Simple framed curve singularities

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Abstract

We obtain a complete list of simple framed curve singularities in $\mathbb{C}^2$ and $\mathbb{C}^3$ up to the framed equivalence. We also find all the adjacencies between simple framed curves.

Natural equivalence of framed parametrised curves in 3-space was introduced in [5] as a tool to study Vassiliev type invariants of framed knots. That context required classification of framed curves up to codimension 3 only. This left the classification problem virtually untouched. In the present paper we fill out this gap and classify simple singularities of framed curves. We do this both in the planar and spatial settings. In particular, we show that simple singularities of planar framed curves are classified by the Weyl groups A_k and B_k. Simple framed space curves in a sense turn out to be suspensions of the planar.

The paper is organised as follows.

Section 1 recalls the definition of the framed equivalence of curves and contains the statement of our main results. In Section 2 we prove most of these results obtaining the classification of simple framed curve singularities. Section 3 describes the discriminants of all our simple singularities. In Section 4 all the adjacencies are obtained. In Section 5 we show that all the singularities of the spatial list are pairwise distinct (similar statement in the planar case is more than obvious). Finally, Section 6 classifies all simple multi-germs of planar framed curves, in the spirit of a more natural approach of [7] rather than [6].

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1 Framed curves

For convenience, we shall work in the holomorphic category. From our classification list, it will be easy to see that the list of real simple normal forms – which could have differed from its complex counterpart by the sign choice – is absolutely the same.

1.1 Framed equivalence

We follow the ribbon approach to framed curves in $\mathbb{C}^n$ as introduced in [5]. Namely, we consider such a curve as a map $f : U \to \mathbb{C}^n$ of an open neighbourhood $U \subset \mathbb{C}^2$ of a fixed line $\mathbb{C} \subset \mathbb{C}^2$. Mapping f defines the map $T_{\mathbb{C}}\mathbb{C}^2 \to T\mathbb{C}^n$ which we denote Tf. The curve $f : \mathbb{C} \to \mathbb{C}^n$ will be called the *core* of the mapping.

Definition 1.1. Two such parametrisations $f_1, f_2 : U \to \mathbb{C}^n$ are the *same* if $Tf_1 = Tf_2$.

Definition 1.2. Two framed curves $f_1, f_2 : U \to \mathbb{C}^n$ are called *$\mathcal{F}$-equivalent* if there exist biholomorphisms of the pair $(U, \mathbb{C})$ and of the target $\mathbb{C}^n$ which send the map Tf_1 to Tf_2.

This is just an infinitesimal version of the standard left-right equivalence of maps. We use $\mathcal{F}$ here for 'framed'.

The obvious local version of the $\mathcal{F}$-equivalence for map-germs $f : (\mathbb{C}^2, \mathbb{C}, 0) \to \mathbb{C}^n$ has the following coordinate description. Let t and ε be coordinates on the source with the $\mathbb{C}$ being the t-axis. We use the notations:

$\mathcal{O}_{t,\varepsilon}$ for the space of holomorphic function-germs on $(\mathbb{C}^2, 0)$;

$\mathcal{O}^n_{t,\varepsilon}$ for the space of holomorphic map-germs of $(\mathbb{C}^2, 0)$ to $\mathbb{C}^n$;

$\mathcal{O}^n_f$ for the space of holomorphic map-germs $(\mathbb{C}^n, 0) \to \mathbb{C}^n$ pulled back to $(\mathbb{C}^2, 0)$ by f.

Putting together the above two definitions, we obtain the extended tangent space to the $\mathcal{F}$-orbit of a map-germ $f \in \mathcal{O}^n_{t,\varepsilon}$:

$$T_f(\mathcal{F}f) = \mathcal{O}^n_f + \mathcal{O}_{t,\varepsilon} \langle \partial f/\partial t, \varepsilon \partial f/\partial \varepsilon \rangle + \varepsilon^2 \mathcal{O}^n_{t,\varepsilon} \,. \tag{1}$$

We will refer to the pseudogroup realising $\mathcal{F}$-equivalences of framed curve-germs as group $\mathcal{F}$. Its non-extended local version, that is, fixing the distinguished points, will be denoted $\mathcal{F}_0$.

The framed equivalence is a geometric equivalence in the sense of Damon [3]. Hence all the standard general theorems of singularity theory apply. For example, the finiteness of the *Tjurina number* (we also call it the *$\mathcal{F}$-codimension*)

$$\tau_{\mathcal{F}}(f) = \dim_{\mathbb{C}} \mathcal{O}^n_{t,\varepsilon} / T_f(\mathcal{F}f)$$

implies finite the $\mathcal{F}$-determinacy. Thus, there is no difference between the formal and holomorphic $\mathcal{F}$-classifications in finite codimension (in particular, a relevant version of Arnold's spectral sequence [1] works for the normal form reduction). Also, the quotient $\mathcal{O}^n_{t,\varepsilon}/T_f(\mathcal{F}f)$ can be viewed as the base of an $\mathcal{F}$-miniversal deformation of the framed curve.

In the set of all germs sending the origin to the origin, the codimension of the $\mathcal{F}_0$-orbit of an individual singularity is $\tau_{\mathcal{F}} + 1$ assuming the singularity is not observed at a generic point of a framed curve. Otherwise the codimension is of course 0.

In this paper we are considering the cases $n = 2, 3$. Our main results are the theorems of the next two sections.

1.2 Simple planar singularities

Theorem 1. *Any $\mathcal{F}$-simple complex framed planar curve-germ is $\mathcal{F}$-equivalent to exactly one germ from the two infinite simple series:*

$$A_k : (t, \varepsilon t^{k+1}) \,, \quad k \geq -1, \qquad \textit{and} \qquad B_k : (t^2, t^{2k+1} + \varepsilon) \,, \quad k \geq 1 \,.$$

In the set of all germs sending the origin to the origin, the codimension of the $\mathcal{F}_0$-orbit of each of these singularities is $k+1$, while the codimension of the set of all non-simple germs is 3.

The notation of the singularities is chosen so that their discriminants coincide with the discriminants of the relevant Weyl groups (see Section 3.1). The index in the notation is the $\mathcal{F}$-codimension of the singularity. The A_{-1} germ (t, ε) is that observed at a generic point of a framed curve, which gives a good reason to say that the codimension of this singularity is -1 (of course, formally $\tau_{\mathcal{F}}(A_{-1}) = 0$). The same applies to the spatial singularity $\widetilde{A}_{-1}$ below.

The A_0 singularity $(t, \varepsilon t)$ is the only stable framing degeneration. The number

$$a_0(f)$$

of A_0 points of a generic pertubation of an arbitrary planar framed curve-germ $f(t, \varepsilon) = (f_1(t, \varepsilon), f_2(t, \varepsilon)) = (f_{10}(t) + \varepsilon f_{11}(t), f_{20}(t) + \varepsilon f_{21}(t))$ is the order of the jacobian

$$\det(\partial f / \partial(t, \varepsilon))|_{\varepsilon=0} = \det \begin{pmatrix} f'_{10} & f_{11} \\ f'_{20} & f_{21} \end{pmatrix} .$$

For example, $a_0(A_k) = k + 1$ and $a_0(B_k) = 1$.

All the adjacencies of the listed singularities can be obtained by composing $A_k \to A_{k-1}$, $B_k \to B_{k-1}$ and $B_1 \to A_0$ (see Section 4.1).

1.3 Simple spatial singularities

Theorem 2. *The complete list of $\mathcal{F}$-equivalence classes of simple framed curve-germs in $\mathbb{C}^3$ consists of two infinite series and two exceptional singularities:*

notation	*normal form*	*range*	$\tau_{\mathcal{F}}$
$\widetilde{A}_k$	$(t, \varepsilon t^{k+1}, 0)$	$k \geq -1$	0 if $k = 0$ $2k + 1$ if $k > 0$
$\widetilde{B}_k^{\ell,n}$	$(t^2, t^{2k+1} + \varepsilon, \varepsilon(t^{2\ell+1} + t^{2n}))$	$k \geq 1$ $0 \leq \ell \leq k$ $0 \leq n \leq 2\ell + 1$	$2k + \ell + n$
$\widetilde{C}^\sigma$	$(t^2 + \sigma\varepsilon, t^3, \varepsilon t)$	$\sigma = 1, 0$	$5 - \sigma$

In the set of all germs sending the origin to the origin, the codimension of the $\mathcal{F}_0$-orbit of $\widetilde{A}_{-1}$ is 0 and $\tau_{\mathcal{F}} + 1$ for any other table singularity, while the codimension of the set of all non-simple germs is 6.

We should notice that in the extreme cases of the $\widetilde{B}$ series, when either $\ell = k$ or $n = 2\ell+1$, the terms $\varepsilon t^{2\ell+1}$ or respectively εt^{2n} may be dropped from the normal form making the singularities quasihomogeneous.

The hierarchy of the spatial singularities is described by

Theorem 3. *All the adjacencies of simple singularities of spatial framed curves are compositions of the following:*

$$\widetilde{A}_k \to \widetilde{A}_{k-1} , \qquad \widetilde{B}_k^{\ell,n} \to \left\{ \begin{array}{c} \widetilde{A}_0 \\ \widetilde{B}_k^{\ell-1,n} \\ \widetilde{B}_k^{\ell,n-1} \\ \widetilde{B}_{k-1}^{\ell,n+1} \end{array} \right. , \qquad \widetilde{C}^0 \to \widetilde{C}^1 \to \left\{ \begin{array}{c} \widetilde{A}_1 \\ \widetilde{B}_1^{0,1} \\ \widetilde{B}_1^{1,0} \end{array} \right. .$$

Here, the range of indices of series $\widetilde{A}$ and $\widetilde{B}$ singularities in the individual adjacencies is such that the less complicated singularities are still in the table of Theorem 2.

The latter Theorem will be proved in Section 4.2. It will help us in Section 5 in proving

Theorem 4. *All germs in the table of Theorem 2 are pairwise inequivalent.*

2 Proofs of the classificational theorems

The proofs will mainly consist of explicit coordinate changes and applications of Arnold's spectral sequence [1] modified for the problem under consideration.

2.1 Proof of Theorem 1

Let $f : (\mathbb{C}^2, \mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be a germ of a framed complex plane curve.

2.1.1 Smooth core: series A

If the core of f has a non-zero 1-jet, then f is equivalent to a germ of the form

$$f(t,\varepsilon) = (t + \varepsilon r(t), \varepsilon s(t)) , \qquad r, s \in \mathcal{O}_t .$$

The reparametrisation $t' = t + \varepsilon r(t)$ belongs to the group $\mathcal{F}$ since it preserves the core of the source. It reduces our curve to the form

$$f(t,\varepsilon) = (t, \varepsilon \hat{s}(t)) .$$

Assuming s is not identically zero, we can write $\hat{s}(t) = t^{k+1} q(t)$, where $q(0) \neq 0$, $k \geq -1$. Now the transformation $\varepsilon' = \varepsilon q(t)$ reduces our curve to

$$f(t,\varepsilon) = (t, \varepsilon t^{k+1}) .$$

This is the nomal form A_k of Theorem 1.

Calculation of the tangent space (1) demonstrates that the Tjurina number $\tau_{\mathcal{F}}$ of $(t, \varepsilon t^{k+1})$ is k, except for the case $k = -1$ when $\tau_{\mathcal{F}}$ is 0, and that

$$(t, \varepsilon(t^{k+1} + q_{k-1} t^{k-1} + \cdots + q_0)) \tag{2}$$

is an $\mathcal{F}$-miniversal deformation of the germ. Thus the germs are inequivalent for different values of k, and all of them are simple.

If in the above argument $s(t) \equiv 0$ then our germ is equivalent to $(t, 0)$ which is non-simple, since it is adjacent to all the A_k.

2.1.2 Singular core with non-trivial 2-jet: series B

We now suppose that the 1-jet of the core of f is zero, but the 2-jet is not. Then, from the classification of $\mathcal{A}$-simple complex plane curves (see [2]), we know that the core of f can be reduced to the form (t^2, t^{2k+1}). Thus we can assume

$$f(t,\varepsilon) = (t^2 + \varepsilon r(t), t^{2k+1} + \varepsilon s(t)). \tag{3}$$

Let us show that the $\mathcal{F}$-simplicity requires $s(0) \neq 0$.

Lemma 1. *The family of quasijets $(a_1 t^2 + a_2 \varepsilon, a_3 t^3 + a_4 \varepsilon t)$, $a_i \in \mathbb{C}$, has a modulus under the action of the group $\mathcal{F}$.*

Proof. Assignment of weights 1 and 2 to t and ε makes the above maps quasihomogenious of the multi-degree $(2,3)$. The left group acts on such quasijets by its 2-dimensional subgroup $\mathcal{L}_d$ of diagonal linear transformations $(f_1, f_2) \mapsto (\lambda_1 f_1, \lambda_2 f_2)$, $\lambda_i \in \mathbb{C}$. The right group preserving the core line $\mathbb{C} \in \mathbb{C}^2$ acts also on the 4-space of our quasijets by its 2-dimensional subgroup $\mathcal{R}_d$ of rescalings of t and ε: $(t,\varepsilon) \mapsto (\lambda_3 t, \lambda_4 \varepsilon)$, $\lambda_j \in \mathbb{C}$. However, the action of $\mathcal{L}_d \times \mathcal{R}_d$ has a 1-dimensional kernel due to the Euler relation. Therefore, the action has $4-(2+2)+1=1$ modulus. □

The Lemma implies that if a map (3) is $\mathcal{F}$-simple then we must have $s(0) \neq 0$ to avoid adjacency to germs with the principal parts as in the Lemma. We claim that in this case f is $\mathcal{F}$-equivalent to the B_k singularity $(t^2, t^{2k+1}+\varepsilon)$. For this, we first take $\varepsilon s(t)$ for new ε reducing our germ to the form

$$f = (t^2 + \varepsilon \hat{r}(t), t^{2k+1} + \varepsilon).$$

The principal quasihomogeneous part of f is $f_0 = (t^2, t^{2k+1}+\varepsilon)$ (setting weights of t and ε to 1 and $2k+1$). The standard calculations show that the quotient $\mathcal{O}^2_{t,\varepsilon}/T_{f_0}(\mathcal{F} f_0)$ is spanned over $\mathbb{C}$ by the elements $(0, t^{2j+1})$, $j = 0, \ldots, k-1$, all of which have filtration lower than f_0. Hence the term $\varepsilon \hat{r}(t)$ in f can be killed (cf. [1]) and thus $f \sim f_0$.

A by-product of our calculations is that the deformation

$$(t^2, t^{2k+1} + q_{k-1}t^{2k-1} + q_{k-2}t^{2k-3} \cdots + q_0 t + \varepsilon)\,, \quad q_i \in \mathbb{C}\,, \tag{4}$$

of B_k is $\mathcal{F}$-miniversal.

2.1.3 Core with trivial 2-jet: non-simplicity

We shall now show that a curve whose core has zero 2-jet cannot be $\mathcal{F}$-simple. Let $f(t,\varepsilon) = (p(t) + \varepsilon r(t), q(t) + \varepsilon s(t))$ be such a curve-germ. Up to obvious coordinate changes we can assume that s has no free term. Then f can be perturbed by an arbitrarily small perturbation to a germ with the principal part $(a_1 t^2 + a_2 \varepsilon, a_3 t^3 + a_4 t\varepsilon)$ as in Lemma 1. Since this family has a modulus, f cannot be simple.

The argument also demostrates that germs with the principal parts of Lemma 1 are the only fencing singularities in the classification. Since the singularities B_k are not adjacent to them and do not have any internal moduli, we conclude that the B_k are simple. □

2.2 Proof of Theorem 2

We will proceed as in the previous proof, except that we will rely more on the tangent space calcultations since explicit coordinate changes become more and more difficult to construct as the complexity of the germs increases. We shall denote coordinates in the target by x_1, x_2, x_3 when it is neccessary to refer to them.

2.2.1 Smooth core: the $\widetilde{A}$ series

If the core of f has a non-zero 1-jet then f is equivalent to a germ of the form

$$f(t,\varepsilon) = (t + \varepsilon p(t), \varepsilon q(t), \varepsilon r(t))\,.$$

We get rid of the ε term in the first coordinate by taking $t+\varepsilon p(t)$ for new t. After this we can completely kill one of the other two coordinate functions (the one of higher order in t) by an obvious action of the left group. By a reparametrisation of the form $\varepsilon' = \varepsilon s(t)$, we finally reduce f to the $\widetilde{A}_k$ normal form $(t, \varepsilon t^{k+1}, 0)$. Calculation of the tangent space (1) yields its $\mathcal{F}$-miniversal deformation

$$(t, \varepsilon(t^{k+1} + p_{k-1}t^{k-1} + \cdots + p_1 t + p_0), \varepsilon(q_k t^k + \ldots q_1 t + q_0)), \quad p_i, q_j \in \mathbb{C}. \qquad (5)$$

2.2.2 Singular core with non-trivial 2-jet: series $\widetilde{B}$

We assume now that the core of f has zero 1-jet, but its 2-jet is non-zero. Then it reduces to $(t^2, t^{2k+1}, 0)$, $k>0$ (see, for example, [4]). Hence our framed curve takes the form

$$f(t,\varepsilon) = (t^2 + \varepsilon p(t), t^{2k+1} + \varepsilon q(t), \varepsilon r(t)). \qquad (6)$$

General case: either q or r are invertible. Up to a transformation $x_2' = x_2 + x_3$ we can assume $q(0) \neq 0$. The proof of Theorem 1 ensures that under this assumption the map-germ given by the first two coordinate functions of f can be reduced to the normal form B_k by transformations not involving the coordinate x_3 in the target. This gives

$$f(t,\varepsilon) = (t^2, t^{2k+1} + \varepsilon, \varepsilon r(t)).$$

Addition to x_3 of $(x_2^2 - x_1^{2k+1})x_1^m$ with an appropriate coefficient kills $t^{2k+1+2m}$ in r (we are working $\mathrm{mod}\, \varepsilon^2$). Similarly $x_2(x_2^2 - x_1^{2k+1})x_1^m$ eliminates $t^{4k+2+2m}$. Hence we may assume that r contains at most $t^{odd<2k}$ and $t^{even\leq 4k}$. Let t^a be the lowest of these powers in r with non-zero coefficients. Normalising the coefficient to 1, we arrive at the principal part

$$f_0(t,\varepsilon) = (t^2, t^{2k+1} + \varepsilon, \varepsilon t^a).$$

Consider Arnold's spectral sequence for our framed curve f (cf. [1]). We have $f_0^*(x_3 x_1^m) = \varepsilon t^{a+2m}$. Therefore the differential d^0 of the spectral sequence kills all the monomials t^{a+2m}, $m>0$, in r:

$$d^0(x_3 x_1^m \partial_{x_3}) = (0, 0, \varepsilon t^{a+2m}).$$

Also

$$(\varepsilon t^{a+2m}\partial_\varepsilon - x_3 x_1^m \partial_{x_2}) f_0 = (0, 0, \varepsilon t^{2a+2m}).$$

Hence d^0 reduces r to $r(t) = t^a +$ higher powers of t of parity opposite to a and of orders either at most $2a-2$ if a is odd or at most $2k-1$ if a is even (these are easily checked not to be in the image of d^0). Let t^b be the lowest of these higher powers in r with non-zero coefficients. The Euler relation e for f_0 is in the kernel of d^0. The first non-trivial positive differential d^+ of the sequence applied to this relation gives a non-zero multiple of $(0,0,\varepsilon t^b)$. The same differential sends $f^*(x_1^m)\, e$ to a non-zero multiple of $(0,0,\varepsilon t^{b+2m})$. Hence the spectral sequence reduces f to

$$f(t,\varepsilon) = (t^2, t^{2k+1} + \varepsilon, \varepsilon(t^a + t^b)).$$

To save on a variety of conditions on a and b here we can sum them up to

$$a = 2\ell + 1, \quad 0 < \ell \leq k, \qquad \text{and} \qquad b = 2n, \quad 0 \leq n \leq 2\ell + 1.$$

Thus we have arrived at the $\widetilde{B}_k^{\ell,n}$ singularities. In fact in the extreme right cases, when either $a = 2k+1$ or $b = 4\ell+2$, the corresponding monomials can be omitted from the normal form – they have been killed by d_0.

Our spectral sequence calculations also show that for an $\mathcal{F}$-miniversal deformation of the $\widetilde{B}_k^{\ell,n}$ singularity we can take

$$\begin{array}{ll}(t^2, & \\ t^{2k+1}+\varepsilon & +p_{k-1}t^{2k-1}+p_{k-2}t^{2k-3}+\cdots+p_0t, \\ \varepsilon(t^{2\ell+1}+t^{2n}) & +q_{k-1}t^{2k-1}+q_{k-2}t^{2k-3}+\cdots+q_0t+ \\ & +\varepsilon(r_{\ell-1}t^{2\ell-1}+r_{\ell-2}t^{2\ell-3}+\cdots+r_0t)+ \\ & +\varepsilon(s_{n-1}t^{2n-2}+s_{n-2}t^{2n-4}+\cdots+s_0))\,. \end{array} \tag{7}$$

In particular, the Tjurina number of the singularity is $2k+\ell+n$.

Both r and q in (6) vanish at the origin. The last case is when the series r and q in $f(t,\varepsilon) = (t^2+\varepsilon p(t), t^{2k+1}+\varepsilon q(t), \varepsilon r(t))$ have no free terms. We start with a simple cutoff statement.

Lemma 2. *The family of quasijets $(a_1t^2+a_2\varepsilon, a_3t^3+a_4\varepsilon t, a_5t^4+a_6\varepsilon t^2)$, $a_i \in \mathbb{C}$, has a modulus under the action of the group $\mathcal{F}$.*

Proof. Similar to Lemma 1, we assign weights 1 to t and 2 to ε and observe that these map-germs are indeed quasihomogenious of multi-degree $(2,3,4)$. The left group acts on the jets by its 4-dimensional subgroup: each coordinate can be multiplied by a scalar, and $x_3' = x_3 + \lambda x_1^2$ gives the fourth dimension. The right group – as in Lemma 1 – contributes 2 dimensions. However, the Euler relation reduces the effective action to a total of $4+2-1=5$ dimensions. Since the family is 6-dimensional, there must be a modulus. □

In fact it is easy to see that a germ which is generic among those having the principal parts of the Lemma can always be reduced to

$$(t^2+\varepsilon, t^3+a_4\varepsilon t, \varepsilon t^2). \tag{8}$$

Since we want to consider only $\mathcal{F}$-simple germs, we shall assume that r in (6) has a non-zero linear term. Otherwise f would be adjacent to a germ having the principal part as in the Lemma and would thus be non-simple.

The Lemma yields a simplicity constraint on k as well. Indeed, since the linear term of r does not vanish, we can eliminate the linear term from q using the coordinate change $x_2' = x_2 - \lambda x_3$. For $k > 1$, this again gives a map adjacent to a germ with the principal part of the Lemma (the second and third coordinate functions must be swapped to make that clearer). Hence we are restricted to $k = 1$ only.

Now with $r(t) = ts(t)$, $s(0) \neq 0$, we take $\varepsilon s(t)$ for new ε and arrive at

$$f(t,\varepsilon) = (t^2+\varepsilon p(t), t^3+\varepsilon q(t), \varepsilon t)\,.$$

Following the previous paragraph, we assume that q has no linear terms. Hence the transformation $t' = t + \varepsilon q(t)/(3t^2)$ kills q completely. After this, addition to x_1 of the monomials $x_3x_1^m$ and $x_3x_2x_1^m$ with appropriate coefficients eliminates t^{1+2m} and t^{4+m} from p giving

$$f(t,\varepsilon) = (t^2+\varepsilon(\sigma+\alpha t^2), t^3, \varepsilon t)\,, \quad \sigma,\alpha \in \mathbb{C}\,.$$

Setting $t' = t(1+\alpha\varepsilon/2)$ kills the last term in the first coordinate function, but introduces εt^3 in the second. However, the latter is readily eliminated by addition to x_2 of a multiple of $x_1 x_3$.

Finally, $\sigma \neq 0$ normalises to 1 giving the singularity $\widetilde{C}^1$ of Theorem 2 while $\sigma = 0$ gives $\widetilde{C}^0$. A straightforward tangent space calculation shows that the Tjurina numbers of the singularities are respectively 4 and 5. Deformations

$$(t^2+\varepsilon, t^3+at+b\varepsilon, \varepsilon t+ct+d\varepsilon)$$

$$\text{and} \qquad (t^2+e\varepsilon, t^3+at+b\varepsilon, \varepsilon t+ct+d\varepsilon) \tag{9}$$

are $\mathcal{F}$-miniversal.

2.2.3 Cores with trivial 2-jet

Lemma 3. *There are no $\mathcal{F}$-simple spatial framed curves whose cores have trivial 2-jet.*

Proof. Set the weights of the variables t and ε to 1 and 3. Let L be the space of quasijets at the origin of weight at most 5 of spatial framed curves whose cores have trivial 2-jet. This space is spanned by the monomials $t^3, t^4, t^5, \varepsilon, \varepsilon t, \varepsilon t^2$ in each coordinate function. Hence $\dim L = 18$.

The group of $\mathcal{F}$-equivalences fixing the origins in the source and target acts on L. The action on the left is that of GL_3. The action on the right is by the subgroup whose Lie algebra is spanned by the fields of weight at most 2: $\varepsilon\partial_t$, $t\partial_t$, $t^2\partial_t$, $t^3\partial_t$, $\varepsilon\partial_\varepsilon$, $\varepsilon t\partial_\varepsilon$, $\varepsilon t^2\partial_\varepsilon$. Hence we get the action on L of the group of dimension $9+7=16$ which means that the action has at least $18-16=2$ moduli. □

Again we can easily see that a generic germ whose 5-quasijet is that from the Lemma can be reduced to the form

$$(t^3, t^4+\varepsilon, t^5+\varepsilon+a\varepsilon t+b\varepsilon t^2), \qquad a, b \in \mathbb{C}. \tag{10}$$

Thus every non-simple germ is adjacent to one with either the principal part as in (8) or with the 5-quasijet as in (10). So these are the only fencing singularities. The codimension of each of the two sets of fencing germs in the space of all germs sending $0 \in \mathbb{C}^2$ to $0 \in \mathbb{C}^3$ is 6, as claimed in Theorem 2. In Section 4 we will show that none of the germs of Theorem 2 is adjacent to these fencing singularities and thus they are simple.

Our list of normal forms of simple germs is now complete. In the next two sections we analyse discriminants and adjacencies of the listed singularities and with their help derive that the germs on our list are pairwise inequivalent. □

3 Discriminants

Definition 3.1. The *discriminant* of a framed curve-germ f is the set Σ of parameter values in the base of its miniversal deformation for which the perturbed germ is non-generic.

The multiplicity of the discriminant is one of basic invariants which may help, for example, to ban certain adjacencies.

3.1 Plane curves

In the planar case the discriminant consists of three components:

Σ_{fr} corresponding to at least A_1 framing degeneration,

Σ_{st} responsible for selftangencies of the core,

Σ_{cusp} which corresponds to a cuspidal core.

$\mathbf{A_k}$. We have already used its standard miniversal deformation

$$(t, \varepsilon q(t)) = (t, \varepsilon(t^{k+1} + q_{k-1}t^{k-1} + \cdots + q_1 t + q_0)) .$$

Clearly $\Sigma_{st} = \Sigma_{cusp} = \emptyset$ and $\Sigma_{fr} = \{(q_0, \ldots, q_{a-2}) \in \mathbb{C}^k : q(t) \text{ has a multiple root}\}$. This is the discriminant of the Weyl group A_k.

$\mathbf{B_k}$. In terms of the miniversal deformation

$$(t^2, q(t^2)t + \varepsilon) = (t^2, t^{2k+1} + q_{k-1}t^{2k-1} + \cdots + q_1 t^3 + q_0 t + \varepsilon) ,$$

the components Σ_{st} and Σ_{cusp} correspond to the polynomials $q(\tau)$ with multiple and respectively zero roots. This gives the discriminant of the Weyl group B_k. The set Σ_{fr} is empty: at an A_1 singularity two A_0 points merge, but $a_0(B_k) = 1$ (see Section 1.2).

3.2 Space curves

This time we have two discriminant components associated with the only two $\tau_{\mathcal{F}} = 1$ events:

Σ_{fr} corresponding to an $\widetilde{A}_1$ framing degeneration,

Σ_c responsible for double points of the core.

The $\widetilde{A}_1$ points are those at which the velocity vector $\partial f/\partial t|_{\varepsilon=0}$ of the core and the framing vector $\partial f/\partial \varepsilon$ are linearly dependent.

$\widetilde{\mathbf{A}}_{\mathbf{k}}$. We have had its miniversal deformation

$$(t, \varepsilon p(t), \varepsilon q(t)) =$$

$$= (t, \varepsilon(t^{k+1} + p_{k-1}t^{k-1} + \cdots + p_1 t + p_0), \varepsilon(q_k t^k + q_{k-1}t^{k-1} + \cdots + q_1 t + q_0)) .$$

So there is no Σ_c, and Σ_{fr} is given by the condition

$$\operatorname{rank} \begin{pmatrix} 1 & 0 & 0 \\ 0 & p(t) & q(t) \end{pmatrix} < 2 ,$$

which yields the resultant of p and q. Therefore, the multiplicity of Σ_{fr} is k.

$\widetilde{\mathbf{B}}_{\mathbf{k}}^{\ell,\mathbf{n}}$. The core of the miniversal deformation (7) is

$$(t^2, tp(t^2), tq(t^2)) =$$

$$= (t^2, t^{2k+1} + p_{k-1}t^{2k-1} + p_{k-2}t^{2k-3} + \cdots + p_0 t, q_{k-1}t^{2k-1} + q_{k-2}t^{2k-3} + \cdots + q_0 t) .$$

Such a curve has a double point, coming from $t = \pm t_0$, if and only if $\tau = t_0^2$ is a root of both $p(\tau)$ and $q(\tau)$. Hence, the component Σ_c is the resultant of these two polynomials.

According to (7) and the notation there, Σ_{fr} should be found from the condition

$$\operatorname{rank}\begin{pmatrix} 2t & \left(tp(t^2)\right)' & \left(tq(t^2)\right)' \\ 0 & 1 & tr(t^2)+s(t^2) \end{pmatrix} < 2\,,$$

where $r(t^2) = t^{2\ell}+r_{\ell-1}t^{2\ell-2}+r_{\ell-2}t^{2\ell-4}+\cdots+r_0$ and $s(t^2) = t^{2n}+s_{n-1}t^{2n-2}+s_{n-2}t^{2n-4}+\cdots+s_0$. This implies $t=0$ and therefore $q_0 = s_0 p_0$. The latter is the equation of Σ_{fr} which turns out to be smooth.

The discriminant is cylindrical in the directions of the parameters $r_{\geq 0}$ and $s_{>0}$. However, its stratification by the equivalence classes of the singularities is not.

$\mathbf{\widetilde{C}^1}$ **and** $\mathbf{\widetilde{C}^0}$. We consider a miniversal deformation of $\widetilde{C}^0$:

$$(t^2+e\varepsilon,\, t^3+at+b\varepsilon,\, \varepsilon t+ct+d\varepsilon)\,.$$

For $\widetilde{C}^1$ we should just set $e=1$ here.

It is clear that Σ_c is the hyperplane $c=0$. For Σ_{fr} we have

$$\operatorname{rank}\begin{pmatrix} 2t & 3t^2+a & c \\ e & b & t+d \end{pmatrix} < 2\,.$$

Calculation of pairwise resultants of the three minors yields an equation of Σ_{fr}:

$$e\hat{a}^2 + 2\hat{b}(\hat{a}d - \hat{b}c) = 0\,, \qquad \text{where } \hat{a} = a + 3ec/2 \quad \text{and} \quad \hat{b} = b + 3de/2\,. \tag{11}$$

Hence the multiplicity of Σ_{fr} of $\widetilde{C}^1$ is 2 and of that of $\widetilde{C}^0$ is 3.

4 Adjacencies of simple singularities

4.1 Plane curves

Their hierarchy is as follows:

$$\begin{array}{ccccccccccc} A_{-1} & \leftarrow & A_0 & \leftarrow & A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & \cdots \\ & & & \nwarrow & & & & & & & \\ & & & & B_1 & \leftarrow & B_2 & \leftarrow & B_3 & \leftarrow & \cdots\ . \end{array}$$

The existence of the horizontal strings is clear as well as the non-existence of any $A_k \to B_\ell$.

For $B_1 \to A_0$ we consider a miniversal deformation of B_1: $(t^2, t^3+\lambda t+\varepsilon)$. If $\lambda \neq 0$, the singularity at the origin is A_0. Indeed, setting $\lambda = 1$ and ignoring t^3, we have

$$(t^2, t+\varepsilon) \sim (t^2 - 2\varepsilon t, t) \sim (-2\varepsilon t, t) \in A_0\,.$$

There are no adjacencies of the B_k singularities to $A_{>0}$. A reason is that the number $a_0(A_k)$ of A_0 singularities collapsing at A_k is $k+1$ while $a_0(B_k) = 1$ (see Section 1.2).

4.2 Space curves: proof of Theorem 3

4.2.1 Adjacencies in the $\widetilde{A}$ series

Just as in the case of plane curves, we have

$$\widetilde{A}_{-1} \leftarrow \widetilde{A}_0 \leftarrow \widetilde{A}_1 \leftarrow \widetilde{A}_2 \leftarrow \widetilde{A}_3 \leftarrow \widetilde{A}_4 \leftarrow \cdots$$

and obviously the $\widetilde{A}_k$ cannot be adjacent to anything else since they are the only germs with non-singular cores.

4.2.2 Adjacencies of the $\widetilde{B}$ series

We have already seen, during the discriminant calculations, that any member of the $\widetilde{B}$ series is adjacent to $\widetilde{A}_0$. On the other hand, no $\widetilde{B}$ series member can be adjacent to $\widetilde{A}_{>0}$ since $\Sigma_{fr}(\widetilde{B}_k^{\ell,n})$ is smooth and $\Sigma_{fr}(\widetilde{A}_{>0})$ is not.

Existence of the adjacencies within the $\widetilde{B}$ series reducing either ℓ or n by 1 is clear from the classification procedure. Let us check all the other possibilities.

We have seen that for a miniversal deformation of $\widetilde{B}_k^{\ell,n}$ we can take

$$\begin{pmatrix} t^2 \\ p(t^2)t + \varepsilon \\ q(t^2)t + \varepsilon\Big(r(t^2)t + s_n(t^2)\Big) \end{pmatrix},$$

where the polynomials p, q, r, s have respectively degrees k, $k-1$, ℓ and n in $\tau = t^2$, all except for q being monic. Let us fix values of the parameters and determine the singularity at $t = 0$ which is the only possible singular point of the core.

The smallest degree term with a non-zero coefficient in p and q will determine the core of the germ. Suppose $p(\tau) = p_i\tau^i + p_{i+1}\tau^{i+1} + \dots$, $q(\tau) = q_j\tau^j + q_{j+1}\tau^{j+1} + \dots$ and $r(\tau) = r_b\tau^b + r_{b+1}\tau^{b+1} + \dots$ where $q_j, p_i, r_b \neq 0$ (or $q(\tau) \equiv 0$).

If either i or j is 0, then the germ at the origin is equivalent to either $\widetilde{A}_{-1}$ or $\widetilde{A}_0$, just as in the plane curve case. Otherwise it will be equivalent to a $\widetilde{B}^{\cdot,\cdot}_{min(i,j)}$ germ.

The process of bringing this germ to a normal form is as follows. First p and q are reduced to t^{2i} and t^{2j} respectively. This is done by coordinate changes $x_2' = x_2 + x_2u(x_1)$ and similarly for x_3. Then the smaller of the two monomials is used to eliminate the other by taking $x_3' = x_3 - x_3x_1^{j-i}$ if $j \geq i$, or $x_2' = x_2 - x_2x_1^{i-j}$ if $j < i$. If $q(t^2) \equiv 0$ then this second step is not neccessary. From here the process follows that described in Section 2.2.2.

Let us introduce some notation to be able to follow the germ throughout the whole reduction. Set $p(t^2) = t^{2i}P(t^2)$ and $q(t^2) = t^{2j}Q(t^2)$, where P and Q are invertible. Then after the first step the germ will be

$$\begin{pmatrix} t^2 \\ t^{2i+1} + \varepsilon P^{-1}(t^2) \\ t^{2j+1} + \varepsilon Q^{-1}(t^2)\Big(r(t^2)t + s(t^2)\Big) \end{pmatrix}.$$

After the second step in the two cases it will become either

$$\begin{pmatrix} t^2 \\ t^{2i+1} + \varepsilon P^{-1}(t^2) \\ \varepsilon\Big(Q^{-1}(t^2)(r(t^2)t + s(t^2)) - t^{2j-2i}P^{-1}(t^2)\Big) \end{pmatrix}$$

or

$$\begin{pmatrix} t^2 \\ \varepsilon\Big(P^{-1}(t^2) - t^{2i-2j}Q^{-1}(t^2)(r(t^2)t + s(t^2))\Big) \\ t^{2j+1} + \varepsilon Q^{-1}(t^2)\Big(r(t^2)t + s(t^2)\Big) \end{pmatrix}.$$

In the first case the last step is to take $\varepsilon = \varepsilon' P(t^2)$ which turns our germ into

$$\begin{pmatrix} t^2 \\ t^{2i+1} + \varepsilon \\ \varepsilon\Big(Q^{-1}(t^2)P(t^2)(r(t^2)t + s(t^2)) - t^{2j-2i}\Big) \end{pmatrix}.$$

In the second case, if s has zero free term then the germ will be equivalent to $\widetilde{B}_j^{b,0}$ where τ^b is the lowest degree term in $r(\tau)$.

To see this note that there is an ε term in x_2 but there is no such in x_3. Moreover, the lowest εt^{odd} term in x_3 is exactly εt^{2b+1} while in x_2 there are no $\varepsilon t^{odd \leq 2b+1}$. Now introduce $x_2' = x_2 + x_3$ and $x_3' = x_2$, and then take everything in the new x_2, except for t^{2j+1}, for new ε. After that, the lowest odd degree term in x_3 will still be εt^{2b+1}. Finally the ε term still existing in x_3 ensures the 0 in $\widetilde{B}_j^{b,0}$.

If, still in the second case, s has a non-zero free term then we set $R(t) = 1/(r(t^2)t + s(t^2))$. The germ will now be further reduced by taking $\varepsilon = \varepsilon' Q(t^2)R(t)$ to

$$\begin{pmatrix} t^2 \\ t^{2j+1} + \varepsilon \\ \varepsilon\Big(P^{-1}(t^2)Q(t^2)R(t) - t^{2i-2j}\Big) \end{pmatrix}.$$

The problem of finding the singularity type of the germ is reduced to the same problem in both cases: to find the lowest even and odd degree terms in $P(t^2)Q^{-1}(t^2)(r(t^2)t+s(t^2)) - t^{2j-2i}$ or respectively in $P^{-1}(t^2)Q(t^2)R(t) - t^{2i-2j}$. As odd degree terms come only from $r(t^2)t$ in both cases (in the second case it is still true, though in a hidden way since $R(t) = (r(t^2)t + s(t^2))^{-1}$), it is always determined by $r(t^2)$ and completely independent of the lowest even degree term.

In the second case finding the lowest even degree term is simpler than finding the odd degree term, as by our assumption $P^{-1}(t^2)Q(t^2)R(t)$ has a non-zero free term and since $i > j$ this cannot be eliminated by subtracting t^{2i-2j}. Thus in this case the germ is again equivalent to $\widetilde{B}_j^{b,0}$.

In the first case we have to find the lowest even degree term in $P(t^2)Q^{-1}(t^2)\Big(r(t^2)t + s(t^2)\Big) - t^{2j-2i}$. This is the same as finding the lowest degree term in $P(\tau)Q^{-1}(\tau)s(\tau) - \tau^{j-i}$. Let $s(\tau) = \tau^c S(\tau)$ where $S(0) \neq 0$. If $c < j - i$ then the answer is just c. If $c > j - i$ then the answer will be $j - i$. The only interesting case is when $c = j - i$.

Lemma 4. *Let P, Q, S be polynomials with non-zero constant terms, and let deg$(P) >$ deg(Q). Then the order of the root at 0 of the function $PS/Q - 1$ is at most deg(P) + deg(S).*

Proof. Let us write $P(\tau)S(\tau)/Q(\tau) = 1 + \tau^K W(\tau)$ where $W(\tau)$ has a non-zero free term. Then $P(\tau)S(\tau) = Q(\tau) + \tau^K W(\tau)Q(\tau)$. We want to prove that $K \leq \deg(P) + \deg(S)$. If $K \leq \deg(Q)$ then this is clear. Otherwise if the right hand side is a polynomial, its degree is obviously at least K. On the other hand it is at most $\deg(P) + \deg(S)$. This finishes the proof. □

Since $\deg(P) = k-i$, $\deg(Q) = k-j-1$ and $j \geq i$ we can apply the lemma. So the lowest degree term in $P(\tau)Q^{-1}(\tau)S(\tau)-1$ has degree at most $\deg(P)+\deg(S) = k-i+(n-j+i) = k+n-j$. Thus in $P(t^2)Q^{-1}(t^2)s(t^2) - t^{2j-2i}$ the lowest even degree term has degree at most $2(k+n-j+j-i) = 2n+2(k-i)$. So in the best case the germ is equivalent to $\widetilde{B}_i^{b,n+k-i}$. This theoretical limit is actually achievable since we have

$\widetilde{B}_k^{\ell,n} \to \widetilde{B}_{k-1}^{\ell,n+1}$:

$$(t^2, t^{2k+1} + \varepsilon - \lambda t^{2k-1}, \varepsilon t^{2\ell+1} + \varepsilon(t^{2n+2} - \lambda^{n+1})/(t^2 - \lambda) - \lambda^{n+1} t^{2k-1})$$

and the rest is handled by the transitivity.

We have gone through all possible cases and the conclusion is that, up to the transitivity, all adjacencies of the $\widetilde{B}$ series are

$$\widetilde{B}_k^{\ell,n} \to \left\{ \begin{array}{l} \widetilde{A}_0 \\ \widetilde{B}_k^{\ell-1,n} \\ \widetilde{B}_k^{\ell,n-1} \\ \widetilde{B}_{k-1}^{\ell,n+1} \end{array} \right. .$$

Remark 4.1. The above considerations demonstrate that if $\widetilde{B}_k^{\ell,n}$ is adjacent to a germ with $\tau_{\mathcal{F}} = 2k+\ell+n-1$ then the latter has normal form either $\widetilde{B}_k^{\ell-1,n}$ or $\widetilde{B}_k^{\ell,n-1}$ or $\widetilde{B}_{k-1}^{\ell,n+1}$. If any of these three is actually not contained in the classification table (if for example $\ell - 1 < 0$, or any other range constraint fails), then that case should be excluded from the adjacency list. For example, this implies that $\widetilde{B}_k^{0,1}$ is not adjacent to any $\widetilde{B}_{k-1}^{\cdot,\cdot}$ singularity with Tjurina number $2k$.

4.2.3 Adjacencies of $\widetilde{C}^0$ and $\widetilde{C}^1$

Miniversal deformations of these singularities are given in (9).

The Tjurina number of $\widetilde{C}^1$ is 4 and the singularity is adjacent to any other simple singularity with lower $\tau_{\mathcal{F}}$. This follows from the existence of the adjacencies to all $\tau_{\mathcal{F}} = 3$ germs:

$\widetilde{C}^1 \to \widetilde{A}_1$: $(t^2 + \varepsilon, t^3 + 3\lambda^2 t + 3\lambda\varepsilon, \varepsilon t - 2\lambda^2 t - 2\lambda\varepsilon)$ has an $\widetilde{A}_1$ singularity at $t = \lambda$;

$\widetilde{C}^1 \to \widetilde{B}_1^{0,1}$: $(t^2 + \varepsilon, t^3 + \lambda\varepsilon, \varepsilon t)$;

$\widetilde{C}^1 \to \widetilde{B}_1^{1,0}$: $(t^2 + \varepsilon, t^3, \varepsilon t + \lambda\varepsilon) \sim (t^2, t^3, \varepsilon)$.

For $\widetilde{C}^0$, a miniversal deformation is $(t^2 + e\varepsilon, t^3 + at + b\varepsilon, \varepsilon t + ct + d\varepsilon)$. Setting here all the parameters except for e to 0, we obtain $\widetilde{C}^0 \to \widetilde{C}^1$.

The only questionable adjacency now remaining is $\widetilde{C}^0 \to \widetilde{B}_1^{1,1}$. For this, the core must be singular which implies $t = 0$ and $a = c = 0$ in the above deformation. The condition $a_0(\widetilde{B}_1^{1,1}) = 1$ implies either $b \neq 0$ or $d \neq 0$. It is easy to see that $d \neq 0$ gives either $\widetilde{B}_1^{0,0}$ or $\widetilde{B}_1^{1,0}$, while $d = 0 \neq b$ yields either $\widetilde{B}_1^{0,0}$ or $\widetilde{B}_1^{0,1}$. Hence the adjacency we have been looking for does not exist.

We collect the adjacencies of the two singularities in a diagram:

$$\widetilde{C}^0 \to \widetilde{C}^1 \to \left\{ \begin{array}{l} \widetilde{A}_1 \\ \widetilde{B}_1^{0,1} \\ \widetilde{B}_1^{1,0} \end{array} \right. .$$

The proof of Theorem 3 is now finished. □

5 Proof of Theorem 4

The multiplicities of various discriminant components and the Tjurina numbers of the germs separate almost all the normal forms from each other. The only possible equivalences could arise between $\widetilde{B}_k^{\ell_1,n_1}$ and $\widetilde{B}_k^{\ell_2,n_2}$ when $\ell_1 + n_1 = \ell_2 + n_2$, $\ell_1 \neq \ell_2$. Assume there are such k, ℓ_1, n_1, ℓ_2 and n_2. Choose the smallest k for which such an equivalence holds. Among these, choose a pair with the least $\tau_{\mathcal{F}}$, and among these a pair with the smallest n_2. If $n_2 \neq 0$ then $\widetilde{B}_k^{\ell_2,n_2}$ is adjacent to $\widetilde{B}_k^{\ell_2,n_2-1}$. On the other hand we have seen that $\widetilde{B}_k^{\ell_1,n_1}$ is adjacent to only two of $\widetilde{B}_k^{\cdot,\cdot}$ singularities with $\tau_{\mathcal{F}} = 2k + \ell_1 + n_1 - 1$. These two are $\widetilde{B}_k^{\ell_1-1,n_1}$ and $\widetilde{B}_k^{\ell_1,n_1-1}$. However if our assumption is true and $\widetilde{B}_k^{\ell_1,n_1}$ is equivalent to $\widetilde{B}_k^{\ell_2,n_2}$ then the former must be adjacent to $\widetilde{B}_k^{\ell_2,n_2-1}$ as well. Thus $\widetilde{B}_k^{\ell_2,n_2-1}$ must be equivalent to either $\widetilde{B}_k^{\ell_1,n_1-1}$ or $\widetilde{B}_k^{\ell_1-1,n_2}$. As $n_1 - 1 > n_2 - 1$ this contradicts the choice of n_2. So n_2 must be 0. Similarly, if $\ell_1 \neq 0$ we can also find an equivalent pair with smaller $\tau_{\mathcal{F}}$.

Now we see that $\ell_1 = n_2 = 0$. Since $n_1 \leq 2\ell_1 + 1$ we get $n_1 = 0$ or 1. The first option is impossible since $\widetilde{B}_k^{0,0}$ is the only normal form with the core $(t^2, t^{2k+1}, 0)$ and $\tau_{\mathcal{F}} = 2k$. Hence $n_1 = 1$ and thus $\ell_2 = 1$. So the two equivalent germs are $\widetilde{B}_k^{0,1}$ and $\widetilde{B}_k^{1,0}$.

Now we can look at the adjacencies once again, to the most degenerate $\widetilde{B}_{k-1}^{\cdot,\cdot}$ singularities. Since $\widetilde{B}_k^{0,1} \to \widetilde{B}_{k-1}^{0,2} \sim \widetilde{B}_{k-1}^{0,1}$ drops $\tau_{\mathcal{F}}$ by 2 while $\widetilde{B}_k^{1,0} \to \widetilde{B}_{k-1}^{1,1}$ drops it by just 1, it follows from Remark 4.1 that the germs $\widetilde{B}_k^{0,1}$ and $\widetilde{B}_k^{1,0}$ cannot be equivalent. This argument works if $k \geq 2$.

The only remainig case to deal with is that $\widetilde{B}_1^{0,1}$ and $\widetilde{B}_1^{1,0}$ should not be equivalent. The normal forms reduce to $(t^2, t^3 + \varepsilon, \varepsilon)$ and $(t^2, t^3 + \varepsilon, t\varepsilon)$ respectively. At a cuspidal point of a core like here, we have a well-defined plane in $\mathbb{C}^3$ spanned by the second and third derivatives of the core. The framing vector $\partial f/\partial\varepsilon|_{\varepsilon=0}$ at such a point is transversal to the plane in the $\widetilde{B}_1^{1,0}$ case and belongs to it if the singularity is $\widetilde{B}_1^{0,1}$. This tells the two singularities apart and finishes the proof. □

6 Multi-singularities

We now consider the multi-germ setting of our planar framed classification. This a framed analog of the classification of simple multi-germs of planar unframed curves which was obtained in [7] and consists of parametrisations of the ADE curves. To save on notations, we introduce them only for series and not for their individual members. The dimension of the base of a miniversal deformation of a framed multi-germ is still denoted $\tau_{\mathcal{F}}$. We do not list the adjacencies – many of them are obvious.

Theorem 5. *Together with the singularities of Theorem 1, the following table provides a complete list of simple equivalence classes of framed complex planar curve multi-germs:*

family	*normal form*	*range*	$\tau_{\mathcal{F}}$
I	$(t, \varepsilon t^a); (\delta u^b, u)$	$a, b \geq 0$	$a + b$
II	$(t, \varepsilon t^a); (u, u^k + \delta u^b)$	$a, b \geq 0,\ k \geq 2$	$a + b + k - 1$
III	$(t, \varepsilon t^a); (u^{2k+1} + \delta, u^2 + \sigma\delta)$	$a \geq 0,\ k \geq 1,\ \sigma \in \{0, 1\}$	$a + k + 2 - \sigma$
IV	$(t, \varepsilon t^a); (u^2, u^3 + \delta + \sigma u^4)$	$a \geq 0,\ \sigma \in \{0, 1\}$	$a + 4 - \sigma$
V	$(t, \varepsilon t^a); (\delta u^b, u); (v, v^k + \xi v^c)$	$a, b, c \geq 0,\ k \geq 1$	$a + b + c + k$

Proof. Four lines passing through a common point in the plane have their cross-ratio as a modulus. Therefore, two cusp germs at one point cannot be simple since by a small perturbation they can be turned into 4 non-singular curves passing through the same point. So we may assume that one of the curves of a bi-germ is non-singular, all three in a tri-germ are non-singular, and there are no simple multi-germs of multipilicity 4 or higher.

In each case we start by reducing one of the non-singular curves to its normal form: $(t, \varepsilon t^a)$. Any further steps must not change its form. This restricts us to a certain subgroup of the left group, namely to those transformations that can be compensated by an action of the right group. It is easily seen that exactly automorphisms of the form

$$x_1' = p(x_1, x_2)\,, \qquad x_2' = x_2 q(x_1, x_2)$$

are permitted. Since this group is independent from a, the value of a does not affect possible normal forms of the other germs. We will denote the variables of the second germ by u, δ.

I If there is a linear u term in the second coordinate function then the germ can be reduced to $(\delta u^b, u)$ by first normalising the core by the left group, and then the framing by the right.

II If there is no linear u term in the second coordinate function but there is one in the first, then let u^k be the smallest degree term in the second coordinate. Then all the higher order terms of the core can be cleared in both coordinates by the left group, and the framing can be reduced by the right group to give $(u, u^k + \delta u^b)$. The parameter k here is the order of contact of the two cores.

III If the core is singular there must be a u^2 term in one of the coordinate functions. If it is in the second coordinate, then the core reduces to (u^{2k+1}, u^2) as before. Since the germ is simple it has to have a δ term in its first coordinate function as in the proof of Theorem 1. The left group reduces this to $(u^{2k+1} + \delta, u^2 + \sigma\delta)$ with $\sigma \in \mathbb{C}$. Finally a rescaling of all coordinates and variables normalises $\sigma \neq 0$ to 1. The cases $\sigma = 1, 0$ are inequivalent since the tangency of the framing of the second germ to the core of the first is preserved by the group action at a cusp point.

IV If there is no u^2 in the second coordinate function, it must be there in the first. The simplicity of the core configuration then implies that the second coordinate function of the second core must start with u^3. Indeed, otherwise, using reparametrisation and diffeomorphisms of $\mathbb{C}^2$ preserving the ∂_{x_1} direction at the origin, we reduce the second core to $(u^2, u^{odd \geq 5})$. This spoils the first core to $(t, p(t))$ where p has zero linear part. Now we can deform the second core to a curve with a self-tangency at the origin so that both branches are tangent to the first core. The latter configuration is modular.

Thus, since the second framed curve-germ must be simple, we may assume that its principal part is $(u^2, u^3 + \delta)$. The standard calculations now show that keeping the first curve in its normal form $(t, \varepsilon t^a)$ we can reduce the second germ to $(u^2, u^3 + \delta + \sigma u^4)$, $\sigma \in \mathbb{C}$. As the last step, $\sigma \neq 0$ easily normalises to 1. The bi-germs with $\sigma = 1, 0$ are distinguished by their $\tau_{\mathcal{F}}$.

V Finally we come to tri-germs. We have seen that in this case all three cores must be non-singular. Since a triple tangency is non-simple, we must have two transversal curves. These can be transformed to

$$(t, \varepsilon t^a) \qquad \text{and} \qquad (\delta u^b, u)\,.$$

Keeping these two fixed restricts the use of the left group to diffeomorphisms of the form

$$x_1' = x_1 p(x_1, x_2)\,, \qquad x_2' = x_2 q(x_1, x_2).$$

We may assume the third germ has a linear term v in its first coordinate function and that the smallest term in its second is v^k. Then it can be easily reduced to $(v, v^k + \xi v^c)$. □

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