Finite order invariants of framed knots in a solid torus and in Arnold's J^+-theory of plane curves

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Interest to the study of plane curve invariants goes back to Gauss. Recently Arnold approached the subject from the point of view of Vassiliev theory of finite order invariants and defined three basic order 1 invariants of regular plane curves [1, 2]. In the present note we consider a theory corresponding to a higher order generalisation of one of Arnold's invariants, J^+, that changes only under direct self-tangency transformations. Our constructions show that indeed, as Arnold remarked in [1], plane curves are much more complicated objects than knots in 3-space.

The legendrian lifting of regular plane curves to the solid torus $ST^*\mathbf{R}^2$ defines a mapping from the space of Vassiliev type invariants of framed knots in a solid torus to the space of Vassiliev type invariants of J^+-theory of plane curves. We show that this is an isomorphism for the complex-valued setting.

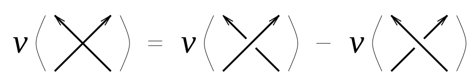
To achieve our goal we introduce chord diagram interpretations of both the invariant spaces. In both cases we arrive to the same function space, namely, to functions on marked chord diagrams subject to the marked 4-term relation. The markings are naturally defined by the fundamental group of a solid torus and by the Whitney winding number of plane curves.

The fact that the invariant spaces coincide with the function space follows from the consideration of a version of the universal Vassiliev-Kontsevich invariant for framed knots in a solid torus. Our approach is different from that by Lê and Murakami [10, 11] and serves arbitrary framings.

For the proofs and details see [6, 7].

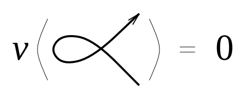
1 Knots in $\mathbf{R}^3$

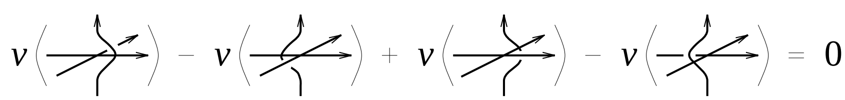
The theory of Vassiliev invariants of unframed oriented knots in $\mathbf{R}^3$ starts with the inductive extension of an invariant of embeddings $S^1 \to \mathbf{R}^3$ to immersions with a finite number of generic double points (such immersions are called *singular knots*) [12, 3, 5, 9, 4]:



The three curves involved in the definition differ only locally, by the shown fragments.

Extensions of invariants are subject to the 1- and 4-term relations:

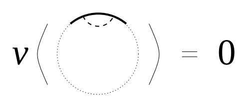


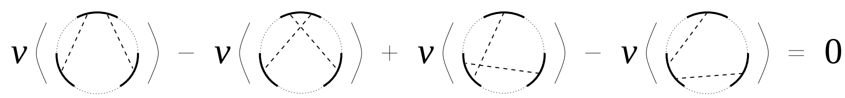


To each singular knot we relate its *chord diagram*: on the oriented source circle we join by chords pairs of points glued together by the parametrising immersion. We identify chord diagrams which might be sent to each other by diffeomorphisms of the circle preserving its orientation.

A knot invariant is said to have *order less than* $n+1$ if its extension vanishes on any singular knot with more than n double points. We denote by $V_n(\mathbf{R}^3)$ the space of all complex-valued invariants of order less than $n+1$.

Consider *the symbol* of an invariant $v \in V_n(\mathbf{R}^3) \setminus V_{n-1}(\mathbf{R}^3)$, that is its restriction to the set of singular knots with exactly n double points. This is a function on n-chord diagrams. The values of this function satisfy the 1- and 4-term relations which follow from those above:





Here we show all the chords based on solid arcs and none of those based on dotted arcs. The four diagrams in the 4-term relation differ only by the chords based on solid arcs. Here and in what follows the circle of a diagram is assumed to be oriented counter-clockwise.

Let $\mathcal{A}_n^*$ be the vector space of $\mathbf{C}$-valued functions on n-chord diagrams satisfying the 4-term relation and $\mathcal{A}_n^{0,*}$ its subspace of functions satisfying the 1-term relation as well.

The 1- and 4-term relations turn out to be the only resrictions on the values on symbols:

Theorem 1.1 [9] $\qquad V_n(\mathbf{R}^3)/V_{n-1}(\mathbf{R}^3) = \mathcal{A}_n^{0,*} \quad .$

For oriented framed knots in $\mathbf{R}^3$ the similar statement holds:

Theorem 1.2 [10] $\qquad V_n^f(\mathbf{R}^3)/V_{n-1}^f(\mathbf{R}^3) = \mathcal{A}_n^* \quad .$

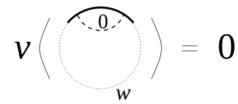
2 Knots in a solid torus

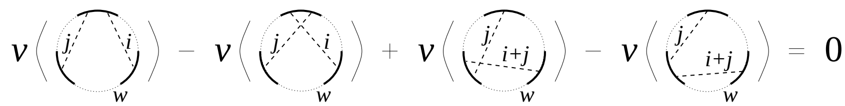
Of course, one can construct Vassiliev type theory for knots in any oriented 3-manifold using the local recursive definition of Section 1 [8]. In the case of a solid torus (ST) we arrive to the following diagram description.

We say that a chord diagram is *marked* if its circle is equipped with an integer number and each of its chords is furnished with a two-side integer labelling such that the sum of the two labels is the marking of the circle.

A choice of a generator of $\pi_1(ST) = \mathbf{Z}$ introduces the marking on the chord diagram of an oriented singular knot in ST. Namely, a double point cuts a knot into two subloops each of which defines an element of $\pi_1(ST)$. We write this element (the integer number) on the side of the chord which faces the arc that parametrises the subloop. The basic circle is marked with the fundamental class of the whole knot.

The 1- and 4-term relations for symbols of invariants of knots in ST are naturally marked:





Here we show only partial markings which allow to restore the complete ones. The marking in the 1-term relation for symbols means the contractibility of the "small" subloop in the 1-term relation for invariants. The markings in the 4-term relation follow from the relation between the fundamental classes of the subloops of a curve with a triple point.

We use $\mathcal{M}$ instead of $\mathcal{A}$ in the notation of the marked versions of the function spaces of the previous Section.

Theorem 2.1 [6]

$$V_n(ST)/V_{n-1}(ST) = \mathcal{M}_n^{0,*}\,,$$
$$V_n^f(ST)/V_{n-1}^f(ST) = \mathcal{M}_n^*\,.$$

3 The universal invariant

As in [12, 9], the difficult part in the proof of Theorem 2.1 is to show that the relations mentioned in the claims are the only restrictions on the values of symbols of invariants. At that point an appropriate version of the universal Vassiliev-Kontsevich invariant [9, 4] is needed. In this Section we define such an invariant for framed knots in ST (the construction for the unframed setting follows from it and [9, 4] in an obvious way). Keeping in mind the application of our universal invariant to the theory of plane curve invariants in Section 4, we introduce a definition which covers arbitrary framings, not only the blackboard one as in [10, 11].

We represent ST as the direct product $\mathbf{C}\times S^1$ with the complex coordinate z and the angular coordinate $\theta \bmod 2\pi$.

We say that a knot in ST is *a Morse knot* if θ is a Morse function on it.

Let K be an oriented non-singular framed Morse knot in ST.

A. For small $\varepsilon > 0$, we shift K in the direction of its framing ν:

$$(z,\theta) \mapsto (z,\theta) + \varepsilon\nu(z,\theta).$$

We denote by K_ε the result of the shift. For all sufficiently small ε, K_ε is a knot that does not intersect K.

B. In order to have a good definition of a chord diagram later on, we make an adjustment of the link $K \cup K_\varepsilon$. Near a local maximum of the function θ on K, θ has the local maximum on K_ε as well. We take the lowest of the two critical levels and remove the small arc of $K \cup K_\varepsilon$ that is locally above this level. In the similar way, we remove the small arc that is locally below the highest of the two critical levels near a local minimum of θ on K. After the surgery at all the local extrema, we remain with the subsets $\widehat{K} \subset K$ and $\widehat{K_\varepsilon} \subset K_\varepsilon$. The shift along the framing provides the one-to-one correspondence between the sets of intervals of monotonicity of the function θ on K and K_ε. For each non-critical point $(z', \theta) \in \widehat{K_\varepsilon}$ this correspondence uniquely defines its *neighbour* $(z'', \theta) \in \widehat{K}$ on the same θ-level.

C. Now we take n different non-critical levels $0 < \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi$. In each section $\theta = \theta_j$ of $\widehat{K} \cup \widehat{K_\varepsilon}$, we choose an *ordered* pair of points $(z_j, z'_j) = (z_j, z'_j)(\theta_j) \in \widehat{K} \times \widehat{K_\varepsilon}$. Let P be a set of n such pairs, one pair per level.

The set P defines the marked n-chord diagram as follows (see Fig.1).

In each pair we substitute $z'_j \in \widehat{K_\varepsilon}$ by its neighbour $z''_j \in \widehat{K}$. The knot K is the image of an immersion of an oriented circle that we take to be a standard counter-clockwise oriented circle on the plane. If $z_j \neq z''_j$, we join the preimages of the points z_j and z''_j on the source circle by the chord. The chord has the two-side marking by the fundamental classes of the two loops obtained by a homotopy of K in ST that glues together z_j and z''_j and is the identity outside a small neighbourhood of the section $\theta = \theta_j$. We assume here that a generator of $\pi_1(\mathrm{ST}) = \mathbf{Z}$ is fixed. Say, it goes once around the torus in the direction of increase of θ.

If $z_j = z''_j$, we draw a small chord between two arbitrary points on the circle that are very close to the preimage of z_j. We mark the side of the chord facing the small arc with 0 and its other side with the class of K in $\pi_1(\mathrm{ST})$.

The whole circle is marked with the class of K in $\pi_1(\mathrm{ST})$ as well.

We denote by $D(P)$ the equivalence class of the obtained marked chord diagram in the space $\mathcal{M}_n$ of all formal $\mathbf{C}$-linear combinations of finitely many marked n-chord diagrams modulo the marked 4-term relation. The latter is the relation of Section 2 for diagrams rather than functions on them.

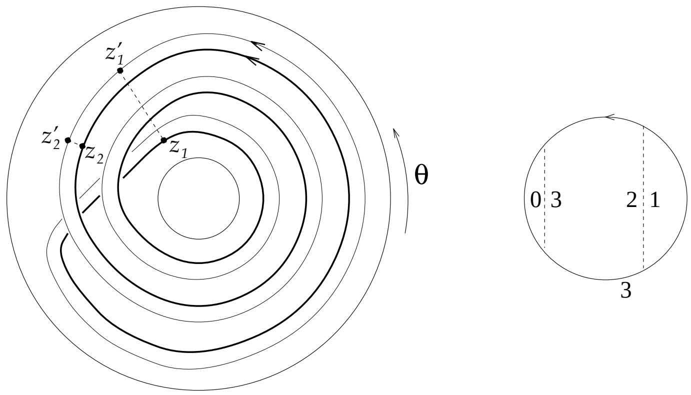


Figure 1: *A pairing on the knot with the blackboard framing and its marked chord diagram.*

D. We introduce

Definition 3.1 $\hat{Z}_n(K, K_\varepsilon) =$

$$= \frac{1}{(2\pi i)^n} \int_{0<\theta_1<\theta_2<\ldots<\theta_n<2\pi} \sum_{P=\{(z_j,z'_j)(\theta_j)\}} (-1)^{P_\downarrow} \bigwedge_{j=1}^{n} \frac{dz_j - dz'_j}{z_j - z'_j} D(P) \in \mathcal{M}_n \ ,$$

where P runs through all possible pairings on $\widehat{K} \cup \widehat{K}_\varepsilon$ and $P_\downarrow$ is the number of points in the n pairs at which the function θ is decreasing along $K \cup K_\varepsilon$.

Definition 3.2 $Z^f_n(K) = \lim_{\varepsilon\to 0} \hat{Z}_n(K, K_\varepsilon)$.

Theorem 3.3 ([6], cf. [9]) *i) The limit that defines $Z^f_n(K)$ is finite.*

ii) $Z^f_n(K)$ is invariant under the homotopy in the class of framed Morse knots.

iii) $Z^f_n(K)$ is an invariant of order less than $n+1$.

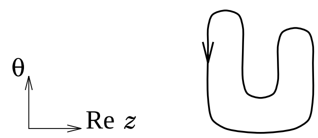
We set

$$Z^f(K) = \sum_{n\geq 0} Z^f_n(K) \in \overline{\mathcal{M}},$$

where $\overline{\mathcal{M}} = \prod_{n\geq 0} \mathcal{M}_n$.

E. The space $\mathcal{M} = \oplus_{n\geq 0}\mathcal{M}_n$ contains a subspace $\mathcal{A}$ spanned by the diagrams with all the markings vanishing. $\mathcal{A}$ is an algebra with respect to the connected sum operation. It is isomorphic to the algebra of non-marked chord diagrams modulo the non-marked 4-term relation [9, 4]. The connected summation provides an $\mathcal{A}$-module structure on the space $\mathcal{M}$ [6].

Let $\mathcal{U}$ be the curve



equipped with the framing $\nu = i\partial_z$. The curve lies in a sector of the annulus $\operatorname{Im} z = 0$ of the solid torus $\mathbf{C} \times S^1$.

The series $Z^f(\mathcal{U}) \in \overline{\mathcal{A}} = \prod_{n\geq 0}\mathcal{A}_n$ is invertible since it starts with $1 \in \mathcal{A}_0$.

Let c be the number of critical points of the function θ on a knot K.

Definition 3.4 The element

$$\widetilde{Z}^f(K) = Z^f(K) \times Z^f(\mathcal{U})^{1-\frac{c}{2}} \in \overline{\mathcal{M}}$$

is called *the universal Vassiliev-Kontsevich invariant* of a framed Morse knot K in the solid torus.

Example 3.5 Let $\omega \in \mathcal{A}_1$ be the one-chord diagram with all three marks zero. Consider an unknot with the framing that makes one positive rotation around it. The value of $\widetilde{Z}^f$ on such unknot in ST is $\exp(\omega)$.

Theorem 3.6 ([6], cf. [9]) *For any framed Morse knot K, $\widetilde{Z}^f(K)$ depends only on the topological type of K and its framing.*

Theorem 3.6 implies Theorem 2.1.

The degree n component $\widetilde{Z}^f_n(K) \in \mathcal{M}_n$ of $\widetilde{Z}^f(K)$ is an invariant of order less than $n+1$.

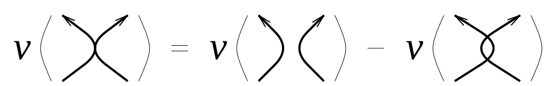
4 Finite type invariants in J^+-theory

A generic plane curve is an immersed curve with a finite number of double points of transversal self-intersection. In the space of all C^∞-immersions

$S^1 \to \mathbf{R}^2$, the complement to the set of all generic curves consists of three hypersurfaces [1, 2]. They correspond to the three possible degenerations in generic 1-parameter families of immersed curves. In such families there can appear either a curve with a triple point or a curve with one of two types of self-tangencies. A self-tangency can be either *direct* (when the two velocity vectors at the self-tangency point have the same direction) or *inverse* (when they are opposite).

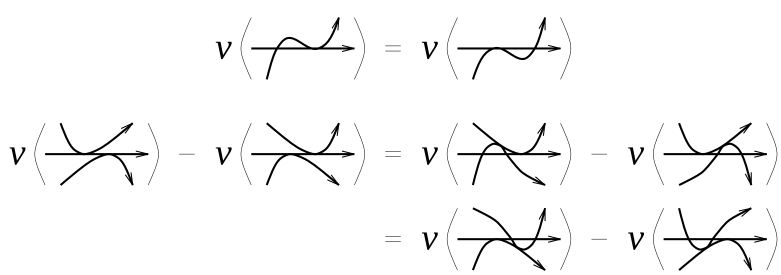
Here we restrict our attention only to invariants of oriented immersed plane curves without direct self-tangencies. The values of such invariants on isotopy classes of curves do not change during inverse-self-tangency and triple-point transformations. The first invariant of this kind was defined by Arnold [1, 2] and called J^+. That is why the whole theory of invariants that we consider is called J^+-theory.

In the spirit of Vassiliev theory for knots, invariants of regular plane curves without direct self-tangencies have a natural extension to curves with finitely many simple (quadratic) direct self-tangencies:



The definitions of the space $V_n(J^+)$ of $\mathbf{C}$-valued invariants of order less than $n+1$ and of symbols of finite order invariants are obvious.

Theorem 4.1 [7] *The values of J^+-theory invariants on plane curves with direct self-tangencies are subject to the 2- and 4-term relations:*



Any oriented regular plane curve rises to an oriented legendrian curve in the solid torus $ST^*\mathbf{R}^2$: we add the direction of the normal vector such that the basis {normal, velocity} gives a fixed (say, counter-clockwise) orientation of the plane. A point of direct self-tangency rises to a generic double point of the legendrian curve. The choice of an orientation of $\mathbf{R}^2$ defines the generator of $\pi_1(ST^*\mathbf{R}^2)$ which is the legendrian lift of an embedded circle of Whitney winding number 1.

Definition 4.2 *The marked chord diagram* of a regular plane curve γ with a finite number of direct self-tangencies is the marked chord diagram of the legendrian lift of γ to $ST^*\mathbf{R}^2$ (Fig.2).

The marking introduced in this way is the same as the marking with the Whitney winding numbers of the "half-curves" into which the plane curve is cut by its self-tangency points.

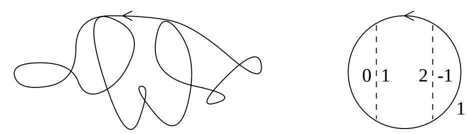


Figure 2: *A plane curve and its marked chord diagram.*

It is easy to see that any marked n-chord diagram is the marked chord diagram of a regular plane curve with n direct self-tangencies. Application of the Whitney-Graustein theorem [13] and the 2-term relation imply

Theorem 4.3 [7] *The value of an invariant $v \in V_n(J^+)$ on a regular plane curve with n direct self-tangencies depends only on the marked chord diagram of this curve.*

The legendrian lifting of the recursive definition of the extension of an invariant in the J^+-theory is exactly the recursive definition of the extension of an invariant of knots in the solid torus $ST^*\mathbf{R}^2$ (with the proper choice of the orientations).

Moreover, on the level of symbols, the legendrian lifting of the 4-term relation of the J^+-theory is exactly the marked 4-term relation for invariants of knots in $ST^*\mathbf{R}^2$.

All this identifies the quotient $V_n(J^+)/V_{n+1}(J^+)$ as a subspace of the function space $\mathcal{M}_n^*$. It turns out that the equality holds:

Theorem 4.4 [7] $\quad V_n(J^+)/V_{n+1}(J^+) = \mathcal{M}_n^*$.

The theorem easily follows from the definition of the universal invariant for the J^+-theory as induced from the universal invariant of framed knots in the solid torus $ST^*\mathbf{R}^2$ via the legendrian lift. The lift of a regular plane curve possesses a canonical framing defined by the lift of the family of plane curves obtained by small shifts of the original curve in the direction of the normals. The crucial point is that the value of the extension of the universal invariant on a regular plane curve with n direct self-tangencies subject to a marked chord diagram $D \in \mathcal{M}_n$ is $2^n D$ modulo higher order terms.

Corollary 4.5 [7] *The space of finite type complex-valued invariants of regular plane curves without direct self-tangencies is isomorphic to that of framed knots in a solid torus.*

Remark 4.6 In the similar way one can show the coincidence of the spaces of complex-valued Vassiliev type invariants of finite order for two other settings. On one side of the equality there is the theory of oriented regular plane curves without self-tangencies (both direct and inverse). This is Arnold's $J^{\pm}$-theory [1, 2]. On the other side we put oriented framed knots in the solid torus $PT^*\mathbf{R}^2$ which define even classes in the fundamental group.

Remark 4.7 The approach of this Section assumes that the Whitney winding number is an invariant of order zero. Setting it to have order 1 makes the situation rather different. For example, for every n, the space of invariants of order less than n becomes finite.

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