GEOMETRY OF BIFURCATION DIAGRAMS OF SIMPLE PROJECTIONS ONTO THE LINE

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A projection of a submanifold V of a fibered space E into the base B is a triple $V \to E \to B$, consisting of an inclusion and a projection (cf. [5]). The germ of a projection is simple if it has no moduli (continuous invariants with respect to the natural equivalence).

Below there is given a list of all simple germs of projections (not necessarily smooth) of manifolds onto the line. The list consists of hypersurfaces and curves in three-dimensional space. Hypersurfaces are given by functions with a simple critical point on a manifold with boundary (cf. [3]): the boundary is the preimage of zero under the projection. It is proved that the germ of the complement of the bifurcation diagram of a simple complex projection onto the line is a space $k(\pi, 1)$, where π is a subgroup of the group of braids. This is a generalization of the theorem of Lyashko–Looijenga (cf. [2, 9]), corresponding to the case when V is a smooth hypersurface. For the boundary singularities B_μ, C_μ and F_4 our generalization differs from the generalization given by Lyashko in [6] (the bifurcation diagrams and groups π are different).

We also prove a theorem on the normal form of the germ of a vector field, which is an extension of a theorem of Lyashko on straightening a vector field by a diffeomorphism, preserving the bifurcation diagram of zeros of a simple function (cf. [6]), to the case when the vector field at a point of a cuspidal edge of the discriminant manifold is tangent to this manifold.

1. Classification of Simple Projections onto a Line

We consider the trivial fibration $\mathbf{C}^n \times \mathbf{C}^p \to \mathbf{C}^p$. We shall denote points of the fiber $\mathbf{C}^n$ by x, and points of the base $\mathbf{C}^p$ by u. A submanifold V is defined by a system of m equations $f_1(x, u) = 0, \ldots, f_m(x, u) = 0$. Below X_μ denotes one of the simple singularities of functions of n variables (X = A, D, or E; cf. [1]). q denotes $x_2^2 + \ldots + x_n^2$. The following proposition constitutes part of the classification obtained by the author of germs of projections onto.

Proposition 1. A germ of a projection onto the line ($p = 1$) is simple if and only if it is stably equivalent with the germ at zero of the projection $(x, u) \to u$ of one of the manifolds $f = 0$, listed in Table 1.

Here $|X_\mu|$ and N are the order and Coxeter number of the Weyl group X_μ. The meaning of the numbers μ_0, μ' and ν is explained below.

One can show that the list of all germs of surfaces of positive dimension in the space $\mathbf{C}^{n+1}$, simple with respect to the group of diffeomorphisms of $\mathbf{C}^{n+1}$, preserving the plane $u = 0$, coincides with our list.

It is convenient to assume that $B_1 = A_1$, $C_{l+1}^{1,l} = C_{l+1}$, $C_2^{1,1} = B_2$ and $F_3 = B_3$.

There exist the following contiguities of projections of curves of Proposition 1:

$$\begin{tikzcd} A_\mu \arrow[r] & A_{\mu-1},\ \mu>0 \\ C_{k+l}^{k,l} \arrow[r] \arrow[d] \arrow[dr] & C_{k+l-1}^{k-1,l},\ 1<k\leqslant l \\ A_{k+l-1} & C_{k+l-1}^{k,l-1},\ 1\leqslant k<l \end{tikzcd} \qquad \begin{tikzcd} B_\mu \arrow[r] & B_{\mu-1},\ \mu>1 \\ F_\mu \arrow[r] \arrow[d] & F_{\mu-1},\ \mu>3 \\ C_{\mu-1}^{s,\mu-s-1},\ s=1,\ldots,\left[\frac{\mu-1}{2}\right] & \end{tikzcd}$$

All possible contiguities for $n = m$ are exhausted by those enumerated up to transitivity ($A \to B$, $B \to C \Rightarrow A \to C$).

2. Definitions of Equivalence and Simplicity

1. By an equivalence of projections $V_i \to E_i \to B_i$, $i = 1, 2$, is meant a commutative 3×2-diagram, whose verticals are diffeomorphisms $h: E_1 \to E_2$, $k: B_1 \to B_2$, such that $hV_1 = V_2$.

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TABLE 1

m	n	Notation	f	Restrictions	μ_0	μ'	ν	$B(\nu+1):\pi$
1	$\geqslant 1$	A_0	x_1	—	0	0	0	1
		X_μ	$X_\mu + u$	$\mu > 0$; $n \geqslant 2$ for D, E	μ	0	$\mu - 1$	$N^\mu \mu! / \mid X_\mu \mid$
		B_μ	$x_1^2 + u^\mu + q$	$\mu \geqslant 2$	1	$\mu - 1$	$\mu - 1$	1
		C_μ	$x_1^\mu + ux_1 + q$	$\mu \geqslant 3$	$\mu - 1$	1	$\mu - 1$	$(\mu - 1)^\mu$
		F_4	$x_1^3 + u^2 + q$	—	2	2	3	3^3
2	2	$C_{k+l}^{k,l}$	$x_1x_2,\ x_1^k + x_2^l + u$	$2 \leqslant k \leqslant l$	$k+l-1$	1	$k+l-1$	$\dfrac{k^k l^l (k+l-1)!}{(k-1)!\,(l-1)!}$
		F_{2k+1}	$x_1^2 + x_2^3,\ x_2^k + u$	$k \geqslant 2$	$2k - 1$	2	$2k$	$\dfrac{\nu^{\nu+1}(\nu^2 - 1)}{24}$
		F_{2k+4}	$x_1^2 + x_2^3,\ x_1x_2^k + u$	$k \geqslant 1$	$2k + 2$	2	$2k + 3$	

2. A suspension of a projection $V \to E \to B$ is a projection $V \to E' \to B$, where V is included in E' as a submanifold of the space of a subbundle $E \subset E'$.

3. A stable equivalence of projections is an equivalence of suspensions.

4. Analogous definitions are given for germs. A germ V at $0 \in \mathbf{C}^n \times \mathbf{C}^p$ is given by a system of m equations $f_1 = 0, \ldots, f_m = 0$, where $f_1, \ldots, f_m$ are germs of holomorphic functions of x and u, equal to 0 at the origin. The system $g = 0$ gives the same germ V as f if $g = Mf$, where M is an appropriate square matrix, holomorphic at zero, and $\det M(0) \neq 0$.

Caution. Two systems of generators of one ideal with different numbers of elements give different germs V.

5. We consider the canonical projection $\mathbf{C}^n \times \mathbf{C}^p \to \mathbf{C}^p$, $(x, u) \to u$. An equivalence of given systems $f = 0$ and $g = 0$ of germs of projections at 0 is a germ of a local diffeomorphism h, leaving the origin fixed, of the form $h(x, u) = (a(x, u), b(u))$, for which $h^*g = Mf$.

6. Analogous to points 4 and 5, definitions are given for formal series f, h, M.

7. A germ of a projection is called simple, if a sufficiently small neighborhood of it intersects only a finite number of equivalence classes.

More precisely, a germ is simple, if there exists a finite collection of equivalence classes such that for any k, any k-jet sufficiently close to the k-jet at the origin, giving a germ of the collection of functions f, is a k-jet of the collection, giving the germ of a projection of one of the classes indicated.

8. A germ of a projection is called a germ of a projection onto, if the number of equations defining the germ V does not exceed the dimension of the fiber: in the notation introduced above $m \leq n$. In this case the projection of the germ V at zero will, in general, be the whole germ $(\mathbf{C}^p, 0)$.

3. Bifurcation Diagrams of Simple Projections onto a Line

Below the germ of the projection $(x, u) \to u$ of the manifold defined by the system of equations $f(x, u) = 0$ will be called the projection f for short.

1. Let $\mathscr{E}^m(n+p) = \mathscr{E}^m_{x,u}$ be the space of germs at zero of holomorphic mappings from $\mathbf{C}^n \times \mathbf{C}^p$ into $\mathbf{C}^m$, $\mathscr{E}^1_{x,u} = \mathscr{E}_{x,u}$; $\mathfrak{m}(m)$ be the maximal ideal in $\mathscr{E}^1(m) = \mathscr{E}(m)$.

For $f \in \mathscr{E}^m_{x,u}$ we set

$$Q(f) = \mathscr{E}^m_{x,u}/\{f^*(\mathfrak{m}(m)) \cdot \mathscr{E}^m_{x,u} + \mathscr{E}_{x,u}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle + \mathscr{E}_u\langle \partial f/\partial u_1, \ldots, \partial f/\partial u_p\rangle\},\quad \nu = \dim_{\mathbf{C}} Q(f).$$

Germs of projections at points q_1 and q_2 are t-equivalent if they become equivalent after translation of the points q_1 and q_2 to 0.

The number ν is the codimension of the t-equivalence class of the projection f in the space $\mathscr{E}^m_{x,u}$.

2. For projections one can, in a natural way, introduce the concept of miniversal deformation (cf. [1]). Let η be the parameter of the deformation. One has the following proposition.

Proposition 2. If $\nu < \infty$ and $\xi_1, \ldots, \xi_\nu$ are representatives of a **C**-basis of space Q(f), then the deformation F,

$$F(x, u, \eta) = f(x, u) + \eta_1\xi_1(x, u) + \ldots + \eta_\nu\xi_\nu(x, u),$$

is a miniversal deformation of the projection f.

3. Let $S = \bigcup S^i$ be a stratified space, $\{S^i\}$ be its strata, $\dim S^i = i$. If L, $\dim L = r$, is a smooth manifold, then by the visible contour of the stratification S under the map $G: S \to L$, $G_i = G|_{S^i}$, is meant the set $\{l \in L \mid l = G_i(s), (\mathrm{rk}\ G_i)|_s < r\}$.

4. We consider a miniversal deformation of the projection f as a $(p + \nu)$-parameter deformation of the map $f_0 = f|_{u=0}$. In the space $\mathbf{C}^{p+\nu}$ there is the bifurcation diagram of zeros Δ (cf. [2]) of the map f_0. The space Δ is stratified by the strata μ = const and their mutual intersections. Let Δ^i be the union of all i-dimensional strata. $\Delta = \bigcup \Delta^i$, $i = 0, \ldots, p + \nu - 1$.

The visible contour Σ of the stratification Δ under the projection $P: \mathbf{C}^{p+\nu} \to \mathbf{C}^\nu$, $(u, \eta) \to \eta$, we call the bifurcation diagram of the projection f.

5. THEOREM 1. If f is a simple projection onto the line, then the germ of the space $\mathbf{C}^\nu \setminus \Sigma$ is a space $k(\pi, 1)$, where π is a subgroup of finite index (indicated in the table) of the group $B(\nu + 1)$ of braids of $(\nu + 1)$ threads.

Proof. Let the projection be reduced to normal form of Proposition 1. We consider its quasihomogeneous (cf. [7]) miniversal deformation F, written in the form indicated in Proposition 2. We shall prove that in this case the space $\mathbf{C}^\nu \setminus \Sigma$ has type $k(\pi, 1)$.

Let $\bar{\eta} \notin \Sigma$. It is easy to show that then the set $P^{-1}(\bar{\eta}) \cap \Delta$ consists of $(\nu + 1)$ distinct points $\{(\bar{u}^0, \bar{\eta}), \ldots, (\bar{u}^\nu, \bar{\eta})\}$, where the point $\bar{x}^i$ such that $F(\bar{x}^i, \bar{u}^i, \bar{\eta}) = 0$, $\mathrm{cork}\,(DF(\bar{x}^i, \bar{u}^i, \bar{\eta})/Dx) = 1$, is uniquely determined by $(\bar{u}^i, \bar{\eta})$. We construct a polynomial $y^{\nu+1} + a_1y^{\nu-1} + \ldots + a_\nu$, $a \in \mathbf{C}^\nu$, with roots $\bar{u}^i - (\bar{u}^0 + \ldots + \bar{u}^\nu)/(\nu + 1)$, $i = 0, \ldots, \nu$ (cf. [6]). We get a map φ from $\mathbf{C}^\nu \setminus \Sigma$ into the space $\mathbf{C}^\nu \setminus \Xi_\nu$ of polynomials of the indicated form without multiple roots, i.e., into a space $k(B(\nu + 1), 1)$. We shall show that this is a covering.

It is not difficult to extend the map φ to $(\mathbf{C}^\nu \setminus \Sigma) \cup \Sigma^{\nu-1}$, and hence to all of $\mathbf{C}^\nu$. We get a quasihomogeneous map $\Phi: (\mathbf{C}^\nu, 0) \to (\mathbf{C}^\nu, 0)$, the arguments and coordinate functions of which have positive weights. Now if we show that $\Phi^{-1}(0) = \{0\}$, then we will be able to deduce that φ is proper.

Let $\Phi^{-1}(0) = \gamma$. Then for any $\eta \in \gamma$ the set $P^{-1}(\eta) \cap \Delta$ consists of one point (u^*, η). Let $\eta \in \gamma$ be a general point and $x^1, \ldots, x^r$ be all those points for which the germ F^i of the projection $F(\ ,\ , \eta)$ at the point (x^i, u^*) (η fixed) is not equivalent with A_0; ν^i be the corresponding codimension. For the codimensions we get $\sum_{i=1}^{r} \nu^i + (r - 1) + \dim\gamma = \nu$. On the other hand, since $\{(u^*, \eta)\} = P^{-1}(\eta) \cap \Delta$ and all the projections $\{F^i\}$ are simple, one has $\sum_{i=1}^{r} (\nu^i + 1) = \nu + 1$. Consequently, $\dim\gamma = 0$, and since Φ is quasihomogeneous, one has $\gamma = \{0\}$.

We shall show that φ is a diffeomorphism at the point $\bar{\eta}$.

Let m = 1. Since $\bar{u}^i \neq \bar{u}^j$, $i \neq j$, the nondegeneracy of φ is equivalent with the nondegeneracy of the matrix $(\partial F/\partial\eta_i|_{\bar{x}^j, \bar{u}^j, \bar{\eta}})_{i,j=0}^{\nu}$, $\eta_0 = u$. But since the function f is quasihomogeneous, for $\eta = 0$, $\{\partial F/\partial\eta_i\}_0^\nu$ is a basis of the space $\mathscr{E}_{x,u}/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

For m = 2, the nondegeneracy of φ is equivalent with the nondegeneracy of the matrix $(H_i(\bar{x}^j, \bar{u}^j, \bar{\eta}))_{i,j=0}^{\nu}$, where $H_i = \det(D(F, \delta)/D(x, \eta_i))$, $\delta = \det(DF/Dx)$. Using the fact that $\delta(\bar{x}^j, \bar{u}^j, \bar{\eta}) = 0$, for each projection of the table we show that outside a set of codimension 2, and consequently also everywhere, φ is a local diffeomorphism.

The index of the group π in $B(\nu + 1)$ can be expressed as the degree of the quasihomogeneous mapping of the formula $\alpha_0^\nu(\nu + 1)! \big/ \prod_{i=1}^{\nu} \alpha_i$, where $\alpha_0, \alpha_1, \ldots, \alpha_\nu$ are quasihomogeneous weights of the variables $u, \eta_1, \ldots, \eta_\nu$.

4. Dynkin Diagrams

1. For a versal deformation F of the projection f, we fix a sufficiently small $\varepsilon > 0$ and a small, in comparison with ε, value η of the parameter of the deformation and we consider inside the ε-ball B_ε with center at

zero in the space $\mathbf{C}^{n+1}$ the surface $V_\eta = \{F(x, u, \eta) = 0\}$. Let $V_\eta^0 = V_\eta \cap \{u = 0\}$. If f gives a simple projection of a curve, then $V_\eta / V_\eta^0 \simeq \bigvee^{\nu+1} S^1$.

2. As was noted in Sec. 1, the list of simple projections for p = 1 coincides with the list of germs of surfaces, simple with respect to the group of diffeomorphisms preserving the plane u = 0. We consider projections onto the line precisely from this point of view and we carry out for them the construction of vanishing cycles and hemicycles, similarly to the way this was done in the case of a function with a critical point on a manifold with boundary (cf. [3]).

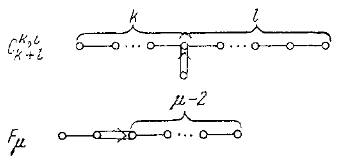
3. Let ε and η be the same as in Point 1; $g_1, \ldots, g_m \in \mathscr{E}_{x,u}$ be general linear combinations of the functions $F_1, \ldots, F_m$ (η fixed). On the surface $\Psi = \{g_1 = \ldots = g_{m-1} = 0\}$ we consider the function g_m. $V_\eta = \Psi \cap \{g_m = 0\}$. At a' critical values of g_m on Ψ and a_0 critical values on $\Psi^0 = \Psi \cap \{u = 0\}$ (Ψ^0 is smooth for general choice of η and $g_1, \ldots, g_{m-1}$) we can define, as for boundary singularities in [3], in the space $H = H_{n-m+1}(V_\eta, V_\eta^0)$, vanishing cycles and hemicycles. We number them from 1 to $a' + a_0$ just as one orders a distinguished homology basis of a nonsingular fiber of a function with an isolated critical point (cf. [2]). H is spanned by the indicated cycles and hemicycles, but if $\operatorname{cork}(Df_0/Dx)|_{x=0} > 1$, then there are relations among them (for the projections $C^{k,l}_{k+l}$, $2 \le k \le l$, the vanishing hemicycles already form a basis of the space H). Let $\mu = \dim H$, $\mu' = \dim H_{n-m+1}(V_\eta)$, $\mu_0 = \dim H_{n-m}(V_\eta^0)$.

4. Setting $u = x_2^0$, we introduce two-sheeted coverings $\tilde{\Psi}$ and $\tilde{V}_\eta$ over Ψ and V_η with branching along Ψ^0 and V_η^0. The function g_m lifts to $\tilde{\Psi}$. Here to each critical point of g_m on Ψ there correspond two critical points of g_m on $\tilde{\Psi}$, and to each critical point of g_m on Ψ^0, one. We get $2a' + a_0$ vanishing cycles in $\tilde{H} = H_{n-m+1}(\tilde{V}_\eta)$, $\dim \tilde{H} = 2\mu' + \mu_0$. These cycles generate $\tilde{H}$. We consider in $\tilde{H}$ the subspace H^- of cycles, antiinvariant with respect to permutations of the sheets of the covering. It is generated by a_0 short cycles (ones which project into V_η in the form of twice traversed vanishing hemicycles) and a' long cycles (which are differences of two interchanged involutions of vanishing cycles, situated on different sheets). Since $\dim H^- = \mu$, one can choose from them a basis of μ cycles. We call such a basis distinguished.

5. Following [3], by the Dynkin diagram of projections we shall mean the graph whose vertices correspond to the elements of a distinguished basis of H^-. The vertices are indexed by μ numbers from 1 to $a' + a_0$ (like the vanishing cycles and hemicycles in H). The i-th and j-th vertices, $i < j$, are joined by k simple (dotted) edges, if the intersection index of the i-th and j-th cycles in H^- is equal to k ($-k$) and at least one of the cycles is short or if the index is equal to 2k ($-2k$) and both cycles are long. Edges, joining the r-th and s-th vertices, corresponding to long and short cycles, are oriented from r to s.

6. One has the following proposition.

Proposition 3. Let f be a projection of the class $C^{k,l}_{k+l}$, $2 \le k \le l$, or F_μ, $\mu \ge 5$. Then one can choose a distinguished basis for which the Dynkin diagram has the form (the indexing of the vertices is omitted)



7. From the existence of the contiguities of projections $F_\mu \to C_{\mu-1}$ (Sec. 1), we get

COROLLARY 1. The Giusti diagram of contiguities of simple curves in $\mathbf{C}^3$ (cf. [7, 8]) is incomplete.

Proof. Let $\tilde{Y}_\mu$ be a curve which is a two-sheeted covering of the curve Y_μ. In the notation of [8]

$$\tilde{F}_5 = T_7,\ \tilde{F}_6 = W_8,\ \tilde{F}_7 = Z_9,\ \tilde{F}_8 = Z_{10}.$$

The remaining curves $\tilde{F}_\mu$ are nonsimple.

On the other hand $\tilde{C}_\mu = D_{\mu+1}$. Thus, lifting the continuities $F_\mu \to C_{\mu-1}$, $\mu \ge 5$, we realize the contiguities

$$T_7 \to D_5,\quad W_8 \to D_6,\ Z_9 \to D_7,\ Z_{10} \to D_8,$$

of which the Giusti diagram contains only the first.

Example. Contiguity of F_6 to C_5:

$$(x_1^2 + x_2^3 - t^2x_2^2,\ x_1x_2 + u + 3tx_2^2 - 4t^2x_1 - 4t^3x_2).$$

Making the substitution $u = x_0^2$, we get the contiguity of W_8 to D_6.

5. Straightening Vector Fields

From Theorem $2C_\mu$ (for $\mu = 3$) formulated below follows

COROLLARY 2. A vector field in general position in three-dimensional space with coordinates (x, y, z) can be reduced, preserving the surface $x^2 = y^3$, by a diffeomorphism in the neighborhood of each point of the cuspidal edge to one of the two formal normal forms: $\partial/\partial x$ (at a general point), $\partial/\partial y + z\partial/\partial x$ (at isolated points).

We consider on the space $(\mathbf{C}^{\mu-1}, \Xi_{\mu-1}) \times \mathbf{C}^1$ (cf. Point 3.5) a vector field in general position. At a general point τ of the line $0 \times \mathbf{C}^1$ it is transversal to the tangent plane to $\Xi_{\mu-1} \times \mathbf{C}^1$, at isolated points it lies in it. In the first case, by Lyashko's theorem [6], in a neighborhood of the point τ, by a diffeomorphism preserving $\Xi_{\mu-1} \times \mathbf{C}^1$, the field can be reduced to the form $\partial/\partial a_{\mu-1}$, if $\Xi_{\mu-1}$ is the discriminant of the polynomial $y^\mu + a_1 y^{\mu-2} + \ldots + a_{\mu-1}$. The normal form of the vector field in the second case is described by the following assertion.

THEOREM $2C_\mu$. Let Δ_μ be the discriminant of the polynomial

$$y^\mu - \sigma_1 y^{\mu-1} + \sigma_2 y^{\mu-2} - \ldots \pm \sigma_\mu, \qquad \sigma \in \mathbf{C}^\mu,$$

v be the germ at the point $0 \in \mathbf{C}^\mu$ of a formal vector field, tangent at zero to the space Δ_μ, and at others in general position (the transversality of the vector field at 0 to the plane $\sigma_{\mu-1} = 0$ in the space $\sigma_\mu = 0$ is necessary). Then by a formal diffeomorphism of the space $(\mathbf{C}^\mu, 0)$ preserving Δ_μ, the field v can be reduced to the form $\partial/\partial\sigma_{\mu-1}$.

Proof. We set $\partial/\partial\sigma_i = \partial_i$.

By [4] the algebra $\mathscr{L}$ of vector fields preserving Δ_μ and leaving 0 in place is $\mathbf{m}_\sigma L_1 + \mathscr{E}_\sigma \langle L_2, \ldots, L_\mu \rangle$, where $\mathbf{m}_\sigma \subset \mathscr{E}_\sigma$ is the maximal ideal, $L_i = \sum_{j=1}^{\mu} L_{ij}\partial_j$,

$$L_{ij} = (\mu - j + 1)\,\sigma_{i-1}\sigma_{j-1} - \sum_{r=0}^{i-2} (i + j - 2r - 2)\,\sigma_r\sigma_{i+j-r-2},$$

where $\sigma_0 = 1$ and $\sigma_k = 0$, if $k > \mu$ [the field L_1 preserves Δ_μ, but $L_1(0) \neq 0$].

The field v which is spoken of in the theorem has the form $v = \sum_{j=1}^{\mu} v_j\partial_j$, $v_\mu(0) = 0$, $v_{\mu-1}(0) \neq 0$. Let $v = \partial_{\mu-1}$. We set $[v, \mathscr{L}] = \{[v, l], l \in \mathscr{L}\}$, $\vartheta = \mathscr{E}_\sigma \langle \partial_1, \ldots, \partial_\mu \rangle$, $\hat\sigma = (\sigma_1, \ldots, \sigma_{\mu-2}, \sigma_\mu)$.

$$\vartheta_k = \mathscr{E}_{\hat\sigma} \langle 1, \sigma_{\mu-1}, \ldots, \sigma_{\mu-1}^{k-1} \rangle \langle \partial_1, \ldots, \partial_\mu \rangle.$$

We need to show that $[\partial_{\mu-1}, \mathscr{L}] = \mathbf{m}_\sigma\partial_\mu + \mathscr{E}_\sigma \langle \partial_1, \ldots, \partial_{\mu-1} \rangle$ or that $[\partial_{\mu-1}, \mathscr{L}'] = \vartheta$, where $\mathscr{L}' = \mathscr{L} + \mathbf{C}L_1$. We denote $[\partial_{\mu-1}, \mathscr{L}']$ by M.

For convenience, as generators of $\mathscr{L}'$ we take $L_1' = (1-\mu)L_1$, $L_2' = L_2, \ldots, L_{\mu-1}' = L_{\mu-1}$, $L_\mu' = L_\mu - \sigma_{\mu-1}L_1$. The matrix $L'|_{\hat\sigma=0}$ has the form

$$\begin{pmatrix} (1-\mu)\mu & & & (1-\mu)\sigma_{\mu-1} \\ & 0 & (1-\mu)\sigma_{\mu-1} & \\ & & \cdot^{\cdot^{\cdot}} & \\ & (1-\mu)\sigma_{\mu-1} & & \\ (1-\mu)\sigma_{\mu-1} & & & 0 \end{pmatrix}.$$

Let $M \supset \vartheta_k$ (for k = 0 this is obvious). We shall show that then $M \supset \vartheta_{k+1}$. Since the map is an $\mathscr{E}_{\hat\sigma}$-morphism, it suffices to prove that $M \supset \sigma_{\mu-1}^k \langle \partial_1, \ldots, \partial_\mu \rangle_{\mathbf{C}} + \vartheta_k$. It is easy to see that $[\partial_{\mu-1}, \,]\colon \mathscr{L} \to \vartheta$, $l \mapsto [\partial_{\mu-1}, l]$

$$[\partial_{\mu-1}, \sigma_{\mu-1}^k L_1'] \equiv (k+1)(1-\mu)\,\sigma_{\mu-1}^k\partial_\mu \mod \vartheta_k,$$

$$[\partial_{\mu-1}, \sigma_{\mu-1}^k L_2'] \equiv (k+1)(1-\mu)\,\sigma_{\mu-1}^k\partial_{\mu-1} \mod (\vartheta_k + \mathscr{E}_{\hat\sigma}\sigma_{\mu-1}^k\partial_\mu), \ldots$$

$$\ldots, [\partial_{\mu-1}, \sigma_{\mu-1}^k L_\mu'] \equiv (k+1)(1-\mu)\,\sigma_{\mu-1}^k\partial_1 \mod (\vartheta_k + \mathscr{E}_{\hat\sigma}\sigma_{\mu-1}^k \langle \partial_2, \ldots, \partial_\mu \rangle).$$

Consequently, $\vartheta_{k+1} \subset M$. The theorem is proved.

In the case F_4 the analog of the assertion proved is the following. We consider in the space $\mathbf{C}^4 = \mathbf{C}^2 \times \mathbf{C}^2$ the surface $\gamma = \Xi_2 \times C^2$. Let v be the germ at the point $0 \in C^4$ of a formal vector field, tangent at 0 to the plane $0 \times \mathbf{C}^2$, and otherwise in general position. We introduce in $\mathbf{C}^4$ coordinates $(u, \eta_1, \eta_2, \eta_3)$ [not preserving the structure of direct product on (C^4, γ)], in which the surface is described as the bifurcation diagram of zeros (cf. Point 3.4) for the projection F_4:

$$\gamma = \{27\,(u^2 + \eta_1)^2 + 4\,(u\eta_3 + \eta_2)^3 = 0\}.$$

THEOREM $2F_4$. By a formal diffeomorphism preserving γ, the field v can be reduced to the form $\partial/\partial u$.

The proof of this assertion is analogous to the previous one: we seek first the stationary algebra of $(\gamma, 0)$, and then, acting on it by the field $\partial/\partial u$, we get the space

$$\mathfrak{m}_{u,\eta}\,\langle \partial/\partial\eta_1,\ \partial/\partial\eta_2\rangle + \mathscr{E}_{u,\eta}\,\langle \partial/\partial u,\ \partial/\partial\eta_3\rangle.$$

Remark. Theorem 2 shows that for miniversal deformations of projections C_μ and F_4, the projection $P\colon \mathbf{C}^{1+\nu} \to \mathbf{C}^\nu$, $(u, \eta) \to \eta$ of the discriminant Δ from Point 3.4 is stable. Whether the corresponding assertion about vector fields and the stability of the projection P for projections $C^{k,l}_{k+l}$, $2 \le k \le l$, and F_μ, $\mu \ge 5$, is true is unknown.

In conclusion, the author expresses profound thanks to V. I. Arnol'd for posing the problem, constant attention to the work, and many useful discussions. In particular, he indicated the connection between projections onto the line and functions with critical points on manifolds with boundary, and also formulated Theorem 2 as a conjecture.

LITERATURE CITED

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