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Regular Legendrian knots and the HOMFLY polynomial of immersed plane curves

S. Chmutov⋆ · **V. Goryunov**⋆⋆ · **H. Murakami** ⋆⋆⋆

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Abstract. We show that every unframed knot type in $ST^*\mathbf{R}^2$ has a representative obtained by the Legendrian lifting of an immersed plane curve. This gives a positive answer to the question asked by V.I.Arnold in [3]. The Legendrian lifting lowers the framed version of the HOMFLY polynomial [20] to generic plane curves. We prove that the induced polynomial invariant can be completely defined in terms of plane curves only. Moreover it is a genuine, not Laurent, polynomial in the framing variable. This provides an estimate on the Bennequin-Tabachnikov number of a Legendrian knot.

Mathematics Subject Classification (1991): 57M25, 53C15

A few years ago Arnold [2,3] gave a new breath to the study of invariants of plane curves, the area which attracted Gauss and Whitney. The approach introduced by Arnold is very similar to that successfully used by Vassiliev in knot theory, which is to describe invariants in terms of their changes in generic homotopies

S. Chmutov
Program Systems Institute, Pereslavl-Zalessky, 152140 Russia
(E-mail: chmutov@math.botik.yaroslavl.su)

V. Goryunov
Department of Mathematical Sciences, Division of Pure Mathematics, The University of Liverpool, Liverpool L69 3BX, UK (E-mail: goryunov@liv.ac.uk)

H. Murakami
Department of Mathematics, Osaka City University, 3-138, Sugimoto 3-chome, Sumiyoshi-ku, Osaka 558, Japan
Current address:
Department of Mathematics, School of Science and Engineering, Waseda University, 4-1, Ohkubo 3-chome, Shinjuku-ku, Tokyo 169, Japan (E-mail: hitoshi@cfi.waseda.ac.jp)

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of curves. Considering invariants which change only in homotopies of plane curves involving *direct self-tangencies* (that is, when the tangent branches have coinciding orientations) we arrive at a situation very reminiscent of knot theory itself. Indeed, one can lift a generic plane curve to a Legendrian knot in the solid torus $ST^*\mathbf{R}^2$ or, if the winding number of the curve is zero, in $\mathbf{R}^3$. Such a knot will experience crossing changes only at the above self-tangencies.

It has been observed that the theory of regular plane curves without direct self-tangencies has in fact a far-reaching parallel with the theory of framed knots. For example, the space of Vassiliev type invariants is the same in both cases [14, 13]. Of course, this does not ensure that any framed knot can be represented by the Legendrian lift of an immersed plane curve equipped with the canonical Legendrian framing. Indeed, this is not true in this generality: Bennequin's inequality [5] shows that the twisting numbers of the canonical framings of Legendrian representatives of a fixed unframed knot type are bounded from one side. On the other hand, while the classical result in the area claims that any unframed knot type in the standard contact solid torus or 3-space has a Legendrian representative (see, e.g., [15]), the canonical projection to the plane of such a representative may have cusps.

In the present paper we are trying to make the parallel between knots and regular plane curves more explicit. We show that, in fact, the Legendrian representatives can be chosen to be the lifts of regular curves. We also investigate restrictions on the Legendrian framings of such lifts. We show that there is another estimate on these framings which is often stronger than Bennequin's inequality (cf. [12]). Our estimate comes from the HOMFLY polynomial of a knot in a solid torus. Other similar estimates provided by Legendrian lowerings of the other polynomial knot invariants to regular plane curves and plane curves with cusps are discussed in [7].

1. Legendrian realisation

1.1. Standard contact spaces

We recall a few basic notions.

A contact element at a point of a plane is a line in the tangent plane. Its *coorientation* is a choice of one of two half-planes into which it divides the tangent plane. The manifold M of all cooriented contact elements of the plane is the spherisation $ST^*\mathbf{R}^2$ of the cotangent bundle of the plane. It is diffeomorphic to the solid torus $\mathbf{R}^2 \times S^1$: the coorienting normal vector is defined by the angle $\varphi \bmod 2\pi$ which it makes with a fixed direction on the plane. The manifold M has the standard contact structure defined as zeros of the 1-form $\alpha = (\cos\varphi)dx + (\sin\varphi)dy$, where (x, y) are coordinates on $\mathbf{R}^2$ with the positive direction of the x-axis being that fixed above (see Fig. 1). We equip M with the orientation $dx \wedge dy \wedge d\varphi = -\alpha \wedge d\alpha$.

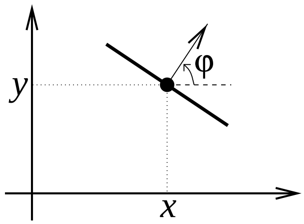

Fig. 1. Coordinates in the solid torus $ST^*\mathbf{R}^2$

A generic oriented curve C in $\mathbf{R}^2$ is an immersed circle whose only singularities are transverse double points. Such a curve lifts to a knot L_C in the solid torus M by setting φ to be the direction of the normal which gives a positive frame on the plane when followed by the orientation of C. The knot L_C will be called *a regular Legendrian knot*. It is everywhere tangent to the contact structure.

Along with the solid torus M we will also be considering its universal cover $\widetilde{M} \simeq \mathbf{R}^3$, with the orientation induced from that of M. Its standard contact form is given by the same formula as α with the only difference that now the angular coordinate φ is not reduced $\mod 2\pi$. A generic closed plane curve lifts to a Legendrian knot in $\widetilde{M}$ only if its winding number (that is the number of rotations made by the coorienting vector during one complete walk along the curve) is zero.

1.2. Knots in $\mathbf{R}^3$

Theorem 1.1 *Any unframed oriented knot type in* $\widetilde{M} \simeq \mathbf{R}^3$ *has a regular Legendrian representative.*

Proof. We have to construct a regular Legendrian knot in $\widetilde{M}$ of a given topological type.

Let $K \subset \widetilde{M}$ be an oriented non-Legendrian knot which represents this type and is generic with respect to the canonical projection $\widetilde{p} : \widetilde{M} \to \mathbf{R}^2$. The plane curve $D = \widetilde{p}(K)$ is generic. Equipping it with the information about the over- and under-crossings we get the knot diagram $D(K)$ of K. We are going to make minor corrections of D to obtain a curve whose Legendrian lifting to $\widetilde{M}$ is topologically equivalent to K.

Choose any non-double point a of D (see Fig. 2). Start the lifting procedure from it sending a to any point of $\widetilde{M}$ that corresponds to the direction of the normal to D at a which agrees with our lifting orientation convention (the ambiguity of the choice is a shift by a multiple of 2π along the fibre of $\widetilde{p}$). Follow D in the direction of its orientation lifting it to $\widetilde{M}$ until nearly the first second-time visit to a double point. Here we have to bother about the type of crossing in $D(K)$: the phase $\varphi \in \mathbf{R}$ which we have gained by this moment may be forcing us to

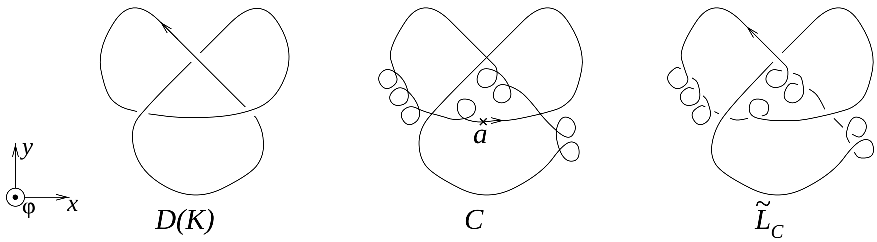

Fig. 2. A knot diagram $D(K)$ of the right-handed trefoil and its adjustment to get a generic plane curve C whose Legendrian lifting $\widetilde{L}_C$ to $\widetilde{M} \simeq \mathbf{R}^3$ is the same trefoil

make a crossing of the wrong type. But we can easily decrease or increase the phase by inserting a certain number of extra small curls (either all clockwise or all counter-clockwise) before our second-time visit and pass the double point in the right way, as prescribed by $D(K)$.

We continue our lifting trip along D in the same fashion adjusting the curve before second-time visits to double points if needed. Just before coming back to the initial point a we may also need to insert a few small curls to make the winding number of the adjusted curve zero. We end up with a regular plane curve C whose Legendrian lifting $\widetilde{L}_C$ to $\widetilde{M}$ has the topological type of K. This is because the two diagrams, $D(K)$ and $D(\widetilde{L}_C)$, differ by only small curls and so give the same knot type (by using the Reidemeister move I). □

1.3. Knots in the solid torus

Theorem 1.2 *Any unframed oriented knot type in the solid torus $M = ST^*\mathbf{R}^2$ has a regular Legendrian representative.*

Proof. Take a generic representative $K \subset M$ of an oriented topological knot type which we have to realise by a regular Legendrian knot. Let $p : M \to \mathbf{R}^2$ be the canonical projection and $D = p(K)$. The curve D is a generic plane curve. As in the proof of the previous theorem we are going to make some changes in D so that its Legendrian lift to M is topologically equivalent to K.

We can assume that K is transversal to the section $\varphi = 0$ of M and that no point of the set $V = K \cap \{\varphi = 0\}$ projects to a double point of D.

Let us first make K “looking in the regular Legendrian way" around the set V. Namely, by a homotopy that fixes V and is trivial outside a small neighbourhood of V in M we can make the velocities of D at all the points of the set $p(V)$ vertically upward. Moreover, we can choose our homotopy so that the p-images of the local (around V) branches of K along which φ is increasing (respectively decreasing) are lying to the left (respectively to the right) of the above vertical velocities (Fig. 3).

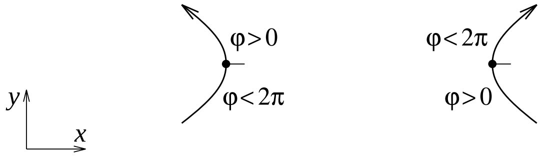

Fig. 3. Projections of increasing and decreasing branches of a knot normalised in a neighbourhood of the section $\varphi = 0$

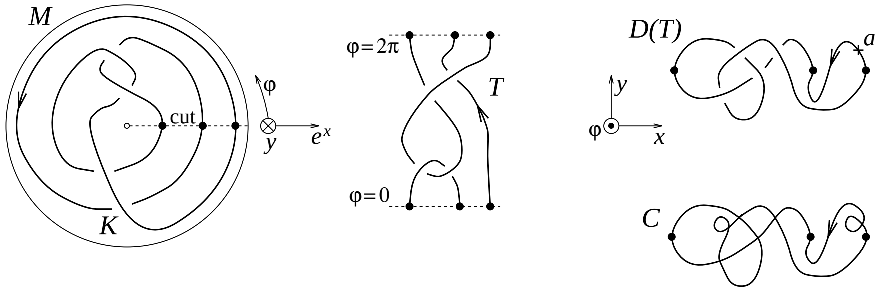

Fig. 4. Breaking a knot in the solid torus M into a tangle T and a modification of the tangle diagram $D(T)$ providing a regular curve C whose Legendrian lift to M is topologically equivalent to K

Now we cut M along the section $\varphi = 0$ and represent it as the direct product $\mathbf{R}^2 \times [0, 2\pi] \subset \mathbf{R}^2_{x,y} \times \mathbf{R}_\varphi = \widetilde{M}$ (Fig. 4). The knot K becomes a tangle T in $\mathbf{R}^2 \times [0, 2\pi]$. Projection $\widetilde{p} : \widetilde{M} \to \mathbf{R}^2$ sends T onto the curve D. The points of the boundary ∂T of T are glued in pairs to become the points of $p(V)$.

The pair $(D, p(V))$, with the additional information about the over- and under-crossings of the tangle T, is the tangle diagram $D(T)$ of T. The way to break D at the points of $p(V)$ to restore the boundary of the tangle is encoded in the local pictures of D shown in Fig. 3.

Let us adjust D and lift the adjusted plane curve C to a Legendrian curve $\widetilde{L}_C \subset \widetilde{M}$ with boundary, such that $\widetilde{p}(\partial \widetilde{L}_C) = p(V)$ and $\widetilde{L}_C$ closes after the canonical projection to M to become a knot equivalent to K .

The adjustment is very similar to that of the previous subsection (see Fig. 4).

We start the straightforward lifting of D to $\widetilde{M}$ at an arbitrary generic point a and go in the direction of the orientation of D. The value of the coordinate $\varphi \in \mathbf{R}$ changes continuously according to the change of the direction of the normal until we arrive at a point of $p(V)$. Having arrived to such a point along an increasing branch (Fig. 3), we subtract 2π from the current phase and continue our further lifting from this reduced φ. Having arrived along a decreasing branch, we add 2π to the current value of φ.

As in the proof of Theorem 1.1, just before a second-time visit to a double point of D we may have to insert some extra curls into D to guarantee the type of the crossing prescribed by the tangle diagram $D(T)$. Now we want to be a bit more accurate than in the case of $\mathbf{R}^3$: we make the absolute value of the difference between the phases of two visits to the same double point less than 2π (unnecessary extra curls, like the one the reader can find in Fig. 2, are not allowed now). Notice that the phases of the two visits to the double point of a curl satisfy this condition.

On the final step we may have to insert some more curls into the adjusted D to close the Legendrian curve in $\widetilde{M}$ above a.

We end up with a modification C of the curve D and its Legendrian lift $\widetilde{L}_C \subset \widetilde{M}$ with boundary. Reduction of φ modulo 2π projects $\widetilde{L}_C$ onto the closed regular Legendrian curve $L_C \subset M$. We claim that L_C is a knot topologically equivalent to K.

Indeed, the condition on the difference of the phases at a double point guarantees that L_C is an embedded curve. Moreover, the same condition implies that there exists a smooth function f, such that the curve $\widetilde{L}_C$ lies in the slice $f(x,y) \leq \varphi \leq f(x,y) + 2\pi$ of $\mathbf{R}^2_{x,y} \times \mathbf{R}_\varphi$ with only the boundary $\partial \widetilde{L}_C$ being on the boundary of the slice.

Homotop the above slice to $\mathbf{R}^2 \times [0, 2\pi]$ along fibres of $\widetilde{p}$ putting the function $(1-t)f$ instead of f into the inequalities. This homotopy sends $\widetilde{L}_C$ to a tangle whose boundary and topological type coincide with those of T: the two tangle diagrams in $\mathbf{R}^2$ provided by the projection $\widetilde{p}$ differ by Reidemeister moves I only. Our homotopy lowers to a family of diffeomorphisms of the solid torus M which therefore sends the knot L_C to a knot equivalent to K. $\square$

Remark 1.3 The link versions of Theorems 1.1 and 1.2 are also valid. In the case of links in $\mathbf{R}^3$ one has to be slightly patient: when starting to lift a component of a link diagram one has to make it clear to which particular φ-level in $\widetilde{M}$ this point is lifted. This equips a starting point on each component of the curve collection C with a real number.

Remark 1.4 The space $PT^*\mathbf{R}^2$ of non-cooriented contact elements of the plane is another standard solid torus of contact geometry. The analogue of Theorem 1.2 for knots defining even classes in $\pi_1(PT^*\mathbf{R}^2)$ (only consideration of such knots makes sense) fails. The place where our proof does not go through for $PT^*\mathbf{R}^2$ is the adjustment of a crossing type by insertion of extra curls. Indeed, in the projective case, we need the curve $\widetilde{L}_C$ lie in a slice of $\mathbf{R}^2_{x,y} \times \mathbf{R}_\varphi$ of thickness π. Therefore, we must be able to change the phase by π which is possible only via insertion of cusps.

Remark 1.5 Theorem 1.2 was proved simultaneously and independently by A. Shumakovich (not published). The method he used is not very much different from ours.

Remark 1.6 E. Ferrand has given results for more general situations [8].

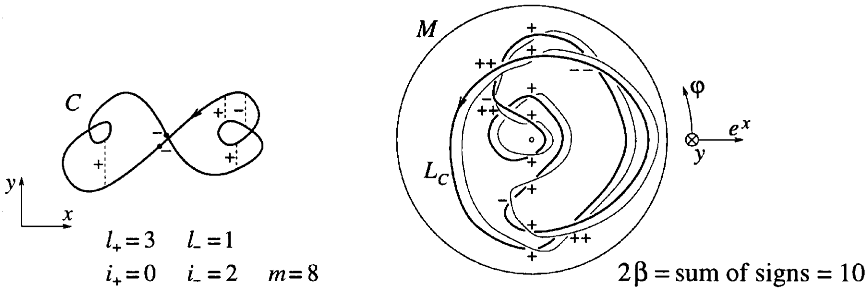

Fig. 5. Example of calculation of the Bennequin-Tabachnikov number β of a regular Legendrian knot in $ST^*\mathbf{R}^2$

2. Framed knots

2.1. The Bennequin-Tabachnikov number

Legendrian knots in a 3-manifold with a cooriented contact structure are canonically framed by a transversal shift in the direction of the coorientation of the structure.

For a regular Legendrian knot in the standard $\mathbf{R}^3 = \widetilde{M} = \mathbf{R}^2 \times \mathbf{R}$ this framing is exactly the blackboard framing with respect to the projection to the base $\mathbf{R}^2$ (notice that here we refer to Figs. 1–3 rather than to the covering of the leftmost fragment of Fig. 4). Here the blackboard framing of a knot diagram is a shift of the diagram in the plane in the direction of the coorienting normals. The writhe β of this framing (the sum of the signs of the crossings of the knot diagram) is called the *Bennequin number* of the knot [5]. It equals the linking number between the Legendrian knot and its shift along the framing.

The analog of the Bennequin number for the standard solid torus $M = ST^*\mathbf{R}^2$ was defined by Tabachnikov [18]. He set it to be the intersection number of a Legendrian knot shifted in the direction of the canonical framing and a 2-film which realises homology between the unshifted knot and the multiple of the fibre over a sufficiently distant point of the plane. To calculate the Tabachnikov number, one can consider the knot diagram of the projection of the Legendrian knot from $M \simeq \mathbf{R}^2_{x,y} \times S^1_\varphi$ to the punctured plane with polar coordinates e^x, φ. An illustration to this is given in Fig. 5. For the projection considered, the canonical framing is not blackboard. Anyway, the Tabachnikov number is still calculated as the ordinary linking number of the knot (L_C, the thick line) and its shift (the thin line) along the framing: this is half the sum of the signs of the crossings of L_C with the shift.

Figure 5 illustrates the following algorithm to evaluate β on a regular Legendrian knot L_C in M.

Take a Cartesian coordinate system on $\mathbf{R}^2$ generic with respect to the curve C.

There are finitely many lines $x = \text{const}$ which intersect C at two points with velocities in the same direction (dashed lines in Fig. 5). Call such a line positive (respectively negative) if the curvature κ of C at the upper of these two points is greater (respectively less) than that at the lower one (the derivative of the unit orienting vector with respect to the natural parameter on C is $-\kappa\nu$, where ν is the unit normal whose direction φ is used for the Legendrian lifting). Let ℓ_+ and ℓ_- be the numbers of all such positive and negative lines respectively.

Now consider an inflection point b of C. Assume that the phase $\varphi(b)$ is either in the first or in the third quadrant. If the phase achieves its local maximum (minimum) at b, call this inflection positive (negative). Use the opposite terminology for the second and forth quadrants. Let i_+ and i_- be the numbers of positive and negative inflections respectively.

Let m be the number of extrema of function y on C.

Proposition 2.1 $\quad 2\beta(L_C) = 2(\ell_+ - \ell_-) + (i_+ - i_-) + m$.

Proof. We will describe how to draw the canonical shift of L_C. It is parallel to L_C except near points corresponding to inflections and extrema of C. Near the point of L_C corresponding to a positive (respectively negative) inflection it makes a positive (respectively negative) crossing with L_C and near the point corresponding to an extrema it makes a positive crossing (Fig. 5). Therefore twice the linking number between L_C and its shift is the sum of $i_+ - i_- + m$ and twice the self-crossing number of L_C, which equals $2(\ell_+ - \ell_-)$. □

Remark 2.2 a) The numbers ℓ_+ and ℓ_- can be split in an obvious way to provide all the coefficients of Arnold's and Aicardi's polynomials [4,1].
b) Other formulas to calculate β can be found in [18] and [9].

2.2. The two invariants of unframed knots

Not every framed knot type in M or $\widetilde{M}$ can be represented by a canonically framed Legendrian knot. The Bennequin-Tabachnikov number β is bounded from one side (according to the chosen orientation) on a set of all Legendrian knots of the same unframed topological type [5]. For our choice of the orientation the number is bounded from below. Indeed, insertion of a small fragment containing two curls with opposite directions of rotation into a generic plane curve C does not affect the unframed type of the Legendrian knot $L_C \subset M$. On the other hand, this operation increases $\beta(L_C)$ by 2.

On a regular Legendrian knot β is odd [2,4] (see also Proposition 3.5 below). To increase $\beta(L_C)$ by 1 within the same unframed knot type in M or $\widetilde{M}$, one can insert into the curve C a small non-self-intersecting fragment with two cusps and

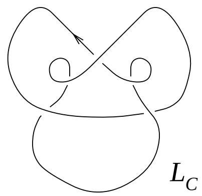

Fig. 6. A Legendrian left-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$ with the minimal Bennequin number -1

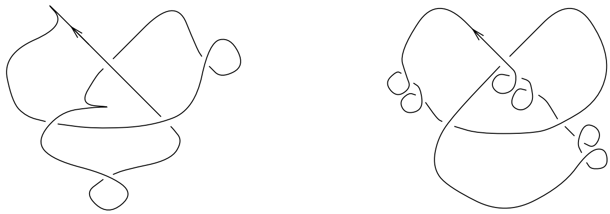

Fig. 7. A Legendrian right-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$ with the minimal Bennequin number 6 and the minimal known example of a regular Legendrian right-handed trefoil knot with $\beta = 9$

zero winding number. In the representation of the solid torus M used in Fig. 5, such a fragment provides a small smooth curl with the blackboard framing of writhe 1.

Thus we arrive at two a priori different characteristics of an unframed knot in M or $\widetilde{M}$. Those are the minimal Bennequin-Tabachnikov numbers of Legendrian knots of the same unframed knot type K realised as Legendrian liftings of either regular plane curves or plane curves with cusps (the latter corresponds to arbitrary Legendrian knots). We denote them by $\beta_{min,reg}(K)$ and $\beta_{min}(K)$ respectively. Of course, $\beta_{min,reg}(K) \geq \beta_{min}(K)$.

Example 2.3 For the left-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$, β_{min} is known to be -1 (see [5]). It is easy to achieve the minimum in a regular way (Fig. 6).

Example 2.4 For the right-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$, $\beta_{min} = 6$ [16,12]. We show the corresponding extreme realisation with cusps in Fig. 7. The best regular Legendrian realisation of the right-handed trefoil we know has $\beta = 9$ (Fig. 7).

Thus the number $\beta_{min,reg}(K)$ does not seem to be completely defined by the parity argument correction of $\beta_{min}(K)$ above.

The main goal of the rest of the paper is to obtain an estimate on $\beta_{min,reg}(K)$ in M and $\widetilde{M}$ (Theorem 5.4). In fact, the lower bound we get here works for $\beta_{min}(K)$ too [7]. The proof in [7] includes a theorem analogous to our Theorem

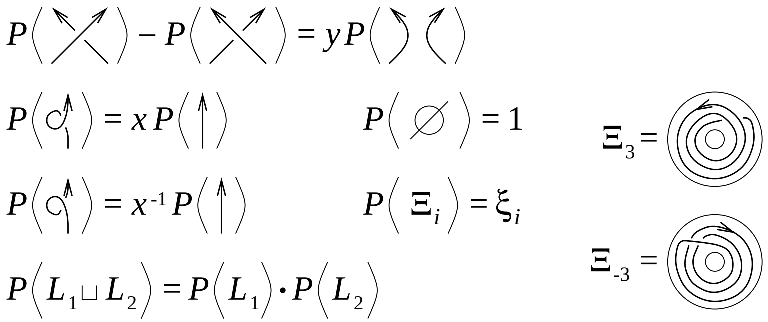

Fig. 8. Definition of the framed version of the HOMFLY polynomial for oriented links with the blackboard framing in a solid torus

3.2 and is heavily based on the arguments of the present paper. Also our proof is more delicate as less moves are allowed.

For other estimates, we refer the reader to [17,10,12,7,19].

3. HOMFLY polynomial

3.1. Legendrian lowering of the polynomial to plane curves

In a generic 1-parameter family of regular plane curves there can appear triple points and points of self-tangency. A self-tangency can be either *direct* (the two velocity vectors have the same directions) or *inverse* (the directions are opposite). Topology of a regular Legendrian knot L_C in M or $\widetilde{M}$ can change only under direct self-tangency perestroikas of the underlying regular curve C.

We call an invariant of collections of regular plane curves a J^+*-type invariant* if it does not change under regular homotopies which involve no direct self-tangencies. Our terminology follows the name of the first invariant of this type introduced by Arnold in [2,3]. Arnold's invariant J^+ of a one-component regular plane curve is basically the Bennequin-Tabachnikov number of its lifting to the solid torus: in [3,4] Arnold shows that $J^+(C) = 1 - \beta(L_C)$.

J^+-type invariants can be induced via the Legendrian lifting from invariants of knots in M or $\widetilde{M}$. In [6] this approach was used to define polynomial invariants of plane fronts. Now we do the same for regular plane curves.

In [20] Turaev defined the HOMFLY polynomial of an unframed oriented link in a solid torus. This is an element of $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \xi_{\pm 1}, \xi_{\pm 2}, \ldots]$. A similar polynomial of a framed oriented link in the same ring is uniquely defined by the relations and initial data of Fig. 8. The links L_1 and L_2 there are disjoint in the projection.

For example, on an unknot with the trivial framing $P = (x - x^{-1})/y$.

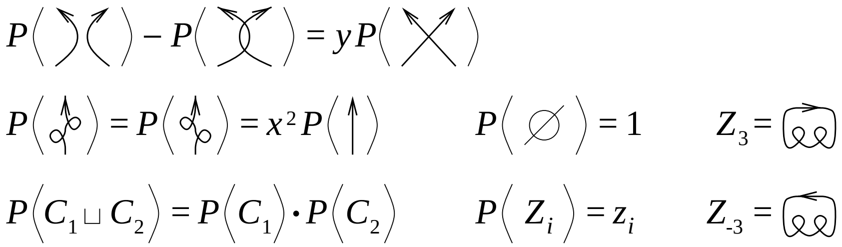

Fig. 9. Legendrian lowering of the definition of Fig. 8 to generic collections of regular oriented plane curves

Definition 3.1 *The HOMFLY polynomial of a plane curve collection C* is that of the Legendrian link L_C in the solid torus $ST^*\mathbf{R}^2$: $P(C) = P(L_C)$.

Thus the Legendrian lifting lowers the polynomial to generic collections of plane curves. Translation of the definition of Fig. 8 to that case is given in Fig. 9. The collections C_1 and C_2 of the last line lie in disjoint half-planes (we call $C_1 \sqcup C_2$ the *split union* of C_1 and C_2). According to the second rule of Fig. 8, the relation between the Legendrian generators z_i we are using now and the blackboard generators ξ_i is $z_i = x^{|i|}\xi_i$: it is easily seen that $L_{Z_i} = \Xi_i$ as unframed knots in the solid torus, and the canonical framing of L_{Z_i} differs from the blackboard one of Ξ_i by $2|i|$ positive half-twists similar to those on the vertical line through the centre of the annulus in Fig. 5.

Theorem 3.2 *There exists a unique J^+–type invariant $P(C) \in \mathbf{Z}[x^2, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$ of a generic collection C of oriented plane curves satisfying the relations and initial data of Fig. 9.*

Thus the HOMFLY polynomial of a regular plane curve turns out to be a genuine polynomial in x, not a Laurent one. Moreover, only even powers of x occur in it.

We prove Theorem 3.2 is Section 4.

3.2. Basic curves

Example 3.3 Consider classes of framed knots represented by the knots Ξ_i of Fig. 8 with the blackboard framing. Since their polynomials are $x^{-|i|}z_i$, Theorem 3.2 implies that they do not have any canonically framed regular Legendrian representatives.

Moreover, according to Theorem 3.2, the Bennequin-Tabachnikov number of the curve L_{Z_i} is the minimum of all such numbers on the set of all regular Legendrian representatives of the unframed knot type of Ξ_i. This minimal number is, thus, $2|i| - 1$. In fact, it is possible to show that $\beta_{min,reg}(\Xi_i) = \beta_{min}(\Xi_i)$ [7].

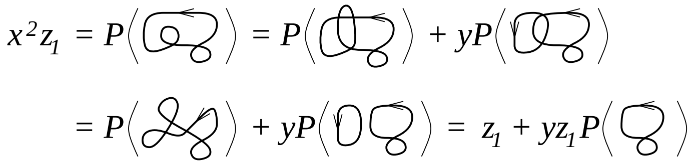

Fig. 10. Calculations of the HOMFLY polynomial of the figure-eight curve

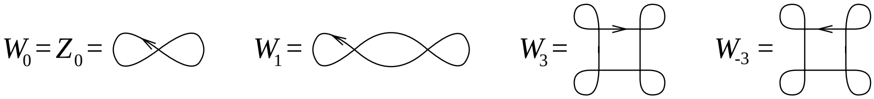

Fig. 11. The curves W_i

Example 3.4 The calculations of Fig. 10 show that the polynomial of the figure-eight curve is $(x^2 - 1)/y$. Indeed, the curve lifts to the Legendrian unknot in M with $\beta = 1$, so its HOMFLY polynomial should be that of an unknot with the trivial framing times x.

In what follows, we will denote the figure-eight curve by Z_0.

The oddness of the $\beta(L_{Z_i})$ implies

Proposition 3.5 *The Bennequin-Tabachnikov number of a regular Legendrian knot in the solid torus $M = ST^*\mathbf{R}^2$ is odd.*

Proof. By the Whitney-Graustein theorem [21], any regular plane curve may be deformed by a regular homotopy to one of the curves Z_i, $i \in \mathbf{Z}$. In a generic regular homotopy, the Bennequin-Tabachnikov number of the corresponding regular Legendrian knot changes only under direct self-tangency perestroikas. Each time the change is ± 2. □

Example 3.6 In some cases, it is convenient to use a different system of generators, w_i, instead of z_i. The w_i is $P(W_i)$, where W_i is the circle equipped with outer $|i| + 1$ α-shaped curls and the orientation providing it with the winding number i (see Fig. 11). For example, $W_0 = Z_0$ and $w_{\pm 1} = z_{\pm 1}$, since the curves $W_{\pm 1}$ can be homotoped without direct self-tangencies to the embedded circles.

The way the two systems of generators are related is shown in Fig. 12. There and below we write the relations on polynomials as relations on the corresponding curves.

Definition 3.7 A *simple curl* of a curve collection is an α-shaped loop that contains no fragments of the collection in its interior.

Example 3.8 A figure-eight curve that has no intersection with other components of a curve collection passes through such a collection freely, with no effect on

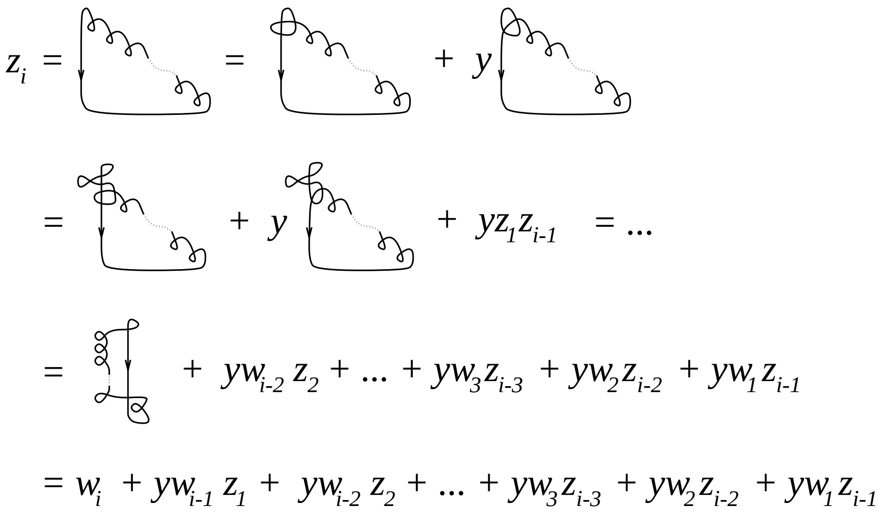

Fig. 12. Recursive relation between the generators z_i and w_i

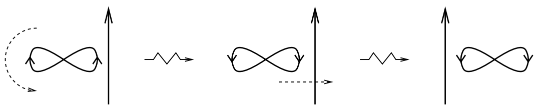

Fig. 13. The figure-eight curve as a neutrino

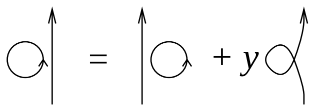

Fig. 14. A circle passes through a line

the polynomial. Indeed, the homotopy of Fig. 13 does not involve any direct self-tangencies.

A circle can also pass through a line, but at the expense of a certain change in P (see Fig. 14).

In general, a basic curve Z_i makes a similar pass generating many extra summands in P (see Figs. 15 and 16). The crucial point for our further considerations is that all these summands can finally be expressed as the polynomials of curve collections that have nothing on that side of the line from which Z_i has been removed and have only basic curves Z_j on the other side. The part of the line involved receives only a number of additional simple curls.

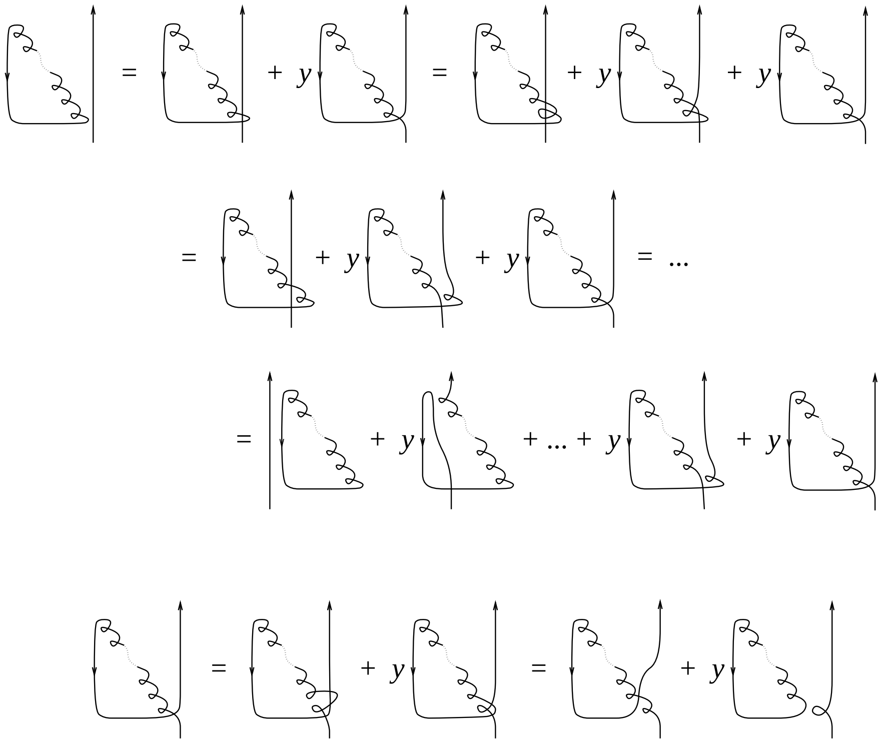

Fig. 15. Removing a basic curve from one side of a line generates a "cloud" of basic curves on the other side. The closed component of the lower right curve has one simple curl less than that of the upper left

4. Proof of Theorem 3.2

The existence of an invariant is guaranteed by [20]. We only need to show that the rules of Fig. 9 are sufficient to define the polynomial of any curve collection uniquely. The restriction on the powers of x that are allowed to appear in the polynomials will immediately follow from the way in which the proof of uniqueness is carried out: we show that in calculations it is sufficient to use the skein relation involving x just to omit pairs of curls.

Our proof is inductive, by a complexity of a curve collection C. The complexity is measured by two quantities. One of them is just the number $k(C)$ of connected components of C (of course, not the same as the number of its irreducible components). The other is as follows.

Definition 4.1 The double point of a simple curl is called a *simple double point.* An *essential double point* is one which is not simple. We denote by $e(C)$ the number of essential double points of a curve collection C.

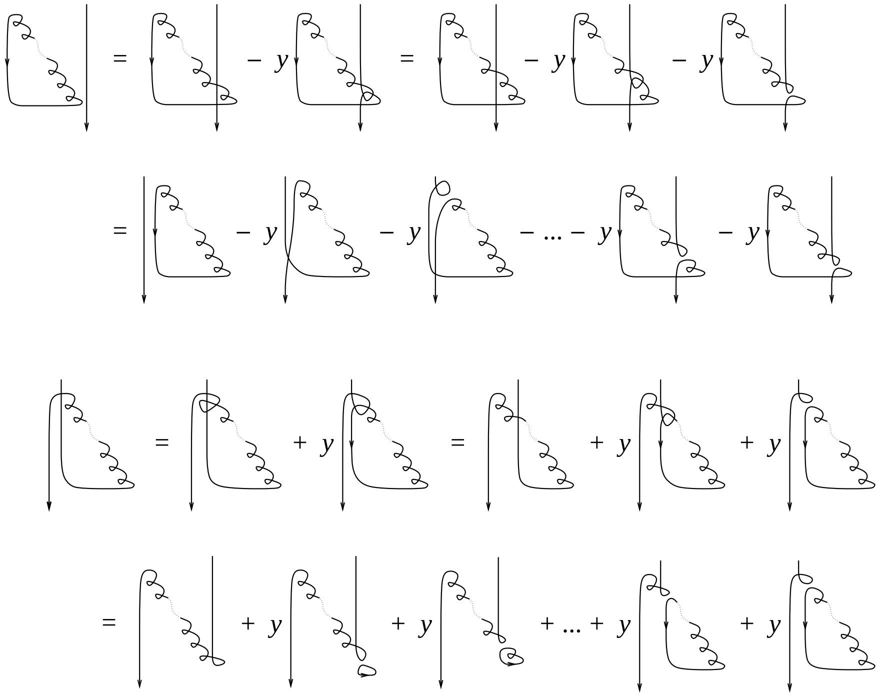

Fig. 16. Removing a basic curve in the case of the other orientation of a line

We define the *complexity* $\chi(C)$ of a generic curve collection to be the pair $(e(C), k(C))$ with lexicographical order.

4.1. The base of induction

If $\chi(C) = (0, 0)$ the curve collection C is empty. According to the rule of Fig. 9 its polynomial is 1.

The rest of the proof shows that the polynomial of any curve collection can be expressed, via our rules, in terms of the polynomials of collections of lower complexity. We construct a homotopy of an arbitrary curve collection C which performs one of the following reductions:

(a) makes one of essential double points simple;

(b) creates a situation in which a self-tangency perestroika is able to kill two double points;

(c) splits a basic curve Z_i or W_i off the rest of the collection.

All the intermediate collections in our homotopies as well as auxiliary collections participating in the calculations of $P(C)$ due to the skein relations are controlled to have complexity lower than $\chi(C)$ (or at least it is shown that their

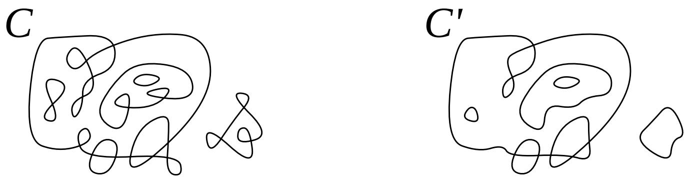

Fig. 17. A curve collection C and its essential part C'

polynomials can be expressed in terms of the polynomials of collections of complexity less than $\chi(C)$). This guarantees that the transformations (a,b,c) provide the inductive reduction.

In our constructions, all the transformations of the above three types are applied to certain elementary domains introduced in the next section.

4.2. Minimal 0- and 1-gons

Consider an arbitrary generic curve collection C. Smooth out all its simple curls. Let C' be the resulting curve collection (see Fig. 17).

Definition 4.2 The collection C' is called the *essential part* of the collection C.

Definition 4.3 A closed disc D' is called an *n-gon* of the collection C' if its boundary $\partial D'$ is contained in C' and has exactly n vertices, that is double points of C' where $\partial D'$ fails to be differentiable.

For example, a 0-gon is bounded by an embedded circle.

The choice of the notation (with dashes) indicates that we are going to use only n-gons of the essential parts of curve collections.

Definition 4.4 A 0- or 1-gon D' of C' is called *minimal* if there are neither 0- nor 1-gons of C' inside D'.

Intersection of the interior of D' with C' may be non-empty.

Any non-empty curve collection contains a 0- or 1-gon. Indeed just make a smooth walk along the curve from an arbitrary point until the first second-time visit to some point q. The loop traced between the two visits to q has no self-intersections and bounds a disc required.

Note that a minimal 1-gon is α-shaped, not heart-shaped: walking inside a heart-shaped 1-gon D' from its only vertex we find a smaller 1-gon in D' (see Fig. 18).

We distinguish two kinds of minimal 0- and 1-gons D' of the essential part C' of a collection C: *simple*, when the interior of D' has no intersection with C',

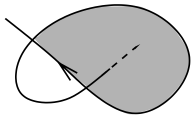

Fig. 18. An α-shaped 1-gon contained in a heart-shaped

and *non-simple* (all the others). For example, the boundary of a simple 1-gon is a simple curl.

The boundary of an n-gon D' of C' comes from the fragment of C after omitting simple curls and smoothing the vertices. This fragment bounds a planar domain which we call the *ancestor* of D' and denote by D. We call the fragment itself the *boundary of D* and denote it ∂D.

4.3. Inductive step: Reduction of the complexity of a curve collection

In four separate subsections below we consider four types of minimal domains at least one of which can be found in any non-empty curve collection C. These are simple and non-simple minimal 0- and 1-gons of the *essential part* C'. In each case, applying a choice of reductions mentioned in Sect. 4.1, we show that it is possible to express $P(C)$ in terms of the polynomials of collections of lower complexity.

4.3.1. Simple minimal 0-gon The aim in this case is to reduce the complexity of a curve collection by passing to a collection with one connected component less.

Up to omitting pairs of opposite curls, we may assume that the boundary of the ancestor D of a simple minimal 0-gon is one of the basic curves, either Z_i or W_i. There are no fragments of C inside the basic curve.

If ∂D is split from the rest of the collection C, the product rule of Fig. 9 gives

$$P(C) = P(\partial D) \cdot P(C \setminus \partial D)$$

where the collection $C \setminus \partial D$ has complexity $(e(C), k(C) - 1)$, and therefore, by the inductive assumption, its polynomial is uniquely defined by the rules of Fig. 9.

Now assume that ∂D is not split from the rest of C and that $\partial D = Z_i$. The reduction algorithm for this case is as follows. We make ∂D small enough and choose a generic path γ to evacuate ∂D into a half-plane separate from the rest of C. According to Example 3.8, pulling Z_i through C once expresses $P(C)$ in terms of the polynomials of collections obtained from $C \setminus \partial D$ by adding a number of simple curls as well as a number of small basic curves Z_j right after the first intersection of γ with C. We iterate the process by pulling all

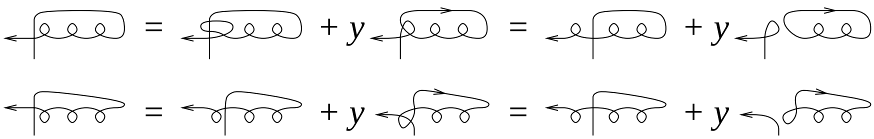

Fig. 19. Making a simple double point of the essential part of a curve simple on the curve itself

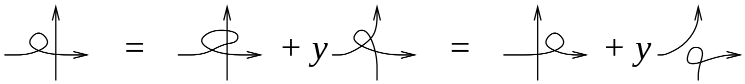

Fig. 20. A simple curl passes through an essential double point

thus obtained small components Z_j of the new collections further along γ. This finally expresses $P(C)$ via the polynomials of collections which are split unions of $C \setminus \partial D$ (equipped with new simple curls) and a number of the curves Z_ℓ. The product rule reduces each of such polynomials to the polynomial of $C \setminus \partial D$ (with extra simple curls) whose complexity is $(e(C), k(C) - 1)$. The inductive assumption finishes the proof of the uniqueness for this case.

Finally, if $\partial D = W_i$ is not split from the rest of C, Fig. 12 expresses $P(C)$ in terms of the polynomials of collections obtained from C by substituting ∂D with a number of the curves Z_j. Evacuation of all of them out from the rest of the collection follows the above pattern for one Z_i.

4.3.2. Simple minimal 1-gon A disc D' being a simple minimal 1-gon means that, after possible omitting pairs of successive curls of opposite orientations, its ancestor D in a curve collection C is bounded by one of the loops of Fig. 19. The figure shows how to make the essential double point of such a loop simple so that the auxiliary collections appearing in the skein relation have less essential double points than C.

By the inductive assumption the proof in this case is complete.

4.3.3. Non-simple minimal 1-gon This is a more complicated case. Now the aim of the reduction is to allow a self-tangency perestroika killing a pair of double points within the ancestor D of a non-simple minimal 1-gon D' of C'.

We start with some preliminary "cleaning" observations.

First of all, from the minimality, there is no irreducible component of C inside D.

Secondly, we can assume that there are no simple curls of C inside D as well as on its boundary. Indeed, due to the relation of Fig. 20, a simple curl move through an essential double point changes the polynomial by the summand corresponding to a collection having less essential double points.

In fact, altogether our assumptions mean that the passing from C to its essential part C' makes no changes within the 1-gon $D = D'$.

Notice that, within our "cleaning" assumptions, the minimality implies that neither branch of $C \cap D$ has a self-intersection in the interior of D.

We call an elementary generic homotopy of a curve collection via a collection with a triple point a *triple-point move*.

Lemma 4.5 *Within the assumptions done, triple-point moves homotop C to a curve collection in which a self-tangency perestroika kills two double points in D.*

A series of triple-point moves followed by a self-tangency perestroika have either no effect on $P(C)$ (for the inverse self-tangency) or represent $P(C)$, by the main skein relation, as the combination of the polynomials of collections with the number of essential double points reduced. Therefore, in the case considered, the inductive step in the proof of Theorem 3.2 is provided by the lemma.

Proof of Lemma 4.5. The proof reduces to the consideration of the following two cases determining if ∂D itself has to be involved in the self-tangency perestroika:

(a) Each pair of branches of $C \cap D$ in the interior of D has at most one point of intersection.

(b) There exists at least one pair of branches of $C \cap D$ having at least two points of intersection in the interior of D.

We start with *Case (a)* and prove the lemma in this situation by induction on the number m of double points of C in the interior of D.

$m = 0$. This implies that D contains a 2-gon adjacent to its boundary ∂D with no other branches of C inside it. This is the 2-gon to be killed by a self-tangency move.

$m > 0$. If there exists a 2-gon adjacent to ∂D we can kill it like in the case $m = 0$.

Suppose that there is no such 2-gon. We use the following lemma to proceed.

Lemma 4.6 *The 1-gon D contains a 3-gon Δ with exactly one of its sides on ∂D and with no fragments of C in the interior of Δ.*

Pushing the inner vertex of such minimal 3-gon Δ through ∂D by the triple-point move we reduce m. By the inductive hypothesis, the proof of Lemma 4.5 in case *(a)* is complete.

Proof of Lemma 4.6. Let B_1 be a branch of $C \cap D$ that intersects some other branches inside D. We may assume that the double point of ∂D and all the branches inside D which do not intersect B_1 are on the same side of B_1. This is a sort of a minimality condition on B_1.

Let $P \in B_1$ be the double point closest to an endpoint N of B_1 (see Fig. 21). Let B_2 be the other branch passing through P. One of its endpoints, Q, is a vertex

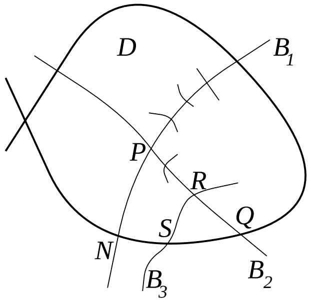

Fig. 21. Search for a minimal 3-gon

of a 3-gon NPQ based on ∂D. This 3-gon may be non-minimal: there can be some other double points of C on the side PQ (due to the minimality of B_1, this is the only possible obstruction to the minimality of NPQ). Choose the one, R, closest to Q. Consider the branch B_3 through R. It cuts the corner piece QRS off NPQ. This is guaranteed by the fact that neither pair of the branches has more than 1 point of intersection.

Now, if QRS is still not minimal, we iterate the descending procedure. Lemma 4.6 is proved.

Now we consider *Case (b)*.

As in case *(a)*, $m > 0$, we assume that there is no empty 2-gon adjacent to ∂D (otherwise we could immediately kill it and make the reduction desired).

Let B^1 and B^2 be branches with more than one common point. Then there exists a 2-gon $T \subset D$ whose boundary lies on these branches and whose vertices are two successive intersections of B^1 and B^2. We may assume the following minimality properties of T:

1) the endpoints of any branch of $C \cap T$ are on different sides of T;
2) any pair of branches of $C \cap T$ has at most one common point.

If there are any double points of C inside T, we move them out using the triple-point moves after finding minimal 3-gons as it was done in the proof of Lemma 4.6. After this we move all the branches of C out from T by the triple-point moves across the vertices of T. Now the 2-gon T is empty and can be killed by a self-tangency perestroika.

This finishes the proof of Lemma 4.5.

4.3.4. Non-simple minimal 0-gon Gathering all the simple curls of the boundary of D in a small neighbourhood of some point $P \in \partial D$, we reduce this case to that of subsection 4.3.3 with the point P playing the role of the double point of the curl.

The proof of Theorem 3.2 is now finished.

Fig. 22. Legendrian lowering of the framed version of the HOMFLY polynomial of links in $\widetilde{M} \simeq \mathbf{R}^3$ to a J_0^+-type invariant of generic collections of plane curves of winding numbers zero. The phases of the two interacting branches in the main skein relation coincide

5. Other versions of the HOMFLY polynomial for regular plane curves

5.1. Regular Legendrian links in the standard 3-space

There exists an obvious analog of Theorem 3.2 that corresponds to links in $\widetilde{M} \simeq \mathbf{R}^3$.

Definition 5.1 A J^+-type invariant of generic one-component plane curves of winding number zero is called a J_0^+*-type invariant* if it changes only under direct self-tangency perestroikas in which the winding numbers of the two subcurves into which the self-tangency point breaks the curve are zero.

This corresponds to a change of the topological type of the lifted Legendrian knot in $\widetilde{M}$.

A multi-component oriented regular Legendrian link in $\widetilde{M}$ is defined by a collection of oriented plane curves in which each of the components has the winding number zero. According to Remark 1.3, on each of the components there should be a point marked by a real number whose reduction modulo 2π is the angle φ of the corresponding normal. The markings define phases $\varphi \in \mathbf{R}$ at all the points of the collection.

Definition 5.2 A J^+-type invariant of the above marked oriented curve collections is called a J_0^+*-type invariant* if it changes only under self-tangencies in which the difference of the phases is zero.

Similar to Theorem 3.2 we have

Theorem 5.3 *There exists a unique* J_0^+*-type invariant* $P_0(C_0) \in \mathbf{Z}[x^2, y^{\pm 1}]$ *of generic collections* C_0 *of marked oriented plane curves of winding numbers zero satisfying the relations and the initial data of Fig. 22.*

5.2. The polynomials of unframed links and the Bennequin-Tabachnikov number estimates

Let β be the Bennequin or Tabachnikov number of an oriented regular Legendrian link $\widetilde{L}_{C_0} \subset \widetilde{M}$ or $L_C \subset M$. The traditional, unframed versions of the HOMFLY

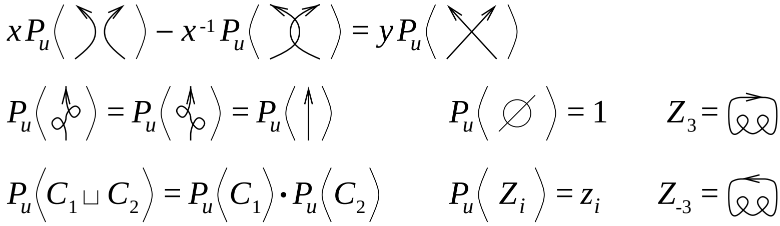

Fig. 23. Legendrian lowering of the unframed version of the HOMFLY polynomial of links in a solid torus to generic plane curve collections

polynomials [11,20] of these links, in terms of the underlying plane curves, are

$$P_{0,u}(C_0) = x^{-\beta} P_0(C_0) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$$

and

$$P_u(C) = x^{-\beta} P(C) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots].$$

Those are topological invariants of the links.

The J^+-type invariant P_u is calculated by the rules of Fig. 23. Omitting the initial data of this figure related to the curves Z_i, one gets the rules to calculate the J_0^+-type invariant $P_{0,u}$. As earlier, the systems of the rules define the plane curve polynomials unambiguously.

Theorems 3.2 and 5.3 immediately imply

Theorem 5.4 *Let $\mathcal{L}$ be an oriented unframed link in the standard contact manifold $\widetilde{M} \simeq \mathbf{R}^3$ or $M = ST^*\mathbf{R}^2$. Let x^k be the minimal power of the framing variable x in the corresponding unframed version of the HOMFLY polynomial of $\mathcal{L}$. Then*

$$\beta_{min,reg}(\mathcal{L}) \geq -k.$$

Remark 5.5 For $\mathbf{R}^3$ this is guaranteed by the theorem of Fuchs and Tabachnikov [12] which derives the same estimate for $\beta_{min}(\mathcal{L})$ from comparison of the results of [5] and [17].

5.3. HOMFLY polynomials of curves with few double points

In Fig. 24 we give the results of calculations of the polynomial P for Arnold's list [2,3] of all the plane curves with at most 3 double points. We set there $z_0 = (x^2 - 1)/y$. The orientations of the curves with non-zero winding numbers are chosen so that these numbers are positive. Change of orientation is covered by the following

Proposition 5.6 *Let C be a generic collection of oriented regular plane curves whose polynomial $P(C)$ is $p(x, y, z_1, z_{-1}, z_2, z_{-2}, \ldots)$. Let C^- (respectively C^r) be the collection obtained from C by the change of orientation of all of its components (respectively by reflection of the plane). Then*

$$P(C^-) = P(C^r) = p(x, y, z_{-1}, z_1, z_{-2}, z_2, \ldots) \ .$$

Proof. One can calculate $P(C^-)$ and $P(C^r)$ following the chain of calculations of $P(C)$. All the curve collections appearing in this chain should be respectively either equipped with the opposite orientations or reflected. The chain for $P(C)$ ends up with disjoint collections of the curves Z_i, $i \in \mathbf{Z}$. Both operations send a curve Z_i to Z_{-i}. □

Corollary 5.7 $P((C^-)^r) = P(C)$.

An illustration to this is seen in the fifth line of Fig. 24.

There is one more rather obvious observation which follows from the coincidence of the total winding numbers of all three curve collections participating in the main skein relation for P.

Proposition 5.8 *The sum of indices of all the z-variables appearing in a particular monomial of $P(C)$ is equal to the winding number of the curve collection C.*

The table of Fig. 24 contains the Bennequin-Tabachnikov numbers of the corresponding regular knots in the solid torus [2–4]. They do not depend on the orientations of the curves and the plane.

Most of the polynomials of Fig. 24 which have no obvious reason to be divisible by x^2 (those are polynomials of the curves with no pairs of simple curls of opposite orientations) are not divisible by it. Non-divisibility of $P(C)$ by x^2 means that the Bennequin-Tabachnikov number of the knot $L_C \subset ST^*\mathbf{R}^2$ is the minimal possible among all the regular knots of the same topological type: $\beta(L_C) = \beta_{min,reg}(L_C)$.

The converse does not seem to be true. For example, for the last curve in the forth line, $P = x^2(\frac{x^2-1}{y} + yz_{-1}z_1)$, but there seems to exist no regular plane curve whose polynomial is that in the brackets of this formula. Another similar example is the first curve of the fifth line. Arnold's tables in [2,3] contain some other curves of the same nature. All of them are certain modifications of those two of Fig. 24. This indicates that the estimate of Theorem 5.4 may not be exact in all the cases. Perhaps there are some special bounds for powers of x in coefficients of various products of z-variables in the HOMFLY polynomials of regular plane curves.

The polynomials $P_0(C)$ of the curves C of Fig. 24 of winding number zero are trivial: each of them is obtained from the $P(C)$ by formally setting $y = 0$

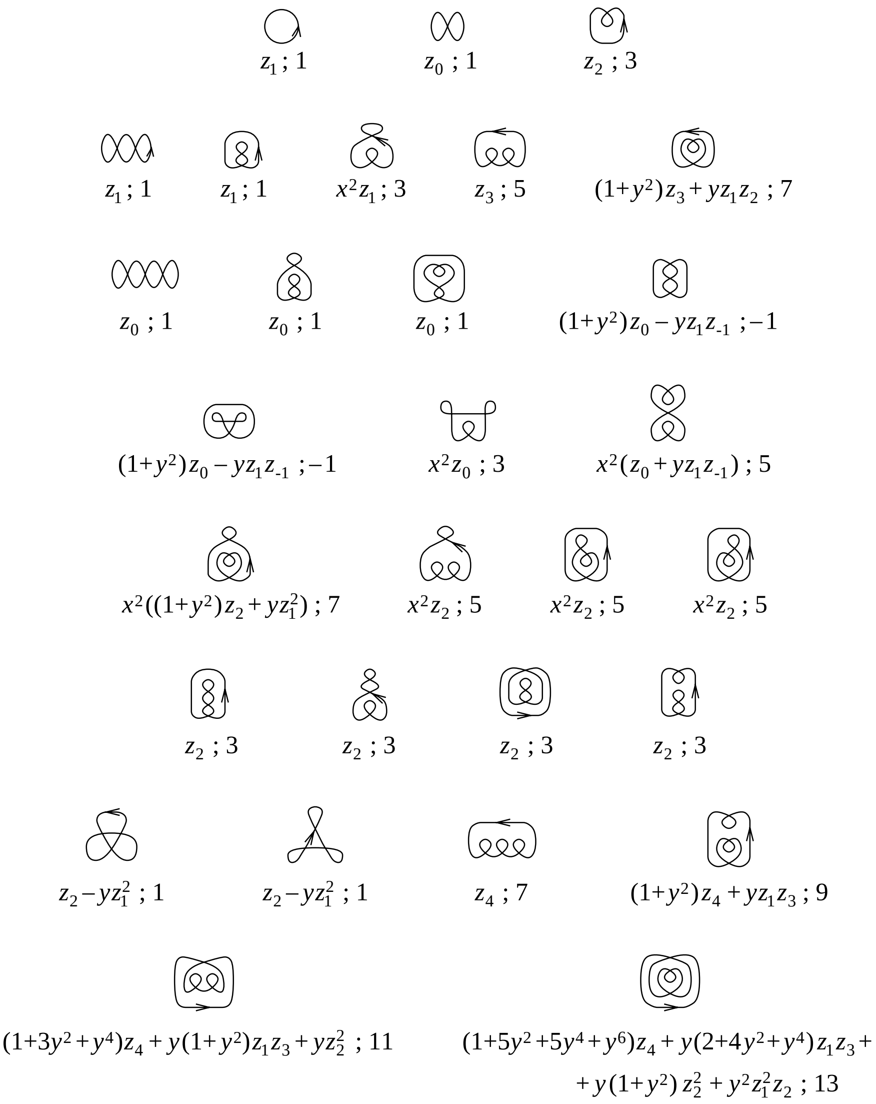

Fig. 24. The HOMFLY polynomials and Bennequin-Tabachnikov numbers of plane curves with at most 3 double points

everywhere except for the relation $z_0 = (x^2 - 1)/y$. Thus, for a table curve, $P_0(C) = x^\alpha z_0$, where $\alpha + 1$ is the Bennequin number of the corresponding Legendrian knot in $\widetilde{M}$. Of course, such a reduction is not correct in general.

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