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Projections of Generic Surfaces with Boundaries

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We meet projections when we are studying various mathematical problems. For instance, let us consider bifurcations of equilibria of a certain dynamical system. In this case we must examine a surface consisting of equilibria for all possible values of parameters in the direct product of the phase space and the parameter space of the system. Singularities of a projection of this surface onto the parameter space correspond to bifurcations of equilibria.

Systematical study of singular projections was started by V. I. Arnold in [A1]. There he classified all singularities which that for a projection of a generic surface in 3-space by any system of parallel rays. In the present paper we consider a problem close to the one studied by Arnold. It is that of classifying projections of surfaces with boundaries embedded in $\mathbb{R}^3$ in a generic way by a system of rays originating at any point outside the surface.

§1. Classifications

Here we formulate the main results of the paper. Their proofs are given in §§3 and 4.

Recall that *a projection of a surface* V, embedded in $\mathbb{R}P^3$, from a point 0 outside the surface, is the diagram $V \hookrightarrow \mathbb{R}P^3 \setminus 0 \to \mathbb{R}P^2$, where the first arrow is the embedding and the second one is the fibration that maps any point into the line passing through this point and the center 0 of the projection. The local version is the germ of the diagram $(V, 0) \hookrightarrow (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ consisting of an embedding and a fibration.

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An *equivalence of* two (germs of) *projections* is a commutative 3×2-diagram whose horizontal lines are the (germs of) projections and the vertical ones are diffeomorphisms.

Let V be a smooth surface with a smooth boundary embedded generically in $\mathbb{R}P^3$. All singularities that may occur at its interior points for any choice of the center of projection in $\mathbb{R}P^3 \setminus V$ are listed in [A2], §15 of Chap. VI. The aim of our work is to get a list of possible singularities for boundary points of V.

The set $\mathbb{R}P^3 \setminus V$ of projection centers is 3-dimensional space. So our classification problem may be divided into two parts. First we have to classify all projection-germs of surfaces with boundaries nonremovable in 3-parameter families. Then, inside the obtained list, we select singularities that may be realized as projections of generic surfaces.

THEOREM 1. *Consider a generic l-parameter family of projections of smooth surfaces with smooth boundaries from 3-space onto a plane, $l \leq 3$. Any projection from this family at any boundary point has a germ that is equivalent to a germ at zero of the projection $(x, y, z) \mapsto (y.z)$ of surfaces $z = f_1(x, y)$, $f_2(x, y, z) \geq 0$ or $z = f_1(x, y)$, $f_2(x, y, z) \leq 0$, where the pair of functions (f_1, f_2) is shown in Table 1 and has codimension $\leq l$.*

The first two columns of the table represent Arnold's list of projections of generic surfaces from $\mathbb{R}^3$ onto $\mathbb{R}^2$ by any system of parallel rays [A1]. Note that we preserve the terms $\alpha x^3 y$ in **7** and $\alpha x^4 y$ in **8**, which may be killed by Arnold's equivalence. We do so in order to make calculations more convenient.

THEOREM 2. *A projection of a generic surface with boundary from 3-space onto a plane by a system of parallel rays may have singularities at boundary points only of the following types:* $\mathbf{1}'$, A_0, A_1, A_2, B_1, B_2, $\mathbf{3}_1$, $\mathbf{3}_2$ $(\alpha \neq -1/3, 1/2, 2/3)$, $\mathbf{4}_1^{\pm}$ $(\alpha \neq 0)$, $\mathbf{6}_1$. *For a projection by a system of rays originating at any center outside the surface, only the following types may additionally occur:* A_3, B_3, $\mathbf{3}_3$ *(*$\alpha \neq -1/3, 1/2, 2/3$*)*, $\mathbf{4}_2^-$ $(\alpha \neq 0)$, $\mathbf{4}^-_{1,1/\sqrt{3}}$, $\mathbf{4}^-_{1,2/\sqrt{3}}$, $\mathbf{5}_1$, $\mathbf{6}_2$. *All the singularities mentioned are observable for a suitable choice of generic surface with boundary and an appropriate system of rays.*

Examples of realizations are given in the last column of Table 1. Here we indicate terms that one may add to the pair (f_1, f_2) in order to get a germ (at zero) of a generic surface with boundary $z = f_2'$, $f_2' \lesseqgtr 0$, which has the prescribed singularity for a projection along the x-axis (i.e., from a center at infinity). "No" means that the singularity is nonrealizable.

TABLE 1

Surface-projection type	f_1	Boundary-projection type	f_2	Restrictions	Codim-ension	Realization
1	x	$1'$	y	–	0	$(0, 0)$
2	x^2	A_k	$x + y^{k+1}$	$k \geq 0$	k	$(y^2, 0)$
		B_k	$y + x^{2k+1}$	$k \geq 1$	k	$(0, x^4)$
3	$x^3 + xy$	3_1	x	–	1	$(0, 0)$
		3_2	$x^2 + \alpha y \pm y^2$	$\alpha \neq 0, 1,$	2	$(0, 0)$ } for $\alpha = -1/3, 1/2,$
		3_3	$x^2 + \alpha y$	$1/3, 4/3$	3	$(y^2, 0)$ } $2/3$ no
		$3_{k,1}$	$x^2 + y \pm y^k$	$k \geq 2$	$k + 1$	no
		$3_{k,\alpha}$	$x^2 + \alpha y \pm x^{k+1}$	$\alpha = \frac{1}{3}, \frac{4}{3}$; $k \geq 2$	$k + 1$	no
		$3_{*,k}$	$y \pm xz + \alpha x^2 z^k$	$\alpha \neq 0$; $k \geq 1$	$k + 2$	no
4^+	$x^3 + xy^2$	4_1^+	$x + \alpha y + y^2$	–	2	$(y^2, 0)$ } for $\alpha = 0$ no
		4_2^+	$x + \alpha y$	–	3	$(y^2, 0)$ }
		4_*^+	$y + x^2 + \alpha x^4$	$\alpha \neq 0$	3	no
4^-	$x^3 - xy^2$	4_1^-	$x + \alpha y + y^2$	$\alpha^2 \neq 1/3, 4/3$	2	$(y^2, 0)$; for $\alpha = 0$ no
		4_2^-	$x + dy$	$\alpha^2 \neq 1/3, 4/3$	3	$(y^2 + By^3, 0)$, $6\alpha^2 - 9B\alpha \neq 4$; for $\alpha = 0$ no
		$4_{k,\alpha}^-$	$x + \alpha y + y^{k+1}$	$\alpha = \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$; $k \geq 1$	$k + 2$	$(y^2, 0)$
		4_*^-	$y + x^2 + 2x^4$	$\alpha \neq 0$	3	no
5	$x^3 + xy^3$	5_1	$x + y$	–	3	$(y^2, 0)$
6	$x^4 + xy$	6_1	$x \pm y$	–	2	} $x := x + x^2$
		6_2	x	–	3	}
		6_*	$x^2 + y + \alpha xy$	–	3	no
7	$x^4 + \alpha x^3 y + x^2 y + xy^2$	7_*	x	$\alpha \neq 2$	3	no
8	$x^5 + \alpha x^4 y \pm x^3 y + xy$	8_*	x	$\alpha \neq 0$	3	no

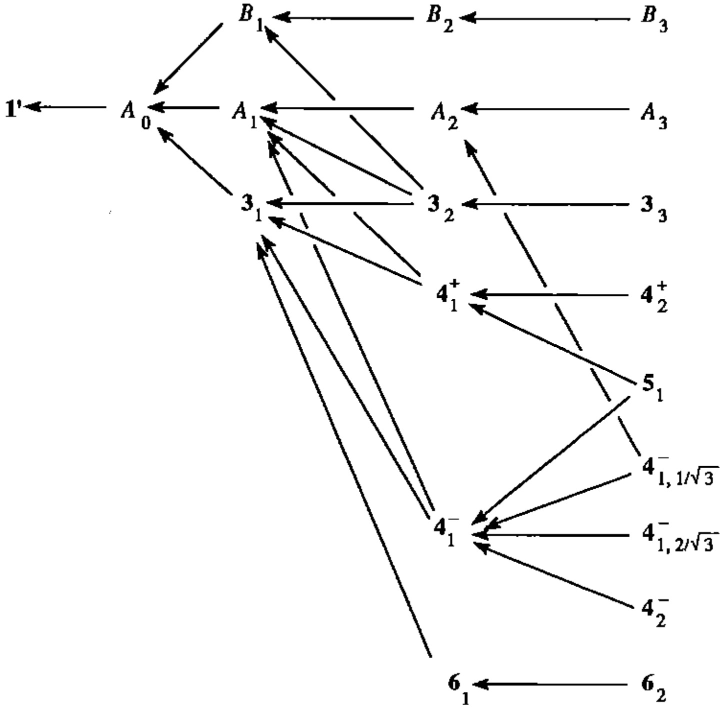
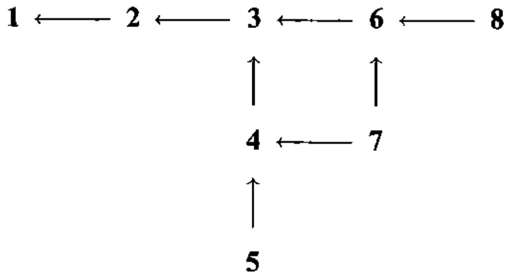


FIGURE 1. Adjacency of realizable boundary projections.

Using the hierarchy of Arnold's singular projections



one can get the hierarchy of boundary projections from Table 1. In Figure 1 we give its part containing the realizable singularities.

In Figure 2 we show the singularities of apparent contours and sets of critical values of realizable boundary projections. Instead of two separate illustrations for the surfaces $z = f_1$, $f_2 \geq 0$ and $z = f_1$, $f_2 \leq 0$, we give an amalgamated one here—for the surface $z = f_1$ with a smooth

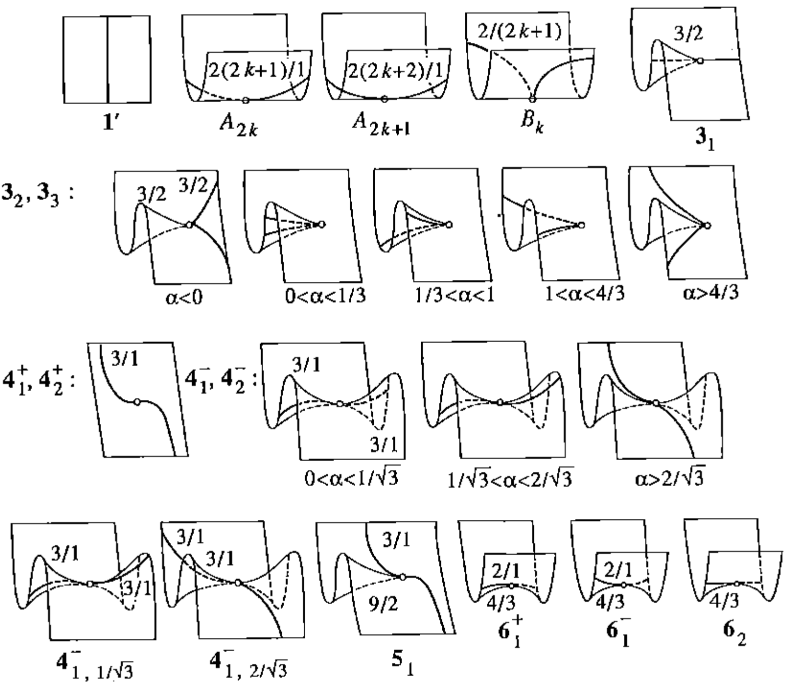


FIGURE 2. Projections of generic surfaces with boundaries.

curve $f_2 = 0$ on it. We also indicate Puiseux exponents for the image of the boundary and the set of critical values of the projection of the surface $z = f_1$. For classes with moduli, these exponents are the same for all possible values of moduli. The drawings for $\mathbf{3}_2$ and $\mathbf{3}_3$, $\mathbf{4}_1$ and $\mathbf{4}_2$ are qualitatively identical. Members of the $\mathbf{4}_1^+$-family with opposite values of α are equivalent—the same is true for all realizable singularities denoted by a $\mathbf{4}$ with various indices.

REMARK. The realizability of a class containing a module in its normal form means that for a projection of a given generic surface we obtain some separate singularity from this class for some specific value of the module. For a projection of a neighboring surface, we also get a singularity from the same class but, generally, with another (neighboring) value of the module.

Recall that a singularity is called *simple* if any of its sufficiently small perturbations only yield singularities from a finite number of orbits of equivalence.

THEOREM 3. *Simple germs of boundary projections of surfaces from 3-space onto a plane are precisely the singularities* $\mathbf{1}'$, A_k, $k \geq 0$, B_k, $k \geq 1$, $\mathbf{3}_1$, $\mathbf{6}_1$, *and* $\mathbf{6}_2$.

Equivalent projections have left-right equivalent composite mappings $V \to \mathbb{R}P^2$ (i.e., one of these mappings may be transformed into the other by diffeomorphisms of the source and the target). Moreover, the classification of projection-germs $V \hookrightarrow \mathbb{R}P^3 \setminus 0 \to \mathbb{R}P^3$ of smooth surfaces is absolutely identical to a left-right classification of germs $V \to \mathbb{R}P^2$ of composite mappings that have corank ≤ 1 at a distinguished point: a composite mapping recorded in certain local coordinates as $(x, y) \mapsto (y, f(x, y))$ corresponds to a germ at zero of the projection $(x, y, z) \mapsto (y, z)$ of the surface $z = f(x, y)$. We prove this quite obvious fact in a general setting in the Appendix.

From the above relation it follows that Table 1 may be easily transformed into a table of normal forms of mappings of a half-plane into a plane. For example, the B_k-projection corresponds to the mapping $(x, y) \mapsto (y - x^{2k+1}, x^2)$ of the half-plane $y \geq 0$.

On the other hand, mappings of a half-plane into a plane correspond to mappings of a plane into a plane invariant under the reflection $(x, u) \mapsto (x, -u)$ in the source. The correspondence is given by the quotient-map $x = x$, $y = u^2$ into the half-plane $y \geq 0$.

The classification of projections of a generic surface with boundary from 3-space onto a plane by parallel rays (as the left-right classification of mappings of the half-plane into a plane of corank ≤ 1) was considered by Bruce and Giblin in [BG]. But in the version of their preprint available to the author of this paper, the singularities that we denote by $\mathbf{4}_{\mathrm{I}}^{\pm}$ were missing.

§2. How to see the singularities

When we project a generic surface from a generic center, we get a $\mathbf{1}'$-singularity at a generic boundary point. Looking at a boundary point along a direction tangent to the surface at this point, we generally get A_0. For the direction tangent to the boundary itself, we obtain B_1. If this is a tangent at a flattening point of the boundary, we get B_2. For a special choice of observation point on such a tangent, one can see a B_3-singularity [Dv].

A critical set of a projection is called a *terminator*. For a generic projection center it meets the boundary of the surface transversally. As we have already stated, the projection is then of type A_0 at the point of this meeting. Now consider a surface consisting of straight lines that are

(a) tangent to the initial surface at boundary points and

(b) orthogonal to the boundary in the sense of the second quadratic form of our surface.

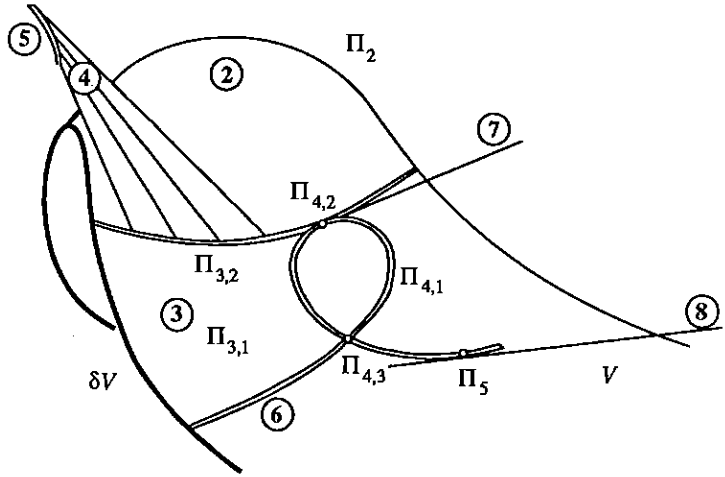


FIGURE 3. Projective classification of points on a generic surface.

For a projection from a point of this surface, a terminator has a contact with the boundary. If the point is regular, this contact is simple and we get an A_1-singularity.

Since the surface we have just described is fibered by (straight) lines, we may expect that its nonregularity will be a cuspidal edge. One can show that this is just the case and that every line-fiber has exactly one point on the edge. Moreover, if the fiber is not tangent to the boundary (or, equivalently, has only simple contact with the initial surface) this point does not lie on the boundary (the converse is also true). Projection from a nonsingular point of the cuspidal edge is of type A_2. Singular points of the edge (i.e., swallow-tail points of our line-fibered surface) provide A_3-projections.

For our further description of visual opportunities, let us recall the projective classification of points on a generic surface in projective 3-space and realizations of singular projections at interior points of such a surface ([P, L, A2]).

A smooth *curve of parabolic points* $\Pi_{3,2}$ divides a surface V into a *region of elliptic points* Π_2 (there are no real tangents of order higher than 1) and a *region of hyperbolic points* $\Pi_{3,1}$ (there are two such tangents called *asymptotic lines*; their directions at the point of contact are called *asymptotic directions*).

In the region of hyperbolic points, there is a distinguished smoothly immersed *curve of inflections* $\Pi_{4,1}$ (at its points the order of contact of an asymptotic line with the surface is higher than 2). On this curve there are separate *points of bi-inflections* Π_5 (where the order of contact is 4) and *points of self-intersections* $\Pi_{4,3}$ (both asymptotic lines have third order contact). At the $\Pi_{4,2}$-points the curve of inflections of the asymptotics is simply tangent to the curve of parabolic points (here the single asymptotic direction is tangent to the curve of parabolic points).

Arnold's list **1–8** is realized in the following way.

For a generic center the projection is nonsingular (type **1**) at almost every interior point of the surface, has a fold **2** along some smooth curve (a terminator) and a cusp **3** at separate points of this curve. Singularities **4** and **6** are obtained by projections from generic points of surfaces consisting of asymptotic lines passing through the points of the parabolic curve and the curve of inflections of the asymptotics respectively. The first of these surfaces has a cuspidal edge. From this edge one sees the singularity **5**. We see **7** from any generic point of an asymptotic line tangent to the curve of parabolic points (i.e., passing through $\Pi_{4,2}$-point), and we see **8** from a generic point of an asymptotic tangent of the fourth order.

Now we return to the description of realizable boundary projections and also point out obstructions for realizability of all forbidden singularities.

All the boundary singularities having the unique index 1 as a subscript are obtained in the same way as the corresponding interior singularities with the single additional condition on the point of interest—that it lies on the boundary. For instance, $\mathbf{3}_1$ is realized by a projection from a point of an asymptotic line applied at a boundary point.

At isolated boundary points asymptotic lines touch the boundary. For a generic point of one of those lines as projection center, we get a $\mathbf{3}_2$-singularity, for a special point we have $\mathbf{3}_3$. It is necessary to note that a value of the module in the normal form of $\mathbf{3}_2$ does not depend on a choice of generic point on the aforementioned line (see §4). By the way, this is the reason for the absence of singularities $\mathbf{3}_{2,1/3}$ and $\mathbf{3}_{2,4/3}$ for projections of generic surfaces with boundary.

In order to see $\mathbf{4}_1^{\pm}$-singularities, we must look from a generic center on an asymptotic line of a parabolic point of the surface lying on the boundary. Note that this time the value of the module in the normal form $\mathbf{4}_1^{\pm}$ depends on the choice of projection center on the line (see §4). For special choices of observation point on the same line, we obtain $\mathbf{4}_2^{\pm}$, $\mathbf{4}^-_{1,1/\sqrt{3}}$ and $\mathbf{4}^-_{1,2/\sqrt{3}}$ (and $\mathbf{5}_1$, too).

Look at the surface from a special point of an asymptotic line passing through a boundary point situated on the curve of inflections of the asymptotics. Then we can realize the $\mathbf{6}_2$-singularity ($\mathbf{6}_1$ is obtained by a projection from a generic point of the same line).

Now we comment on nonrealizability. The forbidden singularities could occur only in the following cases:

$\mathbf{8}_*$—the point of biinflection Π_5 (an isolated point of the surface) is on the boundary;

$\mathbf{7}_*$—the point $\Pi_{4,2}$, where the curve of inflections of the asymptotics is tangent to the curve of parabolic points (also an isolated point of the surface), is on the boundary;

$\mathbf{6}_*$—a projection from a point of an asymptotic line passing through a common point of the boundary and the curve of inflections of the asymptotics. This asymptotic line must in addition be tangent to the boundary;

$\mathbf{4}_*^{\pm}$—the same as in the $\mathbf{6}_*$-case but with the curve of parabolic points instead of the curve of inflections;

$\mathbf{3}_{2,1}$—a projection from a point of an asymptotic line that is tangent to the boundary curve at its flattening point.

It is evident that none of the possibilities listed can occur for a generic surface with boundary.

The $\mathbf{3}_{*,1}$-singularity has a 3/4-parabola as the image of the boundary—this case cannot occur for the projection of a generic curve [DV].

In families $\mathbf{3}_2$ and $\mathbf{3}_3$:

(1) the value $\alpha = 2/3$ of the module is excluded as otherwise there would be second order contact of the boundary with the integral curve of the field of asymptotic lines (see §4);

(2) the bifurcation diagrams for values $\alpha = -1/3$ and $\alpha = 1/2$ are distinct from the diagrams corresponding to generic values of α (§4).

Finally, the value $\alpha = 0$ of the module in the families $\mathbf{4}_{\mathrm{I}}^{\pm}$ means that the curve of parabolic points is tangent to the boundary (§4 again).

§3. Classificational calculations

We treat the problem of the local classification of projections of surfaces with boundary in the spirit of the theory of functions on a manifold with boundary [A3], as a problem of the classification of projections of flags, i.e., germs of diagrams $\mathbb{R}^2, \mathbb{R} \hookrightarrow \mathbb{R}^3 \to \mathbb{R}^2$ consisting of an embedding of a flag and a fibration (of course, one must additionally fix a half-plane in $\mathbb{R}^2$ situated on one side or the other of the distinguished line $\mathbb{R} \subset \mathbb{R}^2$). The equivalence of flag-projections is analogous to the one for boundary projections.

Let $(x, y, z) \mapsto (y, z)$ be the coordinate description of the fibration

$\mathbb{R}^3 \to \mathbb{R}^2$ (we fix this description up to the end of the paper), $g_1(x, y, z) = 0$ – the equation of a plane embedded in 3-space and $g_2(x, y, z) = 0$ – the equation of its distinguished curve $\mathbb{R}^2, \mathbb{R} \hookrightarrow \mathbb{R}^3 \to \mathbb{R}^2$ (in section 1, $g_1 = f_1 - z$, $g_2 = f_2$). The condition of smoothness of the embedded surface and the curve on it implies

$$\operatorname{rk}\left(\frac{\partial(g_1, g_2)}{\partial(x, y, z)}\right) = 2.$$

The classification of flag-projections is the classification of germs at zero of function-pairs (g_1, g_2) up to:

(a) substitutions $x(x', y', z')$, $y(y', z')$, $z(y', z')$ (diffeomorphisms of $\mathbb{R}^3$ fibered over $\mathbb{R}^2$) and

(b) transformations

$$(g_1, g_2) \mapsto (ag_1, bg_1 + cg_2),$$

where a, b, c are germs at zero of functions on $\mathbb{R}^3$, $a(0)\cdot c(0) \neq 0$ (new choice of equations of the surface and the curve).

We are interested in the classification of map-germs $g = (g_1, g_2)$, equivalence classes that have low codimension in the space $\mathscr{O}_3^2$ of all analytic (formal or C^∞-) germs $(\mathbb{R}^3, 0) \to \mathbb{R}^2$. This codimension equals $\dim_{\mathbb{R}} \mathscr{O}_3^2/T_g$, where T_g is the tangent space to the equivalence class of g:

$$T_g = \mathscr{O}_3\left\langle \begin{pmatrix} g_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_1 \end{pmatrix}, \begin{pmatrix} 0 \\ g_2 \end{pmatrix}, \frac{\partial g}{\partial x} \right\rangle + \mathscr{O}_2\left\langle \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle.$$

Here $\mathscr{O}_3$ and $\mathscr{O}_2$ are rings of function-germs in x, y, z and y, z respectively. The mapping g and other elements of $\mathscr{O}_3^2$ are written in columns.

REMARKS. 1°. The codimension of the family mentioned in classification Table 1 equals the codimension of an individual member of this family minus a number of moduli of the family.

2°. For a versal deformation of a boundary projection g (in the sense traditional for singularity theory) one may take $g + \sum \lambda_i e_i$, where $e_i \in \mathscr{O}_3^2$ are representatives of a basis of the linear space $\mathscr{O}_3^2/T_g$ (see, e.g., [Dm]).

It is convenient to carry out classificational calculations in the following order. First reduce to normal form the germ of the surface-projection $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \to \mathbb{R}^2$, i.e., function g_1. Up to the depth of classification we need (as one can easily see, up to surface-projection singularities of codimension 2) we obtain on this step Arnold's list **1–8**: $g_1(x, y, z) = f_1(x.y) - z$ for f_1 from Table 1.

In the second step we reduce to normal form the curve $g_2 = 0$ that belongs to one of the surfaces **1–8**. This reduction is carried out by diffeomorphisms of 3-space, preserving its fibration over the plane and translating the surface $g_1 = 0$ into itself, and also by transformations $g_2 \mapsto bg_1 + cg_2$, $c(0) \neq 0$. Two diffeomorphisms of $\mathbb{R}^3$ of the type indicated, acting on $g_1 = 0$ in the same way, differ by a diffeomorphism, which is the identity on $g_1 = 0$. An action of the latter diffeomorphism on the equation of the curve adds a function divisible by g_1 to it. Hence, instead of diffeomorphisms of $\mathbb{R}^3$ fibered over the plane and preserving the surface $g_1 = 0$, it is sufficient to consider diffeomorphisms χ of the surface itself that can be lowered onto the plane (i.e., such that there exist diffeomorphisms χ' of the plane for which $\pi \circ \chi = \chi' \circ \pi$, where $\pi\colon \{g_1 = 0\} \to \mathbb{R}^2$ is the projection). π has a fold at almost every one of its critical points. So, according to [A4], The Lie algebra $\mathfrak{A}$ of a group of diffeomorphisms of the surface, possessing the property we need, is exactly the algebra of vector fields on the surface $\{g_1 = 0\} \subset \mathbb{R}^3$ that are projected into fields on the plane tangent to the set C of critical values of π.

For a weighted-homogeneous surface $g_1(x, y, z) = 0$, $\mathfrak{A}$ is generated over $\mathcal{O}_2 = \mathcal{O}_{y,z}$ by two elements: the Euler field e and the lifting h onto $g_1 = 0$ of the Hamiltonian field $h' = (\partial\varphi/\partial z)\partial_y - (\partial\varphi/\partial y)\partial_z$, where $\varphi(y, z) = 0$ is the equation of the set C.

The codimension of a flag-projection is the sum of two codimensions:

$c_1 = \dim_{\mathbb{R}} \mathcal{O}_3/\{\mathcal{O}_3\langle g_1, \partial g_1/\partial x\rangle + \mathcal{O}_2\langle \partial g_1/\partial y, \partial g_2/\partial z\rangle\}$, the codimension of the projection of the surface $g_1 = 0$;

$c_2 = \dim_{\mathbb{R}} \mathcal{O}_3/\{\mathfrak{A}g_2 + \mathcal{O}_3\langle g_1, g_2\rangle\}$, the codimension of the curve $g_2 = 0$ in the space of all curves on the fixed surface $g_1 = 0$ in $\mathbb{R}^3$ fibered over $\mathbb{R}^2$.

We denote the denominator of the latter quotient space by T'_{g_2}, the space of i-th coordinate functions of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2$ by $\mathcal{O}_3^{(i)}$, $i = 1, 2$. We write elements of $\mathcal{O}_3^2$ in long formulas as columns of their coordinate functions, in short formulas we write them in line. When we want to show generators of a ring of functions, we write, say, $\mathcal{O}_{y,z}$, instead of $\mathcal{O}_2$.

Now we shall consider the variants **1–8** of the surfaces $g_1 = 0$, one after another. Here we shall use standard techniques of singularity theory [AGLV].

Since $g_1 = f(x, y) - z$, we may take $g_2 = g_2(x, y)$.

1. Let $g_1 = x - z$; then the projection $(x, y, z) \mapsto (y, z)$ is an isomorphism. The distinguished curve is smooth, hence its equation may be reduced to $y=0$, i.e. we get the singularity $\mathbf{1}'$ from Table 1.

2. Let $g_1=x^2-z$; then $c_1=0$, C: $z=0$, $\mathfrak{A}=\mathscr{O}_2\langle\partial_y, x\partial_x+2z\partial_z\rangle$.

If $\partial g_2/\partial x(0)\neq 0$, we may suppose the main quasihomogeneous part of g_2 to be $g_2^0=x+y^{k+1}$ for some integer $k\geq 0$. Then $T'_{g_2^0}=\mathscr{O}_3\langle x, y^k, z\rangle$. Hence g_2 is equivalent to g_2^0 ($g_2\sim g_2^0$) and the flag-projection has normal form A_k and codimension k.

If $\partial g_2/\partial x(0)=0$, the smoothness of the boundary implies $\partial g_2/\partial y(0)\neq 0$. Then $g_2^0\sim y+x^{2k+1}$, $k\geq 1$ (one can kill the even powers x^{2k} in g_2 since it may be replaced by z^k obtained by the substitution $y:=y+az^k$). We have

$$\mathscr{O}_3/T'_{g_2^0}\simeq\mathbb{R}\langle x, x^3, \dots, x^{2k-1}\rangle, \qquad g_2\sim g_2^0 \quad\text{and}\quad g\sim B_k.$$

3. Let $g_1=x^3+xy-z$. Here $c_1=0$, C: $27z^2+4y^3=0$.

The basic vector fields tangent to C are:

$$e'=2y\partial_y+3z\partial_z \quad\text{and}\quad h'=9z\partial_y-2y^2\partial_z.$$

So $\mathfrak{A}=\mathscr{O}_{y,z}\langle x\partial_x+2y\partial_y+3z\partial_z, (3x^2+2y)\partial_x-9z\partial_y+2y^2\partial_z\rangle$.

Let us consider successively all possible main quasihomogeneous parts of g_2 ordered by increasing weight (grading: $wt\,x=1$, $wt\,y=2$, $wt\,z=3$).

(a) $g_2^0=x$. $T'_{g_2^0}=\mathscr{O}_3\langle x, z\rangle+\mathscr{O}_{y,z}\langle x, y\rangle$. So $g_2\sim g_2^0$ is of type $\mathbf{3}_1$ and $c=c_2=1$.

(b) $g_2^0=x^2+\alpha y$, $\alpha\neq 0$.

We have $T'_{g_2^0}=\mathscr{O}_3\langle x^3+xy-z, x^2+\alpha y\rangle+\mathscr{O}_{y,z}\langle(3x^2+2y)2x-9\alpha z\rangle$ (we omit the action of the Euler field as g_2^0 is quasihomogeneous). Substitute x^3+xy for z and then $-x^2/\alpha$ for y in $\mathscr{O}_3/T'_{g_2^0}$. This yields

$$\mathscr{O}_3/T'_{g_2^0}\simeq\mathscr{O}_x/\mathscr{O}_{x^2,(\alpha-1)x^3}\langle(9\alpha^2-15\alpha+4)x^3\rangle.$$

So, for $\alpha\neq 1, 1/3, 4/3$ this factor equals $\mathbb{R}\langle 1, x, x^2, x^4\rangle\simeq\mathbb{R}\langle 1, x, y, y^2\rangle$ and $g_2\sim x^2+\alpha y+\beta y^2$. Normalization of $\beta\neq 0$ to ± 1 by Euler stretching gives $\mathbf{3}_2$. For $\beta=0$, we get $\mathbf{3}_3$ at once.

If $\alpha=1$, $\mathscr{O}_3/T'_{g_2^0}\simeq\mathscr{O}_x/\mathscr{O}_{x^2}\langle x^3\rangle\sim\mathbb{R}\langle x\rangle+\mathscr{O}_y$. This is the cokernel $\operatorname{Coker} d_0$ of the 0-differential of the corresponding spectral sequence [AGLV]. Consideration of the first nontrivial positive differential of this sequence shows that Euler stretching yields g_2 to $x^2+y\pm y^k$, $k\geq 2$, and $\mathscr{O}_3/T'_{g_2^0} \simeq \mathbb{R}\langle x, 1, y, \dots, y^{k-1}\rangle$. So we get $\mathbf{3}_{k,1}$ of codimention $c = c_2 = k+1$.

If $\alpha = 1/3, 4/3$, then: $\operatorname{Coker} d_0 = \mathscr{O}_3/T'_{g_2^0} \simeq \mathscr{O}_x$, g_2 is reduced by d_0 to the form $x^2 + \alpha y + \varphi(x)$ ($\operatorname{order}\varphi \geq 3$) and the kernel $\operatorname{Ker} d_0$ of the 0-differential of the spectral sequence contains the module over $\mathscr{O}_{y,z} \equiv \mathscr{O}_{x^2,x^3}$ generated by e and h ($\operatorname{mod}\mathscr{O}_3\langle g_1, g_2^0\rangle$). The action of this kernel on φ of order $k+1$ gives us $\mathscr{O}_{x^2,x^3}\langle x^{k+1}, x^{k+2}\rangle \subset T'_{g_2}$. Thus, $g_2 \sim x^2 + \alpha y \pm x^{k+1}$, $k \geq 2$, i.e., g belongs to $\mathbf{3}_{k,1/3}$ or $\mathbf{3}_{k,4/3}$, $c = c_2 = k+1$.

(c) Let $g_2^0 = y$. Then $T'_{g_2^0} = \mathscr{O}_3\langle x^3 - z, y\rangle + \mathscr{O}_2\langle z\rangle$. Hence, $g_2 \sim y + xz(\varphi_1(z) + x\varphi_2(z))$.

Calculations of higher differentials of the spectral sequence show that for $\varphi_1(0) \neq 0$, g_2 is reduced to $\mathbf{3}_{*,k}$: $y + xz + \beta x^2 z^k$, $\beta \neq 0$, $k \geq 1$. Here $\mathscr{O}_3/T'_{g_2} \simeq \mathbb{R}\langle 1, x, x^2, x^2 z, \dots, x^2 z^k\rangle$ and $c = c_2 = k+3$.

$\mathbf{4}^-$. $g_1 = x^3 - xy^2 - z$. Then $c_1 = 1$, C: $27z^2 - 4y^6 = 0$. The surface is quasihomogeneous: $wt x = wt y = 1$, $wt z = 3$.

$$e = x\partial_x + y\partial_y + 3z\partial_z, \quad h = (3x^2 - 2y^2)2y\partial_x + 9z\partial_y + 4y^5\partial_z.$$

(a) Let $\partial g_2/\partial x(0) \neq 0$. Hence, $g_2^0 \sim x + \alpha y$. We obtain

$$T'_{g_2^0} = \mathscr{O}_3\langle x^3 - xy^2 - z, x + \alpha y\rangle + \mathscr{O}_{y,z}\langle (3x^2 - 2y^2)2y + 9\alpha z\rangle.$$

Thus, $x \equiv -\alpha y$ and $z \equiv \alpha(1-\alpha^2)y^3$ in $\mathscr{O}_3/T'_{g_2^0} \simeq \mathscr{O}_y/\mathscr{O}_y\langle(-9\alpha^4 + 15\alpha^2 - 4)y^3\rangle$.

So, for $\alpha^2 \neq 1/3, 4/3$ we have $g_2 \sim x + \alpha y + \beta y^2$. If $\beta = 0$, we have $\mathbf{4}_2^-$. If $\beta \neq 0$, we normalize it to 1 by Euler stretching and get $\mathbf{4}_1^-$.

For $\alpha^2 = 1/3, 4/3$ we have $\mathscr{O}_3/T'_{g_2^0} \simeq \mathscr{O}_y$ and $g_2 \sim x + \alpha y + \varphi(y)$, $\operatorname{order}\varphi = k+1 \geq 2$. The Euler transformation takes φ to y^{k+1} and we get $\mathbf{4}_{k,\alpha}^-$ with $\mathscr{O}_3/T'_{g_2} \simeq \mathbb{R}\langle 1, y, \dots, y^k\rangle$ and $e = 1 + c_2 = k+2$.

(b) Let $\partial g_2/\partial x(0) = 0$. Then $g_2^0 = y$. $T'_{g_2^0} = \mathscr{O}_3\langle x^3 - z, y\rangle + \mathscr{O}_{y,z}\langle z\rangle$. $\operatorname{Coker} d_0 \simeq \mathscr{O}_x/\mathscr{O}_{x^3}\langle x^3\rangle$.

In the general case $g_2 \sim y + x^2 + \alpha x^4 + \dots$. Higher differentials of the spectral sequence show that "$+\dots$" may be omitted. Thus, we get $\mathbf{4}_*^-$ with $c = 1 + c_2 = 4$.

$\mathbf{4}^+$. $g_1 = x^3 + xy^2 - z$. The substitution $y := iy$ in $\mathbf{4}^-$ yields the classification for this surface.

5. $g_1 = x^3 + xy^3 - z$. It is preferable here to consider the quasi-homogeneous grading on the space $\mathscr{O}_3^2$ of pairs $g = (g_1, g_2)$ induced by the generic linear part $x + y$ of g_2. Then the main part of g is $g^0 = (x^3 - z, x + y)$. We have:

$$\begin{aligned}\mathscr{O}_3^2/T_{g^0} &= \mathscr{O}_3^2/\left\{\mathscr{O}_3\left\langle\binom{x^3-z}{0}, \binom{0}{x^3-z}, \binom{0}{x+y}, \binom{3x^2}{1}\right\rangle + \mathscr{O}_2\left\langle\binom{0}{1}, \binom{1}{0}\right\rangle\right\}\\ &\simeq \mathscr{O}_3^{(1)}/\{\mathscr{O}_3\langle x^3 - z, x^2(x+y)\rangle + \mathscr{O}_2\langle 1, 3x^2\rangle\}\\ &\simeq \mathscr{O}_{x,y}^{(1)}/\{\mathscr{O}_{x,y}\langle x^2(x+y)\rangle + \mathscr{O}_{y,x^3}\langle 1, x^2\rangle\}\\ &\simeq x\cdot\mathscr{O}_y^{(1)}.\end{aligned}$$

Hence, the differential d_0 takes g to the form $(x^3 - z + x\varphi(y), x + y)$, order $\varphi = k \geq 3$. Since the Euler transformation is in $\operatorname{Ker} d_0$, we may reduce φ to $\pm y^k$. Such a singularity has codimension k. In the nondegenerate case, $k = 3$ and we get $\mathbf{5}_1$.

6. $g_1 = x^4 + xy - z$. Here $c_1 = 1$ and C: $27y^4 + 256z^3 = 0$.

$$e = x\partial_x + 3y\partial_y + 4z\partial_z, \quad h = (16x^6 + 28x^3y + 9y^2)\partial_x - 64z^2\partial_y + 9y^3\partial_z.$$

Grading: $wt\,x = 1$, $wt\,y = 3$, $wt\,z = 4$.

(a) Let $g_2^0 = x$. Then $T'_{g_2^0} = \mathscr{O}_3\langle z, x\rangle + \mathscr{O}_2\langle y^2\rangle$. So $g_2 \sim x + \beta y$. For $\beta \neq 0$ we get $\mathbf{6}_1$ ($c = 2$) and for $\beta = 0$ – $\mathbf{6}_2$ ($c = 3$).

(b) Let $g_2^0 = x^2$. Then

$$\mathscr{O}_3/T'_{g_2^0} = \mathscr{O}_3/\{\mathscr{O}_3\langle xy - z, x^2\rangle + \mathscr{O}_2\langle xy^2\rangle\} \simeq \mathscr{O}_y + \mathbb{R}\langle x, xy\rangle.$$

Thus, $g_2 \sim x^2 + \varphi(y) + \alpha xy$, $\varphi(0) = 0$. Generally $\varphi'(0) \neq 0$ and by Euler transformation $g_2 \sim x^2 + y + \alpha xy$. Since there is no element of degree 1 in $\operatorname{Ker} d_0$, α is a module. So, we get $\mathbf{6}_*$ ($c = 4$).

7. $g_1 = x^4 + x^2y + xy^2 - z$. Generally $\partial g_2/\partial x(0) \neq 0$, and as the main part g^0 of g we may take the pair $(x^4 + x^2y - z, x)$. Then

$$\begin{aligned}\mathscr{O}_3^2/T_{g^0} &= \mathscr{O}_3^2/\left\{\mathscr{O}_3\left\langle\binom{x^4+x^2y-z}{0}, \binom{0}{z}, \binom{0}{x}, \binom{4x^3+2xy}{1}\right\rangle\right.\\ &\qquad \left.+\mathscr{O}_2\left\langle\binom{x^2}{0}, \binom{1}{0}\right\rangle\right\}\\ &\simeq \mathscr{O}_3^{(1)}/\{\mathscr{O}_3\langle x^4 + x^2y - z, 2x^4 + x^2y\rangle + \mathscr{O}_2\langle x^2, 1\rangle\}\\ &\simeq \mathscr{O}_{x,y}^{(1)}/\{\mathscr{O}_{x,y}\langle 2x^4 + x^2y\rangle + \mathscr{O}_{y,x^4+x^2y}\langle x^2, 1\rangle\}\\ &\simeq \mathscr{O}_y\langle x, x^3\rangle.\end{aligned}$$

Thus, $g \sim (x^4 + x^2y - z + xy^2\varphi_1(y) + x^3y\varphi_2(y), x)$.

The kernel $\operatorname{Ker} d_0$ contains the Euler generator and the generator $(2x^3 + xy)\partial_x - 2y^2\partial_y - 8x^2(g_1, 0) - (2x^2 + y)(0, g_2)$. Their action on the terms $(\alpha x^3 y + \beta xy^2, 0)$ of g_2, which have the lowest degree after g^0, gives us modulo $\operatorname{Im} d_0$

$$(\alpha x^3 y + \beta xy^2, 0) \quad \text{and} \quad ((4\alpha - 14\beta)x^3y^2 - 3\beta xy^3).$$

Hence, for

$$\det \begin{vmatrix} \alpha & 4\alpha - 14\beta \\ \beta & -3\beta \end{vmatrix} \neq 0,$$

i.e., $\beta(\alpha - 2\beta) \neq 0$, we obtain $g \sim (x^4 + \alpha x^3 y + x^2 y + xy^2 - z, x)$, $\alpha \neq 0$. Thus g has type $\mathbf{7}_*$. $c = 4$.

8. $g_1 = x^5 \pm x^3 y + xy - z$. Again generally we have $g^0 = (x^5 + xy - z, x)$. Then,

$$\begin{aligned}
\mathscr{O}_3^2/T_{g^0} &= \mathscr{O}_3^2/\left\{\mathscr{O}_3\left\langle \binom{x^5+xy-z}{0}, \binom{0}{x}, \binom{0}{z}, \binom{5x^4+y}{1} \right\rangle \right. \\
&\qquad \left. + \mathscr{O}_2 \left\langle \binom{1}{0}, \binom{x}{0} \right\rangle \right\} \\
&\simeq \mathscr{O}_3^{(1)}/\{\mathscr{O}_3\langle x^5 + xy - z, x(5x^4 + y)\rangle + \mathscr{O}_2\langle 1, x\rangle\} \\
&\simeq \mathscr{O}_{x,y}^{(1)}/\{\mathscr{O}_{x,y}\langle x(5x^4 + y)\rangle + \mathscr{O}_{y, x^5+xy}\langle 1, x\rangle\} \\
&\simeq \mathscr{O}_{x,y}^{(1)}/\{\mathscr{O}_{x,y}\langle x(5x^4 + y)\rangle + \mathscr{O}_{y, xy}\langle 1, x\rangle\} \\
&\simeq \mathbb{R}\langle x^2, x^3, x^3y, x^4, x^4y, x^4y^2\rangle.
\end{aligned}$$

Hence, d_0 takes g to the form $(x^5 + xy - z + \beta x^3 y + \alpha x^4 y + \gamma x^4 y^2, x)$. Higher differentials normalize this mapping by the Euler transformation: if $\alpha\beta \neq 0$, we can achieve $\beta = \pm 1$, $\gamma = 0$. So we have the singularity $\mathbf{8}_*$ of codimension $c = 4$.

We have finished our classificational computations. One of their by-products is Theorem 3 on simple boundary projections. Indeed, our classificational tree shows that any nonsimple singularity is adjacent to one of the families $\mathbf{3}_2$, $\mathbf{4}_1^{\pm}$, $\mathbf{7}_*$, $\mathbf{8}_*$ with moduli in their normal forms. On the other hand, it is quite obvious that the projections, listed in Theorem 3 as simple, are not adjacent to these families.

Another byproduct is a list of possible minimal transversals t_0 to the equivalence classes of classified boundary projections. We have brought together $\mathbb{R}$-linear bases $\{e_i\}$ of these transversals in Table 2. This table provides, for instance, the miniversal deformation of the boundary projection g: $g + \sum \lambda_i e_i$.

TABLE 2

type	$\{e_i\}$
$\mathbf{1}'$	–
A_k	$(0, 1), (0, y), \cdots, (0, y^{k-1})$
B_k	$(0, x), (0, x^3), \cdots, (0, x^{2k-1})$
$\mathbf{3}_1$	$(0, 1)$
$\mathbf{3}_2$	$(0, 1), (0, x), (0, y)$
$\mathbf{3}_3$	$(0, 1), (0, x), (0, y), (0, y^2)$
$\mathbf{3}_{k,1}$	$(0, x), (0, 1), (0, y), \cdots, (0, y^{k-1})$
$\mathbf{3}_{k,1/3}$ $\mathbf{3}_{k,4/3}$	$(0, 1), (0, x), (0, x^2), \cdots, (0, x^{k-1})$
$\mathbf{3}_{*,k}$	$(0, 1), (0, x), (0, x^2), (0, x^2 z), \cdots, (0, x^2 z^k)$
$\mathbf{4}_1^{\pm}$	$(x, 0), (0, 1), (0, y)$
$\mathbf{4}_2^{\pm}$	$(x, 0), (0, 1), (0, y), (0, y^2)$
$\mathbf{4}_*^{\pm}$	$(x, 0), (0, 1), (0, x), (0, x^4)$
$\mathbf{4}^-_{k,1/\sqrt{3}}$ $\mathbf{4}^-_{k,2/\sqrt{3}}$	$(x, 0), (0, 1), (0, y), \cdots, (0, y^k)$
$\mathbf{5}_1$	$(x, 0), (xy, 0), (xy^2, 0)$
$\mathbf{6}_1$	$(x^2, 0), (0, 1)$
$\mathbf{6}_*$	$(x^2, 0), (0, 1), (0, x), (0, xy)$
$\mathbf{6}_2$	$(x^2, 0), (0, 1), (0, y)$
$\mathbf{7}_*$	$(x, 0), (x^3, 0), (xy, 0), (x^3 y, 0)$
$\mathbf{8}_*$	$(x^2, 0), (x^3, 0), (x^4, 0), (x^4 y, 0)$

§4. Realization of Singularities

Consider the projection $(x, y, z) \mapsto (y, z)$ of the surface $g_1(x, y, z) = 0$ along a system of parallel rays. A variation in the direction of this projection corresponds to the substitution $x = x'$, $y = y' + \varepsilon_1 x'$, $z = z' + \varepsilon_2 x'$ into g_1 followed by the projection $(x', y', z') \mapsto (y', z')$.

We may also consider the first projection as a projection from a point of $\mathbb{R}P^3$ situated at infinity. Then a change of the center of the projection corresponds to the substitution $x = x'$, $y = (y' + \varepsilon_1 x')/(1 + \varepsilon_3 x')$, $z = (z' + \varepsilon_2 x')/(1 + \varepsilon_3 x')$, $|\varepsilon_i| \ll 1$ into the equation of the surface.

Thus, a surface-projection singularity is realized as a projection of a generic surface in $\mathbb{R}P^3$ along a system of parallel rays (or rays initiating at one point) iff the corresponding substitution into some (and hence, almost any) representative of its equivalence class provides a versal deformation of this singular projection. This is equivalent to providing a transversal to the equivalence class of surface projections.

A reformulation for the case of boundary projections gives us the following

REALIZATION CRITERION. *The projection $(x, y, z) \mapsto (y, z)$ of the surface $g_1(x, y, z) = 0$, $g_2(x, y, z) \lesseqgtr 0$ with boundary can be realized for a surface with boundary, embedded generically in $\mathbb{R}P^3$,*

(a) *as a projection by a system of parallel rays iff*

$$\mathscr{O}_3^2 = T_g + \mathbb{R}\left\langle x\frac{\partial g}{\partial y}, x\frac{\partial g}{\partial z}\right\rangle;$$

(b) *as a projection from a point iff*

$$\mathscr{O}_3^2 = T_g + \mathbb{R}\left\langle x\frac{\partial g}{\partial y}, x\frac{\partial g}{\partial z}, xy\frac{\partial g}{\partial y} + xz\frac{\partial g}{\partial z}\right\rangle.$$

Here the pair $g = (g_1, g_2)$ is not necessarily one from our classification Table 1. It need only be equivalent to a table singularity (since the table singularity itself may be "too reduced").

When we are dealing with the realization of a family with modulus α, we must to add the term $\mathbb{R}\langle \partial g/\partial\alpha\rangle$ to the right sides of the relations above.

We have already mentioned that every boundary projection $(x, y, z) \mapsto (y, z)$ may be reduced to the form $g = (f_1(x, y) - z, f_2(x, y))$ (all table singularities, except $\mathbf{3}_{*,k}$, are given in this form). Then $\mathscr{O}_3^2/T_g \simeq \mathscr{O}_{x,y}^2/\tilde{T}_g$, where $\tilde{T}_g = \mathscr{O}_{x,y}\langle(0, f_2), \partial g/\partial x\rangle + \mathscr{O}_{y,f_1(x,y)}\langle \partial g/\partial y, \partial g/\partial z\rangle$. For such g, we have the following obvious (but useful)

LEMMA 1. *Let* $g' = g + (\varphi(y), 0)$ *be a boundary projection* (*equivalent to* g). *Then* $\tilde{T}_{g'} = \tilde{T}_g$.

The realization criterion for the g' considered in the lemma may be rewritten in the following form:

(a′) $\mathscr{O}^2_{x,y} = \tilde{T}_g + \mathbb{R}\langle x\partial g'/\partial y, x\partial g'/\partial z\rangle$ or

(b′) $\mathscr{O}^2_{x,y} = \tilde{T}_g + \mathbb{R}\langle x\partial g'/\partial y, x\partial g'/\partial z, xy\partial g'/\partial y + xf_1'\partial g'/\partial z\rangle$, $f_1' = f_1 + \varphi$.

If g is a normal form from the table, then the monomials listed at the end of §3 generate a minimal transversal to $\tilde{T}_g$ in $\mathscr{O}^2_{x,y}$ ($g \notin \mathbf{3}_{*,k}$). Almost all possible generic realizations of the singularities are listed in the classification table in the form $g' = g + (\varphi(y), 0)$. Thus, in order to prove that such g' provides a realization of the singularity with normal form g, we must show that the additional finite-dimensional space in (a′) or (b′) coincides modulo $\tilde{T}_g$ with the aforementioned transversal. We shall usually treat the realization criterion in this way in our subsequent considerations of table singularities.

A_k. Here $g = (x^2 - z, x + y^{k+1})$ and $g' = g + (y^2, 0)$. Then

$$x\frac{\partial g'}{\partial y} = (2xy, (k+1)xy^k), \quad x\frac{\partial g'}{\partial z} = (-x, 0),$$
$$xy\frac{\partial g'}{\partial y} + xf_1'\frac{\partial g'}{\partial z} = (xy^2 - x^3, (k+1)xy^{k+1}).$$

Now g is quasihomogeneous for the grading: $wt x = k+1$, $wt y = 1$, $wt z = 2(k+1)$. As usually done for mappings, we set $wt(1,0) = -2(k+1)$, $wt(0,1) = -(k+1)$ in order to make the weight of g equal to zero. Consider the minimal transversal to $\tilde{T}_g$ in $\mathscr{O}^2_{x,y}$ generated by $(0,1), (0,y), \dots, (0, y^{k-1})$. Then the quasihomogeneity of g implies that $\tilde{T}_g$ contains a subspace $\mathscr{F}_0\mathscr{O}^2_{x,y}$ of $\mathscr{O}^2_{x,y}$ of non-negative filtration. So the three above elements are equal modulo $\tilde{T}_g$ to $(2xy, 0)$, $(-x, 0)$, $(xy^2, 0)$. Since $\partial g/\partial x = (2x, 1)$, we may replace these monomials by $(0, -y)$, $(0, 1/2)$ and $(0, -y^2/2)$ respectively. Thus, the singularities A_1, A_2 and A_3 are realizable in the sense of Theorem 2.

B_k. $g' = (x^2 - z, x^{2k+1} + y + x^4)$ (obviously, the monomial x^4 may be killed and then we get the normal form B_k). We have:

$$\begin{aligned}\mathscr{O}^2_{x,y}/\tilde{T}_g &= \mathscr{O}^2_{x,y}/\left\{\mathscr{O}_{x,y}\left\langle \binom{0}{x^{2k+1}+y+x^4}, \binom{2x}{(2k+1)x^{2k}+4x^3}\right\rangle \right.\\ &\quad \left. +\mathscr{O}_{y,x^2}\left\langle \binom{0}{1}, \binom{-1}{0}\right\rangle\right\}\\ &\simeq x\mathscr{O}^2_{x^2,y}/\mathscr{O}_{x^2,y}\left\langle \binom{0}{x^{2k+1}}, \binom{0}{xy+x^5}, \binom{2x}{4x^3}, \binom{0}{(2k+1)x^{2k+1}}\right\rangle\\ &\simeq x\mathscr{O}^{(2)}_{x^2,y}/x\mathscr{O}_{x^2,y}\langle x^{2k}, y+x^4\rangle\\ &\simeq \mathbb{R}\langle (0,x), (0,x^3), \dots, (0,x^{2k-1})\rangle.\end{aligned}$$

On the other hand,

$$x\frac{\partial g'}{\partial y} = (0, x), \quad x\frac{\partial g'}{\partial z} = (-x, 0) \equiv (0, 2x^3)\mathrm{mod}\tilde{T}_{g'},$$

$$xy\frac{\partial g'}{\partial y} + xf_1'\frac{\partial g'}{\partial z} = (-x^3, xy) \equiv (0, xy+2x^5) \equiv (0, x^5)\mathrm{mod}\tilde{T}_{g'}.$$

So B_1, B_2 and B_3 are realizable.

$\mathbf{3}_1$. By Table 2, a miniversal deformation of the $\mathbf{3}_1$-singularity $g = (x^3+xy-z, x)$ may be chosen in the form $g+(0, \lambda)$. This deformation is equivalent to the deformation

$$((x-\lambda)^3 + (x-\lambda)y - z, x) =$$
$$(x^3 - 3\lambda x^2 + x(y+3\lambda^2) - (z+\lambda y+\lambda^3), x) \sim (x^3 - 3\lambda x^2 + xy - z, x).$$

The velocity $(-3x^2, 0)$ of the latter deformation is proportional to $x\partial g/\partial y = (x^2, 0)$. This implies the realizability of $\mathbf{3}_1$.

$\mathbf{3}_2$. In the proof of the realizability of this family, we show that the modulus of its normal form is a projective invariant. We also discuss why some values of this modulus are exceptional and nonrealizable by projections of generic surfaces with boundaries.

Every boundary projection of type $\mathbf{3}_2$ may be reduced by a projective transformation of $\mathbb{R}P^3$ to the form $g' = (x^3+xy-z+\dots, x^2+\alpha y+\dots)$, where "$+\dots$" denotes terms of positive filtration (grading: $wt x = 1$, $wt y = 2$, $wt z = 3$, $wt(1,0) = -3$, $wt(0,1) = -2$, then the weight of the main part of g' is zero). On the other hand, let g' be a generic mapping of the above form. Then it is $\mathbf{3}_2$-singular (as a consequence of the classification carried out in §3). So $T_{g'}$ contains $\mathscr{F}_1\mathscr{O}^2_3$, a subspace of $\mathscr{O}^2_{x,y}$ of positive filtration. Since we may slightly vary all coefficients of the main part of g', staying in the $\mathbf{3}_2$-class, the space $T_{g'} + \mathbb{R}\langle \partial g'/\partial\alpha\rangle$, tangent to this class, contains $\mathscr{F}_0\mathscr{O}^2_3$.

Consider the grading on $\mathscr{O}_3^2$ and compare it with the grading on

$$T_{g'} + \mathbb{R}\langle x\frac{\partial g'}{\partial y}, x\frac{\partial g'}{\partial z}, xy\frac{\partial g'}{\partial y} + xz\frac{\partial g'}{\partial z}, \frac{\partial g'}{\partial \alpha}\rangle.$$

Poincaré series of $\mathscr{O}_3^2$ is $t^{-3} + 2t^{-2} + 3t^{-1} + 5t^0 + \dots$. The second space has the following linear generators with main parts having negative weights. In weight -3 : $\partial g'/\partial z = (1, 0)$; in weight -2 : $\partial g'/\partial y = (x, \alpha)$, $x\partial g'/\partial z = (-x, 0)$; in weight -1 : $\partial g'/\partial x = (3x^2 + y, 2x)$, $x\partial g'/\partial y = (x^2, \alpha x)$, $y\partial g'/\partial z = (-y, 0)$ (we write only the main parts here). So, variation of the direction of projection provides a transversal at g' to the equivalence class of all $\mathbf{3}_2$-singularities in $\mathscr{O}_3^2$ iff these main parts are linearly independent, i.e., $\alpha(\alpha - 2/3) \neq 0$.

Thus, the $\mathbf{3}_2$-singularity is realized by projections of a generic surface with a boundary along separate directions. Consider one such direction. Take a ray passing through a $\mathbf{3}_2$-point. A projection from almost every point of this ray is again of type $\mathbf{3}_2$ with the same value of the modulus α. Indeed, there are five elements of 0-weight in $\mathscr{O}_3^2$ and five elements with 0-weight main parts in $T_{g'}$. But there is a Euler relation between these main parts. On the other hand, the element $xy\partial g'/\partial y + xz\partial g'/\partial z$ (which corresponds to velocity of deformation of g' by variaton of the projection center on our ray) has filtration 1. So we cannot get a transversal to the equivalence class of g' by variaton of the projection center.

Hence, we cannot realize (as a projection of a generic surface with boundary) the projection with main part $(x^3 + xy - z, x^2 + \alpha y)$ and the prescribed value of the modulus α. Thus, we cannot realize $\mathbf{3}_{2,1/3}$ and $\mathbf{3}_{2,4/3}$. These singularities are really exceptional:

$\mathbf{3}_{2,1/3}$, the critical set of the projection of the surface $g_1 = 0$, has second order contact with the boundary;

$\mathbf{3}_{2,4/3}$, the set of critical values of the surface projection is $4y^3+27z^2 = 0$ and the image of the boundary has the same equation modulo terms of higher weight (for other values of the modulus $\alpha \neq 1/3$, it is a 3/2-parabola with other coefficients).

We have found one more exceptional value – $\alpha = 2/3$. In order to understand this case, consider the field of asymptotic directions on the surface $z = x^3 + xy + \dots$. These directions are self-orthogonal in the sense of the second quadratic form. In a (x, y)-chart, the direction ∂_x at the origin is included in the field $\dot{x} = 1$, $\dot{y} = -3x + \dots$. This field has an integral curve $y = -\frac{3}{2}x^2 + \dots$ passing through the origin. For $\alpha = 2/3$, this curve has second order contact with the boundary.

Two more exceptional values of α will appear when we consider bifurcation diagrams of $\mathbf{3}_2$-singularities in the next section.

$\mathbf{3}_3$. This singularity has quasihomogeneous normal form $g = (x^3 + xy - z, x^2 + \alpha y)$ (the grading on $\mathscr{O}_3^2$ is the same as for $\mathbf{3}_2$: $wt x = 1$, $wt(1, 0) = -3$, etc.). T_g contains the subspace $\mathscr{F}_3\mathscr{O}_3^2 \subset \mathscr{O}_3^2$ of elements with filtration ≥ 3 (see §3). Consider $g' = g + (y^2, 0) \sim g$. By Lemma 1, in order to prove that such a g' gives us an example of a projection of a generic surface with boundary, we have to show that $\mathscr{O}_{x,y}^2$ coincides with

$$\tilde{T}_g + \mathbb{R}\left\langle x\frac{\partial g'}{\partial y}, x\frac{\partial g'}{\partial z}, xy\frac{\partial g'}{\partial y} + x(x^3 + xy + y^2)\frac{\partial g'}{\partial z}, \frac{\partial g'}{\partial \alpha}\right\rangle$$

in weights ≤ 2.

Coincidence in weights ≤ 0 for $\alpha(\alpha - 2/3) \neq 0$ is similar to the case $\mathbf{3}_2$. Computations from §3 show that every element of $\mathscr{O}_{x,y}^2$ of weight 1 lies in $\tilde{T}_g$. Hence, in $xy\partial g'/\partial y + x(x^3 + xy + y^2)\partial g'/\partial z$ the terms having this weight may be omitted modulo $\tilde{T}_g$. Then this element gives the term $(xy^2, 0)$ of weight 2. In weight 2, where $\mathscr{O}_{x,y}^2$ has a 6-dimensional subspace, $\tilde{T}_g$ also has 6 quasihomogeneous generators: $x^3\partial g/\partial x = (3x^5 + x^3y, 2x^4)$, $xy\,\partial g/\partial x = (3x^3y + xy^2, 2x^2y)$, $x^2(0, g_2) = (0, x^4 + \alpha x^2y)$, $y(0, g_2) = (0, x^2y + \alpha y^2)$, $y^2\partial g/\partial y = (xy^2, \alpha y^2)$, $y(x^3 + xy)\partial g/\partial z = (-x^3y - xy^2, 0)$. The 7×6-matrix of coefficients of these generators together with $(xy^2, 0)$ has rank $= 6$ iff $\alpha \neq 0$. This implies that the $\mathbf{3}_3$-family is observable in general position. We see $\mathbf{3}_3$ from a separate point of the asymptotic ray tangent to the boundary. From the other points of this ray we see $\mathbf{3}_2$. The values of the modulus α for $\mathbf{3}_2$- and $\mathbf{3}_3$-singularities, observable from the same ray, are equal.

$\mathbf{4}_1^{\pm}$, $\mathbf{4}_{1,1/\sqrt{3}}^-$, $\mathbf{4}_{1,2/\sqrt{3}}^-$. These singularities have normal form $g = (x^3 \pm xy^2 - z, x + \alpha y + y^2)$. Consider the grading on $\mathscr{O}_3^2$: $wt x = wt y = 1$, $wt z = 1$, $wt(1, 0) = -3$, $wt(0, 1) = -1$. The calculations of §3 show that the subspace $\mathscr{F}_1\mathscr{O}_{x,y}^2$ is contained in $\tilde{T}_g$.

Now take $g' = g + (y^2, 0)$. Again, by Lemma 1, in order to prove that g' provides a general realization of the singularity g, we have to show that $\mathscr{O}^2_{x,y}$ coincides with

$$\tilde{T}_g + \mathbb{R}\left\langle x\frac{\partial g'}{\partial y}, x\frac{\partial g'}{\partial z}, xy\frac{\partial g'}{\partial y} + x(x^3 \pm xy^2 + y^2)\frac{\partial g'}{\partial z}\right\rangle$$

in nonpositive weights. One may check that this coincidence takes place for $\alpha \neq 0$ (since ideologically there is nothing new as compared with the cases $\mathbf{3}_2$ and $\mathbf{3}_3$, we omit the calculations). Thus, the singularities $\mathbf{4}_1^{\pm}$ (for all values of the modulus $\alpha \neq 0$), $\mathbf{4}^-_{1,1/\sqrt{3}}$ and $\mathbf{4}^-_{1,2/\sqrt{3}}$ are observable as projections of certain generic surfaces with boundary from certain centers.

Consideration of the same boundary projection g' allows us to say that the families $\mathbf{4}_1^{\pm}$ are realized as projections of generic surfaces with boundary along separate directions. This follows from the relation

$$\mathscr{O}^2_{x,y} = \tilde{T}_g + \mathbb{R}\left\langle x\frac{\partial g'}{\partial y}, x\frac{\partial g'}{\partial z}, \frac{\partial g'}{\partial \alpha}\right\rangle,$$

which is easily checked for $\alpha \neq 0$.

Why is the value $\alpha = 0$ exceptional? Every $\mathbf{4}_1^{\pm}$- singularity can be reduced by a projective transformation to the form $g = (x^3 \pm xy^2 + y^2 + By^3 - z + \dots, x + \alpha y + \dots)$. Then the curve of parabolic points on the surface $g_1 = 0$ is given by the equation $x = \frac{1}{3}y^2 + \dots$ (zeros of the determinant of the second quadratic form). So, this curve is tangent to the boundary when $\alpha = 0$. This position is not generic.

$\mathbf{4}_2^{\pm}$. Here $g = (x^3 \pm xy^2 - z, x + \alpha y)$ and as a candidate for its generic realization we take $g' = g + (y^2 + By^3, 0)$ with $B \neq 0$ for $\mathbf{4}_2^-$ and $B = 0$ for $\mathbf{4}_2^+$. It is easy to show that

$$\mathscr{O}^2_{x,y} = \tilde{T}_g + \mathbb{R}\left\langle x\frac{\partial g'}{\partial y}, x\frac{\partial g'}{\partial z}, xy\frac{\partial g'}{\partial y} + x(x^3 \pm xy^2 + y^2 + By^3)\frac{\partial g'}{\partial z}, \frac{\partial g'}{\partial \alpha}\right\rangle$$

iff $\alpha(6\alpha^2 - 9B\alpha - 4) \neq 0$ for $\mathbf{4}_2^-$ and $\alpha \neq 0$ for $\mathbf{4}_2^+$ (it suffices to demonstrate the coincidence of two spaces only in weights ≤ 1 for grading as in the case $\mathbf{4}_1^{\pm}$). So we are done.

$\mathbf{5}_1$. Now $g = (x^3 + xy^3 - z, x + y)$ and we take $g' = g + (y^2, 0)$. Consider the same grading on $\mathscr{O}^2_{x,y}$ as for $\mathbf{4}_1^{\pm}$: $wt x = wt y = 1$, $wt z = 3$, $wt(1, 0) = -3$, $wt(0, 1) = -1$. By §3 we have $\mathscr{F}_1\mathscr{O}^2_{x,y} \subset \tilde{T}_g$ and the transversal to $\tilde{T}_g$ in $\mathscr{O}^2_{x,y}$ may be chosen as the linear space generated by $(x, 0)$, $(xy, 0)$, $(xy^2, 0)$. One easily checks (omitting the terms of positive weight) that modulo $\tilde{T}_g$ these three elements are equal to $-x\partial g'/\partial z$, $\frac{1}{2}x\partial g'/\partial y$ and $xy\partial g'/\partial y + x(x^3 + xy^3 + y^2)\partial g'/\partial z$ respectively. Thus, g' is a generic realization of $\mathbf{5}_1$.

$\mathbf{6}_1$. We begin with

LEMMA 2. *Suppose that some germ h of a fibration-preserving diffeomorphism of 3-space takes the boundary projection g to the boundary projection g': $g' = g \circ h$. Then h takes the tangent space $T_g \subset \mathscr{O}_3^2$ and its transversal to $T_{g'}$ and the transversal to $T_{g'}$.*

The proof is by direct comparison of $T_{g'}$ and T_g.

REMARK. Lemma 2 remains valid for transformations $g \mapsto g'$ when we additionally multiply both components of g by the same function a, $a(0) \neq 0$ on the base of the fibration.

The $\mathbf{6}_1$-normal form is $g = (x^4 + xy - z, x + y)$. We want to show that $g' = g\big|_{x:=x+x^2}$ provides a generic realization of $\mathbf{6}_1$ as the projection along a system of parallel rays.

Consider the grading on $\mathscr{O}_3^2$: $wt x = 1$, $wt y = 3$, $wt z = 4$, $wt(1, 0) = -4$, $wt(0, 1) = -1$. According to §3, T_g contains the subspace $\mathscr{F}_0\mathscr{O}_3^2 \subset \mathscr{O}_3^2$ of elements of non-negative filtration. This subspace is invariant with respect to the substitution $x := x+x^2$. So, by Lemma 2, $T_{g'} \supset \mathscr{F}_0\mathscr{O}_3^2$. Thus, in order to show that g' provides the realization, it is enough to demonstrate that the negative subspace $\mathbb{R}\langle(1, 0), (x, 0), (x^2, 0), (x^3, 0), (y, 0), (0, 1)\rangle$ of $\mathscr{O}_3^2$ is equal to the negative subspace of $T_{g'} + \mathbb{R}\langle x\partial g'/\partial y, x\partial g'/\partial z\rangle$. There are six elements in the latter space that have terms of negative weights (we write out these terms only):

$$\frac{\partial g'}{\partial x} = (4x^3, 1), \quad \frac{\partial g'}{\partial y} = (x + x^2, 1), \quad x\frac{\partial g'}{\partial y} = (x^2 + x^3, 0),$$
$$\frac{\partial g'}{\partial z} = (-1, 0), \quad y\frac{\partial g'}{\partial z} = (-y, 0), \quad x\frac{\partial g'}{\partial z} = (-x, 0).$$

The matrix of coefficients of these elements with respect to the chosen basis of negative subspace of $\mathscr{O}_3^2$ is nondegenerate. So we are done.

$\mathbf{6}_2$. Here $g = (x^4 + xy - z, x)$. We put $g' = g\big|_{x:=x+x^2}$ and repeat everything we have done for the singularity $\mathbf{6}_1$. The only difference: now to $T_{g'}$ we also add $\mathbb{R}\langle xy\partial g'/\partial y + xz\partial g'/\partial z\rangle$ and write out the matrix of coefficients corresponding to the terms of weight ≤ 1. This matrix is again of maximal rank.

Theorem 2 is now proved, since the reasons for the nonrealizability of the other singularities in the classification table have been already discussed in §2.

§5. Bifurcation diagrams

Consider the base of a versal deformation of a boundary projection. It contains a hypersurface organized by the values of deformation parameters that correspond to unstable (multi)singularities. We call this hypersurface a *bifurcation diagram* of the boundary projection. It has up to fifteen components (fifteen is a number of possible codimension 1 singular complex projections). Only ten of them appear in bifurcation diagrams of realizable boundary projection. Denote by C and C' the set of critical values of the surface's projection and of the image of its boundary. Then these ten components correspond to: boundary singularities A_1, B_1, $\mathbf{3}_1$; interior singularities $\mathbf{4}^{\pm}$, $\mathbf{6}$; tangency of C and C' (we shall denote this bisingularity by X); self-tangency of C' (X'); C or C' passing through the image of A_0-point (T and T' respectively); C' passing through a cusp of $C(K)$. Flag versions of all these bifurcations are shown in Figure 4.

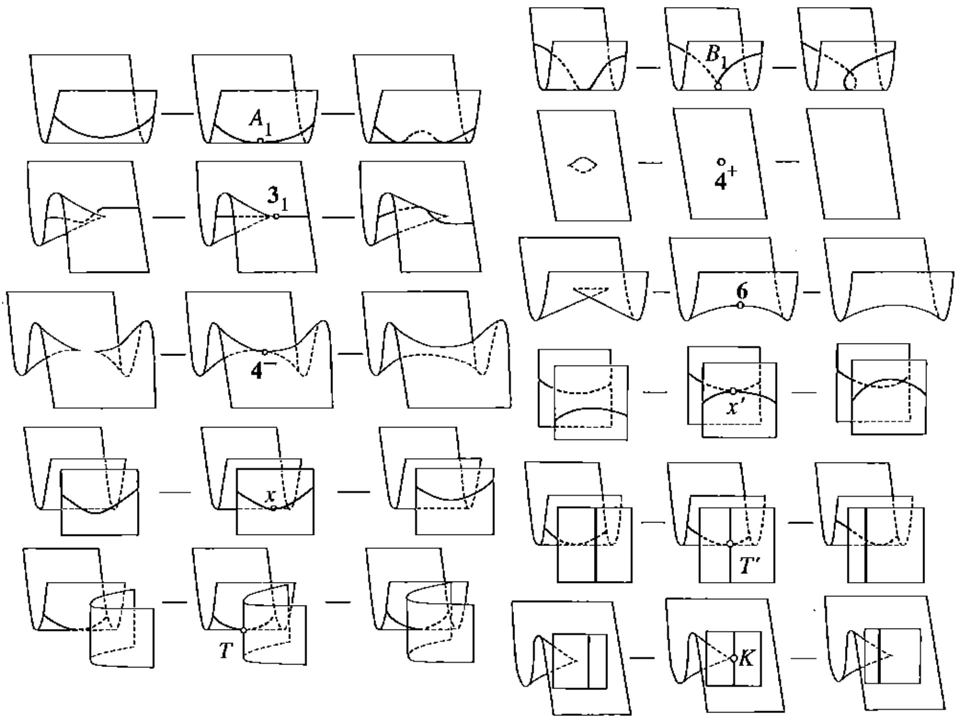


FIGURE 4. Codimension 1 bifurcations of flag-projections.

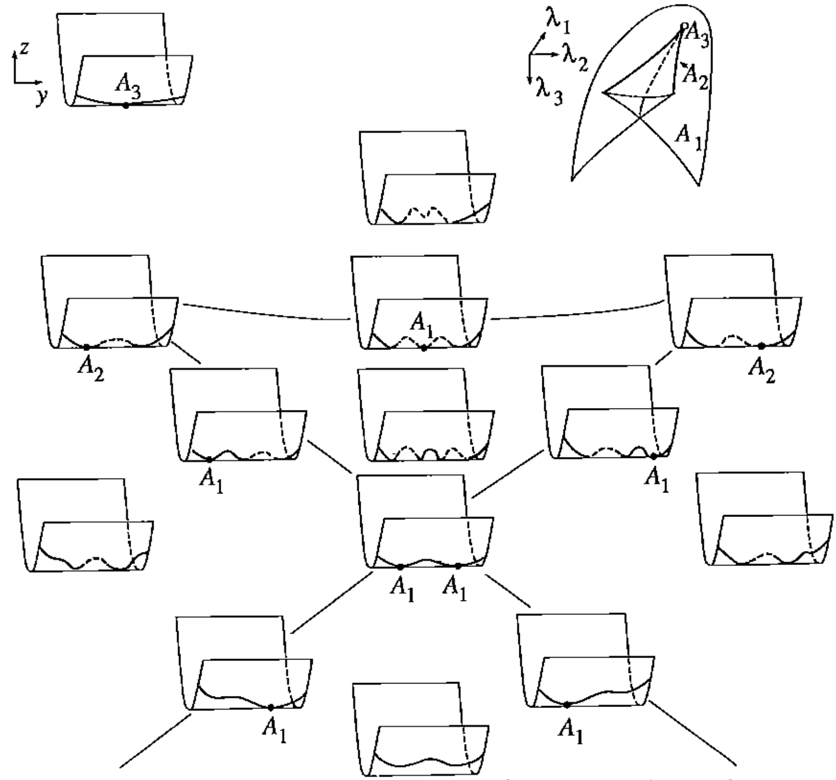


FIGURE 5. Versal deformation $(x^2 - z, x + y^4 + \lambda_1 y^2 + \lambda_2 y + \lambda_3)$ of A_3.

In [BG] bifurcation diagrams of A_2, B_2, $\mathbf{3}_2$ (two variants only), $\mathbf{6}_1$ were given, as well as drawings of $C \cup C'$ corresponding to various strata in the decompositions of the bases of versal deformations by these diagrams. But not all the components of the diagrams were present there.

In this section we present bifurcation diagrams of all boundary singularities realizable by projections of generic surfaces with boundaries from points of 3-space (frankly speaking, we do omit some diagrams when they are easily seen in diagrams of more complicated singularities, e.g., A_2 in A_3). We also show how one sees a surface with a boundary curve on it from points near a point from which a degenerate projection singularity is observable.

A_k, $k \geq 0$. The versal deformation is $(x^2 - z, x + p(y))$, where $p(y) = y^{k+1} + \lambda_1 y^{k-1} + \cdots + \lambda_k$. The bifurcation diagram is an ordinary diagram A_k (p has a multiple real root). For $k = 3$ it is shown in Figure 5. There, on a generic plane section of the A_3-diagram, we show

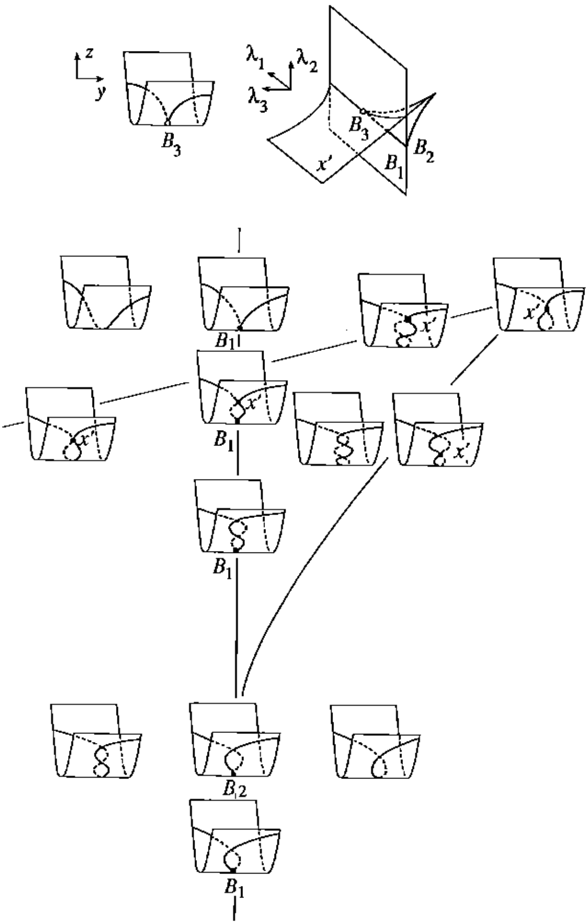


FIGURE 6. Versal deformation $(x^2 - z, y + x^7 + \lambda_1 x^5 + \lambda_2 x^3 + \lambda_3 x)$ of B_3.

the distribution of qualitative flag-projection types by strata of the base of the deformation.

B_k, $k \geq 1$. The versal deformation is $(x^2 - z, y + xp(x^2))$, where $p(x^2) = x^{2k} + \lambda_1 x^{2k-2} + \cdots + \lambda_k$. The bifurcation diagram is the discriminant B_k of a function on a real manifold with a boundary ($p(x^2)$ has a multiple real root). In Figure 6 we represent B_3 in the same way as it was done for A_3.

3_2. The versal deformation is $(x^3+xy-z\,,\,x^2+(\alpha+\lambda_3)y\pm y^2+\lambda_1 x+\lambda_2)$. For a general value of the modulus α, the germ of this deformation is topologically trivial along the λ_3-direction. So we give only the sections $\lambda_3 = 0$ of the bifurcation diagrams (this corresponds to a variation of the projection center for a fixed flag in 3-space). There are five curves in these sections: A_1, B_1, $\mathbf{3}_1$, X, T'. Now $\mathbf{3}_1$ has the equation $\lambda_2 = 0$. The others are given by $\lambda_2 = k\lambda_1^2+$ higher order terms in λ_1. The coefficient k depends on α. It equals $(4-12\alpha)^{-1}$ for A_1, $(1+3\alpha)/4$ for $\mathscr{B}_1$, $(4-3\alpha)^{-1}$ for X and $(3\alpha-2)$ for T' (Figure 7). These four coefficients provide two new exceptional values of α: $-1/3$ and $1/2$. Thus, the exceptional values divide the α-axis into eight intervals where the bifurcation diagram's topological type is constant. In Figures 8–14 we show all of them. In those figures:

1°. We show distributions of the perturbed projections only for a half $\lambda_1 \geq 0$ of the base of the deformation (since there is a symmetry that changes the signs of x, z and λ_1 and preserves y and λ_2).

2°. In Figure 11 we represent the distributions for two cases: $1/3 < \alpha < 1/2$ and $1/2 < \alpha < 2/3$. These distributions differ only in the projections corresponding to the curves A_1 and T and the sector between these curves. So we give two versions of illustrations for three strata.

3_3. In order to get a versal deformation of this singularity, take a versal deformation of $\mathbf{3}_2$ and substitute the term $+\lambda_4 y^2$ for $\pm y^2$. This leads to bifurcation diagrams topologically trivial along λ_4. Hence, we see the general surface with boundary from points near the $\mathbf{3}_3$-center looking similar to the $\mathbf{3}_2$-case.

$4_1^{\pm}$. The versal deformation is $(x^3+x(\lambda_1 \pm y^2)-z\,,\,x+(\alpha+\lambda_3)y+y^2+\lambda_2)$. Again we have topological triviality along λ_3-axis and consider only the sections $\lambda_3 = 0$ of bifurcation diagrams (see figures 15–18). Codimension 1 strata have simple contact at the origin. We consider singularities with positive α (singularities with opposite values of the modulus are equivalent). The illustrations are given only for a half of the base stratification, because there is a qualitative symmetry that changes the signs of all arguments except λ_2.

$4_2^{\pm}$. One reduces this case to $\mathbf{4}_1^{\pm}$ in the same way as $\mathbf{3}_3$ was reduced to $\mathbf{3}_2$.

$4^-_{1,1/\sqrt{3}}$, $4^-_{1,2/\sqrt{3}}$. The versal deformation may be taken in the form of $\mathbf{4}_1^-$ above with $\alpha = 1/\sqrt{3}$ and $\alpha = 2/\sqrt{3}$. This time there is no

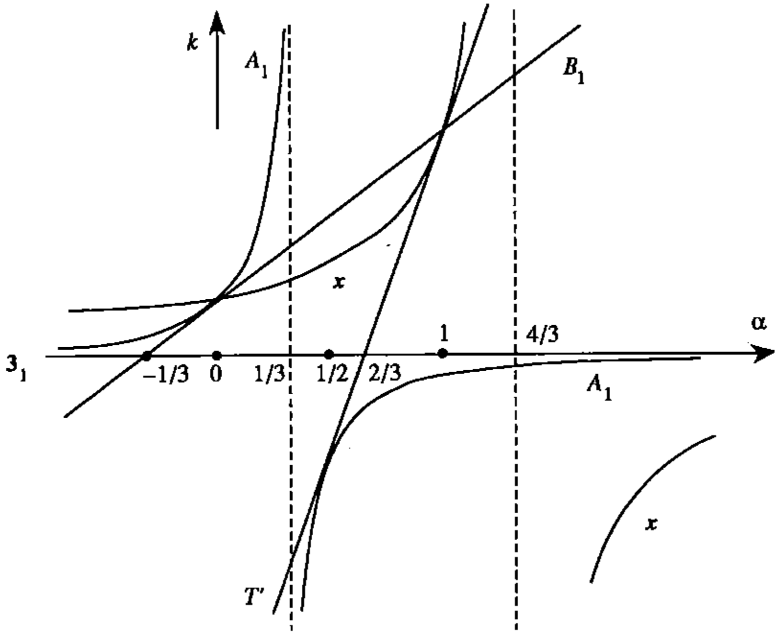


FIGURE 7. Ordering of strata in bases of versal deformations of singularities $\mathbf{3}_2$ for different values of the modulus α.

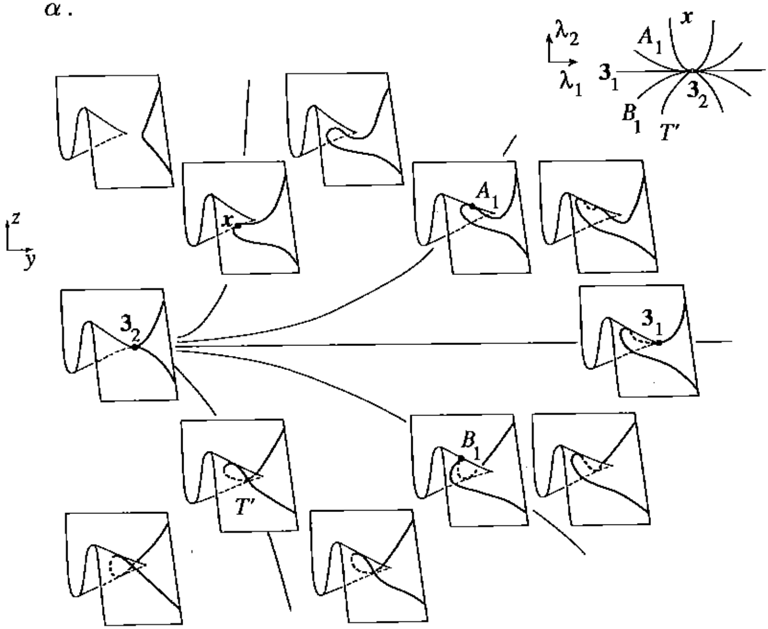


FIGURE 8. Deformation $(x^3+xy-z\,,\ x^2+\alpha y\pm y^2+\lambda_1 x+\lambda_2)$ of $\mathbf{3}_2$, $\alpha < -1/3$.

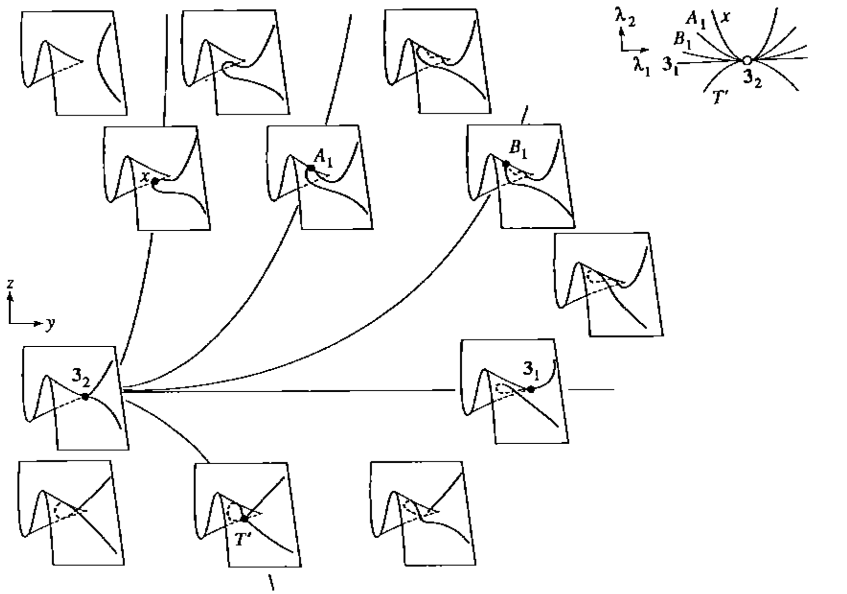


FIGURE 9. Deformation $(x^3+xy-z, x^2+\alpha y\pm y^2+\lambda_1 x+\lambda_2)$ of $\mathbf{3}_2$, $-1/3<\alpha<0$.

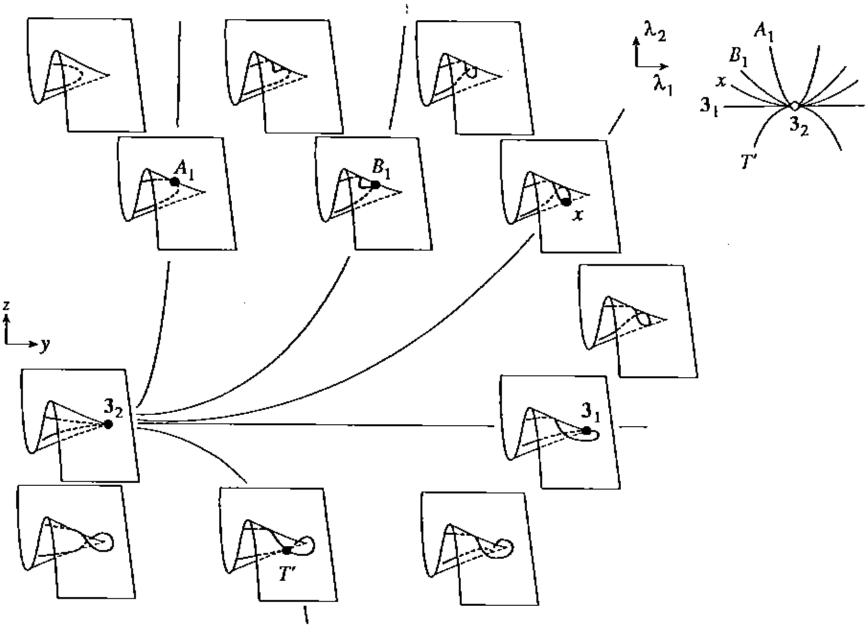


FIGURE 10. Deformation $(x^3+xy-z, x^2+\alpha y\pm y^2+$
$\lambda_1 x+\lambda_2)$ of $\mathbf{3}_2$, $0<\alpha<1/2$

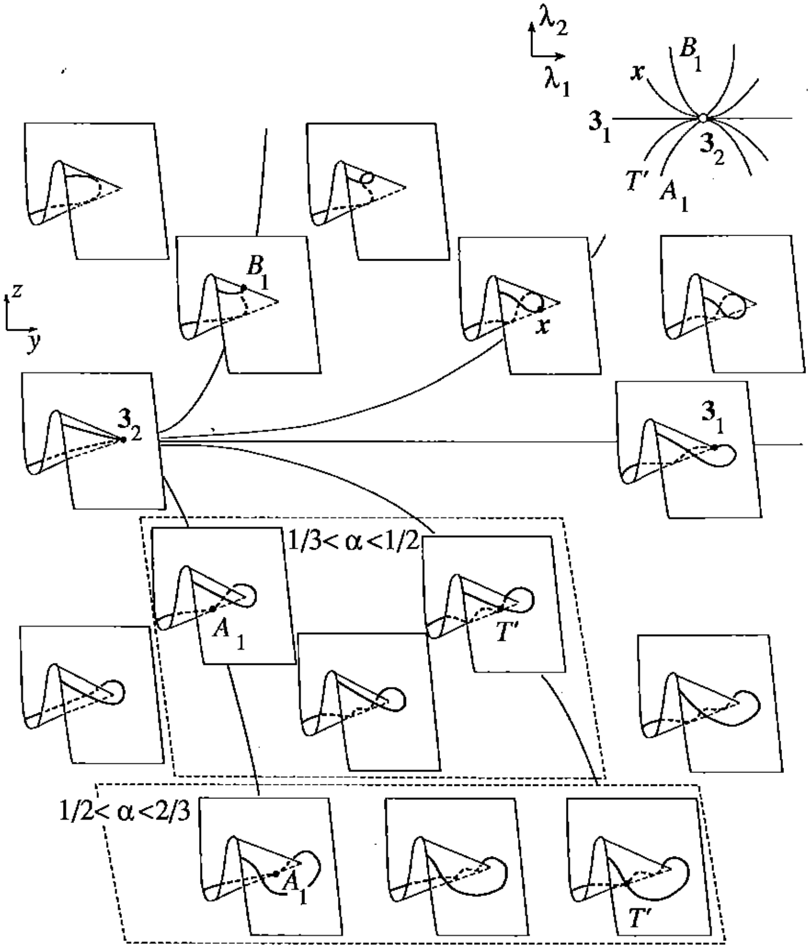


FIGURE 11. Deformation $(x^3 + xy - z, x^2 + \alpha y \pm y^2 + \lambda_1 x + \lambda_2)$ of $\mathbf{3}_2$, $1/3 < \alpha < 1/2$ and $1/2 < \alpha < 2/3$.

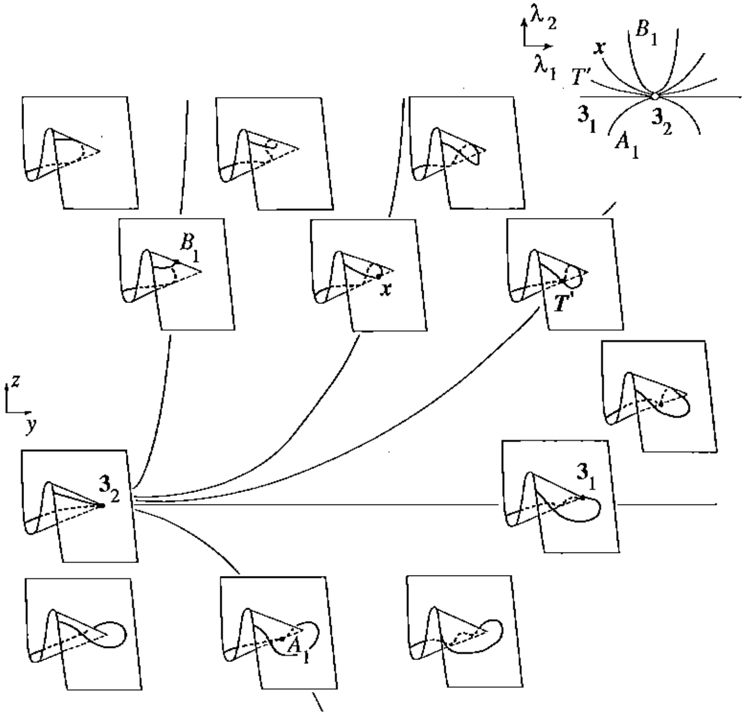


FIGURE 12. Deformation $(x^3 + xy - z, x^2 + \alpha y \pm y^2 + \lambda_1 x + \lambda_2)$ of $\mathbf{3}_2$, $2/3 < \alpha < 1$.

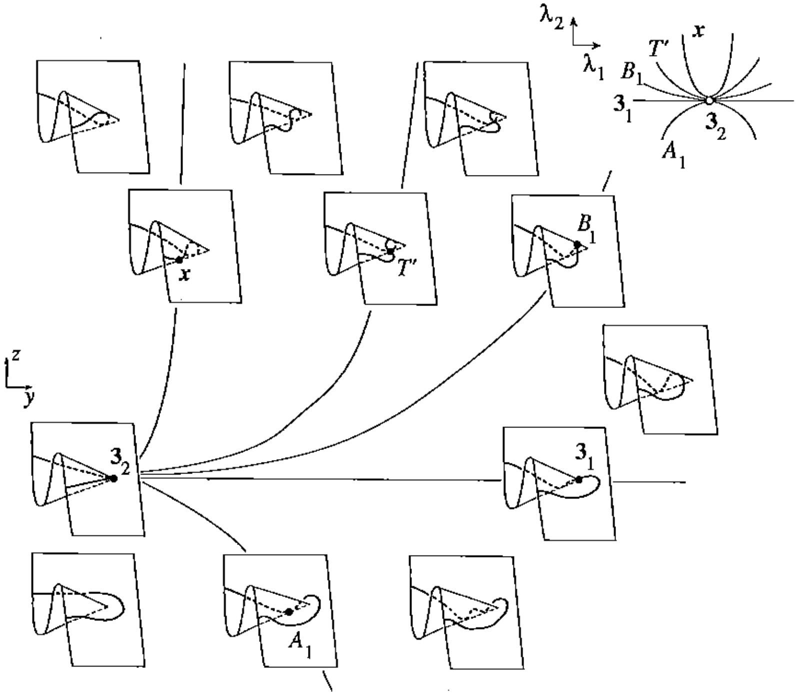


FIGURE 13. Deformation $(x^3 + xy - z, x^2 + \alpha y \pm y^2 + \lambda_1 x + \lambda_2)$ of $\mathbf{3}_2$, $1 < \alpha < 4/3$.

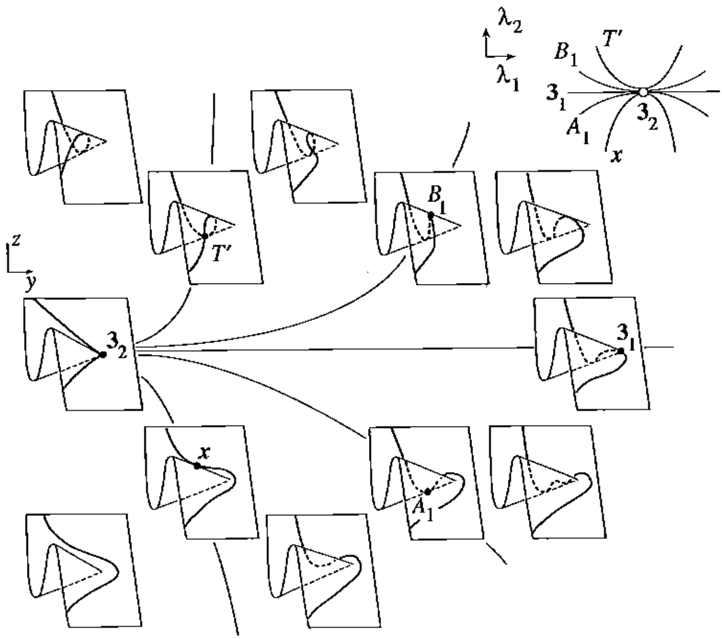


FIGURE 14. Deformation $(x^3 + xy - z, x^2 + \alpha y \pm y^2 + \lambda_1 x + \lambda_2)$ of $\mathbf{3}_2$, $\alpha > 4/3$.

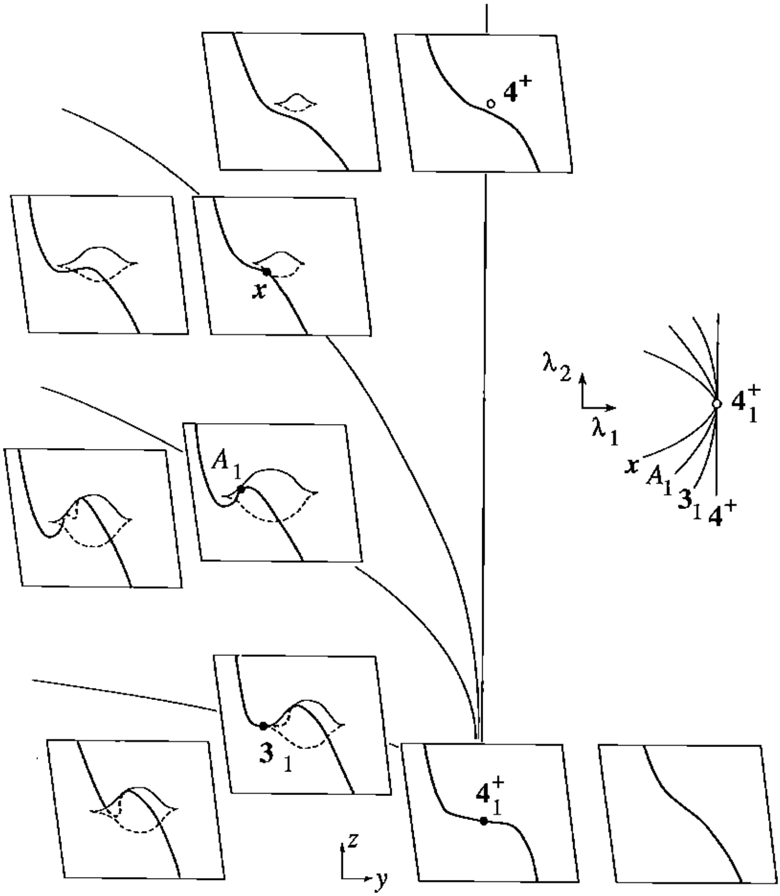


FIGURE 15. Deformation $(x^3 + x(\lambda_1 + y^2) - z, x + \alpha y + y^2 + \lambda_2)$ of $\mathbf{4}_1^+$.

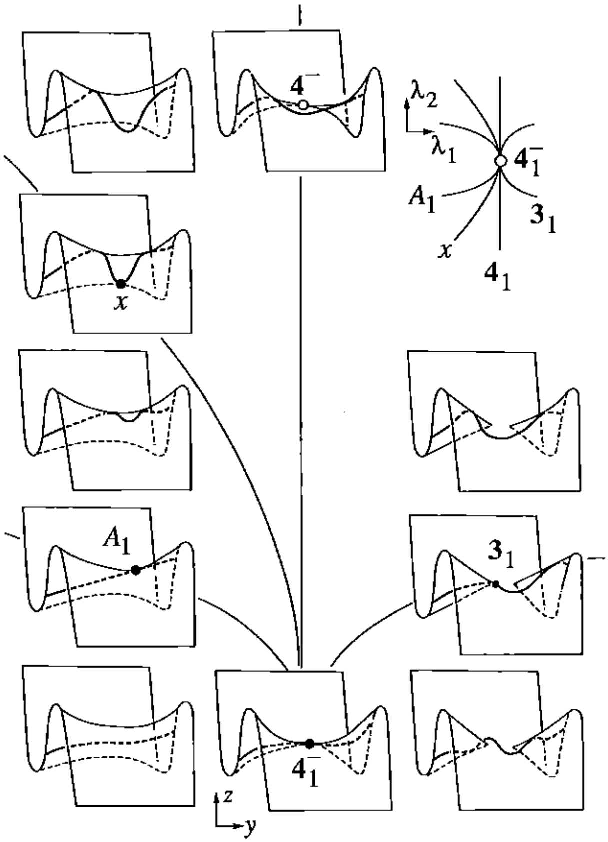


FIGURE 16. Deformation $(x^3 + x(\lambda_1 - y^2) - z\,,\, x + \alpha y + y^2 + \lambda_2)$ of $\mathbf{4}_1^-$, $0 < \alpha < 1/\sqrt{3}$.

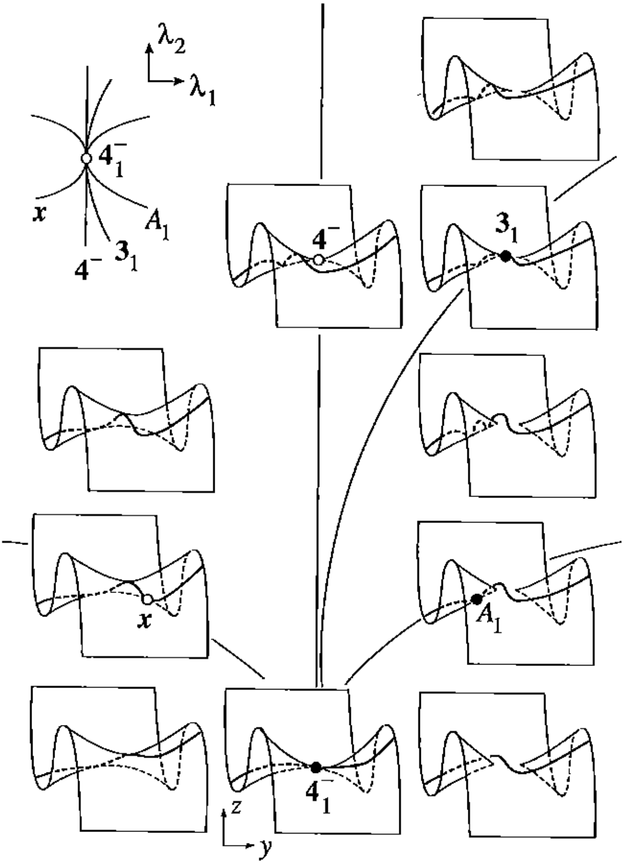


FIGURE 17. Deformation $(x^3 + x(\lambda_1 - y^2) - z, x + \alpha y + y^2 + \lambda_2)$ of 4_1^-, $1/\sqrt{3} < \alpha < 2/\sqrt{3}$.

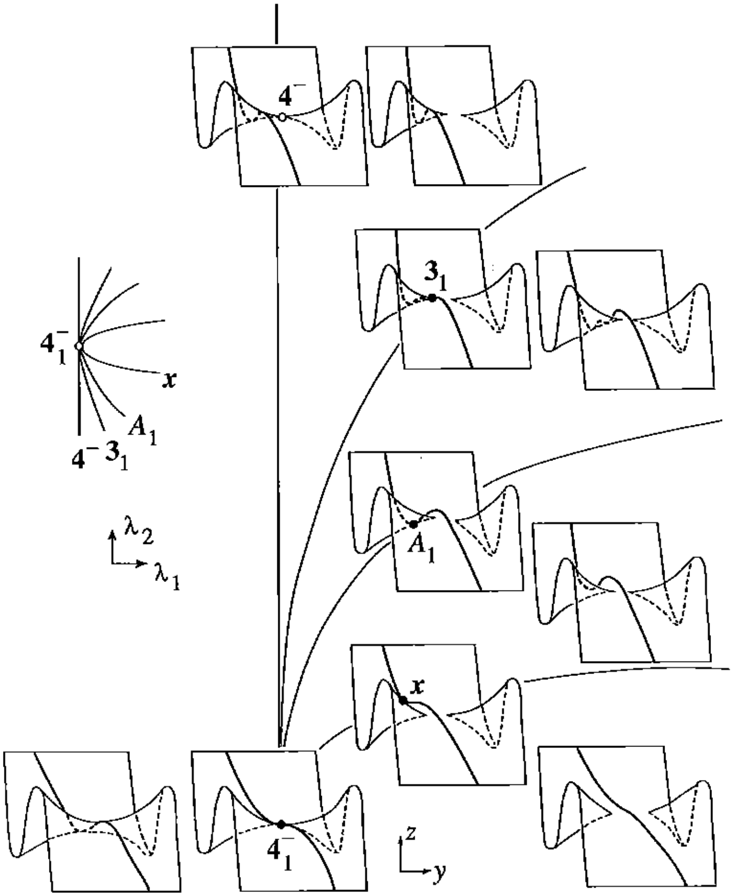


FIGURE 18. Deformation $(x^3 + x(\lambda_1 - y^2) - z, x + \alpha y + y^2 + \lambda_2)$ of 4_1^-, $\alpha > 2/\sqrt{3}$.

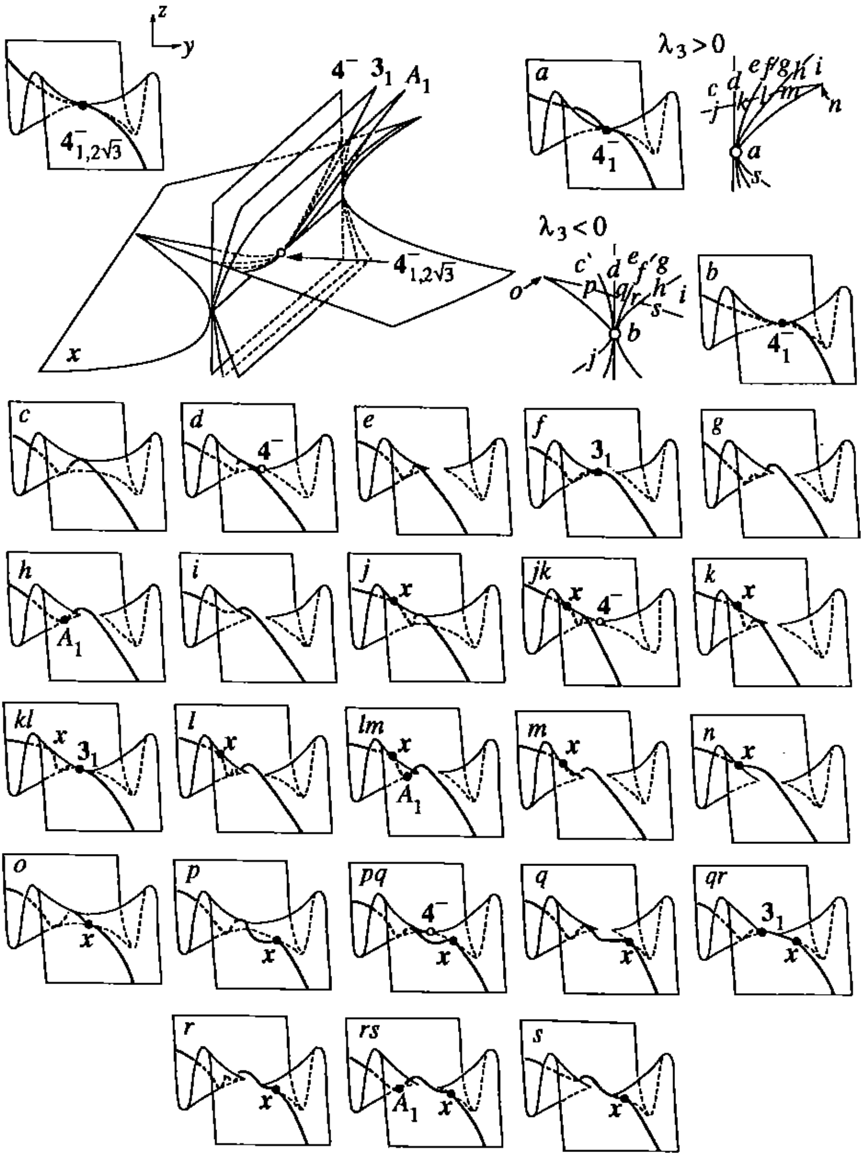


FIGURE 20. Versal deformation $(x^3 + x(\lambda_1 - y^2) - z\,,\, x + (2/\sqrt{3} + \lambda_3) + y^2 + \lambda_2)$ of $\mathbf{4}^-_{1,2/\sqrt{3}}$.

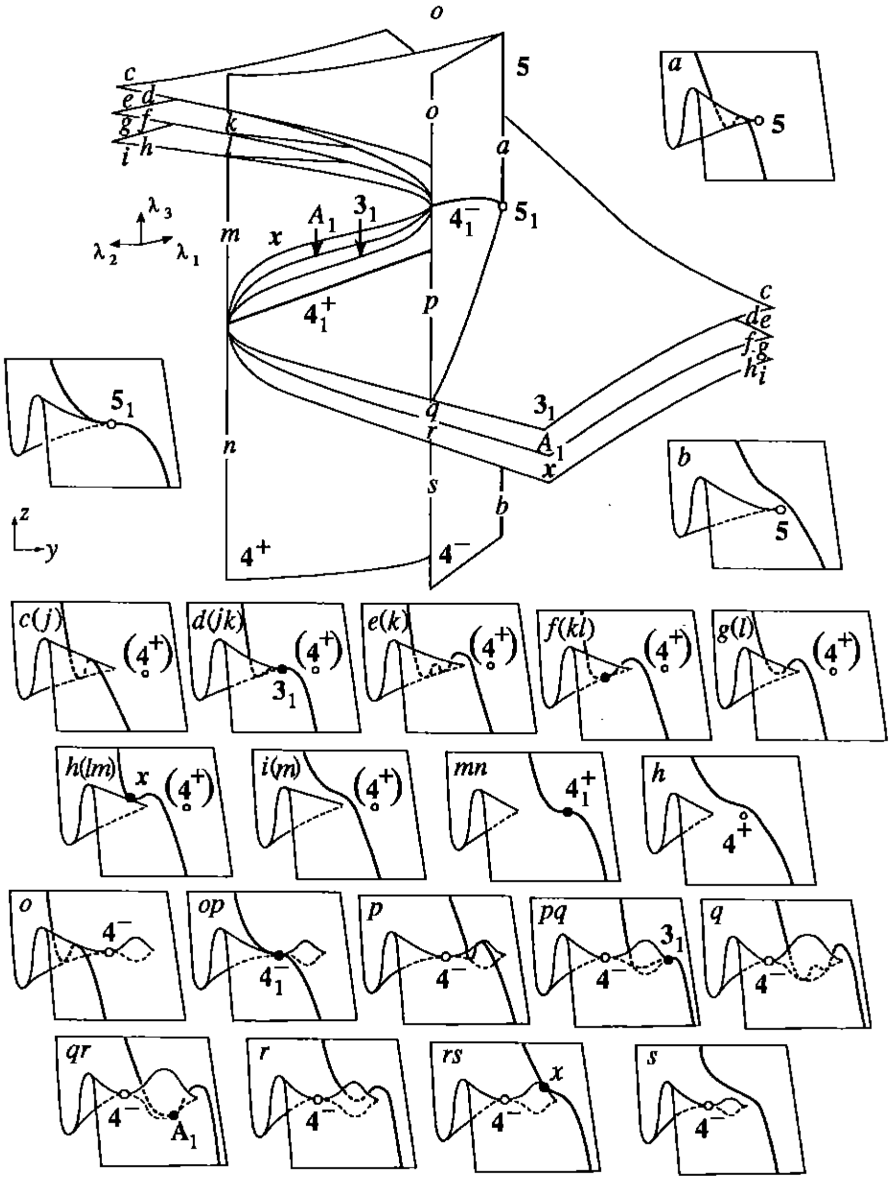


FIGURE 21. Versal deformation $(x^3 + x(y^3 + \lambda_1 y + \lambda_2) - z, x + y + \lambda_2)$ of $\mathbf{5}_1$.

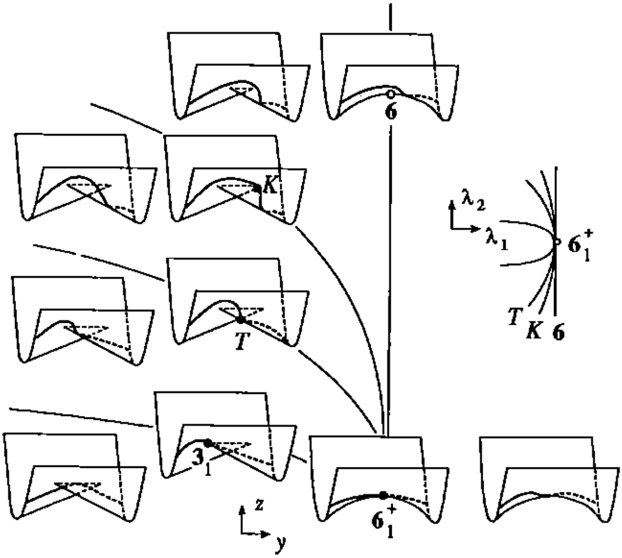


FIGURE 22. Versal deformation $(x^4+xy+\lambda_1x^2-z, x+y+\lambda_2)$ of $\mathbf{6}_1^+$.

Appendix. Projections and left-right equivalence

Recall that in a general setting *a projection of a subvariety* $V \subset \mathbb{R}^{k+p}$ on $\mathbb{R}^p$ is a diagram $V \hookrightarrow \mathbb{R}^{k+p} \to \mathbb{R}^p$ organized by an embedding and a fibration. An equivalence of such objects is a fibration-preserving diffeomorphism of $\mathbb{R}^{k+p}$ that takes one subvariety to the other.

THEOREM. *The classification of projection-germs* $\mathbb{R}^n \hookrightarrow \mathbb{R}^{k+p} \to \mathbb{R}^p$ *is identical to the left-right classification of map-germs* $\mathbb{R}^n \to \mathbb{R}^p$ *with* $\leq$ *k-dimensional kernel at a distinguished point: the equivalence class of a projection corresponds to the equivalence class of its composite mapping.*

PROOF. Indeed, equivalent projections have left-right equivalent composite mappings. Of course, the dimension of the kernel of a composite mapping cannot exceed the dimention of the fiber of the fibration $\mathbb{R}^{k+p} \to \mathbb{R}^p$.

On the other hand, consider any germ $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $\leq k$-dimensional kernel at zero. Take its coordinate record $\varphi \sim (\varphi'(x, y), y)$, $\dim x = k$, $\dim y = n-k$. The projection $(x, y, z) \mapsto (y, z)$, $\dim z = p+k-n$, of a surface $z = \varphi'(x, y)$ has the composite mapping φ.

We have to show that projections with left-right-equivalent composite mappings are equivalent.

Consider the germ of the commutative diagram

$$\begin{array}{ccccc} (\mathbb{R}_1^n, 0) & \longrightarrow & (\mathbb{R}_1^{k+p}, 0) & \longrightarrow & (\mathbb{R}_1^p, 0) \\ {\scriptstyle h}\uparrow & & & & {\scriptstyle \chi}\uparrow \\ (\mathbb{R}_2^n, 0) & \longrightarrow & (\mathbb{R}_2^{k+p}, 0) & \longrightarrow & (\mathbb{R}_2^p, 0) \end{array}$$

with diffeomorphisms h and χ. We want to add a diffeomorphism of $(k+p)$-spaces which preserves commutativity. Consider two cases.

(a) $n \leq k$. Project $\mathbb{R}_i^{k+p}$ (linearly) onto the distinguished fiber $(\mathbb{R}_i^k, 0)$ so that the embedded $(\mathbb{R}_i^n, 0)$ maps isomorphically on its image. The diffeomorphism h induces diffeomorphism h' of these images.

The fibrations over $\mathbb{R}_i^p$ and $\mathbb{R}_i^k$ introduce the structure of a direct product $(\mathbb{R}_i^p, 0) \times (\mathbb{R}_i^k, 0)$ on $(\mathbb{R}_i^{k+p}, 0)$. We define the diffeomorphism we need as an isomorphism of these structures: we set it to be χ on the first factors and for the second ones take any continuation of h' to a diffeomorphism of k-spaces (or h' itself if $n = k$).

(b) $n > k$. Here again we construct an isomorphism of direct products, but a bit more carefully than in the previous case (Figure A).

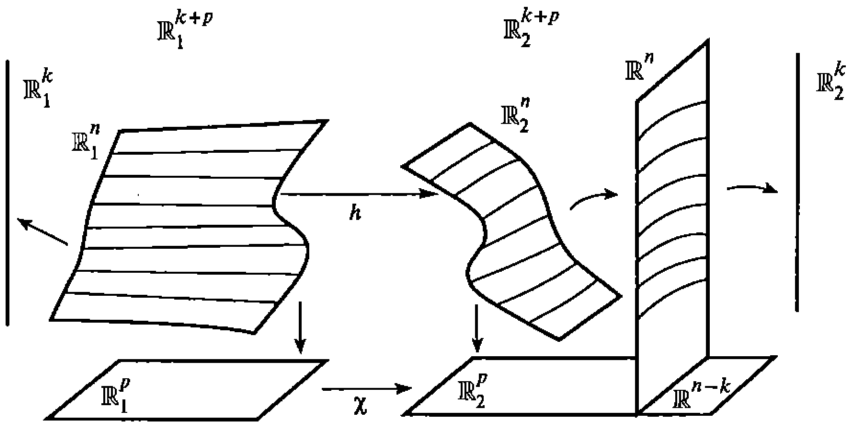


FIGURE A. Construction of the equivalence of projections with left-right equivalent composite mappings.

Define a decomposition $(\mathbb{R}_1^{k+p}, 0) \simeq (\mathbb{R}_1^p, 0) \times (\mathbb{R}_1^k, 0)$ by means of a projection of $\mathbb{R}_1^{k+p}$ onto $(\mathbb{R}_1^k, 0)$ with maximal rank k on the fibered

$(\mathbb{R}_1^n, 0)$. Then $(\mathbb{R}_1^n, 0)$ is fibered over $(\mathbb{R}_1^k, 0)$ with a fiber diffeomorphic to $(\mathbb{R}^{n-k}, 0)$. The diffeomorphism h takes this fibration onto $(\mathbb{R}_2^n, 0)$.

The projection of $\mathbb{R}_2^{k+p}$ on the fiber $(\mathbb{R}_2^k, 0)$ is constructed in two steps. First take a germ of a generic subvariety $(\mathbb{R}^{n-k}, 0)$ in the base $(\mathbb{R}_2^p, 0)$. Let $(\mathbb{R}^n, 0)$ be the space of the restriction of the fibration $\mathbb{R}_2^{k+p} \to \mathbb{R}_2^p$ to this subvariety. The generic projection of $\mathbb{R}_2^{k+p}$ onto $\mathbb{R}^n$ maps the embedded $(\mathbb{R}_2^n, 0)$ isomorphically. Take the fibration by $(n-k)$-dimentional fibers from $(\mathbb{R}_2^n, 0)$ onto $(\mathbb{R}^n, 0)$. By general position, we may suppose that the projection of $(\mathbb{R}^n, 0)$ along these fibers onto the distinguished fiber $(\mathbb{R}_2^k, 0)$ of the fibration $\mathbb{R}^n \to \mathbb{R}^{n-k}$ has rank k. When we take the composition of the two aforementioned projections, we get a projection of $\mathbb{R}_2^{k+p}$ onto the fiber $\mathbb{R}_2^k$ and a direct-product structure $(\mathbb{R}_2^{k+p}, 0) \simeq (\mathbb{R}_2^p, 0) \times (\mathbb{R}_2^k, 0)$. At the same time the fibration of the diffeomorphism h induces a diffeomorphism h' of the spaces $\mathbb{R}_i^k$.

Define the diffeomorphism of $(\mathbb{R}_i^{k+p}, 0) \simeq (\mathbb{R}_i^p, 0) \times (R_i^k, 0)$ as (χ, h').

REMARK. The correspondence between the two classifications may obviously be extended to a correspondence between (mini)versal deformations.

REFERENCES

[A1] V. I. Arnold, *Indices of singular points of 1-forms on a manifold with boundary, convolution of invariants of reflection groups, and singular projections of smooth surfaces*, Russian Math. Surveys **34** (1979), 1–42.

[A2] ——, *Singularities of systems of rays*, Russian Math. Surveys **38** (1983), 87–176.

[A3] ——, *Critical points of functions on a manifold with boundary, the simple Lie groups B_k, C_k, F_4 and singularities of evolutes*, Russian Math. Surveys **33** (1978) 99–116.

[A4] ——, *Wave front evolution and equivariant Morse lemma*, Commun. Pure and Appl. Math. **29** (1976), 557–582.

[AGLV] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vassiliev, *Singularities. I, Local and global theory*, Encyclopaedia Math. Sci., **6** (1990), Springer.

[BG] J. W. Bruce and P. J. Gibblin, *Projections of surfaces with boundaries*, Preprint.

[Dm] J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathscr{A}$ and $\mathscr{K}$*, Mem. Amer. Math. Soc. **50**, 306 (1984).

[Dv] J. M. S. David, *Projection-generic curves*, J. London Math. Soc. **27** (1983), 552–562.

[L] E. E. Landis, *Tangential singularities*, Funct. Anal. Appl. **15** (1981), 103–114.

[P] O. A. Platonova, *Singularities of the mutual disposition of a surface and a line*, Russian Math. Surveys **36** (1981), 248–249.

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