BIFURCATION DIAGRAMS OF SOME SIMPLE AND QUASIHOMOGENEOUS SINGULARITIES

V. V. Goryunov

UDC 513.836+517.919

We recall the construction defining the bifurcation diagram of a function [1].

We consider the space of smooth functions. The group of diffeomorphisms acts on it (change of independent variables). The orbits of this action are infinite-dimensional submanifolds, but if the critical points of the function are not very complicated (namely, are of of finite multiplicity, i.e., are obtained by the confluence of a finite number of Morse critical points), then its orbit has finite codimension. In other words, through such a function as through a point of a function space, one can draw a finite-dimensional transversal to the orbit. The partition of the function space into orbits induces a partition of the transversal into submanifolds (with singularities). The union Σ of the submanifolds whose dimension is less than the dimension of the transversal (they correspond to functions with non-Morsean critical points or coincident critical values), is called the bifurcation diagram of the function.

One can give an analogous definition for germs of holomorphic functions. In this case the transversal is the germ of a μ-dimensional complex space (called the base of the versal deformation), where μ is the multiplicity of the critical point (that is, the number of merged Morse points).

Let us assume that the germ of a holomorphic function is simple, has no continuous invariants with respect to the group of substitutions of arguments. We consider the space $\mathbf{C}^\mu \setminus \Sigma$, the complement of the bifurcation diagram of such a function in the base of the versal deformation. About 10 years ago Lyashko and Looijenga proved that all the homotopy groups except the fundamental group of this space are trivial ([1, 9]).

This theorem was later extended to simple functions on a manifold with boundary [6], and also to simple projections onto a line [4]. In the present paper, using the technique of [6, 9], we show that the analogous assertion is valid in two more cases, for functions with simple linear singularities and for simple projections of a hypersurface with boundary onto a line (all the definitions needed are contained, respectively, in Secs. 1 and 2).

We shall dwell briefly on the objects considered in the paper.

It is known that in the classification of critical points of functions there arise in a natural way infinite series of singularities (A, D, T, etc., cf. [1]). Arnol'd indicated in [1] that the classification of the series themselves reflects the classification of singularities with nonisolated critical points (for example, the series of functions A_k, $k \geqslant 0$, having the form $x_1^{k+1} + x_2^2 + \ldots + x_n^2$ in suitable coordinates, corresponds to the function A_∞: $x_2^2 + \ldots + x_n$ with singularity on the line $x_2 = \ldots = x_n = 0$). In his recent paper [7], Siersma singled out the simplest among the nonisolated singularities, the so-called linear ones, with smooth one-dimensional critical set. Siersma also gave the classification of simple linear singularities, which is cited in Paragraph 1 of Sec. 1. We shall show that the start of the classification of linear singularities of functions onto the plane coincides with the classification of critical points of functions on a manifold with boundary. But the bifurcation diagrams which arise naturally in our case differ from the bifurcation diagrams of boundary singularities considered in [6]. Namely, for the complements of these new diagrams we shall also prove in Sec. 1 a theorem on the triviality of the higher homotopy groups.

In Sec. 2 we consider projections of a hypersurface with boundary onto a line, i.e., of hypersurfaces in $\mathbf{C}^m$ on which there is singled out a submanifold of codimension 1. Projections of such manifolds are a natural generalization of projections of hypersurfaces [4] and functions with a critical point on a manifold with boundary [2]. Hence it is not surprising that the start of the classification of projections of hypersurfaces with boundary which we give coincides in part with the classification of projections of hypersurfaces (without boundary), and in part with the classification of functions on a manifold with boundary. Here the

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 17, No. 2, pp. 23-37, April-June, 1983. Original article submitted June 4, 1982.

0016-2663/83/1702-0097$07.50

TABLE 1

n	Notation	Restriction	$f(x, y_1, \ldots, y_n)$	codim f	$BB_\sigma : \pi$
1	A_∞	—	y^2	0	—
	D_∞	—	y^2x	0	—
	$J_{\mu+1,\infty}$	$\mu\geqslant 1$	$y^2(y+x^{\mu+1})$	μ	$(\mu+1)3^{\mu-1}$
	$T_{\infty,\mu+2,2}$	$\mu\geqslant 2$	$y^2(y^\mu+x^2)$	μ	$2^\mu(\mu+2)^\mu$
	$Z_{\mu-2,\infty}$	$\mu\geqslant 3$	$y^2(xy+x^\mu)$	μ	$(3\mu-2)^\mu$
	$W_{1,\infty}$	—	$y^2(y^2+x^3)$	4	$2^8\cdot 3^3$
2	$T_{\infty,q,r}$	$q\geqslant r\geqslant 3$	$xy_1y_2+y_1^q+y_2^r$	$q+r-4$	$C_{q+r-4}^{q-2}r^{r-2}q^{q-2}$
	$Q_{\mu-1,\infty}$	$\mu\geqslant 3$	$xy_2^2+y_1^3+x^{\mu-1}y_1^2$	μ	$2\cdot 3^{\mu+1}(\mu-1)^\mu$
	$S_{1,\infty}$	—	$xy_2^2+y_1^2y_2+x^2y_1^2$	4	10^4

new bifurcation diagrams of simple singularities do not differ from the previous ones, the complements of which, as we already noted, are Eilenberg–MacLane spaces. But for those singularities which occur in our classification first, it turns out to be possible to prove the theorem that the germ of the complement of the bifurcation diagram is a space $k(\pi, 1)$.

Finally, in Sec. 3 we study the bifurcation diagrams of zeros of quasihomogeneous projections of complete intersections onto a line (here a complete intersection can have a boundary). It turns out that the analog of Zakalyukin's theorem about the stability of a vector field, transverse to the bifurcation diagram of zeros of a quasihomogeneous function with respect to the group of diffeomorphisms preserving the diagram [5] is also true for them. From this there follows as a corollary, Zakalyukin's theorem itself and its version for functions on a manifold with boundary.

The author expresses profound thanks to Arnol'd for posing the problem and for his constant interest in the work.

1. Simple Linear Singularities

1. In [7], Siersma introduced the concept of a function with linear singularity as a function whose critical set is a line. He got a list of simple germs of functions with isolated linear singularities. Each term of this list is obtained as the limit in a certain infinite series of isolated points singularities (cf. [1]). Up to stable equivalence [the function $f(x, y_1,\ldots,y_n)$ is stably equivalent with the function $f(x, y_1,\ldots,y_n) + y_{n+1}^2$] Siersma's list is given in Table 1.

Here $(x, y_1, \ldots, y_n) \in \mathbf{C}^{n+1}$, $\sigma = \operatorname{codim} f$ (cf. Paragraph 2), BB_σ is the generalized braid group of the series B [3]. The meaning of the last column is explained below.

In this section we define the bifurcation diagram $\Sigma \subset \mathbf{C}^\sigma$ of a function with an isolated linear singularity and we prove the theorem about the homotopy type of the space $\mathbf{C}^\sigma \setminus \Sigma$ for simple singularities.

THEOREM 1. The germ at zero of the complementary space to the bifurcation diagram of a simple function with an isolated linear singularity is a space $k(\pi, 1)$, where π is a subgroup of finite index (indicated in the last column of Table 1) in the group BB_σ.

2. We recall the definition of a function with isolated linear singularity. Let $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be the germ of a holomorphic function with smooth one-dimensional critical set L. We introduce in $\mathbf{C}^{n+1}$ coordinates $(x, y) \in \mathbf{C}^1 \times \mathbf{C}^n$, such that $L = \{y = 0\}$.

Definition. f has an isolated linear singularity if for any $x \neq 0$ the germ at the point $(x, 0)$ of the restriction of f to the plane $x = \text{const}$ is equivalent with the germ of the function $y_1^2 + \ldots + y_n^2$.

We denote by $\mathscr{E}_{x,y}$ the space of germs at 0 of holomorphic functions on $\mathbf{C}^{n+1}$, $(y) \subset \mathscr{E}_{x,y}$ is the ideal generated by the elements $y_1,\ldots,y_n$. If $f(0) = 0$ and f has L as critical set, then $f \in (y)^2$.

Let $\mathscr{G}$ be the (pseudo)group of germs of diffeomorphisms of the space $\mathbf{C}^{n+1}$, preserving the line $y = 0$. The tangent space to the $\mathscr{G}$-orbit of the function f at the point f in $(y)^2$

is $T_f = \mathcal{E}_{x,y} f_x + (y)\langle f_y\rangle$, where $f_x = \partial f/\partial x$, $\langle f_y\rangle = \langle f_{y_1}, \ldots, f_{y_n}\rangle$. We set $\operatorname{codim} f = \dim_{\mathbf{C}} (y)^2/T_f$.

3. We consider the question of the connection of the classifications of boundary and isolated linear singularities.

It is easy to see that the quasihomogeneous singularities of functions h on $\mathbf{C}^2$ with boundary $\mathbf{C}^1 = L$ (cf. [2]) and of quasihomogeneous isolated linear singularities of f for n = 1 are in one-to-one correspondence:

$$h(x, y) \sim f(x, y) = y^2 h(x, y).$$

Here $\operatorname{codim}_{\mathcal{E}_{x,y}} \mathcal{G}h = \operatorname{codim}_{(y)^2} \mathcal{G}f$.

For simple singularities the indicated correspondence is this:

$$1 \sim A_\infty,\quad A_0 \sim D_\infty,\quad A_\mu \sim J_{\mu+1,\infty},$$
$$B_\mu \sim T_{\infty,\mu+2,2},\quad C_\mu \sim Z_{\mu-2,\infty},\quad F_4 \sim W_{1,\infty}.$$

The coincidence considered of the classifications of quasihomogeneous functions extends to boundary singularities $\mathbf{C}^{m+1}$ and functions on $\mathbf{C}^{m+1}$, whose critical set is a hyperplane and whose restriction to almost any transversal to it is a Morse function.

Further, we consider a function on a manifold with boundary with isolated critical point 0 and critical value 0, stably equivalent with a function of two variables:

$$h'(v, u_1, \ldots, u_n) = h(v, u_1) + u_2^2 + \ldots + u_n^2,$$
$$(v, u_1, \ldots, u_n) \in \mathbf{C}^{n+1},\quad v = 0 \text{ — in the boundary}$$

The function v^2h' has critical set the hyperplane $v = 0$. Contracting this hyperplane to a line by the map $x = u_1$, $y_1 = v$, $y_2 = vu_2, \ldots, y_n = vu_n$, we get a function $y_1^2 h(x, y_1) + y_1^2 + \ldots + y_n^2$ with isolated linear singularity on $y = 0$, stably equivalent with a function of two variables.

By the same method, from the boundary singularity D_μ, $\mu \geqslant 3$: $v + u_1u_2^2 + u_1^{\mu-1}$ one makes a singularity $Q_{\mu-1,\infty}$, and from C_4: $vu_2 + u_1^2 + u_1u_2^2$ one makes the singularity $S_{1,\infty}$. We note that the codimensions of the singularities which correspond to one another coincide.

4. Definition. By a k-parametric deformation of the function $f \in (y)^2$ is meant the germ of a holomorphic map from $(\mathbf{C}^k, 0)$ to $((y)^2, f)$.

One can introduce in a natural way the concept of versal deformation of a function f with isolated linear singularity (in this case $\operatorname{codim} f < \infty$, cf. [7]). One can show that

$$F(x, y, \lambda) = f(x, y) + \sum_{i=1}^{\sigma} \lambda_i e_i(x, y),\quad \lambda \in \mathbf{C}^\sigma,$$

is a miniversal deformation, where $\sigma = \operatorname{codim} f$, $e_1, \ldots, e_\sigma \in (y)^2$ are representatives of a $\mathbf{C}$-basis of the space $(y)^2/T_f$.

Let $\tilde{F}$ be a representative of F. For a generic value of the parameter λ the function $\tilde{F}^\lambda = \tilde{F}|_{\lambda=\text{const}}$ has on L only singularities A_∞ and D_∞ (the critical value zero corresponds to them), and outside L only A_1 (while the corresponding critical values of $\tilde{F}^\lambda$ are distinct and different from zero). For almost all sufficiently small λ the number of critical values of the function $\tilde{F}^\lambda$ is the same. Let this number be a. We shall show below that at least for n = 1, $a = \sigma + 1$.

Definition. The germ at zero Σ of the set $\tilde{\Sigma} \subset \mathbf{C}^\sigma$ of those values of the parameter λ, for which the function $\tilde{F}^\lambda$ has less than a critical values, is called the bifurcation diagram of the function f with isolated linear singularity.

The diagram Σ consists of four components, $\Sigma = \bigcup_1^4 \Sigma_i$: Σ_1 corresponds to the appearance in $\tilde{F}^\lambda$ of a singularity A_2 outside L; Σ_2 to a singularity $J_{2,\infty}$ on L; Σ_3 corresponds to the critical value 0 of the function $\tilde{F}^\lambda$ outside L; Σ_4 to the coincidence of critical values $\tilde{F}^\lambda$ outside L.

Examples.

a) $J_{2,\infty}$. $F = y^2(y + x^2 + \lambda_1)$. $\Sigma = \{0\}$.

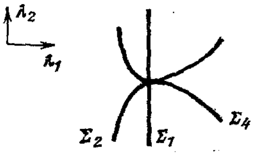


Fig. 1

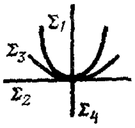


Fig. 2

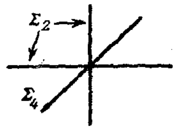


Fig. 3

b) $J_{3,\infty}$. $F = y^2(y + x^3 + \lambda_1 x + \lambda_2)$. $\Sigma_1 = \{\lambda_1 = 0\}$, $\Sigma_2 = \{27\lambda_2^2 + 4\lambda_1^3 = 0\}$, $\Sigma_3 = \varnothing$, $\Sigma_4 = \{81\lambda_2^2 - 4\lambda_1^3 = 0\}$. (Fig. 1).

c) $T_{\infty,4,2}$. $F = y^2(x^2 + y^2 + \lambda_1 y + \lambda_2)$. $\Sigma_1 = \{32\lambda_2 = 9\lambda_1^2\}$, $\Sigma_2 = \{\lambda_2 = 0\}$, $\Sigma_3 = \{4\lambda_2 = \lambda_1^2\}$, $\Sigma_4 = \{\lambda_1 = 0\}$. (Fig. 2).

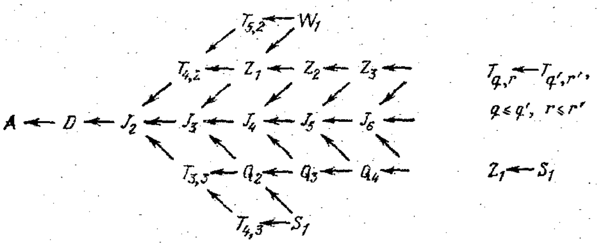
d) $T_{\infty,3,3}$. $F = xy_1y_2 + y_1^3 + y_2^3 + \lambda_1 y_1^2 + \lambda_2 y_2^2$: $\Sigma_1 = \varnothing$, $\Sigma_2 = \{\lambda_1\lambda_2 = 0\}$. $\Sigma_3 = \varnothing$, $\Sigma_4 = \{\lambda_1^3 = \lambda_2^3\}$. (Fig. 3).

Remark. Let f be a quasihomogeneous function of two variables with isolated linear singularity or $f \in Q_{\mu-1,\infty}$, $S_{1,\infty}$, and h be its dual function in the sense of Paragraph 3 on a manifold with boundary. It is easy to see that the correspondence indicated in Paragraph 3 extends to miniversal quasihomogeneous deformations of f and h. Here the spaces of parameters are mapped isomorphically and it turns out that $\Sigma_2 \cup \Sigma_3$ is the bifurcation diagram of zeros of the boundary singularity h.

If $f \in T_{\infty,q,r}$, then $\sigma = q + r - 4$ and $\mathbf{C}^\sigma \setminus (\Sigma_2 \cup \Sigma_3) = (\mathbf{C}^{q-2} \setminus W_{q-2}) \times (\mathbf{C}^{r-2} \setminus W_{r-2})$, where $W_\nu \subset \mathbf{C}^\nu$ is the bifurcation diagram of zeros of the boundary singularity B_ν.

5. Using the lists of contiguities of point and boundary singularities [1, 2], one can show that one has

Proposition 1. All contiguities of simple isolated linear singularities with one another are obtained by composition of a finite number of contiguities indicated in the diagram (the index ∞ is omitted).



One also has

Proposition 2. All contiguities of simple linear singularities to point singularities are obtained by composition of contiguities of linear singularities to one another, of simple point singularities to one another, and contiguities listed below:

$A_\sigma \leftarrow X$, where X is any of the simple linear singularities except $T_{\infty,q,r}$, $q \geqslant r \geqslant 3$, σ is the codimension of X;

$$W_{1,\infty} \rightarrow D_4 \leftarrow S_{1,\infty}, \quad D_\mu \leftarrow Q_{\mu-1,\infty}.$$

6. Let an isolated linear singularity of f split into r singular points of type D_∞ on $L = \{y = 0\}$ and s singular points of type A_1 outside L.

Conjecture (Siersma [7]).

$$s = \dim_{\mathbf{C}} (y)^2/\{\mathscr{E}_{x,y} f_x + (y)\langle f_y\rangle\}, \text{ i.e., } \quad s = \sigma;$$
$$r + s = \dim_{\mathbf{C}} (y)/\mathscr{E}_{x,y}\langle f_x, f_y\rangle.$$

Proposition 3. The conjecture is true for n = 1.

Proof. Let $F: (\mathbf{C}^{n+1} \times \mathbf{C}^\sigma, 0) \rightarrow (\mathbf{C}, 0)$, $\lambda \in \mathbf{C}^\sigma$, be a versal deformation of the function f. For n = 1, $f(x, y) = y^2 h(x, y)$ and $F(x, y, \lambda) = y^2 H(x, y, \lambda)$, where $H \in \mathscr{E}_{x,y,\lambda}$, $h = H|_{\lambda=0}$.

s is the intersection index of the plane $\lambda = 0$ with the germ of the surface $M \subset \mathbf{C}^{n+1} \times \mathbf{C}^{\sigma}$, which is the closure of the germ of the set $\{F_x = 0,\ F_y\ 0,\ y \neq 0\}$; r is the intersection index of $\lambda = 0$ and the germ of $M' = \{y = 0,\ H = 0\}$.

$$M = \{H_x = 0,\ 2H + yH_y = 0\},$$
$$M \cup M' = \{yH_x = 0,\ 2H + yH_y = 0\}.$$

Whence

$$s = \dim_{\mathbf{C}} \mathscr{E}_{x,y,\lambda}/\mathscr{E}_{x,y,\lambda} \langle H_x,\ 2H + yH_y,\ \lambda\rangle =$$
$$= \dim_{\mathbf{C}} \mathscr{E}_{x,y}/\mathscr{E}_{x,y} \langle h_x,\ 2h + yh_y\rangle = \dim_{\mathbf{C}} (y)^2/\mathscr{E}_{x,y} \langle y^2 h_x,\ \ 2y^2 h + y^3 h_y\rangle = \dim_{\mathbf{C}} (y)^2/\mathscr{E}_{x,y} \langle f_x,\ yf_y\rangle = \sigma;$$
$$r + s = \dim_{\mathbf{C}} \mathscr{E}_{x,y,\lambda}/\mathscr{E}_{x,y,\lambda} \langle yH_x,\ 2H + yH_y,\ \lambda\rangle = \dim_{\mathbf{C}} (y)/\mathscr{E}_{x,y} \langle y^2 h_x,\ 2yh + y^2 h_y\rangle = \dim_{\mathbf{C}} (y)/\mathscr{E}_{x,y} \langle f_x,\ f_y\rangle.$$

Direct calculation shows that Siersma's conjecture is also valid for simple linear singularities.

7. Proof of Theorem 1. It is evident from Table 1 that any simple linear singularity has quasihomogeneous normal form f, having the following property. If $e_1,\dots,e_\sigma$ is a monomial basis of the space $(y)^2/T_f$, then the weights of all arguments of the corresponding miniversal deformation $F(x, y, \lambda) = f(x, y) + \sum_1^{\sigma} \lambda_i e_i(x, y)$ are positive.

The quasihomogeneous map F and diagram $\Sigma \subset \mathbf{C}^{\sigma}$ are defined globally. We shall show that $\mathbf{C}^{\sigma} \setminus \Sigma$ is a space $k(\pi, 1)$.

If $\lambda \notin \Sigma$, then by Paragraph 5 the function $F^{\lambda} = F|_{\lambda=\mathrm{const}}$ has exactly σ critical points of type A_1 and at them it assumes distinct and nonzero values $z_1,\dots,z_\sigma$. We construct a polynomial of degree σ with leading coefficient 1 and roots $z_1,\dots,z_\sigma$. We get a quasihomogeneous map $\varphi\colon \mathbf{C}^{\sigma} \setminus \Sigma \to \mathbf{C}^{\sigma} \setminus \Xi$, where $\mathbf{C}^{\sigma} \setminus \Xi$ is the space of polynomials of the form $z^{\sigma} + \alpha_1 z^{\sigma-1} + \dots + \alpha_\sigma$, $\alpha \in \mathbf{C}^{\sigma}$, without multiple and zero roots. $\mathbf{C}^{\sigma} \setminus \Xi$ is the classifying space of the generalized braid group of the series B: $\mathbf{C}^{\sigma} \setminus \Xi = k(BB_\sigma, 1)$ [3].

We shall show that φ is a covering.

φ extends continuously to strata of the highest dimension of the diagram Σ, and hence also to a quasihomogeneous map $\Phi\colon (\mathbf{C}^{\sigma}, 0) \to (\mathbf{C}^{\sigma}, 0)$.

LEMMA. $\Phi^{-1}(0) = \{0\}$.

Proof. a) $f \notin T_{\infty,q,r}$, $q \geqslant r \geqslant 3$.

Let $H(v, u, \lambda)$ be the quasihomogeneous miniversal deformation corresponding to F of the boundary singularity h, dual to f (cf. Paragraph 3 and the remark of Paragraph 4). $\Phi^{-1}(0)$ belongs to the bifurcation diagram of zeros of h. It is easy to see that if 0 is the unique critical value of the function F^{λ}, then 0 is also the unique critical value of H^{λ} as a function on a manifold with boundary. Since h is simple and the Dynkin diagram of a simple function on a manifold with boundary is connected, H^{λ} has a unique critical point of multiplicity $\operatorname{codim} h = \operatorname{codim} f = \sigma$, while at it $H^{\lambda} = 0$. The miniversality and quasihomogeneity of H now imply that $\lambda = 0$.

b) $f \in T_{\infty,q,r}$, $q \geqslant r \geqslant 3$.

$$F(x, y, \lambda) = xy_1y_2 + y_1^q + \lambda_1 y_1^{q-1} + \dots + \lambda_{q-2} y_1^2 + y_2^r + \lambda_{q-1} y_2^{r-1} + \dots + \lambda_{q+r-4} y_2^2 = xy_1y_2 + \alpha(y_1, \lambda) y_1^2 + \beta(y_2, \lambda) y_2^2.$$

For $\lambda \notin \Sigma$ the critical points of F^{λ} outside L are

$$\{x = 0, y_1 = 0,\ (\beta^{\lambda})' y_2 + 2\beta^{\lambda} = 0\} \cup \{x = 0,\ y_2 = 0,\ (\alpha^{\lambda})' y_1 + 2\alpha^{\lambda} = 0\}.$$

On the first of these subsets $F^{\lambda} = \beta^{\lambda} y_2^2$, on the second $F^{\lambda} = \alpha^{\lambda} y_1^2$. Hence $\lambda \in \Phi^{-1}(0)$ if and only if $\beta^{\lambda} y_2^2 = 0$ everywhere on $(\beta^{\lambda} y_2^2)' = 0$, and $\alpha^{\lambda} y_1^2 = 0$ on $(\alpha^{\lambda} y_1^2)' = 0$. But the polynomial p(t), which vanishes at all zeros of its derivative, has the form $A(t - a)^k$, $A, a \in \mathbf{C}$. Consequently, $\beta^{\lambda} y_2^2 = y_2^r$ and $\alpha^{\lambda} y_1^2 = y_1^q$, i.e., $\Phi^{-1}(0) = \{0\}$.

The lemma is proved.

From the lemma and the positivity of the weights of the arguments and the coordinate functions of the map Φ it follows that φ is proper.

To prove the theorem now it suffices to show that φ is a local diffeomorphism.

In the space $\mathbf{C}^1 \times \mathbf{C}^1 \times \mathbf{C}^n \times \mathbf{C}^\sigma$ with coordinates (z, x, y, λ) we consider the surface N which is the closure of the set $\{z = F(x, y, \lambda),\ F_x = 0,\ F_y = 0,\ y \neq 0\}$. We set $N^\lambda = N \cap \{\lambda = \mathrm{const}\}$.

Let $\lambda \notin \Sigma$. Then $N^\lambda = \{p_i\}_1^\sigma$, while for all $i \neq j$, $z(p_i) \neq z(p_j) \neq 0$, $y(p_i) \neq 0$. The map φ is constructed from the numbers $z_i = z(p_i)$. It is easy to see that since $z_i \neq z_j$, the map φ is degenerate at the point $\lambda \in \mathbf{C}^\sigma \setminus \Sigma$ if and only if there exist vectors $v_i = (0, (dx)_i, (dy)_i, d\lambda)$, $d\lambda \neq 0$ tangent to N at the points p_i, $i = 1,\ldots,\sigma$.

The tangency condition has the form

$$v_i \in \operatorname{Ker}\begin{pmatrix} -1 & F_x & F_y & F_\lambda \\ 0 & F_{xx} & F_{xy} & F_{x\lambda} \\ 0 & F_{xy} & F_{yy} & F_{y\lambda} \end{pmatrix}\Bigg|_{p_i}.$$

Since F^λ has a singularity A_1 at the point $(x(p_i), y(p_i))$, one has $F_x(p_i) = F_y(p_i) = 0$ and $\det\begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy}\end{pmatrix}\Big|_{p_i} \neq 0$. Hence the condition of degeneracy of φ assumes the form

$$F_\lambda(p_i)d\lambda = 0, \quad i = 1, \ldots, \sigma, \quad d\lambda \neq 0.$$

This is equivalent with the degeneracy of the matrix $(F_{\lambda_j}(p_i))_{i,j=1}^\sigma = (e_j(p_i))_{i,j=1}^\sigma$, i.e., the linear dependence of the functions $e_1,\ldots,e_\sigma$ on N^λ.

We shall show that as a matter of fact, $\{e_j\}$ is a $\mathbf{C}$-basis of the σ-dimensional function space $\mathcal{O}^\lambda$ on N^λ.

a) $n = 1$. $f = y^2h$; $F = y^2H$, h and H are polynomials. $N^\lambda = \{H_x^\lambda = 0,\ 2H^\lambda + yH_y^\lambda = 0\}$. $\mathcal{O} = \mathbf{C}[x, y]/\mathbf{C}[x, y]\langle H_x^\lambda, 2H^\lambda + yH_y^\lambda\rangle$, where $\mathbf{C}[x, y]$ is the space of polynomials in the variables x, y.

Since the linear singularity f is isolated, the ideal $T_f = \mathcal{E}_{x,y}\langle f_x, yf_y\rangle$ contains some power of the maximal ideal $\mathfrak{m}_{x,y} \subset \mathcal{E}_{x,y}$, multiplied by y^2 [7]. Hence the monomials $e_1,\ldots,e_\sigma$ are a basis not only of the space $(y)^2/T_f$, but also of the space $\mathbf{C}[x, y]y^2/\mathbf{C}[x, y]\langle f_x, yf_y\rangle$. Consequently, $e_1y^{-2},\ldots,e_\sigma y^{-2}$ is a basis of $\mathbf{C}[x, y]/\mathbf{C}[x, y]\langle h_x, 2h + yh_y\rangle$, and hence also of the space $\mathcal{O}^\lambda$ for sufficiently small (by quasihomogeneity also for any) $\lambda \notin \Sigma$. Since the function y is invertible on N^λ, one has that $\{e_j\}$ is a basis of $\mathcal{O}^\lambda$.

b) $n = 2$. We consider $f \in Q_{\sigma-1,\infty}$. The cases $f \in T_{\infty,\,,r}$ and $f \in S_{1,\infty}$ are analyzed analogously.

$$F = y_1^3 + xy_2^2 + gy_1^2 + \lambda_\sigma y_1 y_2,$$
$$g = x^{\sigma-1} + \lambda_1 x^{\sigma-2} + \ldots + \lambda_{\sigma-1}.$$

Let $\lambda_\sigma \neq 0$. The coordinates of the points $p_1,\ldots,p_\sigma$ of the set N^λ are determined by the conditions

$$g'(x)\,y_1^2 + y_2^2 = 0, \quad 3y_1^2 + 2g(x)\,y_1 + \lambda_\sigma y_2 = 0,$$
$$2xy_2 + \lambda_\sigma y_1 = 0, \quad y \neq 0$$

or $4g'(x)\,y_1^2 + \lambda_\sigma^2 = 0$, $y_1 = (\lambda_\sigma^2 - 4g(x)\,x)/6x$, $y_2 = -\lambda_\sigma y_1/2x$.

The condition $\lambda \notin \Sigma \cup \{\lambda_\sigma = 0\}$ implies for all $i \neq k$, $x(p_i) \neq x(p_k) \neq 0 \neq y_1(p_i)$.

Hence the functions x and y_1 are invertible on N^λ and

$$\det(e_j(p_i)) \neq 0 \Leftrightarrow \det((e_jxy_1^{-2})(p_i)) \neq 0.$$

But $\{e_jxy_1^{-2}\} = \{x, x^2, \ldots, x^{\sigma-1}, xy_2y_1^{-1}\}$ and on N^λ $xy_2y_1^{-1} = -\lambda_\sigma/2$. Since $x(p_i) \neq x(p_k)$ for $i \neq k$, one has $\det((e_jxy_1^{-2})(p_i)) \neq 0$.

Now let $\lambda_\sigma = 0$. Then

$$N^\lambda = \left\{g'(x) = 0,\ y_1 = -\frac{2}{3}g(x),\ y_2 = 0\right\} \cup \{x = 0,\ 3y_1 + 2\lambda_{\sigma-1} = 0,\ \lambda_{\sigma-2}y_1^2 + y_2^2 = 0\}.$$

For $\lambda \notin \Sigma$ the function y_2 is invertible on N^λ and

$$\det(e_j(p_i)) \neq 0 \Leftrightarrow \det((e_j y_1^{-2})(p_i)) \neq 0,$$

$$\{e_j y_1^{-2}\} = \{1, x, \ldots, x^{\sigma-2}, \quad y_2 y_1^{-1}\}.$$

$$((e_j y_1^{-2})(p_i)) = \begin{pmatrix} 1 & x_k^r & 0 \\ 1 & 0 & (-\lambda_{\sigma-2})^{1/2} \\ 1 & 0 & -(-\lambda_{\sigma-2})^{1/2} \end{pmatrix}, \qquad k, r = 1, \ldots, \sigma-2.$$

Here $\{x_k\}$ are the roots of the polynomial g', $\deg g' = \sigma - 2$.

Since for $\lambda \notin \Sigma \cup \{\lambda_{\sigma-2} = 0\}$ the roots of g' are distinct and different from zero, for $\lambda_{\sigma-2} \neq 0$ the matrix written down is nondegenerate.

Thus, on $\mathbf{C}^\sigma \setminus (\Sigma \cup \{\lambda_\sigma = \lambda_{\sigma-2} = 0\})$, and hence also everywhere in $\mathbf{C}^\sigma \setminus \Sigma$, φ is a local diffeomorphism.

Thus, we have shown that $\mathbf{C}^\sigma \setminus \Sigma = k(\pi, 1)$, where π is a subgroup of finite index in the group BB_σ. The index of π is sought as the degree of the quasihomogeneous map φ. If $\beta_1, \ldots, \beta_\sigma$ are the weights of the parameters $\lambda_1, \ldots, \lambda_\sigma$, β_0 is the weight of f, then β_0, $2\beta_0, \ldots, \sigma\beta_0$ are the weights of the coordinate functions of φ and

$$(BB_\sigma : \pi) = \sigma!\, \beta_0^\sigma \Big/ \prod_1^\sigma \beta_i.$$

2. Projections of Hypersurfaces with Boundary onto a Line

In this section we consider the problem of classification of projections of hypersurfaces with boundary onto a line, we list all simple objects of this classification and for them we prove a theorem on the homotopy type of the complementary space to the bifurcation diagram, analogous to Theorem 1 on linear singularities.

1. Definition. By the boundary of a hypersurface $S \subset \mathbf{C}^{n+1}$, $n > 1$, is meant a submanifold of ∂S of codimension 1.

By a projection of a hypersurface S with boundary ∂S onto a line is meant a diagram

$$\partial S \to S \to \mathbf{C}^{n+1} \xrightarrow{\Pi} \mathbf{C}^1,$$

where the first two arrows are imbeddings, Π is a projection. An equivalence of two such projections is a commutative diagram

$$\begin{array}{ccccccc} \partial S & \to & S & \to & \mathbf{C}^{n+1} & \xrightarrow{\Pi} & \mathbf{C}^1 \\ & & & & \downarrow h & & \downarrow k \\ \partial S_1 & \to & S_1 & \to & \mathbf{C}^{n+1} & \xrightarrow{\Pi_1} & \mathbf{C}^1, \end{array}$$

where h and k are diffeomorphisms and $h(S, \partial S) = (S_1, \partial S_1)$.

One can give analogous definitions for germs.

2. We introduce in $\mathbf{C}^{n+1}$ coordinates $(x, u) \in \mathbf{C}^n \times \mathbf{C}^1$, in which the projection Π can be written as $\Pi(x, u) = u$. It will be assumed that the germs at 0 of the manifolds S and ∂S are complete intersections: $S = \{f_1 = 0\}$, $\partial S = \{f_1 = f_2 = 0\}$, where f_1 and f_2 are elements of the maximal ideal $\mathfrak{m}_{x,u} \subset \mathscr{E}_{x,u}$. The germ at 0 of the projection $(x, u) \to u$ of the hypersurface $f_1(x, u) = 0$ with boundary $f_1(x, u) = f_2(x, u) = 0$ will be called a boundary projection f.

Let $\mathscr{E}^m_{x,u}$ be the set of all germs at 0 of holomorphic mappings from $\mathbf{C}^{n+1}$ to $\mathbf{C}^m$. The space $\mathfrak{m}_{x,u}\mathscr{E}^2_{x,u}$ splits into equivalence classes of boundary projections: projections $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are considered equivalent if and only if there exist $a, b, c \in \mathscr{E}_{x,u}$, $ac \notin \mathfrak{m}_{x,u}$, and a germ of a diffeomorphism $h: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$, $h(x, u) = (h_0(x, u), k(u))$, such that $h^*g_1 = af_1$, $h^*g_2 = bf_1 + cf_2$.

Definition. The germ of a boundary projection is simple if it has no moduli (continuous invariants) with respect to the equivalence introduced.

We set $T'_f = \mathscr{E}_{x,u}\langle (f_1, 0), (0, f_1), (0, f_2), f_x \rangle + \mathscr{E}_u f_u$ and we introduce the codimension of the boundary projection f: $\nu = \dim_{\mathbf{C}} \mathscr{E}^2_{x,u}/T'_f$.

One can prove by traditional methods of the theory of singularities

Proposition 4. Any germ of a projection of a hypersurface with boundary onto a line, which is simple, is equivalent the germ at zero of a projection $(x, u) \to u$ of a hypersurface $f = 0$ with boundary $f_1 = f_2 = 0$, where $f = (f_1, f_2)$ is one of the maps of Table 2.

Remarks. a) The problem considered here can be posed not only for hypersurfaces, but also for complete intersections of arbitrary codimension. One can show that up to stable equivalent of projections [4] any simple projection of a complete intersection with boundary onto a line is a projection of a hypersurface with boundary.

b) The whole classification of germs at 0 of projections of hypersurfaces $f_1 = 0$ with boundary $f_1 = f_2 = 0$ from $\mathbf{C}^{n+1}$ onto $\mathbf{C}^1$, $(x, u) \to u$, such that $f_{1x}(0) \neq 0$, is equivalent with the classification of projections of hypersurfaces without boundary from $\mathbf{C}^n$ to $\mathbf{C}^1$. Now the classification of boundary projections (f_1, f_2), such that $f_{1x}(0) = 0$, but $f_{2x}(0) \neq 0$ and $f_{1u}(0) \neq 0$, is equivalent with the problem of RL-classification of germs of functions onto an n-dimensional with boundary.

Definition. The boundary projection f abuts the boundary projection g, $g \leftarrow f$, if f lies in the closure of the equivalence class of g.

Proposition 5. All abutments of simple boundary projections are obtained by compositions of a finite number of the following:

a) $Y_\mu \leftarrow Z_{\mu'}$, $Y, Z = A, B, C, D, E, F \Leftrightarrow$ there is just such an abutment of projections of hypersurfaces [4];

b) $Y^*_\mu \leftarrow Z^*_{\mu'}$, $Y, Z = A, B, C, F$ (here $A^*_\mu = A_\mu$) $\Leftrightarrow Y_\mu \leftarrow Z_{\mu'}$ as functions on a manifold with boundary [2];

c)
$$\begin{array}{ccc} B_l \leftarrow & X_{k,l} & \rightarrow B^*_k \\ & \downarrow & \\ & X_{k',l'}, & k \geqslant k', \quad l \geqslant l'. \end{array}$$

3. The concepts of deformation and versal deformation of a boundary projection can be introduced in the traditional way. Here it turns out that, for example, the deformation

$F(x, u, \lambda) = f(x, u) + \sum_1^\nu \lambda_i e_i(x, u)$, where $\{e_i\} \subset \mathscr{E}^2_{x,u}$ are representatives of a $\mathbf{C}$-basis of the space $\mathscr{E}^2_{x,u}/T'_f$ is a miniversal deformation of the boundary projection $f = (f_1, f_2)$ of finite codimension ν.

Let F be a representative of F, $\tilde{F}^{u,\lambda} = \tilde{F}|_{(u,\lambda)=\text{const}}$. We denote by $\tilde{W}_1 \subset \mathbf{C}^{1-\nu}$ the set of those values of the parameters (u, λ) for which the function $\tilde{F}^{u,\lambda}_1$ has critical value 0, and

TABLE 2

n	Notation	$f=(f_1, f_2)$	ν
$\geqslant 2$	A_0	(x_n, x_1)	0
$\geqslant 1$	A_1, B_μ	$(x_n, f_2(x_1, \ldots, x_{n-1}, u))$, f_2 is a simple projection $(x_1, \ldots, x_{n-1}, u) \mapsto u$ of a hypersurface onto a line, which is of the same name as the boundary projection [4]	$\mu-1$
$\geqslant 2$	A_μ, $\mu \geqslant 2$, C_μ, F_4		
$\geqslant 3$	D_μ, E_μ		
$\geqslant 1$	B^*_μ, $\mu \geqslant 2$	$(x_1^\mu + x_2^2 + \ldots + x_n^2 + u, x_1)$	
$\geqslant 2$	C^*_μ, $\mu \geqslant 3$	$(x_1 x_2 + x_2^\mu + x_3^2 + \ldots + x_n^2 + u, x_1)$	
	F^*_4	$(x_1^2 + x_2^3 + x_3^2 + \ldots + x_n^2 + u, x_1)$	
1	$X_{k,l}$, $k, l \geqslant 2$	$(x^k + u, x^l)$	$k+l-2$

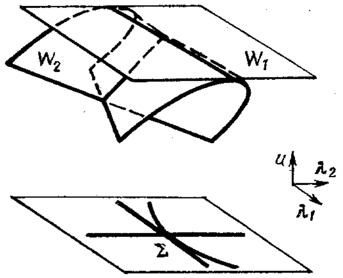


Fig. 4

by $\tilde{W}_2$ the analogous set for the mappings $\tilde{F}^{u,\lambda}$. For example, for n = 1, $W_2 = \{(u, \lambda) | \tilde{F}^{u,\lambda}$ assumes the value 0}.

Definition. The germ at $0 \in \mathbf{C}^{1+\nu}$, W of the set $\tilde{W} = \tilde{W}_1 \cup \tilde{W}_2$ is called the bifurcation diagram of zeros of the boundary projection f.

For almost all sufficiently small values of the parameter λ the set $\tilde{W} \cap \{\lambda = \mathrm{const}\}$ consists of the same number of points. We denote by $\tilde{\Sigma} \subset \mathbf{C}^\nu$ the set of those λ, for which this number is smaller.

Definition. The germ at $0 \in \mathbf{C}^\nu$, Σ of the set $\tilde{\Sigma}$ is called the bifurcation diagram of the boundary projection f.

Example. $f \in X_{2,2}$. $F = (x^2 + u,\ x^2 + \lambda_1 x + \lambda_2)$. $W = \{u = 0\} \cup \{(\lambda_2 - u)^2 + u\lambda_1^2 = 0\}$, $\Sigma = \{\lambda_1\lambda_2 \times (\lambda_2 - 1/4\lambda_1^2) = 0\}$ (cf. Fig. 4).

We note that for boundary projections A_μ, B_μ, C_μ, D_μ, E_μ, F_4 the bifurcation diagrams (of zeros) we have introduced coincide with the bifurcation diagrams (of zeros) of mononomial projections of hypersurfaces [4], and for singularities B_μ^*, C_μ^*, F_4^*, with the bifurcation diagrams (of zeros) of functions $f_1|_{u=0}$ on the manifold $\mathbf{C}^n$ with boundary $x_1 = 0$ (these are, respectively, functions B_μ, C_μ, F_4, cf. [2]).

4. THEOREM 2. For a simple boundary projection the germ at zero of the space $\mathbf{C}^\nu \setminus \Sigma$ is a space $k(\pi, 1)$, where π is a subgroup of finite index in the braid group on $(\nu + 1)$ strings.

Proof. From what was said at the end of the preceding paragraph and [4, 6] the validity of the assertion of the theorem follows for all singularities except the series $X_{k,l}$.

We shall show that the theorem is also true for $f \in X_{k,l}$. In this case $\nu = k + l - 2$, and as miniversal deformation one can take

$$F = (x^k + \lambda_1 x^{k-1} + \ldots + \lambda_{k-1}x + u,\ x^l + \lambda_k x^{l-2} + \ldots + \lambda_{k+l-2}) = (p + u,\ q).$$

The quasihomogeneous map F, the weights of all of whose arguments are positive, and the diagrams W and Σ are defined globally. We shall show that $\mathbf{C}^{k+l-2} \setminus \Sigma$ is a space $k(\pi, 1)$.

The bifurcation diagram of zeros of $X_{k,l}$ is $W = \{(u, \lambda) \in \mathbf{C}^{k+l-1} \mid \exists x : p^\lambda(x) + u = 0,\ p_x^\lambda(x) q^\lambda(x) = 0\}$, where $p^\lambda = p|_{\lambda=\mathrm{const}}$, $p_x = \partial p/\partial x$, etc. We see that for $\lambda \notin \Sigma$ the set $W^\lambda = W \cap \{\lambda = \mathrm{const}\}$ consists of $k + l - 1$ distinct points: $W^\lambda = \{(u_i, \lambda)\}$. We construct the polynomial of degree $k + l - 1$ with roots $\{u_i - (k+l-1)^{-1} \sum_1^{k+l-1} u_j\}$ and leading coefficient 1.

We get a map φ: $\mathbf{C}^{k+l-2} \setminus \Sigma \to \mathbf{C}^{k+l-2} \setminus \Xi$, where the latter space is the set of polynomials of the form $z^{k+l-1} + \alpha_1 z^{k+l-3} + \ldots + \alpha_{k+l-2}$, $\alpha \in \mathbf{C}^{k+l-2}$, without multiple roots. $\mathbf{C}^{k+l-2} \setminus \Xi = k(B(k+l-1), 1)$.

We shall show that φ is a covering.

As in Sec. 1, φ extends to a map Φ: $\mathbf{C}^{k+l-2} \to \mathbf{C}^{k+l-2}$ and the fact that φ is proper follows from the fact that $\Phi^{-1}(0) = \{0\}$: $\lambda \in \Phi^{-1}(0) \Leftrightarrow p^\lambda = c = \mathrm{const}$ on $p_x^\lambda q^\lambda = 0$, whence $p^\lambda = (x - a)^k + c$, $a, c \in \mathbf{C}$, and $q^\lambda = (x - a)^l$; since the sum of the roots of q^λ is zero and $p^\lambda(0) = 0$, one has $a = c = 0$, i.e., $\lambda = 0$.

Repeating the argument of Theorem 1, we see that the degeneracy of φ at some point $\lambda \notin \Sigma$ is equivalent with the existence of the vector $d\lambda \neq 0$ and vectors du, $(dx)_j$ such that

$$(du + p_\lambda d\lambda)|_{x_i,\lambda} = 0, \quad i = 1, \ldots, k-1,$$
$$(du + p_\lambda d\lambda + p_x (dx)_j)|_{x_j\lambda} = 0, \quad (q_\lambda d\lambda + q_x (dx)_j)|_{x_j,\lambda} = 0,$$
$$j = k, \ldots, k+l-1,$$

where $\{x_i\}_1^{k-1}$ are the roots of the polynomial p_x^λ, and $\{x_j\}_k^{k+l-1}$ are the roots of q^λ.

We shall show that this cannot hold.

For fixed λ, du and $d\lambda$, $du + p_\lambda d\lambda$ is a polynomial in x of degree not higher than $k-1$. $x_1, \ldots, x_{k-1}$ are its roots. All these roots are distinct and are roots of the polynomial p_x^λ, having degree $k-1$. Hence $(du + p_\lambda d\lambda)^\lambda = \beta p_x^\lambda$, $\beta \in \mathbf{C}$.

Since $\lambda \notin \Sigma$, at the points $x_j, j = k, \ldots, k+l-1$, $p_x^\lambda q_x^\lambda \neq 0$. Consequently, $(dx)_j = (-q_\lambda d\lambda / q_x)|_{x_j,\lambda}$ and $(\beta p_x - p_x q_\lambda d\lambda/q_x)|_{x_j,\lambda} = 0$, whence $(\beta q_x - q_\lambda d\lambda)|_{x_j,\lambda} = 0$. We conclude from the form of the polynomial $(\beta q_x - q_\lambda d\lambda)^\lambda$, which has degree in x no higher than $l-1$ and l distinct roots $\{x_j\}$, we conclude that $\beta = d\lambda_k = \ldots = d\lambda_{k+l-2} = 0$. Whence $(du + p_\lambda d\lambda)^\lambda \equiv 0$ and $d\lambda_1 = \ldots = d\lambda_{k-1} = du = 0$.

Thus, φ is a covering.

The index of the group π in the braid group can be calculated as the degree of the quasihomogeneous map φ: $(B(k+l-1):\pi) = C_{k+l-1}^l k^{k+l-2}$.

3. Stability of Vector Fields

Now we analyze the question of the stability of a vector field defined in a neighborhood of the bifurcation diagram of zeros of a quasihomogeneous projection onto $\mathbf{C}^1$. Everything said below in this connection transfers in an obvious way to the case of a projection of a complete intersection with boundary onto a line.

1. We consider the germ at zero of the projection from $\mathbf{C}^{n+1}$ to $\mathbf{C}^1$, $(x, u) \to u$ of the surface $f(x, u) = 0$ (without boundary), $f \in \mathcal{E}_{x,u}^m, n+1 \geqslant m$ [4]. We shall call it the projection f.

Let $F: (\mathbf{C}^{n+1+\nu}, 0) \to (\mathbf{C}^m, 0)$ be a miniversal deformation of the projection f, $\lambda \in \mathbf{C}^\nu$ be the parameter of the deformation. For example, $F(x,u,\lambda) = f(x,u) + \sum_1^\nu \lambda_i e_i(x,u)$, where $e_1, \ldots, e_\nu \in \mathcal{E}_{x,u}^m$ are representatives of a $\mathbf{C}$-basis of the space $\mathcal{E}_{x,u}^m / \{f^*(\mathfrak{m}(m))\mathcal{E}_{x,u}^m + \mathcal{E}_{x,u}\langle f_x\rangle + \mathcal{E}_u f_u\}, \mathfrak{m}(m) = \{h: (\mathbf{C}^m, 0) \to (\mathbf{C}, 0)\}$.

We recall that the *bifurcation diagram of zeros* of the projection f is the set $W \subset \mathbf{C}^{1+\nu}$ of critical values of the projection $(x, u, \lambda) \to (u, \lambda)$, restricted to $F = 0$ (cf. the definition of the bifurcation diagram of zeros of a boundary projection). For example, for $m = n+1$, W is the image of this restriction.

For a quasihomogeneous map F the diagram W is defined globally.

2. THEOREM 3. Let the miniversal deformation F of the projection f be quasihomogeneous. Then the germ at $0 \in \mathbf{C}^{1+\nu}$ of the field ∂_u is stable: If v is a vector field which is sufficiently close to ∂_u, then there exists a point $q \in \mathbf{C}^{1+\nu}$ close to zero and a germ of a diffeomorphism $H: (\mathbf{C}^{1+\nu}, W, 0) \to (\mathbf{C}^{1+\nu}, W, q)$, carrying the germ at zero of the field ∂_u into the germ of v at the point q.

Proof. We shall not impose the requirement of quasihomogeneity on F yet.

Let $\tilde{F}$ and $\tilde{W}$ be representatives of F and W, v be a vector field without singular points, defined on $\mathbf{C}^{1+\nu}$ in the same neighborhood U of the point 0 as $\tilde{W}$. In U we introduce coordinates (u', λ'), $u'(0) = 0$, $\lambda'(0) = 0$: $\lambda' \in \mathbf{C}^\nu$ is a parameter indexing the phase curves of the field v, u' is the time of motion along the field v from some smooth hypersurface passing through $0 \in \mathbf{C}^{1+\nu}$ and transverse to the field v; $v = \partial_{u'}$.

We denote by $\tilde{G}$ the new coordinate description of the map $\tilde{F}$: $\tilde{G}(x, u', \lambda') = F(x, u(u', \lambda'), \lambda(u', \lambda'))$. This description does not preserve the projection $\mathbf{C}^{1+\nu} \to \mathbf{C}^{\nu}$, $(u, \lambda) \mapsto \lambda$.

We consider $\tilde{G}$ as a representative of a ν-parametric deformation of the projection $(x, u') \to u'$ of the surface $G^{\lambda'} = 0$. $\tilde{W}$ is a representative of the bifurcation diagram of zeros of this projection.

Let the field v be sufficiently close to ∂_u. Then the coordinates (u', λ') can be chosen to differ slightly from (u, λ), and since F is a versal deformation of the projection f, close to $0 \in \mathbf{C}^{n+1+\nu}$ there exists a point (x_0, u_0, λ_0') such that:

a) the germ G of the map $\tilde{G}$ at the point (x_0, u_0', λ_0') is a versal deformation of the germ at the point (x_0, u_0') of the projection $(x, u') \to u'$ of the surface $\tilde{G}^{\lambda_0} = 0$;

b) the indicated germ of a projection is equivalent with the germ at zero of the projection $(x, u) \to u$ of the surface $f(x, u) = 0$.

This follows from the property of stability of a versal deformation of a projection, which can be proved using the technique of Wasserman [8].

The terms G and F are equivalent as miniversal deformations of equivalent projections. We note that it follows in particular from this that the germ at zero of the direction field ∂_u on $\mathbf{C}^{1+\nu}$ is stable (the quasihomogeneity of F is not required).

An equivalence of G and F carries the germ of $\tilde{W}$ at (u_0', λ_0') into the germ of $\tilde{W}$ at zero. Hence in what follows in clarifying the question of stability of the vector field ∂_u we shall consider only germs at $0 \in \mathbf{C}^{1+\nu}$ close to ∂_u of vector fields $v = \partial_{u'}$, having the same singularity as ∂_u, i.e.:

i) the germs at zero of the projections f and $g = \tilde{G}|_{\lambda'=0}$ are equivalent;

ii) the germ at zero G of the map $\tilde{G}$ is a miniversal deformation of the projection g.

When are ∂_u and $\partial_{u'}$ carried into one another by the germ at 0 of a diffeomorphism of the pair $(\mathbf{C}^{1+\nu}, W)$? This holds, for example, if on $(\mathbf{C}^{1+\nu}, W)$ there exists a change of coordinates $u = u' + a(\lambda')$, $\lambda = \lambda(\lambda')$, $a(0) = 0, \lambda(0) = 0$, i.e., if one has

$$G(x', u', \lambda') = M(x', u', \lambda')\, F(x(x', u', \lambda'),\ u + a(\lambda'),\ \lambda(\lambda')) \qquad (1)$$

for some germ of a matrix M, $\det M(0) \neq 0$, and some change of variables $x = x(x', u', \lambda')$, $x(0) = 0$ (the coordinate description of the diagram W is independent of multiplication of F by M and changes of x-coordinates).

The tangent space at the point $F \in \mathcal{E}^m_{x,u,\lambda}$ to the set of mappings which are induced from F by (1) is

$$R = F^*(\mathfrak{m}(m))\,\mathcal{E}^m_{x,u,\lambda} + \mathfrak{m}_{x,u,\lambda}\langle F_x\rangle + \mathfrak{m}_\lambda \langle F_u, F_\lambda\rangle.$$

We compare R with the tangent space T to the set of those germs G for which i) and ii) hold. Here we should again take into account the arbitrariness in the choice of coordinates x and the fact that the germ F is defined up to multiplication by a germ of a matrix.

We have the obvious inclusion

$$T \supseteq F^*(\mathfrak{m}(m))\,\mathcal{E}^m_{x,u,\lambda} + \mathfrak{m}_{x,u,\lambda}\langle F_x\rangle + \mathfrak{m}_{u,\lambda} F_u + \mathfrak{m}_\lambda \langle F_\lambda\rangle. \qquad (2)$$

The versality of F implies $T + \langle F_x, \quad F_u, F_\lambda\rangle_{\mathbf{C}} = \mathcal{E}^m_{x,u,\lambda}$, i.e., $\operatorname{codim}_{\mathcal{E}^m_{x,u,\lambda}} T \leqslant n + 1 + \nu$. On the other hand, by i)

$$T|_{\lambda=0} = f^*(\mathfrak{m}(m))\,\mathcal{E}^m_{x,u} + \mathfrak{m}_{x,u}\langle f_x\rangle + \mathfrak{m}_u f_u.$$

From the miniversality of F $T|_{\lambda=0} + \langle f_x, f_u, F_\lambda|_{\lambda=0}\rangle_{\mathbf{C}} = \mathcal{E}^m_{x,u}$ and $\operatorname{codim}_{\mathcal{E}^m_{x,u}} T|_{\lambda=0} = n + 1 + \nu$.

Whence $\operatorname{codim}_{\mathcal{E}^m_{x,u,\lambda}} T \geqslant n + 1 + \nu$.

Hence the inclusion of (2) is in fact an equality.

The modality of the vector field ∂_u with respect to the group of formal diffeomorphisms $(\mathbf{C}^{1+\nu}, W, 0)$ in the class of fields for which i) and ii) hold, does not exceed the number $\delta = \dim_{\mathbf{C}} T/R$, which coincides with the number $\dim_{\mathbf{C}} \mathcal{E}^m_{x,u,\lambda} / \{F^*(\mathfrak{m}(m))\,\mathcal{E}^m_{x,u,\lambda} + \mathcal{E}_{x,u,\lambda}\langle F_x\rangle + \mathcal{E}_\lambda \langle F_u, F_\lambda\rangle\}$.

Let us assume that F is a quasihomogeneous. Then $\mathfrak{m}_u f_u \subset f^*(\mathfrak{m}(m))\,\mathcal{E}^m_{x,u} + \mathcal{E}_{x,u}\langle f_x\rangle$ and $\mathcal{E}^m_{x,u} = f^*(\mathfrak{m}(m))\,\mathcal{E}^m_{x,u} + \mathcal{E}_{x,u}\langle f_x\rangle + \langle f_u, F_\lambda|_{\lambda=0}\rangle_{\mathbf{C}}$.

Hence, $\mathscr{E}^m_{x,u,\lambda} = F^*(\mathfrak{m}(m))\mathscr{E}^m_{x,u,\lambda} + \mathscr{E}_{x,u,\lambda}\langle F_x\rangle + \mathscr{E}_\lambda\langle F_u, F_\lambda\rangle + \mathfrak{m}_\lambda \mathscr{E}^m_{x,u,\lambda}$.

By Wasserman's lemma [8] the last summand can be omitted. Hence $\delta = 0$.

Thus, in the formal case Theorem 3 is proved. Its validity in the holomorphic situation follows from the fact that if

$$\dim_{\mathbf{C}} \mathscr{E}^m_{x,u,\lambda} / \{F^*(\mathfrak{m}(m))\mathscr{E}^m_{x,u,\lambda} + \mathscr{E}_{x,u,\lambda}\langle F_x\rangle + \mathscr{E}_\lambda\langle F_u, F_\lambda\rangle\} < \infty,$$

then F is finitely defined with respect to the group of germs at $0 \in \mathbf{C}^{n+1+v}$ of biholomorphisms of the form $(x, u, \lambda) \mapsto (x'(x, u, \lambda), u + a(\lambda), \lambda'(\lambda))$ and multiplication of F by germs of matrices $M(x, u, \lambda)$, $\det M(0) \neq 0$. But again the last assertion is proved with the help of the technique of [8].

3. COROLLARY (Zakalyukin [5]). Let $F(x, \lambda) = f_0(x) + \lambda_1 + \sum_2^\mu \lambda_i \varphi_i(x)$ be a quasihomogeneous miniversal deformation of the function f_0 with isolated critical point 0, $W \subset \mathbf{C}^\mu$ be the bifurcation diagram of zeros of f_0. Then the germ at $0 \in \mathbf{C}^\mu$ of the vector field ∂_{λ_1} is stable with respect to the group of germs at zero of diffeomorphisms of the pair $(\mathbf{C}^\mu, W)$.

Proof. F is a miniversal deformation of the germ at zero of the projection $(x, \lambda_1) \to \lambda_1$ of the surface $f_0(x) + \lambda_1 = 0$, $\lambda_2, \ldots, \lambda_n$ are the parameters of the deformation. The bifurcation diagram of zeros of this projection is precisely W.

We note that an assertion analogous to the corollary is also true for quasihomogeneous functions on a manifold with boundary. To prove it one must note that if $F(x, \lambda) = f_0(x) + \lambda_1 + \sum_2^\mu \lambda_i \varphi_i(x)$ is a quasihomogeneous miniversal deformation of the function f_0 on the manifold $\mathbf{C}^n$ with boundary $x_1 = 0$, then $(F(x, \lambda), x_1)$ is a miniversal deformation of the projection $(x, \lambda_1) \to \lambda_1$ of the hypersurface $f_0(x) + \lambda_1 = 0$ with boundary $f_0(x) + \lambda_1 = x_1 = 0$ (cf. Remark b) of Paragraph 2 of Sec. 2). Here the bifurcation diagrams of zeros of the function on the manifold with boundary and of the boundary projection coincide. It remains to apply the version of Theorem 3 for boundary projections.

LITERATURE CITED

1. V. I. Arnol'd, "Critical points of smooth functions and their normal forms," Usp. Mat. Nauk, 30, No. 5, 3-65 (1975).
2. V. I. Arnol'd, "Critical points of functions on a manifold with boundary, simple Lie groups B_k, C_k, F_4 and singularities of evolutes," Usp. Mat. Nauk, 33, No. 5, 91-107 (1978).
3. E. Brieskorn, "Braid groups (a la V. I. Arnol'd)," Matematika, 18, No. 3, 46-59 (1974).
4. V. V. Goryunov, "Geometry of bifurcation diagrams of simple projections onto a line," Funkts. Anal., 15, No. 2, 1-8 (1981).
5. V. M. Zakalyukin, "Legendre mappings in Hamiltonian systems," in: Some Problems of Mechanics [in Russian], Moscow Aviation Inst. (1979), pp. 11-15.
6. O. V. Lyashko, "Geometry of bifurcation diagrams," Usp. Mat. Nauk, 34, No. 3, 205-206 (1979).
7. D. Siersma, "Isolated line singularities," Utrecht Univ., Dept. of Math. (1981), Preprint 217.
8. G. Wassermann, "Stability of unfoldings in space and time," Acta Math., 135, 58-108 (1975).
9. E. Looijenga, "The complement of the bifurcation variety of a simple singularity," Invent. Math., 32, 105-116 (1974).