PROJECTION AND VECTOR FIELDS, TANGENT TO THE DISCRIMINANT OF A COMPLETE INTERSECTION

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If a system of equations depends on parameters, then in the space of parameters one singles out the discriminant surface, which corresponds to bifurcations of the manifold of solutions of the system. Among the parameters there may be distinguished ones. For example, when one of the parameters is time and one studies reconstructions which occur in the system in the course of time. In such a situation it is necessary to be able to reduce a function of time on the space of parameters to normal form by diffeomorphisms which preserve the discriminant. At the infinitesimal level this requires knowledge of the algebra of vector fields on the discriminant [5, 7].

The basic result of the present paper is a description of generators of the algbera of holomorphic vector fields tangent to the discriminant of a complete intersection with an isolated singularity. Earlier, Looijenga proved that this algebra is a free module over the ring of holomorphic functions on the ambient space [13]. Unfortunately, this theorem is nonconstructive and consequently inapplicable for example, to the problem of classification of functions or mappings defined on the space containing the discriminant.

The proof of the theorem on basic fields on the discriminant is based on a number of properties of a one-parameter deformation of a complete intersection (Secs. 1 and 2). Such a deformation defines a projection to a line of a complete intersection Y of dimension one greater than the dimension of the deformed manifold. We recall that by projection to a line is meant a diagram $Y \subset \mathbf{C}^{n+1} \xrightarrow{p} \mathbf{C}$ where the first arrow is an imbedding and the second is a nondegenerate linear projection [3]. The restriction of p to Y (height function) can have critical points (a singular point of the submanifold is considered critical for p). In Sec. 1, where we give all the necessary definitions, we introduce the multiplicity μ of a critical point of a height function as the maximal number of Morse points at which it splits under a small deformation of the complete intersection Y.

It turns out that the number μ is closely connected with the codimension τ of the orbit of the projection with respect to an equivalence which we call R_+-equivalence, since it is a natural generalization of the corresponding concept for functions on smooth manifolds [2, Vol. 1]. Projections of two submanifolds are considered R_+-equivalent if the submanifolds are carried into one another by a biholomorphism of the ambient space which commutes with the projections and induces a translation on the base of p.

In Sec. 2 we show that just as for functions on smooth manifolds, for a height function $\mu = \tau + 1$ (Theorem 2.1). This fact is basic for the constructions of Sec. 3, where we consider vector fields tangent to the discriminant of complete intersections and projections. The discriminant of a projection is the discriminant of the complete intersection $Y \cap p^{-1}(0)$ multiplied by a complex linear space (the definition is given in point 1.4).

The basic vector fields on the discriminant of a projection are given by decompositions of products of the height function by the velocity of an R_+-versal deformation into velocities of deformation with coefficients which depend only on the parameters of the deformation (Theorem 3.1).

The basic vector fields on the discriminant of a complete intersection can be found in the following way (Theorem 3.2). We single one out from the parameters of a versal deformation of a complete intersection, λ_0. Let the axis $0\lambda_0$ have finite intersection index μ with the discriminant. The restriction of the versal deformation to the $0\lambda_0$ axis defines a projection to a line, while μ is exactly the multiplicity of the critical point of the function λ_0 (the height function in the present case). We write generators of the module of vector

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fields which preserve the discriminant in factorizations as products of powers of the parameter λ_0 by the velocity of the versal deformation into velocities of deformation with coefficients which are polynomials of bounded degree in λ_0. The relation $\mu = \tau + 1$ guarantees us that the fields found in this way really generate the whole module of vector fields tangent to the discriminant.

In Sec. 4 we consider the bifurcation diagram Σ of a projection to a line. This object can be considered the bifurcation diagram of a complete intersection, because one gets it from the discriminant of a complete intersection (possibly multiplied by a complex linear space) in the same way as the bifurcation diagram of a smooth function from the discriminant of the smooth function — as the ramification manifold of a stable projection of the discriminant along a line. We show that the algebra of vector fields preserving Σ is a free module over the ring of holomorphic functions on the ambient space, i.e., like the discriminant, the bifurcation diagram of a projection is a free divisor in the sense of Saito [14] (Theorem 4.1). Generators of this module can be constructed by expansions analogous to the expansions of Terao—Bruce [14; 8] for fields on the bifurcation diagram of a function.

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1. R_+-Equivalence of Projections

1. Suppose given on $\mathbf{C}^{n+1}$ a nondegenerate linear projection $p: \mathbf{C}^{n+1} \to \mathbf{C}$.

Definition. By a projection of a submanifold $Y \subset \mathbf{C}^{n+1}$ to a line is meant a diagram $Y \subset \mathbf{C}^{n+1} \xrightarrow{p} \mathbf{C}$ [3].

We fix a coordinate function u on the base of p. Its restriction to Y will be called the height function.

Definition. Projections of submanifolds Y_1, $Y_2 \subset \mathbf{C}^{n+1}$ are R_+-equivalent if there exists a biholomorphism of the ambient space carrying Y_1 into Y_2, which preserves the projection p, and on the base of p induces a translation $u \to u + \text{const}$.

Taking the graph of a function on $\mathbf{C}^n$ as Y (u is the value of the function), we see that the equivalence introduced is a natural generalization of the concept of R_+-equivalence of smooth functions [2, Vol. 1]. The previously considered equivalence of projections [3], under which an arbitrary biholomorphism was induced on the base of the projection, corresponds to RL-equivalence of functions.

2. As projected submanifolds we consider complete intersections of positive dimension: $Y = f^{-1}(0)$, where $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^m, 0)$, $m = \operatorname{codim} Y \leqslant n$. For brevity we shall speak about a projection of a complete intersection $f^{-1}(0)$ to a line as a projection f.

We shall use the following notation:

$\mathcal{O}^m_{Z,z}$ is the space of germs at the point $z \in Z$ of holomorphic maps from Z to $\mathbf{C}^m$;

$\mathcal{O}_{Z,z} = \mathcal{O}^1_{Z,z}$;

$\mathfrak{m}_{Z,z}$ is the maximal ideal in $\mathcal{O}_{Z,z}$;

$(x, u) = (x_1, \ldots, x_n, u) \in \mathbf{C}^{n+1}$.

For $f \in \mathcal{O}^m_{\mathbf{C}^{n+1},0}$ we set

$$T_f = f^*(\mathfrak{m}_{\mathbf{C}^m,0})\mathcal{O}^m_{\mathbf{C}^{n+1},0} + \mathcal{O}_{\mathbf{C}^{n+1},0}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle.$$

$T_f^+ = T_f + \mathbf{C}\,\partial f/\partial u$ is the tangent space to the germ of the R_+-equivalence class of the projection f.

We introduce $Q_f = \mathcal{O}^m_{\mathbf{C}^{n+1},0}/T_f^+$.

Definition. By the R_+-codimension of the projection f is meant the number $\tau = \dim_{\mathbf{C}} Q_f$.

Obviously

Proposition 1.1. If $\tau < \infty$ then the complete intersections $\{f(x, u) = 0\} \subset \mathbf{C}^{n+1}$ and $\{f(x, 0) = 0\} \subset \mathbf{C}^n$ have isolated singularities.

Projection of the manifold $f(x, u) = 0$ to a line is a one-parameter deformation of the complete intersection $f(x, 0) = 0$. R_+-equivalence of projections corresponds to a more rigid equivalence of deformations than usual: change of the parameter of deformation is prohibited.

We shall not consider the R_+-classification of projections to a line. We only note that the list of simple singularities is the same as in [3].

3. We consider a k-parameter deformation of the projection $F: (\mathbf{C}^{n+1+k}, 0) \to (\mathbf{C}^m, 0)$, $\lambda \in \mathbf{C}^k$ being the parameter of deformation, $F|_{\lambda=0} = f$.

Definition. A deformation F is called infinitesimally R_+-versal, if its initial speeds $\partial F/\partial \lambda_i|_{\lambda=0}$, $i = 1, \ldots, k$, generate the linear space Q_f.

In the obvious way one also defines an R_+-versal deformation of a projection. By an obvious theorem the concepts of R_+-versality and infinitesimal R_+-versality of deformations are equivalent. The dimension of the space of parameters of an R_+-miniversal deformation is τ.

A k-parameter R_+-versal deformation of a projection f is also:

a) a k-parameter versal deformation of the complete intersection $f(x, u) = 0$;

b) a $(k + 1)$-parameter versal deformation of the complete intersection $f(x, 0) = 0$ (u being an additional parameter).

We shall call the space $\mathbf{C}^{1+k}$, $(u, \lambda) \in \mathbf{C}^{1+k}$, an extension of the space of parameters.

4. Definition [4]. By the discriminant Δ of the projection f is meant the discriminant of the complete intersection $f(x, 0) = 0$, which lies in the extended space of parameters of an R_+-versal deformation F.

Thus, $\Delta \subset \mathbf{C}^{1+k}$ is the set of critical values of the projection $(x, u, \lambda) \to (u, \lambda)$, restricted for $F = 0$.

5. Let $\tilde{F}$ be a representative of the germ of an R_+-versal deformation $F \in \mathcal{O}^m_{\mathbf{C}^{n+1+k},0}$ of the projection f. Let us assume that on the set $\tilde{Y}_0 = \tilde{f}^{-1}(0)$ [a representative of $f^{-1}(0)$] the height function u has a unique critical point $(x, u) = (0, 0)$. Here a singular point of a complete intersection is considered critical for the height function.

Definition. For almost all sufficiently small values of the deformation parameter λ the function u has the same number of critical values on the complete intersection $\tilde{Y}_\lambda = \{\tilde{F}|_{\lambda=\text{const}} = 0\}$. We call this number the multiplicity of the critical point $(x, u) = (0, 0)$ of the height function on the germ $f = 0$ and we shall denote the multiplicity by μ.

Example. If $f = x^2 + u^2$, then $\mu = 2$.

To critical values of the height function on the manifold $\tilde{Y}_\lambda$ there correspond in $\mathbf{C}^{1+k}$ the u-coordinates of the points of intersection of the line $\lambda = \text{const}$ with $\tilde{\Delta}$ (a representative of Δ). Hence μ is the index of intersection of the line $\lambda = 0$ and Δ in $\mathbf{C}^{1+k}$, which coincides with the index of intersection in $\mathbf{C}^{n+1+k}$ of the plane $\lambda = 0$ and the set C of critical points of the projection $(x, u, \lambda) \to (u, \lambda)$ restricted to $F = 0$ [13, Sec. 4]. If $I_{C_0} \subset \mathcal{O}_{\mathbf{C}^{n+1},0}$ is the ideal generated by the coordinate functions of the map f and all m-minors of the matrix $(\partial f/\partial x)$ then by [13, Sec. 4] we get

Proposition 1.2. $\mu = \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^{n+1},0}/I_{C_0}$.

COROLLARY 1.3. The discriminant Δ is defined by the Weierstrass polynomial of degree μ in the variable u:

$$\Delta = \{u^\mu + \alpha_{\mu-1}(\lambda)\, u^{\mu-1} + \ldots + \alpha_0(\lambda) = 0\}, \quad \alpha_i \in \mathcal{O}_{\mathbf{C}^k,0}.$$

The corollary follows from the propriety of the map $\Delta \to \mathbf{C}^k$, $(u, \lambda) \mapsto \lambda$.

6. Finally, we recall the definition of another object considered below. Let $\tilde{F}$ be from point 5.

Definition [3]. By the bifurcation diagram $\Sigma \subset \mathbf{C}^k$ of the projection f is meant the germ at zero of the set $\tilde{\Sigma}$ of those values of the parameter λ for which the height function has, on $\tilde{Y}_\lambda$, less than μ critical values.

For $\lambda \notin \tilde{\Sigma}$ the height function is a Morse function on $\tilde{Y}_\lambda$. The diagram Σ consists of three components in general:

Σ_d corresponds to the degenerate critical points of the height function on the smooth manifold $\tilde{Y}_\lambda$;

Σ_m corresponds to the coincidence of critical values of the height function on smooth $\tilde{Y}_\lambda$ (Maxwell stratum);

Σ_s corresponds to nonsmooth sets $\tilde{Y}_\lambda$.

Under the projection $(u, \lambda) \to \lambda$ these components are the images, respectively, of three subsets of the discriminant: cuspidal edge, set of self-intersections, closure of the set of critical points of the restriction of the projection to a stratum of higher dimension.

Σ is the branching manifold of the covering $\Delta \to \mathbf{C}^k$, $(u, \lambda) \mapsto \lambda$.

2. Multiplicity of a Singular Point of the Height Function and Codimension of the Projection

THEOREM 2.1. $\mu = \tau + 1$ for $0 < \mu < \infty$.

The theorem follows from several assertions proved below.

1. LEMMA 2.2. If $0 < \mu < \infty$ then $\partial f/\partial u \notin T_f$.

Proof. Let this not be so. Then there exists a germ at zero of a vector field $v = \partial_u + v_1\partial_{x_1} + \ldots + v_n\partial_{x_n}$ on $\mathbf{C}^{n+1}$, such that $vf = Bf$, where B is a germ of an $m \times m$-matrix. The field v is tangent to the manifold $f = 0$ and can be dropped with respect to the projection $p: \mathbf{C}^{n+1} \to \mathbf{C}$. It follows easily from this that v is also tangent to the set C_0 of critical points of the height function on $f = 0$ (C_0 is the manifold of zeros of the ideal I_{C_0} of point 1.5). Since $\mu > 0$ one has $C_0 \supset \{0\}$. Consequently, C_0 also contains the germ of the phase curve $\Gamma \neq \{0\}$ of the field v, passing through 0. Hence, $\dim C_0 > 0$. But this is impossible because $\dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^{n+1},0}/I_{C_0} = \mu < \infty$.

2. LEMMA 2.3. Let $f = (f_1 \ldots f_m) = (x_1^r \ldots x_n^r u^r)M$, where M is a constant $(n + 1) \times m$-matrix, $r > 1$. For generic choice of M, $\mu \geqslant \tau + 1$.

Proof. We consider a small deformation $f: \tilde{f} = (q_1(x_1) \ldots q_n(x_n)\, q_{n+1}(u))M$, where $q_1, \ldots, q_{n+1}$ are polynomials of degree r, whose derivatives have no multiple roots.

The critical points of the height function on $\tilde{f} = 0$ are defined for generic choice of the matrix M by the equations

$$q'_{i_1}(x_{i_1}) \cdot \ldots \cdot q'_{i_m}(x_{i_m}) = 0,\ 1 \leqslant i_1 < \ldots < i_m \leqslant n.$$

Straightforward combinatorics give the number of these points:

$$\mu = C_n^{m-1}(r-1)^{n-(m-1)} r^m.$$

To estimate the codimension τ we consider complete intersections $f = 0$ and $f|_{u=0} = 0$. Let τ' and τ'' be their Tyurina numbers. Considering the corresponding quotient spaces and using the (quasi)homogeneity of f (due to which $T_f \ni u\partial f/\partial u$), we get: $\tau + 1 \leqslant \tau' + \tau''$ (in fact it follows from what follows that for $n > m$ here the equality holds, as for functions on a manifold with boundary [1]). For a quasihomogeneous complete intersection of positive dimension, the Tyurina number coincides with the Milnor number [10]. One can calculate the latter for $f = 0$, for example, as $\sum_{s=1}^{m} (-1)^{s+1}\mu_s$, where μ_s is the number of critical points of the function $\bar{f}_{m-st+1}$ on the set $\bar{f}_1 = \ldots = \bar{f}_{m-s} = 0$ [2, Vol. 2]. μ_s is the number of common zeros of all $(m - s + 1)$-minors of the matrix $(\partial(\bar{f}_1, \ldots, \bar{f}_{m-s+1})/\partial(x, u))$ on $\bar{f}_1 = \ldots = \bar{f}_{m-s} = 0$. Combinatorics gives

$$\mu_s = C_{n+1}^{m-s}(r-1)^{n+1-(m-s)} r^{m-s}.$$

Consequently, for $n > m$

$$\tau + 1 \leqslant \sum_{s=1}^{m} (-1)^{s+1} C_{n+1}^{m-s}(r-1)^{n+1-(m-s)} r^{m-s} + \sum_{t=1}^{m} (-1)^{t+1} C_n^{m-t}(r-1)^{n-(m-t)} r^{m-t}.$$

It is easy to see that the number obtained from the summation coincides with the number μ calculated previously.

For $n = m$, considering that $f_i = x_i^r + \alpha_i u^r$, $\alpha_i = \mathrm{const}$, $i = 1, \ldots, m$, we see by the direct calculation that $\tau + 1 = m(r - 1)r^m = \mu$.

3. We choose a representative $\tilde{F}$ of the R_+-miniversal deformation $F \in \mathcal{O}^m_{\mathbf{C}^{n+1+\tau},0}$ of an arbitrary projection f. Let the map $\tilde{F}$ be defined on the product $X \times U \times \Lambda$ of neighborhoods

of zero in the spaces $\mathbf{C}^n$, $\mathbf{C}$, and $\mathbf{C}^\tau$. On these neighborhoods we impose the following conditions:

a) the height function has, on $\tilde{Y}_0 = \tilde{f}^{-1}(0)$, a unique critical point (x, u) = (0, 0);

b) for any $\lambda \in \Lambda$ all critical points of the height function on $\tilde{Y}_\lambda = \{\tilde{F}|_{\lambda=\mathrm{const}} = 0\}$ lie in the product $X' \times U'$, $\bar{X}' \times \bar{U}' \subset X \times U$.

We denote by W the image of $\tilde{F}$. $0 \in W \subset \mathbf{C}^m$.

LEMMA 2.4. $\mu \leqslant \tau + 1$.

Proof. Let $\mathcal{O}_Z$ be the sheaf of holomorphic functions on Z. We consider on $\tilde{F} = 0$ the coherent sheaf of $\mathcal{O}_\Lambda$-modules

$$\mathcal{Y} = \mathcal{O}^m_{\tilde{F}^{-1}(0)}/\mathcal{O}_{\tilde{F}^{-1}(0)}\langle \partial \tilde{F}/\partial x_1, \ldots, \partial \tilde{F}/\partial x_n\rangle.$$

Its support is the set $\tilde{C}$ of critical points of the restriction to $\tilde{F} = 0$ of the projection $(x, u, \lambda) \to (u, \lambda)$. In fact, if $(x, u) \in \tilde{Y}_\lambda$, but $(x, u, \lambda) \notin \tilde{C}$, then the stalk of the sheaf in the numerator coincides with the stalk of the sheaf in the denominator. Now if $(x, u, \lambda) \in \tilde{C}$, then as minimum $\mathcal{Y}_{(x,u,\lambda)} \simeq \mathcal{O}_{\Lambda,\lambda}$ [the minimum is achieved if $(x, u) \in \tilde{Y}_\lambda$ is a Morse critical point of the height function].

The map π: $\tilde{C} \to \Lambda$, $(x, u, \lambda) \mapsto \lambda$, is proper of multiplicity μ. Hence the direct image $\pi_*\mathcal{Y}$ is a coherent sheaf of $\mathcal{O}_\Lambda$-modules. We show that in some neighborhood of the point $O \in \mathbf{C}^\tau$ the sheaf $\pi_*\mathcal{Y}$ is generated by $\tau + 1$ elements.

In fact, it follows from the versality of F that $\partial f/\partial u$, $\partial F/\partial \lambda_1|_{\lambda=0}, \ldots, \partial F/\partial \lambda_\tau|_{\lambda=0}$ generate the linear space

$$\mathcal{O}^m_{X\times U,(0,0)}/\{f^*(\mathfrak{m}_{W,0})\, \mathcal{O}^m_{X\times U,(0,0)} + \mathcal{O}_{X\times U,(0,0)}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle\}.$$

By the preparation theorem this means that $\partial \tilde{F}/\partial u, \partial \tilde{F}/\partial \lambda_1, \ldots, \partial \tilde{F}/\partial \lambda_\tau$ generate the stalk $\mathcal{Y}_{(0,0,0)}$ as an $\mathcal{O}_{\Lambda,0}$-module. It follows from the coherence that $\pi_*(\partial \tilde{F}/\partial u)$, $\pi_*(\partial \tilde{F}/\partial \lambda_1), \ldots$, $\pi_*(\partial \tilde{F}/\partial \lambda_\tau)$ are generators of the sheaf $\pi_*\mathcal{Y}$ over a neighborhood of zero in $\Lambda' \subset \Lambda \subset \mathbf{C}^\tau$.

Thus, for $\lambda \in \Lambda'$ the rank of the module $(\pi_*\mathcal{Y})_\lambda$ is not greater than $\tau + 1$. Lemma 2.4 now follows from the fact that for a generic value λ the height function has, on $\tilde{Y}_\lambda$, only Morse critical points, there are μ of them, and each of them makes a contribution of 1 to the rank of $(\pi_*\mathcal{Y})_\lambda$.

4. Proof of Theorem 2.1. We continue the study of the representative $\tilde{F}$ of a miniversal deformation of an arbitrary projection. For some sufficiently small value λ, let the height function have, on the manifold $\tilde{Y}_\lambda l$, critical points of multiplicity $\mu_1, \ldots, \mu_l$. Since μ is the intersection index of the line $\lambda = 0$ with the discriminant in the extended parameter space $\mathbf{C}^{1+\tau}$, one has $\mu = \mu_1 + \ldots + \mu_l$.

It follows from the coherence of the sheaf $\pi_*\mathcal{Y}$ and the preparation theorem that for the codimensions of the corresponding projections $(\tau_1 + 1) + \ldots + (\tau_l + 1) \leqslant \tau + 1$.

Using Lemma 2.4, we have $\mu = \mu_1 + \ldots + \mu_l \leqslant (\tau_1 + 1) + \ldots + (\tau_l + 1) \leqslant \tau + 1$, while $\mu_i \leqslant \tau_i + 1$, $i = 1, \ldots, l$.

Taking as the deformed projection f, the singularity of Lemma 2.3, we get: $\mu = \tau + 1$. Consequently, $\mu_i = \tau_i + 1$ for all i.

Since any projection can be obtained by a small deformation of the projection of Lemma 2.3 (for suitable r), this finishes the proof of the theorem.

5. COROLLARY 2.5. Let $\tilde{F}$ be a representative of a miniversal deformation of an arbitrary projection f, and the value $\lambda' \in \mathbf{C}^\tau$ of the deformation parameter be sufficiently small. Then $\partial \tilde{F}/\partial u|_{\lambda=\lambda'}$, $\partial \tilde{F}/\partial \lambda_1|_{\lambda=\lambda'}, \ldots, \partial \tilde{F}/\partial \lambda_\tau|_{\lambda=\lambda'}$ is a basis for the $(\tau + 1)$-dimensional linear space $(\pi_*\mathcal{Y})_{\lambda'}/\mathfrak{m}_{\mathbf{C}^\tau,\lambda'}(\pi_*\mathcal{Y})_{\lambda'}$.

6. We consider representatives $\tilde{\Delta}$ and $\tilde{\Sigma}$ of the discriminant and bifurcation diagram of the projection f, and also the covering $\tilde{\Delta} \to \Lambda$, $(u, \lambda) \to \lambda$, where $\Lambda \subset \mathbf{C}^\tau$ is a sufficiently small neighborhood of zero.

COROLLARY 2.6. The tangent planes to the manifold $\tilde{\Delta}$ at points lying over a nonbifurcation value $\lambda' \subset \Lambda$, are in general position (form a coordinate cross after parallel transport to one point).

An analogous assertion has long been known for singularities of functions on smooth manifolds [12].

Proof. Let (x', u') be a Morse critical point of the height function on the manifold $\tilde{Y}_{\lambda'}$. The tangent space to $\tilde{\Delta}$ at (u', λ') is the image of the tangent space to $\tilde{C} \subset \mathbf{C}^{n+1+\tau}$ under projection along the x-direction. $\tilde{F} = 0$ is a part of the equations which define C. Hence the vector $\beta = (\beta_0, \beta_1, \ldots, \beta_\tau)$ lies in $T_{(u', \lambda')}\tilde{\Delta}$ only for

$$\beta_0 \partial \tilde{F}/\partial u\,|_{(x', u', \lambda')} + \sum_{i=1}^{\tau} \beta_i \partial \tilde{F}/\partial \lambda_i\,|_{(x', u', \lambda')} \equiv 0 \bmod (\partial \tilde{F}/\partial x_1\,|(x', u', \lambda'), \ldots, \partial \tilde{F}/\partial x_n\,|_{(x', u', \lambda')}).$$

Consequently, the vector β is tangent to $\tilde{\Delta}$ at all $\mu = \tau + 1$ points corresponding to critical values of the height function on $\tilde{Y}_{\lambda'}$, $\lambda' \notin \tilde{\Sigma}$, only if, in the stalk $(\pi_* \mathcal{Y})_{\lambda_1}$

$$\beta_0 \partial \tilde{F}/\partial u + \sum_{i=1}^{\tau} \beta_i \partial \tilde{F}/\partial \lambda_i \in \mathfrak{m}_{\Lambda', \lambda'} (\pi_* \mathcal{Y})_{\lambda'}.$$

But by Corollary 2.5, this is only possible for $\beta_0 = \beta_1 = \ldots = \beta_\tau = 0$.

3. Vector Fields Tangent to the Discriminant

1. Let $F \in \mathcal{O}^m_{\mathbf{C}^{n+1+\tau}, 0}$ be an R_+-miniversal deformation of the projection f. By virtue of the R_+-versality there exist decompositions

$$u\, \partial F/\partial \lambda_j \equiv \sum_{i=0}^{\tau} v_{ij}\, \partial F/\partial \lambda_i + \sum_{s=1}^{n} h_{sj} \partial F/\partial x_s \bmod F^* (\mathfrak{m}_{\mathbf{C}^m, 0}) \mathcal{O}^m_{\mathbf{C}^{n+1+\tau}, 0}, \quad j = 0, \ldots, \tau.$$

Here $v_{ij}(\lambda)$ and $h_{sj}(x, u, \lambda)$ are germs of holomorphic functions; $\lambda_0 = u$ but $\lambda = (\lambda_1, \ldots, \lambda_\tau)$.

2. Let δ_{ij} be the Kronecker symbol.

THEOREM 3.1. The algebra $\mathfrak{A}_\Delta$ of germs at $0 \in \mathbf{C}^{1+\tau}$ of holomorphic vector fields tangent to the discriminant Δ of the projection f, is generated by the fields $v_j = \sum_{i=0}^{\tau} (v_{ij} - \delta_{ij} u) \partial_{\lambda_i}$, $j = 0, \ldots, \tau$ as a free $\mathcal{O}_{\mathbf{C}^{1+\tau}, 0}$-module.

Proof. a) We lift the field v_j on $\mathbf{C}^{n+1+\tau}$ to a germ of a field $v_j' = v_j + \sum_{s=1}^{n} h_{sj} \partial_{x_s}$. It follows from the decompositions of point 1 that v_j' is tangent to the manifold $F = 0$. Hence, v_j is tangent to Δ.

b) The fields $v_0, \ldots, v_\tau$ are independent over the ring $\mathcal{O}_{\mathbf{C}^{1+\tau}, 0}$, because

$$v_0 \wedge v_1 \wedge \ldots \wedge v_\tau = \det (v_{ij} - \delta_{ij} u) \partial_u \wedge \partial_{\lambda_1} \wedge \ldots \wedge \partial_{\lambda_\tau},$$

where $\delta = \det(v_{ij} - \delta_{ij} u)$ is not identically zero on $\mathbf{C}^{1+\tau}$ (the functions v_{ij} do not depend on u).

c) $\delta = 0$ is the equation on the discriminant.

In fact, since the $\tau + 1$ vector fields $v_0, \ldots, v_\tau$ are tangent to the τ-dimensional manifold Δ one has $\{\delta = 0\} \supseteq \Delta$ and δ is divisible by the equation of Δ. But δ is a Weierstrass polynomial of degree $\mu = \tau + 1$ in the variable u. By Corollary 1.3, precisely this polynomial defines the discriminant.

d) We show that any field v, tangent to Δ, belongs to the module $\mathcal{O}_{\mathbf{C}^{1+\tau}, 0} \langle v_0, \ldots, v_\tau \rangle$.

Analogously to point c,

$$v_0 \wedge \ldots \wedge v_{j-1} \wedge v \wedge v_{j+1} \wedge \ldots \wedge v_\tau = \beta_j \delta \partial_u \wedge \partial_{\lambda_1} \wedge \ldots \wedge \partial_{\lambda_\tau},$$

where $\beta_j(u, \lambda)$ is a holomorphic function.

Then $v = \beta_0 v_0 + \ldots + \beta_\tau v_\tau$. In fact, $v - (\beta_0 v_0 + \ldots + \beta_\tau v_\tau) = 0$ outside Δ (since at any point outside Δ the vector fields $v_0, \ldots, v_\tau$ form a frame), and hence, by continuity this is also true everywhere in $\mathbf{C}^{1+\tau}$.

3. Acting by analogy with points 1 and 2, we give generators of the algebra of vector fields which preserve the discriminant of a complete intersection.

Let the map $G\colon (\mathbf{C}^{n+1+k}, 0) \to (\mathbf{C}^m, 0)$ give a miniversal deformation of the complete intersection $g_0 = 0$, $\lambda = (\lambda_0, \ldots, \lambda_k)$ be the deformation parameters, $x \in \mathbf{C}^n$. The discriminant of g_0: $\Delta \subset \mathbf{C}^{1+k}$ lies in the parameter space.

Let the deformation parameters be chosen so that the $0\lambda_0$ axis has finite intersection index μ with Δ. We denote by g the restriction of G to $\lambda_1 = \ldots = \lambda_k = 0$. By Theorem 2.1 we have

$$\mu = \dim_{\mathbf{C}} \mathcal{O}^m_{\mathbf{C}^{n+1},0} / \{g^* (\mathfrak{m}_{\mathbf{C}^m,0}) \mathcal{O}^m_{\mathbf{C}^{n+1},0} + \mathcal{O}_{\mathbf{C}^{n+1},0} \langle \partial g/\partial x_1, \ldots, \partial g/\partial x_n \rangle\}.$$

Since G is a miniversal deformation of the complete intersection $g_0 = 0$ one can take a basis of the μ-dimensional space considered in the form of the restrictions to $\lambda_1 = \ldots = \lambda_k = 0$ of the elements

$$\partial G/\partial\lambda_0, \ldots, \lambda_0^{\mu_0-1}\partial G/\partial\lambda_0,\ \partial G/\partial\lambda_1, \ldots, \lambda_0^{\mu_1-1}\partial G/\partial\lambda_1, \ldots, \partial G/\partial\lambda_k, \ldots, \lambda_0^{\mu_k-1}\partial G/\partial\lambda_k,$$

where $\mu_0 + \mu_1 + \ldots + \mu_k = \mu$.

This follows from the preparation theorem.

Now by the preparation theorem there exist decompositions

$$\lambda_0^{\mu_j}\partial G/\partial\lambda_j \equiv \sum_{i=0}^{k} v_{ij}\partial G/\partial\lambda_i + \sum_{s=1}^{n} h_{sj}\partial G/\partial x_s \bmod G^*(\mathfrak{m}_{\mathbf{C}^m,0})\mathcal{O}^m_{\mathbf{C}^{n+1+k},0}, \quad j = 0, \ldots, k,$$

where $h_{sj}(x, \lambda)$ are germs of holomorphic functions, $v_{ij}(\lambda)$ are polynomials in the variable λ_0 of degree strictly less than μ_i.

THEOREM 3.2. The algebra $\mathfrak{A}_\Delta$ of germs at $0 \in \mathbf{C}^{1+k}$ of holomorphic vector fields tangent to the discriminant Δ of the complete intersection $g_0 = 0$, is generated by fields $v_j = \sum_{i=0}^{k} (v_{ij} - \delta_{ij}\lambda_0^{\mu_j})\partial_{\lambda_i}$, $j = 0, \ldots, k$, as a free $\mathcal{O}_{\mathbf{C}^{1+k},0}$-module.

The proof completely repeats the proof of Theorem 3.1 (one uses the fact that $\delta = \det \times (v_{ij} - \delta_{ij}\lambda_0^{\mu_j})$ has degree μ in λ_0 and consequently $\delta = 0$ is the equation of the discriminant).

Remark. For m = 1 and $g = g_0 + \lambda_0$, Theorem 3.2 is Zakalyukin's theorem on vector fields which preserve fronts [5].

4. Example. On $\mathbf{C}^2$ we consider a multiple point of $g_0(x) = (x_1^p + x_2^q, x_1x_2) = 0$, $p \geqslant q \geqslant 2$. This is a complete intersection $I_{p,q}$ from Giusti's list [11]. A miniversal deformation is:

$$G(x, \lambda) = (x_1^p + \lambda_{p-1}x_1^{p-1} + \ldots + \lambda_1x_1 + \lambda_0 + x_2^q + \lambda_{p+q-2}x_2^{q-1} + \ldots + \lambda_px_2,\ x_1x_2 + \lambda_{p+q-1}).$$

This same map (with λ_0 replaced by u) defines an R_+-miniversal deformation of the projection $C_{2,2}$ [3] of a curve from $\mathbf{C}^3$ to a line. Hence $\mu = p + q$ and all $\mu_j = 1$. One can get a basis for $\mathfrak{A}_\Delta$ by Theorem 3.1 as well as by Theorem 3.2. For the initial singularities of the series we have

$$p = q = 2$$

$$\begin{aligned}
v_0 &= (2\lambda_0, \lambda_1, \lambda_2, 2\lambda_3);\\
v_1 &= (-6\lambda_2\lambda_3, 4\lambda_0 - \lambda_1^2, -8\lambda_3, \lambda_1\lambda_3);\\
v_2 &= (-6\lambda_1\lambda_3, -8\lambda_3, 4\lambda_0 - \lambda_2^2, \lambda_2\lambda_3);\\
v_3 &= (4\lambda_3 - 2\lambda_1\lambda_2, -3\lambda_2, -3\lambda_1, \lambda_0);
\end{aligned}$$

$$p = 3,\ q = 2$$

$$\begin{aligned}
v_0 &= (6\lambda_0, 4\lambda_1, 2\lambda_2, 3\lambda_3, 5\lambda_4);\\
v_1 &= (-12\lambda_3\lambda_4, 9\lambda_0 - \lambda_1\lambda_2, 6\lambda_1 - 2\lambda_2^2, -15\lambda_4, \lambda_2\lambda_4);\\
v_2 &= (-3\lambda_2\lambda_3\lambda_4 + 90\lambda_4^2, -6\lambda_1^2 + 2\lambda_1\lambda_2^2 - 36\lambda_3\lambda_4, 27\lambda_0 - 15\lambda_1\lambda_2 + 4\lambda_2^3,\\
&\qquad -6\lambda_2\lambda_4, 6\lambda_1\lambda_4 - 2\lambda_2^2\lambda_4);\\
v_3 &= (-6\lambda_1\lambda_4, -8\lambda_2\lambda_4, -10\lambda_4, 4\lambda_0 - \lambda_3^2, \lambda_3\lambda_4),\\
v_4 &= (-2\lambda_1\lambda_3 + 4\lambda_2\lambda_4, -3\lambda_2\lambda_3 + 5\lambda_4, -4\lambda_3, -3\lambda_1, \lambda_0).
\end{aligned}$$

The deformation of the singularity $I_{p,q}$ we have described is quasihomogeneous. Considering the weights and linear parts of the basis vector fields it is easy to prove the following assertions about normal forms with respect to the group of biholomorphisms of the space $\mathbf{C}^{p+q}$, which preserve the discriminant of $I_{p,q}$ (cf. [4, 5, 7]).

Proposition 3.3. The germ at $0 \in \mathbf{C}^{p+q}$ of a holomorphic function in general position for q = 2 reduces to normal form λ_{p-1} and for q > 2 has moduli.

Proposition 3.4. The germ at $0 \in \mathbf{C}^{p+q}$ of a hypersurface in general position for q = 2 reduces to normal form $\lambda_{p-1} = 0$ for q = 3 < p to normal form $\lambda_{p-1} + \lambda_{p+1} = 0$, and for the remaining values of p and q has moduli.

5. We formulate an assertion which gives more convenient formulas for generators of the module $\mathfrak{A}_\Delta$ in the case of a quasihomogeneous complete intersection.

This time let $G(x, \lambda)$ be a quasihomogeneous $(k + 1)$-parameter miniversal deformation of quasihomogeneous complete intersection $g_0 = 0$ of positive dimension, w_i be the weight of the parameter λ_i. The Euler field $e = w_0\lambda_0\partial_{\lambda_0} + \ldots + w_k\lambda_k\partial_{\lambda_k}$ preserves the discriminant. Let $\Phi(\lambda)$ be the matrix of multiplication by a function $\varphi(x)$ and the $\mathcal{O}_{\mathbf{C}^n,0}$-module

$$\mathcal{O}^m_{\mathbf{C}^{n+k+1},0}/\{G^*(\mathfrak{m}_{\mathbf{C}^m,0})\mathcal{O}^m_{\mathbf{C}^{n+k+1},0} + \mathcal{O}_{\mathbf{C}^{n+k+1},0}\langle\partial G/\partial x_1, \ldots, \partial G/\partial x_n\rangle\}$$

in the generators $\partial G/\partial\lambda_i$, $i = 0, \ldots, k$:

$$\varphi\partial G/\partial\lambda_j \equiv \sum_{i=0}^{k} \Phi_{ij}\partial G/\partial\lambda_i.$$

Φ exists due to the versality of G, but is defined up to addition to its columns, of columns composed of the components of any fields from $\mathfrak{A}_\Delta$.

Identifying a vector field on $\mathbf{C}^{k+1}$ with the column of height $k + 1$ of its components, it is easy to see that $\Phi e \in \mathfrak{A}_\Delta$.

Let I be the ideal in $\mathcal{O}_{\mathbf{C}^n,0}$, generated by the coordinate functions of the map g_0 and all m-minors of its Jacobian matrix. It follows from the quasihomogeneity of g_0 and the condition $n > m$ that the linear space $\mathcal{O}_{\mathbf{C}^n,0}/I$ is $(k + 1)$-dimensional [10]. Let $\varphi_0, \ldots, \varphi_k$ be representatives of a basis of this space, and $\Phi_0, \ldots, \Phi_k$ be the corresponding matrices of multiplication.

THEOREM. The fields $\Phi_0 e, \ldots, \Phi_k e$ are free generators of the $\mathcal{O}_{\mathbf{C}^{k+1},0}$-module $\mathfrak{A}_\Delta$ of vector fields tangent to the discriminant of an isolated singularity of the complete intersection $g_0 = 0$ of positive dimension.

The proof of this fact is given in the author's paper in Vol. 33 of the series "Current Problems of Mathematics" (Itogi Nauki i Tekhniki VINITI AN SSSR).

4. Vector Fields Tangent to the Bifurcation Diagram of a Projection

1. We continue to consider an R_+-miniversal deformation F of the projection f. Since F is versal, there exist decompositions

$$u^j\partial F/\partial u = \sum_{i=0}^{\tau} w_{ij}\partial F/\partial\lambda_i + \sum_{s=1}^{n} h_{sj}\partial F/\partial x_s \bmod F^*(\mathfrak{m}_{\mathbf{C}^m,0})\mathcal{O}^m_{\mathbf{C}^{n+1+\tau},0}, \quad j = 1, \ldots, \tau.$$

Here $w_{ij}(\lambda)$ and $h_{sj}(x, u, \lambda)$ are germs of holomorphic functions, $\lambda_0 = 0$ but $\lambda = (\lambda_1, \ldots, \lambda_\tau)$.

THEOREM 4.1. The algebra $\mathfrak{A}_\Sigma$ of germs at $0 \in \mathbf{C}^\tau$ of vector fields tangent to the bifurcation diagram Σ of the projection f, is generated by the fields $w_j = \sum_{i=1}^{\tau} w_{ij}\partial_{\lambda_i}$, $j = 1, \ldots, \tau$, as a free $\mathcal{O}_{\mathbf{C}^\tau,0}$-module.

The theorem follows from the three lemmas proved below.

2. LEMMA 4.2. The field w_j is tangent to Σ.

Proof. We lift w_j to a germ of a vector field on $\mathbf{C}^{n+1+\tau}$ $w''_j = w_j + (w_{0j} - u^j)\partial_u + \sum_{s=1}^{n} h_{sj}\partial_{x_s}$. It follows from the decompositions of point 1 that w''_j is tangent to the manifold $F = 0$. Consequently, the image of this field under projection in the x-direction on $\mathbf{C}^{1+\tau}$, the field $w'_j = w_j + (w_{0j} - u^j)\partial_u$, is tangent to the discriminant of the projection f. Consequently, the field w_j itself, obtained by projection of w'_j on $\mathbf{C}^\tau$ in the u-direction, is tangent to the branching manifold of the covering $\Delta \to \mathbf{C}^\tau$, i.e., to the set Σ.

3. LEMMA 4.3. The algebra $\mathfrak{A}_\Sigma$ is generated by the fields $w_1, \ldots, w_\tau$ as an $\mathcal{O}_{\mathbf{C}^\tau,0}$-module.

Proof. By a theorem of Lyashko [6], any germ of a vector field w, tangent to Σ lifts from $\mathbf{C}^\tau$ to $\mathbf{C}^{\tau+1}$, to a germ of a field w', preserving the discriminant Δ

$$w' = w - \chi\partial_u, \quad \chi = \chi(u, \lambda).$$

The function χ is defined at least up to the equation of the discriminant. By Corollary 1.3, χ can be assumed to be a polynomial in the variable u, of degree no higher than $\tau: \chi = \sum_{j=0}^{\tau} \chi_j(\lambda)u^j$.

We show that $w=\sum_{j=1}^{\tau}\chi_j w_j$.

For this, we consider on $\mathbf{C}_\tau^{1+\tau}$, the germ of a vector field

$$\eta=w'-\sum_{j=1}^{\tau}\chi_j w_j'.$$

This field is independent of u. In fact, for its λ-component this is obvious, and the coefficient of ∂_u is equal to

$$-\chi-\sum_{j=1}^{\tau}\chi_j(w_{0j}-u^j)=-\chi_0-\sum_{j=1}^{\tau}\chi_j w_{0j}.$$

On the other hand η is tangent to Δ. Hence, if we pass to representatives of the terms, then for $\lambda\notin\tilde{\Sigma}$ the vector $\tilde{\eta}(\lambda)$ is tangent to the discriminant at all $\mu=\tau+1$ points of the set $\tilde{\Delta}\cap\{\lambda=\text{const}\}$. It follows from this, by Corollary 2.6, that $\tilde{\eta}(\lambda)=0$. Since $\tilde{\Sigma}$ is a hypersurface in $\mathbf{C}^\tau$, one has that η is identically zero.

4. LEMMA 4.4. Among the fields $w_1,\ldots,w_\tau$ there are no relations.

Proof. Let us assume that a relation exists:

$$\sum_{j=1}^{\tau}\chi_j(\lambda)w_j=0.$$

We lift it to a relation on $\mathbf{C}^{1+\tau}$ among fields from $\mathfrak{A}_\Delta$:

$$\sum_{j=1}^{\tau}\chi_j(\lambda)w_j'-\chi(u,\lambda)\partial_u=0.$$

The fields $w_1',\ldots,w_\tau'$ are tangent to Δ. Hence the field $\chi\partial_u$ is also tangent to Δ. But for $\lambda\notin\tilde{\Sigma}$ the direction of the vector ∂_u is transverse to the manifold $\tilde{\Delta}$. Hence for $\lambda\notin\tilde{\Sigma}$, $\tilde{\chi}$ vanishes at any point of $\tilde{\Delta}\cap\{\lambda=\text{const}\}$. Consequently, $\chi\equiv0$ on Δ.

Now we consider, on the discriminant, the u-component of the relation among the fields

$$\sum_{j=1}^{\tau}\chi_j(\lambda)(w_{0j}(\lambda)-u^j).$$

This polynomial of degree no higher than τ in the variable u, vanishes identically on Δ. Hence, by Corollary 2.6, $\chi_1=\ldots=\chi_\tau=0$.

This finishes the proof of Lemma 4.4, and with it, that of Theorem 4.1 also.

Remark. For m = 1 and $f(x,u)=f_0(x)=u$ the assertion of Theorem 4.1 coincides with the assertion of a theorem of Bruce [8] on vector fields, tangent to the bifurcation diagram of the function f_0.

5. For practical calculations it is convenient to express the generators of the module $\mathfrak{A}_\Sigma$ in terms of the generators of the module $\mathfrak{A}_\Delta$. We show how to do this (cf. [14]).

Let $\pi\colon(x,u,\lambda)\mapsto\lambda$, $\pi_1\colon(u,\lambda)\mapsto\lambda$, $\lambda_0=u$, $\lambda=(\lambda_1,\ldots,\lambda_\tau)$ Let $V=(v_{ij})_{i,j=0}^{\tau}$ be the matrix of the decompositions of point 3.1, i.e., the matrix of multiplication by u in the $\mathcal{O}_{\mathbf{C}^\tau,0}$-module

$$\pi_*(\mathcal{O}^m_{\mathbf{C}^{n+1+\tau},0}/\{F^*(\mathfrak{m}_{\mathbf{C}^m,0})\mathcal{O}^m_{\mathbf{C}^{n+1+\tau},0}+\mathcal{O}_{\mathbf{C}^{n+1+\tau},0}\langle\partial F/\partial x_1,\ldots,\partial F/\partial x_n\rangle\}):$$

$$u\partial F/\partial\lambda_j\equiv\sum_{i=0}^{\tau}v_{ij}\partial F/\partial\lambda_i.$$

Writing a vector field as the column of its components, we get from Theorem 4.1

COROLLARY 4.5. Generators of the module $\mathfrak{A}_\Sigma$ are given by the formulas

$$w_j=\pi_{1*}[V^{j-1}\mathbf{v}_0],\ j=1,\ldots,\tau,$$

where $\mathbf{v}_0=\sum_{i=0}^{\tau}v_{i0}\partial_{\lambda_i}=v_0+u\partial_u$.

6. We consider the projection $C_{2,2}(x_1^2+x_2^2+u,\ x_1x_2)$.

Using the algorithm of Corollary 4.5 and the vector fields of point 3.4 ($p = q = 2$, $\lambda_0 = u$) we get basis fields which preserve Σ: $w_0 = (\lambda_1, \lambda_2, 2\lambda_3)$ is the Euler field;

$$w_1 = (\lambda_1^3 + 32\lambda_2\lambda_3,\ \lambda_2^3 + 32\lambda_1\lambda_3,\ -\lambda_1^2\lambda_3 - \lambda_2^2\lambda_3);$$

$$w_2 = (\lambda_1^5 - 36\lambda_1^2\lambda_2\lambda_3 + 288\lambda_1\lambda_3^2 - 4\lambda_2^3\lambda_3,\ \lambda_2^5 - 36\lambda_1\lambda_2^2\lambda_3 + 288\lambda_2\lambda_3^2 \cdot 4\lambda_1^3\lambda_3,\ 64\lambda_3^3 - 176\lambda_1\lambda_2\lambda_3^2 - \lambda_1^4\lambda_3 - \lambda_2^4\lambda_3).$$

The equation of the bifurcation diagram of $C_{2,2}$ (cf. [9]):

$$\det(w_{ij}) = (4096\lambda_3^3 + 768\lambda_1\lambda_2\lambda_3^2 + 27\lambda_1^4\lambda_3 - 6\lambda_1^2\lambda_2^2\lambda_3 + 27\lambda_2^4\lambda_3\ \lambda_1^3\lambda_2^3)(\lambda_1^2 - \lambda_2^2)\lambda_3 = 0.$$

The first, second, and third factors correspond, respectively, to the components Σ_d, Σ_m, and Σ_s of point 1.6.

Proposition 4.6. The germ at $0 \in \mathbf{C}^3$ in general position can be reduced to the normal form $\lambda_1 + \lambda_3 + \alpha\lambda_2 + \beta\lambda_2^2 = 0$, $\alpha, \beta =$ const by a biholomorphism of the space $\mathbf{C}^3$, which preserves the bifurcation diagram of the projection $C_{2,2}$.

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