Sectional singularities and geometry of families of planar quadratic forms

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Abstract. We show that, for hypersurface sections (in the sense of Damon) of isolated functions singularities, the Tjurina and Milnor numbers coincide. An application of this to the families of 2×2 symmetric and arbitrary matrices proves the conjectures naturally arising from the results of [2] and [3]. In addition, we study the vanishing homology of the determinantal curves of two-parameter families of symmetric order 2 matrices and construct Dynkin diagrams of simple singularities of such families.

Recently the first author and F. Tari ([2] and [3]) obtained classifications of simple singularities of families of symmetric and arbitrary square matrices (see also [11] for a partial result). The equivalences they used are the most natural and, similar to [7, 8], involve the following.

A family of symmetric matrices determines a mapping of the parameter space into the space of quadratic forms equipped with the standard action of the general linear group.

A family of arbitrary square matrices is a mapping of the parameter space into the set of linear operators between equidimensional vector spaces equipped with the action of the product of the corresponding general linear groups.

Diffeomorphisms of the parameter space along with the families of transformations from the linear groups form the equivalences considered in [2, 3].

The results of the two papers lead to a series of conjectures related to the questions traditionally being asked in singularity theory: what is the relation between the Tjurina and Milnor numbers? what can we say about the vanishing topology of a singularity?

The present paper is devoted to the analysis of these questions.

The very first observation about the lists of simple classes is that, for some special choices of the dimensions, the Tjurina number of a matrix singularity coincides with the Milnor number of the determinantal hypersurface.

Attempts to generalise this observation lead to a conjecture that this must be a particular case of a universal property of sections of an isolated function singularity f considered up to a version of Damon's $\mathcal{K}_V$-equivalence preserving all the levels of f rather than just the hypersurface $V = \{f = 0\}$ (it was first

*Supported by INTAS-00259, NWO047008005 and RBRF99010147 research grants

introduced in [6]). The sections to consider are by arbitrary mappings while the earlier investigations have been mostly concentrating on sections by embeddings (see [4, 5]).

The main result of the first half of the paper is Theorem 1.3 describing the relation between the Tjurina and Milnor numbers of sections of isolated hypersurface singularities. The best case turns out to be that when the section is done by a hypersurface too (Corollary 1.6).

The second half of the paper (Sections 3 and 4) studies vanishing homology of the discriminantal curve of a two-parameter family of order 2 symmetric matrices. We describe the types of vanishing cycles and construct the Dynkin diagrams of all the simple singularities.

1. Sections of isolated hypersurface singularities

1.1. The equivalences

Consider a diagram of holomorphic map-germs

$$(\mathbb{C}^m,0) \xrightarrow{F} (\mathbb{C}^n,0) \xrightarrow{f} (\mathbb{C},0)\,, \qquad (1)$$

in which function f has an isolated critical point at the origin. Let V be the zero set of f. In what follows we fix f, but vary F.

Definition 1.1. Two map-germs $F, F' : (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ are called *$\mathcal{K}_V$-equivalent* if the preimages of V under F and F' are diffeomorphic.

The tangent space $T_{\mathcal{K}_V}F$ to the $\mathcal{K}_V$-equivalence class of a germ F has the following description (see [4]).

Let θ_m be the space of germs of holomorphic vector fields on $(\mathbb{C}^m,0)$. The differential dF sends an element $\xi \in \theta_m$ to the derivative $dF(\xi)$ of F along ξ.

Now let $\mathrm{Derlog}_V \subset \theta_n$ be the algebra of vector fields on $(\mathbb{C}^n,0)$ tangent to V, that is, sending the function f to its multiple. Every vector field $\eta \in \theta_n$ on the target of F provides a variation $F^*(\eta)$ of F.

We have

$$T_{\mathcal{K}_V}F = dF(\theta_m) + F^*(\mathrm{Derlog}_V)\,.$$

There exists a more restrictive version of this kind of equivalence requiring preserving all the level-sets of the composed map.

Definition 1.2. Two map-germs $F, F' : (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ are called *$\mathcal{K}_f$-equivalent* if there exists a diffeomorphism of $(\mathbb{C}^m,0)$, which, for each ε, sends the fibre $f \circ F = \varepsilon$ to the fibre $f \circ F' = \varepsilon$.

As we shall see, this equivalence is in a sense nicer than the previous one: it has general properties shared only by quasi-homogeneous maps within the $\mathcal{K}_V$-context (cf. [6]).

The corresponding tangent space

$$T_{\mathcal{K}_f} F = dF(\theta_m) + F^*(\mathrm{Derlog}_f) \tag{2}$$

is obtained by taking the vector fields $\mathrm{Derlog}_f \subset \theta_n$ annihilating f. Over the functions on $\mathbb{C}^n$, the module Derlog_f is generated by *elementary hamiltonian fields* $f_i\partial_{y_j} - f_j\partial_{y_i}$, where $y_1, \ldots, y_n$ are coordinates on $\mathbb{C}^n$ and $f_i = \partial f/\partial y_i$.

Denote by $\mathcal{O}_m$ the space of holomorphic function-germs on $(\mathbb{C}^m, 0)$. The space of all variations of a particular map from $(\mathbb{C}^m, 0)$ to $(\mathbb{C}^n, 0)$ is the module $\mathcal{O}_m^n = \mathcal{O}_m\langle\partial_{y_1}, \ldots, \partial_{y_n}\rangle$ (this can be identified with the space of all map-germs from $(\mathbb{C}^m, 0)$ to $\mathbb{C}^n$). We introduce two *Tjurina numbers*:

$$\tau_V(F) = \dim_{\mathbb{C}} \mathcal{O}_m^n / T_{\mathcal{K}_V} F \qquad \text{and} \qquad \tau_f(F) = \dim_{\mathbb{C}} \mathcal{O}_m^n / T_{\mathcal{K}_f} F\,.$$

1.2. The Milnor number

Suppose $m \le n$ and $\tau_V(F) < \infty$. Then the function $\varphi = f \circ F$ has an isolated singularity at the origin. Denote by $\mu(\varphi)$ its Milnor number.

Theorem 1.3. *Let the β_r be the Betti numbers of the Koszul complex of the elements $F^*(f_1), \ldots, F^*(f_n) \in \mathcal{O}_m$. Then*

$$\mu(\varphi) = \tau_f(F) - \beta_1 + \beta_0\,.$$

Example 1.4. Assume $m = n$. Then the only non-zero Betti number is β_0 which is the degree of the composition of F with the gradient map of f. Thus, the theorem claims that $\mu(\varphi) - \mu(f) \cdot \deg F = \tau_f(F)$, which turns out to be a sort of a $\mu = \tau$ statement once the left hand side of the equality is interpreted as an appropriate Milnor number. For such an iterpretation, consider a generic perturbation $\widetilde{F}$ of F. The hypersurface $W = \widetilde{F}^{-1}(f = 0)$ has $\deg F$ singularities isomorphic to the singularity of f at the origin. Being a perturbation of the singular level of the function φ, the hypersurface W is homotopic to a wedge of $\mu(\varphi) - \mu(f) \cdot \deg F$ copies of the $n - 1$ sphere.

Proof of Theorem 1.3. To shorten the notations, we write Λ for the module of vertical vector fields $\mathcal{O}_m\langle\partial_{y_1}, \ldots, \partial_{y_n}\rangle$, and Λ^i for its ith exterior power.

The Koszul complex of the n-tuple $F^*(f_1), \ldots, F^*(f_n)$ is the complex of $\mathcal{O}_m$-modules

$$\mathcal{C}: \qquad \cdots \to \Lambda^2 \xrightarrow{d_2} \Lambda^1 \xrightarrow{d_1} \mathcal{O}_m \to 0\,, \tag{3}$$

in which the operators d_i are the convolutions i_{df} with the differential $df = F^*(f_1)dy_1 + \cdots + F^*(f_n)dy_n$ of the function f.

The tangent space $T = T_{K_f}F$ is a subspace of codimension $\tau_f(F)$ in Λ^1. It is mapped by d_1 onto the Jacobi ideal $d\varphi(\theta_m)$ of φ: indeed, the second summand in (2) is clearly the image of d_2 and hence is annihilated by d_1, while the d_1-image $df\Big(dF(\theta_m)\Big)$ of the first summand is exactly $d\varphi(\theta_m)$.

Thus, we have, taking the dimensions of the $\mathbb{C}$-vector spaces:

$$\mu(\varphi) = \dim \mathcal{O}_m/d\varphi(\theta_m) = \dim \mathcal{O}_m/d_1(\Lambda^1) + \dim d_1(\Lambda^1)/d\varphi(\theta_m)$$
$$= \beta_0 + \dim d_1(\Lambda^1)/d_1(T)\,.$$

The last summand is the same as

$$\dim \Lambda^1/(T + \operatorname{Ker} d_1) = \dim \Lambda^1/T - \dim\,(T + \operatorname{Ker} d_1)/T$$
$$= \tau_f(F) - \dim \operatorname{Ker} d_1/(T \cap \operatorname{Ker} d_1)\,.$$

Therefore, the claim of the theorem is equivalent to the last term here being β_1, that is, to the following assertion.

Lemma 1.5. $T_{K_f}F \cap \operatorname{Ker} d_1 = \operatorname{Im} d_2\,.$

Proof. Since the image of d_2 is the second summand in (2), we just need to check that the part of the first summand, $dF(\theta_m)$, contained in $\operatorname{Ker} d_1$ is in $\operatorname{Im} d_2$.

So, take $\zeta = dF(\xi)$, $\xi \in \theta_m$, such that $df(\zeta) = 0$. Then $d\varphi(\xi) = (df \circ dF)(\xi) = 0$. Since function φ has an isolated singularity at the origin, this implies that ξ is an $\mathcal{O}_m$-linear combination of elementary hamiltonian fields $\varphi_i\partial_{x_j} - \varphi_j\partial_{x_i}$ of φ (here the φ_i are the derivatives $\partial\varphi/\partial x_i$ with respect to some coordinates on $\mathbb{C}^m$).

It is enough to consider just $\xi = \varphi_i\partial_{x_j} - \varphi_j\partial_{x_i}$. In this case

$$\zeta = dF(\xi) = dF(\varphi_i\partial_{x_j} - \varphi_j\partial_{x_i}) = \varphi_i \cdot dF(\partial_{x_j}) - \varphi_j \cdot dF(\partial_{x_i})$$
$$= df\Big(dF(\partial_{x_i})\Big) \cdot dF(\partial_{x_j}) - df\Big(dF(\partial_{x_j})\Big) \cdot dF(\partial_{x_i})$$
$$= i_{df}\Big(dF(\partial_{x_i}) \wedge dF(\partial_{x_j})\Big) \in \operatorname{Im} d_2\,.$$

This proves the lemma and hence the theorem.

Corollary 1.6. *For* $m = n - 1,\quad \mu(\varphi) = \tau_f(F)$.

Proof. This is a well-known fact that $\beta_1 = \beta_0$ for the Koszul complex of $m+1$ elements $g_1, \ldots, g_{m+1} \in \mathcal{O}_m$ provided they generate the ideal of finite codimension β_0 in $\mathcal{O}_m$. However, to be self-contained, we shall prove this.

We are still considering the complex $\mathcal{C}$ of (3), with $\Lambda = \mathcal{O}_m\langle\partial_{y_1}, \ldots, \partial_{y_{m+1}}\rangle$ and with slightly more general differentials which are the convolutions with $dg = g_1dy_1 + \cdots + g_{m+1}dy_{m+1}$, where the y_i are formal variables.

Passing to linear combinations of the g_i if needed, we can assume that the ideal I generated by $g_1, \ldots, g_m$ already has a finite codimension in $\mathcal{O}_m$. Now calculate the homology of the complex $\mathcal{C}$ using the spectral sequence associated with the filtration by the number of the $\partial_{y_{m+1}}$'s in the exterior fields (the number is at most either 0 or 1). Setting $\Lambda_0 = \mathcal{O}_m\langle\partial_{y_1}, \ldots, \partial_{y_m}\rangle$, we have

$$\Lambda^r = \Lambda_0^r \oplus \left(\partial_{y_{m+1}} \wedge \Lambda_0^{r-1}\right).$$

The differential d^0 is the convolution with $g_1dy_1 + \cdots + g_mdy_m$. The regularity of the sequence $g_1, \ldots, g_m$ implies that the E^1 is just the complex

$$0 \to Q\partial_{y_{m+1}} \overset{d_1^1}{\to} Q \to 0\,, \qquad Q = \mathcal{O}_m/I\,,$$

with the differential d^1 being the convolution with $g_{m+1}dy_{m+1}$. The Euler characteristic of E^1 is zero. Thus, $\beta_0 = \beta_1$.

Remark 1.7. Another case when the homology of the Koszul complex is rather simple, is when $n-m=2$. Then $\beta_0 = \beta_2 = \beta_1/2$ and all the higher Betti numbers vanish. If the difference between n and m is greater, the situation becomes more complicated. See, for example, [9, 10].

For the $\mathcal{K}_V$-equivalence, the analog of Theorem 1.3 is also true, but just in the quasi-homogeneous setting.

Corollary 1.8. *Assume the map-germs F and f in (1) are quasi-homogeneous, with the weights of the coordinate components of F coinciding with the weights of the arguments of f. Then*

$$\mu(\varphi) = \tau_V(F) - \beta_1 + \beta_0 \,.$$

Proof. Indeed, for a quasi-homogeneous function f,

$$\mathrm{Derlog}_V = \mathrm{Derlog}_f \oplus \mathcal{O}_n E \,,$$

where E is an Euler vector field on $\mathbb{C}^n$. For F as in the assumption of the corollary, the element $dF(E)$ is in $dF(\theta_m)$. Therefore, $\tau_V(F) = \tau_f(F)$.

2. Families of 2×2 matrices

Now we apply the results of Section 1 to the study of mappings into the spaces of 2×2 matrices. Of course, the equivalences we are considering below exist for families of matrices of all orders, but only in the case of order 2 square matrices the determinant is a function with an isolated singularity on the entire matrix space.

2.1. Symmetric matrices

Let $S_2 \simeq \mathbb{C}^3$ be the space of complex symmetric 2×2-matrices. Consider a holomorphic map-germ $S : (\mathbb{C}^m, 0) \to S_2$:

$$x \mapsto S(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix} . \tag{4}$$

We call S a *family of (symmetric) matrices*.

Two such families, S and S', are said to be *$\mathcal{SG}$-equivalent* ($\mathcal{S}$ for 'symmetric', $\mathcal{G}$ for 'general linear') if there exist a biholomorphism-germ h of $(\mathbb{C}^m, 0)$ and a map-germ $A : (\mathbb{C}^m, 0) \to GL(2, \mathbb{C})$ such that

$$S' \circ h = A^T S A \,, \tag{5}$$

where A^T is the transpose of A.

The $\mathcal{SG}$-equivalence provides the right equivalence of the hypersurfaces $\det S = 0$ in $\mathbb{C}^m$, that is of the inverse images of the cone $\mathcal{S}K = \{ac - b^2 = 0\}$ of degenerate matrices. In fact, it is easy to show (see [2]) that the $\mathcal{SG}$-equivalence is the same as Damon's $\mathcal{K}_{SK}$-equivalence [4] of such sections.

The latter is well-seen on the level of the tangent spaces to the orbits. Indeed, let us identify the space of map-germs (4) with the space $\mathcal{O}_m^3$ of the 3-columns of the functions a, b, c and denote $\partial a/\partial x_i$ by a_i as before. The extended tangent space to the $\mathcal{SG}$-orbit of a germ (4) is the $\mathcal{O}_m$-submodule in $\mathcal{O}_m^3$ generated by the elements

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_m \\ b_m \\ c_m \end{pmatrix}, \begin{pmatrix} 2a \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ 2b \end{pmatrix}, \begin{pmatrix} 2b \\ c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ 2c \end{pmatrix}.$$

From the set of the last four generators here, the second, the third and the difference between the two others can be considered as the results of the action on the mapping $(a, b, c)^T$ by the basic hamiltonian vector fields on $\mathbb{C}^3$ preserving the cone $\mathcal{S}K$. The sum of the first and the forth generators is the result of the similar action of the Euler field.

Therefore, for example, deformations and $\mathcal{SG}$-miniversal deformations of families of matrices (defined and constructed in the standard way) are just those for the $\mathcal{K}_{SK}$-equivalence.

Allowing in (5) families A of special linear matrices only, we obtain equivalence of matrix families which we shall call $\mathcal{SS}$-equivalence (the second $\mathcal{S}$ stays here for 'special linear'). Clearly, this is the same as the $\mathcal{K}_{sdet}$-equivalence on $\mathcal{O}_m^3$ where *sdet* is the function $ac - b^2$ on $\mathbb{C}^3$ (in the rather abusive notation *sdet*, the prefix *s*- is used for 'symmetric' consistently with the notation of the equivalences).

The results of Section 1 in particular imply the following for the obvious Tjurina numbers $\tau_{\mathcal{SS}}$ and $\tau_{\mathcal{SG}}$.

Corollary 2.1. *(i) For a two-parameter family S of symmetric 2×2 matrices,*

$$\tau_{\mathcal{SS}}(S) = \mu(\delta),$$

where δ is the function $ac - b^2$ on $\mathbb{C}^2$.

(ii) If the family S is quasi-homogeneous and such that $weight(a) + weight(c) = 2\,weight(b)$, then

$$\tau_{\mathcal{SG}}(S) = \mu(\delta).$$

2.2. Arbitrary matrices and the stabilisation

Now let $M_2 \simeq \mathbb{C}^4$ be the space of all order 2 square matrices and $M : (\mathbb{C}^m, 0) \to M_2$,

$$x \mapsto M(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

a family of such matrices.

The natural equivalence, we call it the $\mathcal{G}$-*equivalence*, of the families in this case consists of biholomorphisms of the source and of the two-side multiplication

$M \mapsto AMB$ by m-parameter families of non-degenerate 2×2 matrices. If A and B here are allowed to be families of just special linear matrices, we obtain what we call the $\mathcal{S}$-*equivalence*.

Let $det = ad - bc$ be the function on M_2 and $K = \{det = 0\} \subset M_2$ the set of degenerate matrices. Then the $\mathcal{G}$- and $\mathcal{S}$-equivalences coincide with respectively $\mathcal{K}_K$- and $\mathcal{K}_{det}$-equivalences. The straightforward analog of Corollary 2.1 on the equality of the relevant Tjurina and Milnor numbers holds for three-parameter matrix families.

It turnes out that a family of symmetric order 2 matrices possesses a natural stabilisation by a family of arbitrary 2×2 matrices. This is absolutely analogous to addition of the square of a new variable to a function singularity and is a particular case of the stabilisation construction for sections of isolated hypersurface singularities. A separate paper on this is in preparation now. However, we formulate the result for the matrix families here.

Theorem 2.2. *1. A germ $M : \mathbb{C}^{m+1} \to M_2$, $M(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, such that the image of the first differential contains a matrix of rank two, is $\mathcal{S}$-equivalent to a germ of the form*

$$g(M) = \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix} + S_M(x_1, \dots, x_m)\,.$$

Here the x_i are local coordinates on $\mathbb{C}^{m+1}$ and S_M is a family of symmetric matrices.

2. The equivalence g above can be chosen so that M' is $\mathcal{S}$-equivalent to M'' if and only if the related symmetric families $S_{M'}$ and $S_{M''}$ are $\mathcal{SS}$-equivalent.

The same holds for the $\mathcal{G}$-equivalence versus the $\mathcal{SG}$-equivalence. This, for example, explains the one-to-one correspondence between simple singularities of two-parameter families of symmetric order 2 matrices and simple singularities of three-parameter families of arbitrary 2×2 matrices (one immediately conjectures existence of such by comparing the lists from [2] and [3]).

3. Vanishing homology

From now on we concentrate on the vanishing topology of two-parameter families of order 2 symmetric matrices, from the point of view of the $\mathcal{SG}$-equivalence.

The base $\mathbb{C}^\tau$ of an $\mathcal{SG}$-miniversal deformation of a mapping $S : (\mathbb{C}^2, 0) \to S_2$, $\tau = \tau_{\mathcal{SG}}(S)$, contains the *discriminant* hypersurface Δ formed by those values of the parameters for which the corresponding perturbation of M is not transversal to the cone $\mathcal{S}K$. The discriminant consists of two components:

- Σ_s, which corresponds to mappings with the image passing through the origin in S_2, and
- Σ_ℓ, corresponding to the non-transversality to the smooth part of $\mathcal{S}K$.

The indexation of the components is based on the types, short and long, of the cycles vanishing on the inverse image of the cone $\mathcal{S}K$ under a generic perturbation of M when the deformation parameters tend to the component. The cycles are defined, rather traditionally, as follows.

Consider a generic line $L \simeq \mathbb{C}$ in the base $\mathbb{C}^\tau$. It meets Δ at a finite number of points $c_1, \ldots, c_\nu$ and does this transversally. Mark a generic point $* \in L$. The inverse image in $\mathbb{C}^2$ of the cone $\mathcal{S}K$ under the mapping S_* corresponding to $*$ is a smooth curve. Connect point $*$ by a system of ν paths γ_i, without mutual- and self-intersections, with all the points c_i. In what follows we assume that the indexation of the paths and critical values (and of the corresponding vanishing cycles) is actually done in the counter-clockwise order in which the paths leave point $*$.

Approaching a point $c_i \in \Sigma_s$ along the path γ_i, we contract on the curve $V_* = \{\det S_* = 0\}$ a 1-cycle (we call it a *short vanishing cycle*) in the same way as we contract the 1-cycle on a generic section of $\mathcal{S}K$ when the sectional plane moves through the vertex. The latter is isomorphic to the contraction of the real 1-cycle in the family

$$\det \begin{pmatrix} x & y \\ y & -x+\lambda \end{pmatrix} = 0$$

when the real parameter λ tends to zero.

Approaching a point $c_i \in \Sigma_\ell$, we define a *long vanishing cycle* on the curve $V_* \subset \mathbb{C}^2$which can be described locally as the real 1-cycle vanishing inthe family

$$\det \begin{pmatrix} x^2+y^2+\lambda & 0 \\ 0 & 1 \end{pmatrix} = 0$$

when the negative real number λ tends to zero.

Thus obtained, from a system of paths γ_i, set of ν vanishing short and long cycles on the curve V_* is called a *distinguished set of vanishing cycles*.

Theorem 3.1. *A distinguished set of vanishing cycles generates* $H_1(V_*)$.

The proof is absolutely traditional (cf., for example, a similar theorem for complete intersections in [1]).

The Picard-Lefschetz operator on $H_1(V_*)$ corresponding to a long vanishing cycle e is the standard one:

$$\sigma \mapsto \sigma - (\sigma, e)e\,,$$

where the brackets denote the intersection number. For a short cycle, the monodromy is clearly the square of the standard operator, thus being

$$\sigma \mapsto \sigma - 2(\sigma, e)e\,.$$

Remark 3.2. The number s of short cycles in a distinguished set of the family (4) is easily seen to be $\dim \mathcal{O}_2/I$ where I is the ideal generated by two generic linear combinations of the functions a, b, c.

The number l of long cycles in a distinguished set can be obtained as follows. Choose generic constants $\alpha, \beta, \gamma \in \mathbb{C}$, and consider functions $A = a + \alpha\lambda$, $B =$

$b + \beta\lambda$ and $C = c + \gamma\lambda$ in three variables x, y, λ. Set $D = AC - B^2$. Let $J \subset \mathcal{O}_3$ be the ideal generated by D, D_x, D_y. Then $\dim \mathcal{O}_3/J = l + 2s$.

4. Dynkin diagrams of simple families of symmetric matrices

4.1. Simple singularities

$\mathcal{SG}$-simple two-parameter families of symmetric 2×2 matrices have been classified in [2]. They are listed in the table below. Here we denote the matrix singularities by the types of their determinantal functions. Such notations are slightly delicate since, in fact, no two families in the table are equivalent: for example, singularity D_{5+2} is different from $D_{2\cdot 3+1}$.

Since all the singularities in the table are quasihomogeneous, the table also serves as a complete list of $\mathcal{SS}$-simple two-parameter families of symmetric order 2 matrices.

TABLE 1. *$\mathcal{SG}$-simple two-parameter families of order 2 symmetric matrices*

$A_{k+\ell-1}$ $\ell \geq k \geq 1$	$\begin{pmatrix} y^k & x \\ x & y^\ell \end{pmatrix}$	D_{k+2} $k \geq 2$	$\begin{pmatrix} x & 0 \\ 0 & y^2 + x^k \end{pmatrix}$
D_{2k+1} $k \geq 2$	$\begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix}$	D_{2k} $k \geq 3$	$\begin{pmatrix} x & 0 \\ 0 & xy + y^k \end{pmatrix}$
E_6	$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$	E_7	$\begin{pmatrix} x & 0 \\ 0 & x^2 + y^3 \end{pmatrix}$

In the remaining subsections, for each of the simple families we construct a distinguished set of vanishing cycles in the vanishing homology of its determinantal curve and calculate the corresponding Dynkin diagram. In the diagrams, white vertices represent short vanishing cycles and black vertices represent long. A directed edge $a \to b$ of multiplicity r stays for the positive intersection number $(a, b) = r$.

4.2. A_{k+k-1}

Consider a deformation of this singularity of the form

$$\begin{pmatrix} -y & \lambda \\ \lambda & y + 2p(x) \end{pmatrix},$$

where p is a monic degree k polynomial with all its roots real and simple.

Choose a non-bifurcational value λ_0 of λ as shown in Figure 1 and take a system of paths corresponding to the clusters of the bifurcation values of λ (all $k-1$ positive values are in the cluster I, all $k-1$ negative in III, and there are k vanishing short cycles at $\lambda = 0$).

We denote the long cycles $I_1, \dots, I_{k-1}$ and $III_1, \dots, III_{k-1}$, and the short cycles $II_1, \dots, II_k$. The Roman number corresponds to the ordering of the clusters of the critical values. The short cycles II_i are those vanishing at the nodes of the curve $\lambda = 0$ of Figure 1, with II_1 vanishing at the rightmost node. The long cycles III_i are the real ovals of the curve $\lambda = \lambda_0$, with III_1 also being the rightmost. The curves $\lambda = \lambda_0$ and $\lambda = -\lambda_0$ coincide. The real ovals of the latter are the cycles I_i.

For each $i = 1, \dots, k-1$, we have a relation

$$I_i = III_i + II_i - II_{i+1}\,, \tag{6}$$

that is

$$I_i = III_i + (III_i, II_i)II_i + (III_i, II_{i+1})II_{i+1}\,.$$

Indeed, III_i is the image of I_i under the square root of the monodromy operator corresponding to the cluster II consisting of all short cycles.

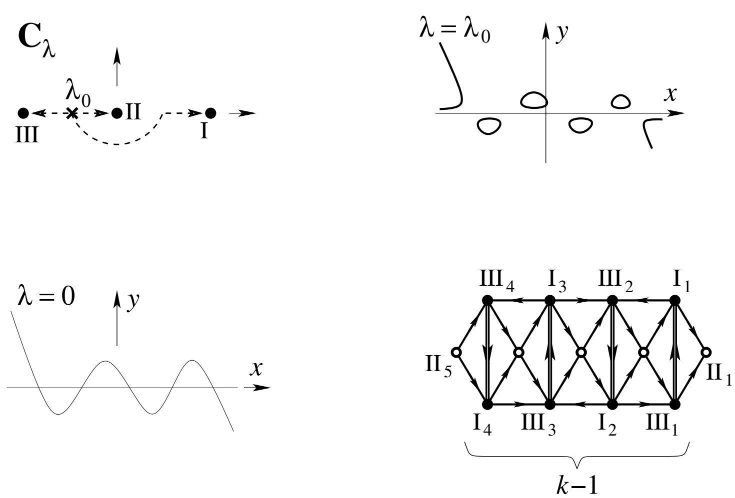

FIGURE 1. *Singularity A_{k+k-1}, $k = 5$. Top left: a distinguished system of paths. Top right: the marked Milnor fibre $\lambda = \lambda_0$. Bottom left: all the distinguished short vanishing cycles II_i contracted. Bottom right: the Dynkin diagram.*

4.3. $A_{k+\ell-1}$

Now we consider the $k < \ell$ case. We deform the singularity

$$\begin{pmatrix} p(x) - \lambda & y \\ y & q(x) + \lambda \end{pmatrix},$$

choosing the polynomials p and q (monic, of degrees k and ℓ respectively) with all their roots real and simple, and the graphs as shown in Figure 2. Taking the same system of paths in $\mathbb{C}_\lambda$ as for A_{k+k-1}, we obtain the Dynkin diagram also shown in Figure 2. The $k-1$ relations between the distinguished cycles are absolutely the same as in the A_{k+k-1} case.

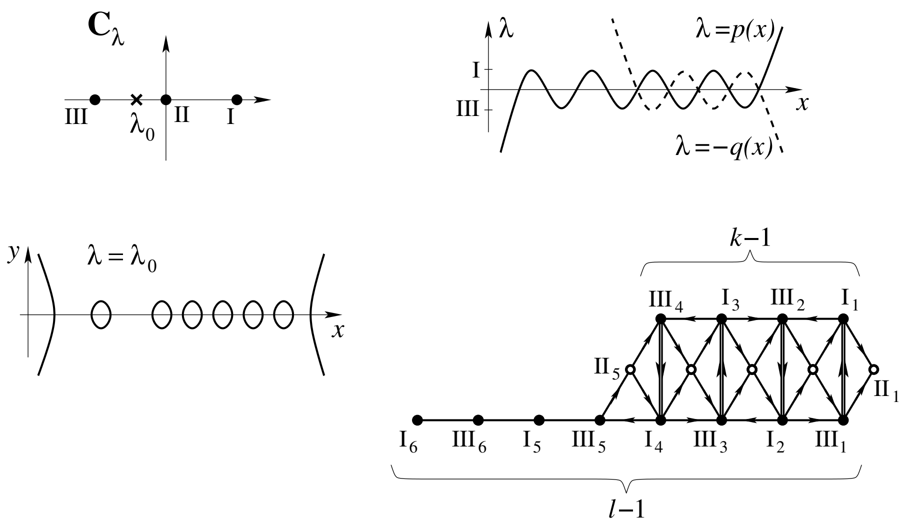

FIGURE 2. *Singularity $A_{k+\ell-1}$, $k = 5$, $\ell = 9$: the marked and critical values of the deformation parameter, graphs of the polynomials participating in the deformation, the marked Milnor fibre and the Dynkin diagram.*

4.4. D_{k+2}

Consider the following deformation of a singularity of this class:

$$\begin{pmatrix} x & \varepsilon\lambda \\ \varepsilon\lambda & p(x) - y^2 + \lambda \end{pmatrix},$$

where ε is a small positive constant, and the degree k monic polynomial p is chosen so that it has all its roots double (except for at most one) and real positive, and the graph of the function $xp(x)$ is as shown in Figure 3. The slopes of the dashed tangents are the bifurcational values of λ. The lower left diagram of Figure 3 shows the distinguished Milnor fibre (the bold curve whose ovals are the vanishing cycles I) and the curve $\lambda = 0$ (fine). Here we also indicate the way how the movement of the branches of the Milnor fibre generates the saddle point corresponding to the third bifurcational value of λ. The rhombus on the left of the Dynkin diagram is again a relation of type (6).

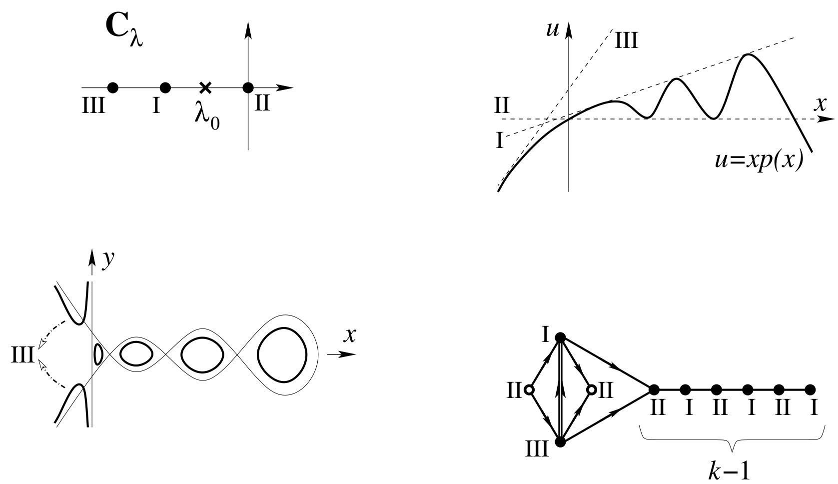

FIGURE 3. *Singularity D_{k+2}, $k = 7$.*

4.5. D_{2k+1}

We take a deformation of the form:

$$\begin{pmatrix} y & p(x) \\ p(x) & xy + \lambda \end{pmatrix},$$

where a degree k monic polynomial $p(x)$ is such that all its roots are real and simple, and the function $xp^2(x)$ has just two critical values, 0 and c. Then the first and third discriminantal values of the family are $(-c)^{1/2}$. The $k-1$ standard relations of type (6) are easily seen in the Dynkin diagram in Figure 4.

4.6. D_{2k}

We choose a deformation of a singularity of this class in the form

$$\begin{pmatrix} y & i\varepsilon\lambda \\ i\varepsilon\lambda & x(y + p_{k-1}(x)) + \lambda \end{pmatrix},$$

where the monic degree $k-1$ polynomial $p(x)$ has all its roots simple and real positive. For $\lambda = 0$, the determinantal curve is shown in Figure 5. It differs from the A_{k+k-1} curve of Figure 1 just by one extra node. Further comparison with the

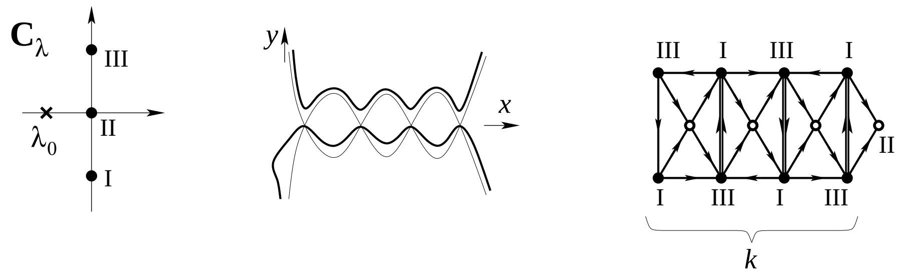

FIGURE 4. *Singularity* D_{2k+1}, $k = 4$.

D_{2+2} curve implies that D_{2k} has a Dynkin diagram as in Figure 5, with the $k-1$ standard relations.

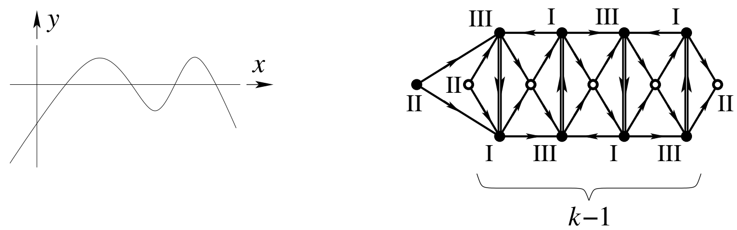

FIGURE 5. *Singularity* D_{2k}, $k = 5$*: the curve with all the short cycles contracted and the Dynkin diagram.*

4.7. E_6

Consider the following deformation of this singularity:

$$\begin{pmatrix} y & x^2+\alpha \\ x^2+\alpha & y^2+\beta y+\lambda \end{pmatrix}. \tag{7}$$

Take $\alpha < 0$ and $\beta > 0$ such that $27\alpha^2 = 4\beta^3$. For $\lambda = 0$, the determinantal curve of (7) is the standard E_6 trefoil, with three nodes, two of which correspond to the short vanishing cycles and one to a long cycle (see Figure 6).

Relation: $e_1 - e_7 - e_2 + e_3 = 0$.

4.8. E_7

We take its deformation in the form

$$\begin{pmatrix} y & \varepsilon\lambda \\ \varepsilon\lambda & y^2+x^3-ax+\lambda \end{pmatrix}.$$

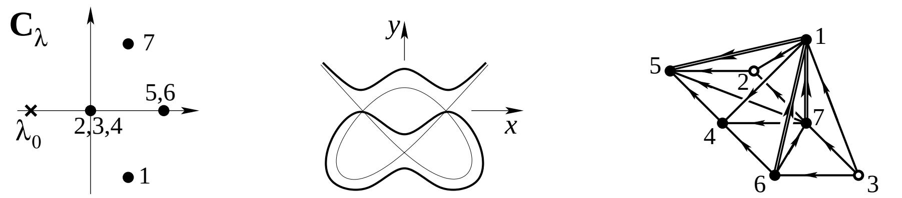

FIGURE 6. *Singularity* E_6.

Here $a > 0$ and ε is a small real non-zero number.

Calculations of the discriminantal values of λ and of the vanishing cycles provide the results represented in Figure 7. We have two standard relations:

$$e_6 - e_5 - e_2 + e_1 = 0 \qquad \text{and} \qquad e_9 - e_8 - e_3 + e_2 = 0\,.$$

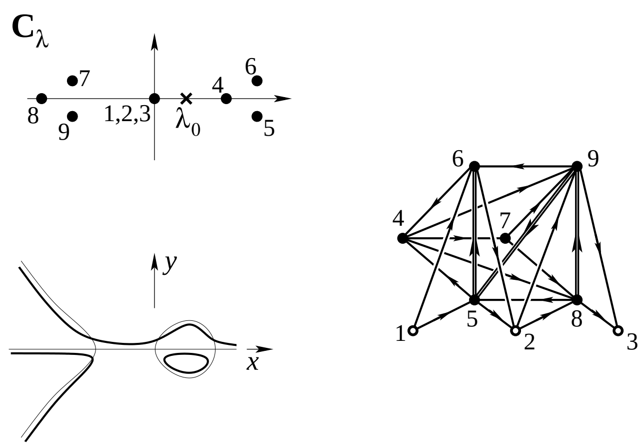

FIGURE 7. *Singularity* E_7.

Remark 4.1. The stabilisation of section 2.2 of two-parameter families of symmetric matrices to the three-parameter families of arbitrary matrices affects the Dynkin diagrams in absolutely the same way as addition of the square of a new variable affects the intersections within a set of distinguished vanishing cycles of a function of two variables (see, for example, [1]). The suspensions $\widetilde{e}$ of both long and short vanishing cycles have self-intersections -2 in our case. The Picard-Lefschetz operator for a long $\widetilde{e}$ is the standard reflection. However, the operator for a short

$\widetilde{e}$ is just the identity since it is the square of the standard reflection. The latter was the major reason for us to consider here in detail the families of symmetric 2×2 matrices rather than of arbitrary.

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