\mathbb{C}^\tau$ of an $\mathscr{R}_c$-miniversal deformation of a singularity (M,f). The *discriminant* $\Delta(M,f)\subset\mathbb{C}^\tau$ of (M,f) is the set of those values of the deformation parameters for which the function on the curve has the critical value 0. The *bifurcation diagram of functions* $\Sigma(M,f)\subset\mathbb{C}^\tau$ is the set of those values of the parameters for which the function on the curve has fewer than $\mu(M,f)$ distinct critical values.

Note that a singular point of a curve must be treated as a critical point for a function on the curve.

Definition 3.3. A hypersurface H in $\mathbb{C}^N$ is called a *free divisor* if the algebra Θ_H of vector fields on $\mathbb{C}^N$ tangent to H (that is, preserving its ideal) is generated by N elements as a module over functions on $\mathbb{C}^N$.

Theorem 3.4. *Suppose that $\tau(M,f)=\mu(M,f)$. Then both the discriminant $\Delta(M,f)\subset\mathbb{C}^\tau$ and the bifurcation diagram of functions $\Sigma(M,f)\subset\mathbb{C}^\tau$ are free divisors.*

Systems of generators of Θ_Δ and Θ_Σ can be constructed as follows (cf. [10, 5, 4]). Let $(\mathscr{M},F)=(\mathscr{M}(x,\lambda),F(x,\lambda))$ be an $\mathscr{R}_c$-miniversal deformation of (M,f) with parameters $\lambda=(\lambda_1,\dots,\lambda_\tau)\in\mathbb{C}^\tau$. The versality implies that, for any $i=1,\dots,\tau$, there exist decompositions.

$$F\frac{\partial}{\partial\lambda_i}(\mathscr{M},F)=(\mathscr{A}_i\mathscr{M}\mathscr{B}_i,\mathscr{G}_i)+\sum_{s=1}^{3}h_{is}\frac{\partial}{\partial x_s}(\mathscr{M},F)+\sum_{j=1}^{\tau}v_{ij}\frac{\partial}{\partial\lambda_j}(\mathscr{M},F),$$

$$F^{i-1}\frac{\partial}{\partial\lambda_i}(\mathscr{M},F')=(\mathscr{A}'_i\mathscr{M}\mathscr{B}'_i,\mathscr{G}'_i)+\sum_{s=1}^{3}h'_{is}\frac{\partial}{\partial x_s}(\mathscr{M},F)+\sum_{j=1}^{\tau}w_{ij}\frac{\partial}{\partial\lambda_j}(\mathscr{M},F),$$

where $\mathscr{A}_i(x,\lambda)$ and $\mathscr{B}_i(x,\lambda)$ are matrix-germs, $h_{is}(x,\lambda)$ and $v_{ij}(\lambda)$ are function-germs, $\mathscr{G}_i(x,\lambda)$ is an element of the ideal generated by the maximal minors of $\mathscr{M}$ in the ring of functions in x and λ, and similar settings are done in the second formula. A function factor or differentiation in front of a pair (matrix, function) means the multiplication by the function or the differentiation of both members of the pair.

The vector fields

$$\nu_i=\sum_{j=1}^{\tau}v_{ij}(\lambda)\partial_{\lambda_j}\quad\text{and}\quad\omega_i=\sum_{j=1}^{\tau}w_{ij}(\lambda)\partial_{\lambda_j}$$

form bases of Θ_Δ and Θ_Σ, respectively.

4. The Lyashko–Looijenga mapping. Let $\mathbb{C}^\mu$ be the space of all monic polynomials in one variable of degree μ and $\Xi\subset\mathbb{C}^\mu$ the set of polynomials with multiple roots.

For a singularity (M,f) with Tyurina number τ and Milnor number μ, consider the mapping $\mathbb{C}^\tau\setminus\Sigma(M,f)\to\mathbb{C}^\mu\setminus\Xi$ of the complement of the bifurcation diagram of functions sending a Morse function on a smooth curve to the unordered set of its critical values, that is, to the monic polynomial whose roots are the critical values. This mapping is extendible to that between the ambient complex linear spaces. The extension will be called the *Lyashko–Looijenga mapping.*

Theorem 4.1. *For an $\mathscr{R}_c$-simple function on a space curve, the Lyashko–Looijenga mapping is a finite covering. As a mapping of $\mathbb{C}^\tau\setminus\Sigma(M,f)$ into $\mathbb{C}^\mu\setminus\Xi$, it has no branching.*

Corollary 4.2. *For an $\mathscr{R}_c$-simple singularity (M,f), the complement of its bifurcation diagram of functions in the base of its $\mathscr{R}_c$-miniversal deformation is a $k(\pi,1)$-space, where π is a subgroup of finite index in the braid group on $\mu(M,f)$ threads.*

The values of the index are given in the last column of the classification table. The theorem and the corollary are similar to the classical theorem on simple functions on smooth manifolds [1, 9, 2, 3] and generalize similar assertions about simple functions on plane curves [7, 4].

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