LAGRANGIAN AND LEGENDRIAN SINGULARITIES

V.V.GORYUNOV and V.M.ZAKALYUKIN

Department of Mathematical Sciences
University of Liverpool
Liverpool L69 3BX, UK
e-mail: goryunov@liv.ac.uk
Department of Mechanics and Mathematics
Moscow State University
Leninskie gory, 1
119992 Moscow,Russia
e-mail: vzakal@liv.ac.uk

These are notes of the introductory courses we lectured in Trieste in 2003 and Luminy in 2004. The lectures contain basic notions and fundamental theorems of the local theory of singularities of wave fronts and caustics with some recent applications to geometry.

Keywords: Symplectic and contact geometry, Lagrangian and Legendrian submanifolds and singularities, generating families

R.Thom and V.Arnold noticed that the singularities that can be visualized in many physical models are of special nature.

This was the starting point of the theory of Lagrangian and Legendrian mappings developed by V.I.Arnold and his school some thirty years ago. Since then the significance of Lagrangian and Legendrian submanifolds of symplectic and respectively contact spaces has been recognised throughout all mathematics, from algebraic geometry to differential equations, optimisation problems and physics.

Alternatively these singularities are called singularities of caustics and wave fronts.

Suppose, for example, that a disturbance (such as a shock wave, light, an epidemic or a flame) is propagating in a medium from a given submanifold (called *initial wave front*). To determine where the disturbance will be at time t (according to the Huygens principle) we must lay a segment of length t along every normal to the initial front. The resulting variety is called an *equidistant* or a wave front.

Along with wave fronts, ray systems may also be used to describe propagation of disturbances. For example, we can consider the family of all normals to the initial front. This family has the envelope, which is called *caustic* – "burning" in Greek – since the light concentrates at it. A caustic is clearly visible on the inner surface of a cup put in the sunshine. A rainbow in the sky is the caustic of a system of rays which have passed through drops of water with total internal reflection.

Generic caustics in three-dimensional space have only standard singularities. Besides regular surfaces, cuspidal edges and their generic (transversal) intersections, these are: the swallowtail, the 'pyramid' (or 'elliptic umbilic') and the 'purse' (or 'hyperbolic umbilic'). They are a part of R.Thom's famous list of simple catastrophes. It is not so difficult to see that the singularities of a propagating wave front slide along the caustic and trace it out.

Symplectic space is essentially the phase space (space of positions and momenta) of classical mechanics, inheriting a rich set of important properties.

It turns out that caustics and wave fronts are the loci of critical values of special non-generic mappings of manifolds of equal dimensions or mappings from n to $n+1$ dimensional manifolds. The general definition of these mapping was given by V.Arnold via the projections of Lagrangian and Legendre submanifolds embedded into symplectic and contact spaces.

These construction describes many special classes of mappings: Gauss mapping, gradient mappping, etc.

In fact, Lagrangian or Legendre mapping is determined by a single family of fuctions. This crucial fact makes the theory transparent and constructive.

In particular, stable wave fronts and caustics are discriminants and bifurcation diagrams of singularities of functions. That is why their generic low dimensional singularities are governed by famous simple Weyl groups.

Recently new areas in the theory of integrable systems in mathematical physics (Frobenuous structures, D-modules) yield new field of applications of Lagrangian and Legendre singularities.

In these lecture notes, we do not touch the fascinating results in symplectic and contact topology, a young branch of mathematics which answers questions on global behaviour of Lagrangian and Legendrian submanifolds. An interested reader may be addressed to the paper [[4]]. Our lectures is an introduction to the original local theory, with an accent on applications in geometry. We hope that they will inspire the reader to do more extensive

reading. Items on our bibliography list [[1–3]]. may be rather useful for this.

1. Symplectic and contact geometry

1.1. *Symplectic geometry*

A symplectic form ω on a manifold M is a closed 2-form, non-degenerate as a skew-symmetric bilinear form on the tangent space at each point. So $d\omega = 0$ and ω^n is a volume form, $\dim M = 2n$. Manifold M equipped with a symplectic form is called symplectic. It is necessarily even-dimensional.

If the form is exact, $\omega = d\lambda$, the manifold M is called *exact symplectic.*

Examples.

1. The basic model of a symplectic space is the vector space $K = \mathbf{R}^{2n} = \{q_1, \dots, q_n, p_1, \dots, p_n\}$ with the form

$$\lambda = pdq = \sum_{i=1}^{n} p_i dq_i\,, \qquad \omega = d\lambda = dp \wedge dq\,.$$

In these coordinates the form ω is constant. The corresponding bilinear form on the tangent space at a point is given by the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Any non-degenerate skew-symmetric bilinear form on a linear space, has a Darboux basis in which the form has this matrix.

2. $M = T^*N$. $\lambda = pdq$ - Take for λ the *Liouville form* defined in a coordinate free way as

$$\lambda(\alpha) = \pi(\alpha)(\rho_*(\alpha))\,,$$

where

$$\alpha \in T(T^*N)\,, \quad \pi : T(T^*N) \to T^*N \quad \text{and} \quad \rho : T^*N \to N\,.$$

The manifold M, $d\lambda$ is exact symplectic. For local coodinates $q_1, \dots, q_n$ on N, the dual coordinates $p_1, \dots, p_n$ are the coefficients of the decomposition of a covector into a linear combination of the differentials dq_i:

$$\lambda = \sum_{i=1}^{n} p_i dq_i\,.$$

3. On a Kähler manifold M, the imaginary part of its Hermitian structure $\omega(\alpha, \beta) = Im(\alpha, \beta)$ is a skew-symmetric 2-form which is closed.

4. Product of two symplectic manifolds. Given two symplectic manifolds (M_i, ω_i), $i = 1, 2$, their product $M_1 \times M_2$ equipped with the 2-form $(\pi_1)_*\omega_1 - (\pi_2)_*\omega_2$, where the π_i are the projections to the corresponding factors, is a symplectic manifold.

A diffeomorphism $\varphi : M_1 \to M_2$ which sends the symplectic structure ω_2 on M_2 to the symplectic structure ω_1 on M_1,

$$\varphi^*\omega_2 = \omega_1 ,$$

is called a symplectomorphism between (M_1, ω_1) and (M_2, ω_2). When the (M_i, ω_i) are the same, a symplectomorphism preserves the symplectic structure. In particular, it preserves the volume form ω^n.

Symplectic group.
For $K = (\mathbf{R}^{2n}, dp \wedge dq)$ of our first example, the group $Sp(2n)$ of linear symplectomorphisms is isomorphic to the group of matrices S such that

$$S^{-1} = -JS^tJ .$$

Here t is for transpose. The characteristic polynomial of such an S is reciprocal: if α is an eigenvalue, then α^{-1} also is. The Jordan structures for α and α^{-1} are the same.

Introduce an auxiliary scalar product $(\cdot, \cdot)$ on K, with the matrix I_{2n} in our Darboux basis. Then

$$\omega(a, b) = (a, \widetilde{J}b) ,$$

where $\widetilde{J}$ is the operator on K with the matrix J. Setting $q = \operatorname{Re} z$ and $p = \operatorname{Im} z$ makes K a complex Hermitian space, with the multiplication by $i = \sqrt{-1}$ being the application of $\widetilde{J}$. The Hermitian structure is

$$(a, b) + i\omega(a, b) .$$

From this,

$$Gl(n, \mathbf{C}) \bigcap O(2n) = Gl(n, \mathbf{C}) \bigcap Sp(2n) = O(2n) \bigcap Sp(2n) = U(n) .$$

Remark. The image of the unit sphere $S_1^{2n-1} : q^2 + p^2 = 1$ under a linear symplectomorphism can belong to a cylinder $q_1^2 + p_1^2 \le r$ only if $r \ge 1$.

The non-linear analog of this result is rather non-trivial: $S_1^{2n-1} \in T^*\mathbf{R}^n$ (in the standard Euclidean structure) cannot be symplectically embedded into the cylinder $\{q_1^2 + p_1^2 < 1\} \times T^*\mathbf{R}^{n-1}$. This is Gromov's theorem on symplectic camel.

Thus, for $n > 1$, symplectomorphisms form a thin subset in the set of diffeomorphisms preserving the volume ω^n.

The dimension k of a linear subspace $L^k \subset K$ and the rank r of the restriction of the bilinear form ω on it are the complete set of $Sp(2n)$-invariants of L.

Define the skew-orthogonal complement $L^\angle$ of L as

$$L^\angle = \{v \in K | \omega(v, u) = 0 \quad \forall u \in L\} .$$

So $\dim L^\angle = 2n - k$. The kernel subspace of the restriction of ω to L is $L \bigcap L^\angle$. Its dimension is $k - r$.

A subspace is called isotropic if $L \subset L^\angle$ (hence $\dim L \le n$).
Any line is isotropic.

A subspace is called co-isotropic if $L^\angle \subset L$ (hence $\dim L \ge n$).
Any hyperplane H is co-isotropic. The line $H^\angle$ is called the characteristic direction on H.

A subspace is called Lagrangian if $L^\angle = L$ (hence $\dim L = n$).

Lemma. *Each Lagrangian subspace $L \subset K$ has a regular projection to at least one of the 2^n co-ordinate Lagrangian planes (p_I, q_J), along the complementary Lagrangian plane (p_J, q_I). Here $I \bigcup J = \{1, \ldots, n\}$ and $I \bigcap J = \emptyset$.*

Proof. Let L_q be the intersection of L with the q-space and $\dim L_q = k$. Assume $k > 0$, otherwise L projects regularly onto the p-space. The plane L_q has a regular projection onto some q_I-plane (along q_J) with $|I| = k$. If L does not project regularly to the p_J-plane (along (q, p_I)) then L contains a vector $v \in (q, p_I)$ with a non-trivial p_I-component. Due to this non-triviality, the intersection of the skew-orthogonal complement $v^\angle$ with the q-space has a $(k-1)$-dimensional projection to q_I (along q_J) and so does not contain L_q. This contradicts to L being Lagrangian.

A Lagrangian subspace L which projects regularly onto the q-plane is the graph of a self-adjoint operator S from the q-space to the p-space with its matrix symmetric in the Darboux basis.

Splitting $K = L_1 \bigoplus L_2$ with the summands Lagrangian is called a polarisation. Any two polarisations are symplectomorphic.

The Lagrangian Grassmanian $Gr_L(2n)$ is diffeomorphic to $U(n)/O(n)$. Its fundamental group is $\mathbf{Z}$.

The Grassmanian $Gr_k(2n)$ of isotropic k-spaces is isomorphic to $U(n)/(O(k)+U(n-k))$.

Even in a non-linear setting a symplectic structure has no local invariants (unlike a Riemannian structure) according to the classical

Darboux Theorem. *Any two symplectic manifolds of the same dimension are locally symplectomorphic.*

Proof. We use the homotopy method. Let ω_t, $t \in [0,1]$, be a family of germs of symplectic forms on a manifold coinciding at the distinguished point A. We are looking for a family $\{g_t\}$ of diffeomorphisms such that $g_t^*\omega_t = \omega_0$ for all t. Differentiate this by t:

$$\mathcal{L}_{v_t}\omega_t = -\gamma_t$$

where $\gamma_t = \partial\omega_t/\partial t$ is a known closed 2-form and $\mathcal{L}_{v_t}$ is the Lie derivative along the vector field to find. Since $\mathcal{L}_v = i_v d + d i_v$, we get

$$d i_{v_t}\omega_t = -\gamma_t\,.$$

Choose a 1-form α_t vanishing at A and such that $d\alpha_t = -\gamma_t$. Due to the non-degeneracy of ω_t, the equation $i_{v_t}\omega_t = \omega(\cdot, v_t) = \alpha_t$ has a unique solution v_t vanishing at A. □

Weinstein's Theorem. *A submanifold of a symplectic manifold is defined, up to a symplectomorphism of its neighbourhood, by the restriction of the symplectic form to the tangent vectors to the ambient manifold at the points of the submanifold.*

In a similar local setting, the inner geometry of a submanifold defines its outer geometry:

Givental's Theorem. *A germ of a submanifold in a symplectic manifold is defined, up to a symplectomorphism, by the restriction of the symplectic structure to the tangent bundle of the submanifold.*

Proof of Givental's Theorem. It is sufficient to prove that if the restrictions of two symplectic forms, ω_0 and ω_1, to the tangent bundle of a submanifold $G \subset M$ at point A coincide, then there exits a local diffeomorphism of M fixing G point-wise and sending one form to the other. We may assume that the forms coincide on T_AM.

We again use the homotopy method, aiming to find a family of diffeomorphism-germs g_t, $t \in [0,1]$, such that

$$g_t|_G = id_G\,, \quad g_0 = id_M\,, \qquad g_t^*(\omega_t) = \omega_0 \quad (*) \qquad \text{where} \quad \omega_t = \omega_0 + (\omega_1 - \omega_0)t\,.$$

Differentiating $(*)$ by t, we again get

$$\mathcal{L}_{v_t}(\omega_t) = d(i_{v_t}\omega_t) = \omega_0 - \omega_1$$

where v_t is the vector field of the flow g_t. Using the "relative Poincare lemma", it is possible to find a 1-form α so that $d\alpha = \omega_0 - \omega_1$ and α vanishes on G. Then the required vector field v_t exists since ω_t is non-degenerate. □

Darboux theorem is a particular case of Givental's theorem: take a point as a submanifold.

If at each point x of a submanifold L of a symplectic manifold M the subspace T_xL is Lagrangian in the symplectic space T_xM, then L is called Lagrangian.

Examples.

1. In T^*N, the following are Lagrangian submanifolds: the zero section of the bundle, fibres of the bundle, graph of the differential of a function on N.

2. The graph of a symplectomorphism is a Lagrangian submanifold of the product space (it has regular projections onto the factors). An arbitrary Lagrangian submanifold of the product space defines a so-called Lagrangian relation which, in a sense, is a multivalued generalization of a symplectomorphism.

Weinstein's theorem implies that a tubular neighbourhood of a Lagrangian submanifold L in any symplectic space is symplectomorphic to a tubular neighbourhood of the zero section in T^*N.

A fibration with Lagrangian fibres is called Lagrangian.
Locally all Lagrangian fibrations are symplectomorphic (the proof is similar to that of the Darboux theorem).
A cotangent bundle is a Lagrangian fibration.

Let $\psi : L \to T^*N$ be a Lagrangian embedding and $\rho : T^*N \to N$ the fibration. The product $\rho \circ \psi : L \to N$ is called a Lagrangian mapping. It critical values

$$\Sigma_L = \{q \in N | \exists p : (p, q) \in L,\ \operatorname{rank} d(\rho \circ \psi)|_{(p,q)} < n\}$$

form the caustic of the Lagrangian mapping. The equivalence of Lagrangian mappings is that up to fibre-preserving symplectomorphisms of the ambient symplectic space. Caustics of equivalent Lagrangian mappings are diffeomorphic.

Hamiltonian vector fields.
Given a real function $h : M \to \mathbf{R}$ on a symplectic manifold, define a Hamiltonian vector field v_h on M by the formula

$$\omega(\cdot, v_h) = dh\,.$$

This field is tangent to the level hypersurfaces $H_c = h^{-1}(c)$:

$$\forall a \in H_c \quad dh(T_a H_c) = 0 \qquad \Longrightarrow \qquad T_a H_c = v_h^{\angle}\,, \quad \text{but} \quad v_h \in v_h^{\angle}\,.$$

The directions of v_h on the level hypersurfaces H_c of h are the characteristic directions of the tangent spaces of the hypersurfaces.

Associating v_h to h, we obtain a Lie algebra structure on the space of functions:

$$[v_h, v_f] = v_{\{h,f\}} \qquad \text{where} \quad \{h, f\} = v_h(f)\,,$$

the latter being the Poisson bracket of the Hamiltonians h and f.

A Hamiltonian flow (even if h depends on time) consists of symplectomorphisms. Locally (or in $\mathbf{R}^{2n}$), any time-dependent family of symplectomorphisms that starts from the identity is a phase flow of a time-dependent Hamiltonian. However, for example, on a torus $\mathbf{R}^2/\mathbf{Z}^2$ (the quotient of the plane by an integer lattice) the family of constant velocity displacements

are symplectomorphisms but they cannot be Hamiltonian since a Hamiltonian function on a torus must have critical points.

Given a time-dependent Hamiltonian $\widetilde{h} = \widetilde{h}(t, p, q)$, consider the extended space $M \times T^*\mathbf{R}$ with auxiliary co-ordinates (s, t) and the form $pdq - sdt$. An auxiliary (extended) Hamiltonian $\widehat{h} = -s + \widetilde{h}$ determines a flow in the extended space generated by the vector field

$$\dot{p} = -\frac{\partial \widehat{h}}{\partial q}, \quad \dot{q} = \frac{\partial \widehat{h}}{\partial p}, \quad \dot{t} = -\frac{\partial \widehat{h}}{\partial s} = 1, \quad \dot{s} = \frac{\partial \widehat{h}}{\partial t}.$$

The restrictions of this flow to the $t = const$ sections are essentially the flow mappings of $\widetilde{h}$.

The integral of the extended form over a closed chain in $M \times \{t_o\}$ is preserved by the $\widehat{h}$-Hamiltonian flow. Hypersurfaces $-s + \widetilde{h} = const$ are invariant. When $\widetilde{h}$ is autonomous, the form pdq is also a relative integral invariant.

A (transversal) intersection of a Lagrangian submanifold $L \subset M$ with a Hamiltonian level set $H_c = h^{-1}(c)$ is an isotropic submanifold L_c. All Hamiltonian trajectories emanating from L_c form a Lagrangian submanifold $exp_H(L_c) \subset M$. The space Ξ_{H_c} of the Hamiltonian trajectories on H_c inherits, at least locally, an induced symplectic structure. The image of the projection of $exp_H(L_c)$ to Ξ_{H_c} is a Lagrangian submanifold there. This is a particular case of a symplectic reduction which will be discussed later.

Example. The set of all oriented straight lines in $\mathbf{R}^n_q$ is T^*S^{n-1} as a space of characteristics of the Hamiltonian $h = p^2$ on its level $p^2 = 1$ in $K = \mathbf{R}^{2n}$.

1.2. *Contact geometry*

An odd-dimensional manifold M^{2n+1} equipped with a maximally non-integrable distribution of hyperplanes (contact elements) in the tangent spaces of its points is called a contact manifold.

The maximal non-integrability means that if locally the distribution is determined by zeros of a 1-form α on M then $\alpha \wedge (d\alpha)^n \neq 0$ (cf. the Frobenius condition $\alpha \wedge d\alpha = 0$ of complete integrability).

Examples.
1. A projectivised cotangent bundle PT^*N^{n+1} with the projectivisation of

the Liouville form $\alpha = pdq$ is a contact manifold. This is also called the space of contact elements on N. The spherisation of PT^*N^{n+1} is a 2-fold covering of PT^*N^{n+1} and its points are co-oriented contact elements.

2. The space J^1N of 1-jets of functions on N^n is another standard model of contact space. (Two functions have the same m-jet at a point x if their Taylor polynomials of degree k at x coincide). The space of all 1-jets at all points of N has local coordinates $q \in N$, $p = df(q)$ which are the partial derivatives of a function at q, and $z = f(q)$. The contact form is $pdq - dz$.

Contactomorphisms are diffeomorphisms preserving the distribution of contact elements.

Contact Darboux theorem. *All equidimensional contact manifolds are locally contactomorphic.*

An analog of Givental's theorem also holds.

Symplectisation.
Let $\widetilde{M}^{2n+2}$ be the space of all linear forms vanishing on contact elements of M. The space $\widetilde{M}^{2n+2}$ is a "line" bundle over M (fibres do not contain the zero forms). Let $\widetilde{\pi} : \widetilde{M} \to M$ be the projection. On $\widetilde{M}$, the symplectic structure (which is homogeneous of degree 1 with respect to fibres) is the differential of the canonical 1-form $\widetilde{\alpha}$ on $\widetilde{M}$ defined as

$$\widetilde{\alpha}(\xi) = p(\widetilde{\pi}_*\xi), \qquad \xi \in T_p\widetilde{M}.$$

A contactomorphism F of M lifts to a symplectomorphism of $\widetilde{M}$:

$$\widetilde{F}(p) := (F^*_{F(x)})^{-1}p.$$

This commutes with the multiplication by constants in the fibres and preserves $\widetilde{\alpha}$. The symplectisation of contact vector fields (= infinitesimal contactomorphisms) yields Hamiltonian vector fields with homogeneous (of degree 1) Hamiltonian functions $h(rx) = rh(x)$.

Assume the contact structure on M is defined by zeros of a fixed 1-form β. Then M has a natural embedding $x \mapsto \beta_x$ into $\widetilde{M}$.

Using the local model $J^1\mathbf{R}^n$, $\beta = pdq - dz$, of a contact space we get the following formulas for components of the contact vector field with a homogeneous Hamiltonian function $K(x) = h(\beta_x)$ (notice that $K = \beta(X)$ where X is the corresponding contact vector field):

$$\dot{z} = pK_p - K, \quad \dot{p} = -K_q - pK_z, \quad \dot{q} = K_p.$$

where the subscripts mean the partial derivations.

Various homogeneous analogs of symplectic properties hold in contact geometry (the analogy is similar to that between affine and projective geometries).

In particular, a hypersurface (transversal to the contact distribution) in a contact space inherits a field of characteristics.

Contactisation.
To an exact symplectic space M^{2n} associate $\widehat{M} = \mathbf{R} \times M$ with an extra co-ordinate z and take the 1-form $\alpha = \lambda - dz$. This gives a contact space.

Here the vector field $\chi = -\frac{\partial}{\partial z}$ satisfies $i_\chi\alpha = 1$ and $i_\chi d\alpha = 0$. Such a field is called a Reeb vector field. Its direction is uniquely defined by a contact structure. It is transversal to the contact distribution. Locally, projection along χ produces a symplectic manifold.

A Legendrian submanifold $\widehat{L}$ of M^{2n+1} is an n-dimensional integral submanifold of the contact distribution. This dimension is maximal possible for integral submanifolds due to maximal non-integrability of the contact distribution.

Examples.
1. To a Lagrangian $L \subset T^*M$ associate $\widehat{L} \subset J^1M$:

$$\widehat{L} = \{(z,p,q) \mid z = \int pdq, \ (p,q) \in L\}.$$

Here the integral is taken along a path on L joining a distinguished point on L with the point (p,q). Such an $\widehat{L}$ is Legendrian.

2. The set of all covectors annihilating tangent spaces to a given submanifold (or variety) $W_0 \subset N$ form a Legendrian submanifold (variety) in PT^*N.

3. If the intersection I of a Legendrian submanifold $\widehat{L}$ with a hypersurface Γ in a contact space is transversal, then I is transversal to the characteristic vector field on Γ. The set of characteristics emanating from I form a Legendrian submanifold.

A Legendrian fibration of a contact space is a fibration with Legendrian fibres. For example, $PT^*N \to N$ and $J^1N \to J^0N$ are Legendrian. Any two Legendrian fibrations of the same dimension are locally contactomorphic.

The projection of an embedded Legendrian submanifold $\widehat{L}$ to the base of a Legendrian fibration is called a Legendrian mapping. Its image is called the wave front of $\widehat{L}$.

Examples.

1. Embed a Legendrian submanifold $\widehat{L}$ into J^1N. Its projection $W(\widehat{L})$ to J^0N, which is the wave front, is a graph of a multivalued action function $\int pdq + c$ (again we integrate along paths on the Lagrangian submanifold $L = \pi_1(\widehat{L})$, where $\pi_1 : J^1N \to T^*N$ is the projection dropping the z coordinate). If $q \in N$ is not in the caustic Σ_L of L, then over q the wave front $W(\widehat{L})$ is a collection of smooth sheets.

If at two distinct points $(p', q), (p'', q) \in L$ with a non-caustical value q, the values z of the action function are equal, then at (z, q) the wave front is a transversal intersection of graphs of two regular functions on N.

The images under the projection $(z, q) \mapsto q$ of the singular and transversal self-intersection loci of $W(\widehat{L})$ are respectively the caustic Σ_L and so-called Maxwell (conflict) set.

2. To a function $f = f(q)$, $q \in \mathbf{R}^n$, associate its Legendrian lifting $\widehat{L} = j^1(f)$ (also called the 1-jet extension of f) to $J^1\mathbf{R}^n$. Project $\widehat{L}$ along the fibres parallel to the q-space of another Legendrian fibration

$$\pi_1^\wedge(z, p, q) \mapsto (z - pq, p)$$

of the same contact structure $pdq - dz = -qdp - d(z - pq)$. The image $\pi_1^\wedge(\widehat{L})$ is called the Legendre transform of the function f. It has singularities if f is not convex.

This is an affine version of the projective duality (which is also related to Legendrian mappings). The space PT^*P^n (P^n is the projective space) is isomorphic to the projectivised cotangent bundle $PT^*P^{n\wedge}$ of the dual space $P^{n\wedge}$. Elements of both are pairs consisting of a point and a hyperplane, containing the point. The natural contact structures coincide. The set of all hyperplanes in P^n tangent to a submanifold $S \subset P^n$ is the front of the dual projection of the Legendrian lifting of S.

Wave front propagation.

Fix a submanifold $W_0 \subset N$. It defines the (homogeneous) Lagrangian submanifold $L_0 \subset T^*N$ formed by all covectors annihilating tangent spaces to W_0.

Consider now a Hamiltonian function $h : T^*N \to \mathbf{R}$. Let I be the intersection of L_0 with a fixed level hypersurface $H = h^{-1}(c)$. Consider the Lagrangian submanifold $L = exp_H(I) \subset H$ which consists of all the characteristics emanating from I. It is invariant under the flow of H.

The intersections of the Legendrian lifting $\widehat{L}$ of L into J^1N ($z = \int pdq$) with co-ordinate hypersurfaces $z = const$ project to Legendrian submanifolds (varieties) $\widehat{L}_z \subset PT^*N$. In fact, the form pdq vanishes on each tangent vector to $\widehat{L}_z$. In general, the dimension of $\widehat{L}_z$ is $n - 1$.

The wave front of $\widehat{L}$ in J^0N is called the big wave front. It is swept out by the family of fronts W_z of the $\widehat{L}_z$ shifted to the corresponding levels of the z-co-ordinate. Notice that, up to a constant, the value of z at a point over a point (p, q) is equal to $z = \int p\frac{\partial h}{\partial p}dt$ along a segment of the Hamiltonian trajectory going from the initial I to (p, q).

When h is homogeneous of degree k with respect to p in each fibre, then $z_t = kct$. Let $I_t \subset L$ be the image of I under the flow transformation g_t for time t. The projectivised I_t are Legendrian in PT^*N. The family of their fronts in N is $\{W_{kct}\}$. So the W_t are momentary wave fronts propagating from the initial W_0. Their singular loci sweep out the caustic Σ_L.

The case of a time-depending Hamiltonian $h = h(t, p, q)$ reduces to the above by considering the extended phase space $J^1(N \times \mathbf{R})$, $\alpha = pdq - rdt - dz$. The image of the initial Legendrian subvariety $\widehat{L}_0 \subset J^1(N \times \{0\})$ under g_t is a Legendrian $L_t \subset J^1(N \times \{t\})$.

When z can be written locally as a regular function in q, t it satisfies the Hamilton-Jacobi equation $-\frac{\partial z}{\delta t} + h(t, \frac{\partial z}{\partial q}, q) = 0$.

2. Generating families

2.1. *Lagrangian case*

Consider a co-isotropic submanifold $C^{n+k} \subset M^{2n}$. The skew-orthogonal complements $T_c^{\angle}C$, $c \in C$, of tangent spaces to C define an integrable distribution on C. Indeed, take two regular functions whose common zero level set contains C. At each point $c \in C$, the vectors of their Hamiltonian fields belong to $T_c^{\angle}C$. So the corresponding flows commute. Trajectories of all such fields emanating from $c \in C$ form a smooth submanifold I_c integral for the distribution.

By Givental's theorem, any co-isotropic submanifold is locally symplectomorphic to a co-ordinate subspace $p_I = 0$, $I = \{1, \dots, n-k\}$, in $K = \mathbf{R}^{2n}$. The fibres are the sets $q_J = const.$

Proposition. Let L^n and C^{n+k} be respectively Lagrangian and co-isotropic submanifolds of a symplectic manifold M^{2n}. Assume L meets C transversally at a point a. Then the intersection $X_0 = L \bigcap C$ is transversal to the isotropic fibres I_c near a.

The proof is immediate. If T_aX_0 contains a vector $v \in T_aI_c$, then v is skew-orthogonal to T_aL and also to T_aC, that is to any vector in T_aM. Hence $v = 0$.

Isotropic fibres define the fibration $\xi : C \to B$ over a certain manifold B of dimension $2k$ (defined at least locally). We can say that B is the manifold of isotropic fibres.

It has a well-defined induced symplectic structure ω_B. Given any two vectors u, v tangent to B at a point b take their liftings, that is vectors $\widetilde{u}, \widetilde{v}$ tangent to C at some point of $\xi^{-1}(b)$ such that their projections to B are u and v. The value $\omega(\widetilde{u}, \widetilde{v})$ depends only on the vectors u, v. For any other choice of liftings the result will be the same. This value is taken for the value of the two-form ω_B on B.

Thus, the base B gets a symplectic structure which is called a symplectic reduction of the co-isotropic submanifold C.

Example. Consider a Lagrangian section L of the (trivial) Lagrangian fibration $T^*(\mathbf{R}^k \times \mathbf{R}^n)$. The submanifold L is the graph of the differential of a function $f = f(x, q)$, $x \in \mathbf{R}^k$, $q \in \mathbf{R}^n$. The dual coordinates y, p are given on L by $y = \frac{\partial f}{\partial x}$, $p = \frac{\partial f}{\partial q}$. Therefore, the intersection $\widetilde{L}$ of L with the co-isotropic subspace $y = 0$ is given by the equations $\frac{\partial f}{\partial x} = 0$. The intersection is transversal iff the rank of the matrix of the derivatives of these equations, with respect to x and q, is k. If so, the symplectic reduction of $\widetilde{L}$ is a Lagrangian submanifold L_r in $T^*\mathbf{R}^n$ (it may not be a section of $T^*\mathbf{R}^n \to \mathbf{R}^n$).

This example leads to the following definition of a generating function (the idea is due to Hörmander).

Definition. A generating family of the Lagrangian mapping of a submanifold $L \subset T^*N$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ \quad (p, q) \quad \middle| \quad \exists x : \frac{\partial F(x,q)}{\partial x} = 0, \quad p = \frac{\partial F(x,q)}{\partial q} \quad \right\}.$$

Here $q \in N$, and x is in the fibre over q. We also assume that the following Morse condition is satisfied:

$$0 \text{ is a regular value of the mapping } \quad (x, q) \mapsto \frac{\partial F}{\partial x}.$$

The latter guarantees L being a smooth manifold.

Remark. The points of the intersection of L with the zero section of T^*N are in one-to-one correspondence with the critical points of the function F. In symplectic topology, when interested in such points, it is desirable to avoid a possibility of having no critical points at all (as it may happen on a non-compact manifold E).

Therefore, dealing with global generating families defining Lagrangian submanifolds globally, generating families with good behaviour at infinity should be considered.

A generating family F is said to be quadratic at infinity (QI) if it coincides with a fibre-wise quadratic non-degenerate form $Q(x, q)$ outside a compact.

On the topological properties of such families and on their rôle in symplectic topology see the papers by C.Viterbo, for example [4].

Existence and uniqueness (up to a certain equivalence relation) of QI generating families for Lagrangian submanifolds which are Hamiltonian isotopic to the zero section in T^*N of a compact N was proved by Viterbo, Laundeback and Sikorav in the 80s:

Given any two QI generating families for L, there is a unique integer m and a real ℓ such that $H^k(F_b, F_a) = H^{k-m}(F_{b-\ell}, F_{a-\ell})$ for any pair of

$a < b$. Here F_a is the inverse image under F of the ray $\{t \le a\}$.

However, we shall need a local result which is older and easier.

Existence.
Any germ L of a Lagrangian submanifold in $T^*\mathbf{R}^n$ has a regular projection to some (p_J, q_I) co-ordinate space. In this case there exists a function $f = f(p_J, q_I)$ (defined up to a constant) such that

$$L = \left\{ \quad (p,q) \quad | \quad q_J = -\frac{\partial f}{\partial p_J}, \quad p_I = \frac{\partial f}{\partial q_I} \quad \right\}.$$

Then the family $F_J = xq_J + f(x, q_I)$, $x \in \mathbf{R}^{|J|}$, is generating for L. If $|J|$ is minimal possible, then $\mathrm{Hess}_{xx} F_J = \mathrm{Hess}_{p_J p_J} f$ vanishes at the distinguished point.

Uniqueness.
Two family-germs $F_i(x,q)$, $x \in \mathbf{R}^k$, $q \in \mathbf{R}^n$, $i = 1, 2$, at the origin are called R-equivalent if there exists a diffeomorphism $\mathcal{T} : (x,q) \mapsto (X(x,q), q)$ (i.e. preserving the fibration $\mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$) such that $F_2 = F_1 \circ \mathcal{T}$.

The family $\Phi(x,y,q) = F(x,q) \pm y_1^2 \pm \ldots, \pm y_m^2$ is called a stabilisation of F.

Two family-germs are called stably R-equivalent if they are R-equivalent to appropriate stabilisations of the same family (in a lower number of variables).

Lemma. *Up to addition of a constant, any two generating families of the same germ L of a Lagrangian submanifold are stably $\mathcal{R}$-equivalent.*

Proof. Morse Lemma with parameters implies that any function-germ $F(x,q)$ (with zero value at the origin which is taken as the distinguished point) is stably $\mathcal{R}_0$-equivalent to $\widetilde{F}(y,q) \pm z^2$ where $x = (y,z)$ and the matrix $\mathrm{Hess}_{yy}\widetilde{F}|_0$ vanishes. Clearly $\widetilde{F}(y,q)$ is a generating family for L if we assume that $F(x,q)$ is.

Since the matrix $\partial^2\widetilde{F}/\partial y^2$ vanishes at the origin, the Morse condition for $\widetilde{F}$ implies that there exists a subset J of indices such that the minor

$\partial^2\widetilde{F}/\partial y \partial q_J$ is not zero at the origin. Hence the mapping

$$\Theta : (y,q) \mapsto (p_J, q) = (\partial\widetilde{F}/\partial q_J, q)$$

is a local diffeomorphism. The family $G = \widetilde{F} \circ \Theta^{-1}$, $G = G(p_J, q)$, is also a generating family for L.

The variety $\partial\widetilde{F}/\partial y = 0$ in the domain of Θ is mapped to the Lagrangian submanifold L in the (p,q)-space by setting $p = \partial\widetilde{F}/\partial q$ and forgetting y. Therefore, the variety $X = \{\partial G/\partial p_J = 0\}$ in the (p_J, q)-space is the image of L under its (regular) projection $(p,q) \mapsto (p_J, q)$.

Compare now G and the standard generating family F_J defined above (with p_J in the role of x). We may assume their values at the origin coinciding. Then the difference $G - F_J$ has vanishing 1-jet along X. Since X is a regular submanifold, $G - F_J$ is in the square of the ideal $\mathcal{I}$ generated by the equations of X, that is by $\partial F_J/\partial p_J$.

The homotopy method applied to the family $A_t = F_J + t(G - F_J)$, $0 \le t \le 1$, shows that G and F_J are $\mathcal{R}_0$-equivalent. Indeed, it is clear that the homological equation

$$-\frac{\partial A_t}{\partial t} = F_J - G = \frac{\partial A_t}{\partial p_J}\dot{p}_J$$

has a smooth solution $\dot{p}_J$ since $F_J - G \in \mathcal{I}^2$ while the $\partial A_t/\partial p_J$ generate $\mathcal{I}$ for any fixed t. □

2.2. ***Legendrian case***

Definition. A generating family of the Legendrian mapping $\pi|_L$ of a Legendrian submanifold $L \subset PT^*(N)$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ \quad (p,q) \quad | \quad \exists x : \quad F(x,q) = 0, \quad \frac{\partial F(x,q)}{\partial x} = 0, \quad p = \frac{\partial F(x,q)}{\partial q} \quad \right\},$$

where $q \in N$ and x is in the fibre over q, provided that the following Morse condition is satisfied:

$$0 \text{ is a regular value of the mapping } (x,q) \mapsto \{F, \frac{\partial F}{\partial x}\}.$$

Definition. Two function family-germs $F_i(x,q)$, $i = 1, 2$, are called V-equivalent if there exists a fibre-preserving diffeomorphism $\Theta : (x,q) \mapsto (X(x,q), q)$ and a function $\Psi(x,q)$ not vanishing at the distinguished point such that $F_2 \circ \Theta = \Psi F_1$.

Two function families are called stably V-equivalent if they are stabilisations of a pair of V-equivalent functions (may be in a lower number of variables x).

Theorem. *Any germ $\pi|_L$ of a Legendrian mapping has a generating family. All generating families of a fixed germ are stably V-equivalent.*

Proof. For an n-dimensional N, we use the local model $\pi_0 : J^1N' \to J^0N'$, $N' = \mathbf{R}^{n-1}$, for the Legendrian fibration.

Consider the projection $\pi_1 : J^1N' \to T^*N'$ restricted to L. Its image is a Lagrangian germ $L_0 \subset T^*N$. If $F(x,q)$ is a generating family for L_0, then $F(x,q) - z$ considered as a family of functions in x with parameters $(q,z) \in J^0N' = N$ is a generating family for L and vice versa. Now the theorem follows from the Lagrangian result and an obvious property: multiplication of a Legendrian generating family by a function-germ not vanishing at the distinguished point gives a generating family. After multiplication by an appropriate function Ψ, a generating family (satisfying the regularity condition) takes the form $F(x,q) - z$ where (q,z) are local coordinates in N. □

Remarks.
A symplectomorphism φ preserving the bundle structure of the standard Lagrangian fibration $\pi : T^*\mathbf{R}^n \to \mathbf{R}^n$, $(q,p) \mapsto q$ has a very simple form

$$\varphi : (q,p) \mapsto \left(Q(q), DQ^{-1*}(q)(p + df(q))\right) ,$$

where $DQ^{-1*}(q)$ is the dual of the derivative of the inverse mapping of the base of the fibration, $Q \circ \pi = \pi \circ \varphi$, and f is a function on the base.

To see this, it is sufficient to write in the coordinates the equation $\varphi_*\lambda - \lambda = df$.

The above formula shows that fibres of any Lagrangian fibration posses a well-defined affine structure.

Consequently, a contactomorphism ψ of the standard Legendrian fibration $PT^*\mathbf{R}^n \to \mathbf{R}^n$ acts by projective transformations in the fibres:

$$\psi : (q,p) \mapsto (Q(q), DQ^{-1*}(q)p) .$$

Hence, there is a well-defined projective structure on the fibres of any Legendrian fibration.

We also see that Lagrangian equivalences act on generating families as R-equivalences $(x,q) \mapsto (X(x,q), Q(q))$ and additions of function in parameters q.

Legendrian equivalences act on Legendrian generating families just as R-equivalences.

2.3. *Examples of generating families*

The importance of the constructions introduced above for various applications is illustrated by the following examples.

1. Consider a Hamiltonian $h : T^*\mathbf{R}^n \to \mathbf{R}$ which is homogeneous of degree k with respect to the impulses p: $h(\tau p, q) = \tau^k h(p,q)$, $\tau \in \mathbf{R}$.

An initial submanifold $W_0 \subset \mathbf{R}^n$ (initial wave front) defines an exact isotropic $I \subset H_c = h^{-1}(c)$. Assume I is a manifold transversal to v_h. Put $c = 1$.

The exact Lagrangian flow-invariant submanifold $L = exp_h(I)$ is a cylinder over I with local coordinates $\alpha \in I$ and time t from a real segment (on which the flow is defined).

Assume that in a domain $U \subset T^*R^n \times \mathbf{R}$ the restriction to L of the phase flow g_t of v_h is given by the mapping $(\alpha,t) \mapsto (Q(\alpha,t), P(\alpha,t))$ with $\frac{\partial P}{\partial \alpha, t} \neq 0$. Then the following holds.

Proposition. a) The family $F = P(\alpha,t)(q - Q(\alpha,t)) + kt$ of functions in α, t with parameters $q \in \mathbf{R}^n$ is a generating family of L in the domain U.

b) For any fixed t, the family $\widetilde{F}_t = P(\alpha,t)(q - Q(\alpha,t))$ is a Legendrian generating family of the momentary wave front W_t.

The proof is an immediate verification of the Hörmander definition using the fact that value of the form pdq on each vector tangent to $g_t(I)$ vanishes and on the vector v_h it is equal to $p\frac{\partial h}{\partial p} = kh = k$.

2. Let $\varphi : T^*\mathbf{R}^n \to T^*\mathbf{R}^n$, $(q,p) \mapsto (Q,P)$ be a symplectomorphism close to the identity. Thus the system of equations $q' = Q(q,p)$ is solvable for q. Write its solution as $q = \widetilde{q}(q',p)$.

Assume the Lagrangian mapping of a Lagrangian submanifold L has a generating family $F(x,q)$. Then the following family G of functions in x, q, p with parameters q' is a generating family of $\varphi(L)$:

$$G(x,p,q;q') = F(x,\widetilde{q}) + p(\widetilde{q} - q) + S(p,q') .$$

Here $S(q',p)$ is the "generating function" in the sense of Hamiltonian mechanics of the canonical transformation φ, that is

$$dS = PdQ - pdq\,.$$

Notice that, if φ coincides with the identity mapping outside a compact, then G is a quadratic form at infinity with respect to the variables (q,p).

The expression $p(\widetilde{q}-q)+S(p,q')$ from the formula above is the generating family of the symplectomorphism φ.

3. Represent a symplectomorphism φ of $T^*\mathbf{R}^n$ into itself homotopic to the identity as a product of a sequence of symplectomorphisms each of which is close to the identity. Iterating the previous construction, we obtain a generating family of $\varphi(L)$ as a sum of the initial generating family with the generating families of each of these transformations. The number of the variables becomes very large, $\dim(x)+2mn$, where m is the number of the iterations. Namely, consider a partition of the time interval $[0,T]$ into m small segments $[t_i,t_{i+1}]$, $i=0,\ldots,m-1$. Let $\varphi=\varphi_m\circ\varphi_{m-1}\circ\ldots\varphi_1$ where $\varphi_i:(Q_i,P_i)\mapsto(Q_{i+1},P_{i+1})$ is the flow map on the interval $[t_i,t_{i+1}]$. Then the generating family is

$$G(x,Q,P,q)=F(x,Q_0)+\sum_{i=0}^{m-1}\left(P_i(U_i(Q_{i+1},P_i)-Q_i)+S_i(P_i,Q_{i+1})\right)\,,$$

where: $Q=Q_0,\ldots,\ Q_{m-1},\ q=Q_m,\ Q_i\in\mathbf{R}^n,\quad q\in\mathbf{R}^n$, S_i is a generating function of φ_i, and $U_i(Q_{i+1},P_i)$ are the solutions of the system of equations $Q_{i+1}=Q_{i+1}(Q_i,P_1)$ defined by φ_i.

One can show that if φ is a flow map for time $t=1$ of a Hamiltonian function which is convex with respect to the impulses then the generating family G is also convex with respect to the P_i and these variables can be removed by the stabilisation procedure. This provides a generating family of $\varphi(L)$ depending just on x,Q,q for the image of Lagrangian submanifold L admitting a generating family itself. Usually these variables x,Q,q are taken from a compact domain. In this case, the generating family has nice properties. For example, the family attains minimal and maximal values on the fibre over point q. This means, that for any point q from the image of the projection of $\varphi(L)$ among all projections of the Hamilton vector field trajectories emanating at $t=0$ from L and coming to $\pi^{-1}(q)$ at time $t=1$ there are some which provide global minimal value of the action function.

In particular, this implies that going from the initial point along a

generic geodesic the first point where the geodesic segment fails to be minimal is a a conflict point. At generic (regular) point of the caustic the generating family does not have minimum value.

3. Applications

3.1. *Singularities of wave fronts and caustics*

Famous results of Arnold and Thom relating stable singularities of low-dimensional wave fronts to the discriminants of the Weyl groups are based on relation between caustics and wave fronts and discriminants and bifurcation diagrams of families of functions depending on parameters.

Caustics

Singularities of Lagrangian projections are essentially the singularities of their generating families treated as families of functions depending on parameters. In particular, the caustic $\Sigma(L)$ of Lagrangian submanifold L projection coincides with the stratum of the bifurcation diagram of the generating family $f(x,q)$ which is the collection of parameter q values such that the restriction $f(\cdot,q)$ has a non-Morse critical point.

Stability of Lagrangian projection with respect to symplectomorphisms preserving the fibration structure corresponds to the versality of the generating family with respect to the R_+- equivalence group (diffeomorphisms of the source space and additions of the function with constants).

In the space of germs of functions in k variables there are only finitely many orbits of codimension $k+n$ with $n\le 5$ of the R_+-equivalence group [[3]]. Those are the orbits of simple A,D,E singularity classes. This fact implies the following

Theorem. *Let $E(L)$ be the space of Lagrangian embeddings of a compact manifold L (of the dimension $n\le 5$) into a Lagrangian fibration space, equipped with C^∞ topology. Then a Lagrangian projection $\pi\circ i$ for an embedding i from an open and dense subset of $E(L)$ at any point is equivalent to a Lagrangian projection determined by the germ at the origin of some of the following standard versal deformations of simple singularities of functions* *with* $m \leq n+1$:

$$A_m: \quad F = \pm x^{m+1} + q_1 x^{m-1} + \cdots + q_{m-1}x;$$
$$D_m: \quad F = x_1^2 x_2 \pm x_2^{m-1} + q_1 x_2^{m-2} + \cdots + q_{m-2}x_2 + q_{m-1}x_1;$$
$$E_6: \quad F = x_1^3 \pm x_2^4 + q_1 x_1 x_2^2 + q_2 x_1 x_2 + q_3 x_2^2 + q_4 x_1 + q_5 x_2.$$

Remarks.

1. In particular, the caustics of generic local Lagrangian projections to 3-space are diffeomorphic of the caustics of A_2(smooth surface), A_3 (cuspidal ridge), A_4 (swallowtail), $D_4^{\pm}$ (purse or pyramid). Germs of generic caustics can have several several components of these types which are mutually transversal.

2. Starting from $n = 6$ some R_+ orbits have continuous invariants (moduli). Therefore, respective Lagrangian projections have invariants which are functional moduli (invariants depending on parameters. However even in this cases generating families provide some useful information of topological structure of caustics.

3. For the dimensions $n \geq 3$ the list of generic singularities of Lagrangian projections differs from the list of singularities of arbitrary mappings of spaces of equal dimensions. Lagrangian mapping are special. However, they arise in many physical and geometrical problems. For example, Gauss map is Lagrangian. Envelope of geodesics emanating from an initial point on a Riemannian manifold is the caustic of so-called exponential Lagrangian mapping. The intensity of light at caustic points of a family of optical rays increases. The asymptotics of the intensity given by the oscillation integral was studied by A.Varchenko, P.Pham and others. It is related to the spectrum and mixed Hodge structure of respective function singularity [[1]].

4. Some specific applied problems involve non-generic Lagrangian singularities. They can be symmetric, or even determined by a non-smooth Lagrangian varieties projections. The study of the correponding generating families require special singularity theory techniques (equivariant mappings, non-isolated singularities, etc.). A recent example of a caustic related to a generating family with non-isolated singularities is given by the exponential mapping on subriemannian 3-space with contact distribution [[5]].

Wave fronts

The wave front of a local Legendrian projection is the discriminant $D(F)$ of its generating family $F(x, q)$ that is the set of parameters q such that

the zero level set of $F(cdot, q)$ contains a critical point.

Legendre equivalent Legendre projections have V-equivalent generating families and diffeomorphic wave fronts. Under some mild conditions, the converse holds also [[6]].

A Legendre submanifold germ L embedded into a Legendre fibration is called regular if the regular points of the Legendre projection are dense in L and the projection is proper.

Proposition. *If the front of a Legendre submanifold germ* $\widetilde{L}$ *coincides with the front of the regular germ* L, *then* $\widetilde{L}$ *coincides with* L.

The proof follows from the fact that near the regular point (p, q) the Legendre submanifold $L \subset PT^*M$ coincides with the set of contact elements p annihilating tangent vectors to wavefront $\pi(L)$. The entire Legendre submanifold is the closure of its regular points.

So, regular Legendrian submanifolds having diffeomorphic wavefronts are Legendre equivalent and the respective generating families are V-equivalent. Hence the classification of the generic singularities of Legendre projections is essentially the classification (up to diffeomorphisms) of generic singularities of wave fronts. Notice that in the Lagrangian case the similar statement is false.

Theorem. *Let* $\widehat{E}(L)$ *be the space of Lagendrian embeddings of a compact manifold* L *(of the dimension* $n \leq 5$*) into a Lagrangian fibration space, equipped with* C^∞ *topology. Then a Legendrian projection* $\pi \circ i$ *for an embedding* i *from an open and dense subset of* $\widehat{E}(L)$ *at any point is equivalent to a Legendrian projection determined by the germ at the origin of some of the following standard* V*-versal deformations of simple singularities of functions with* $m \leq n+1$:

$$A_m: \quad F = \pm x^{m+1} + q_1 x^{m-1} + \cdots + q_{m-1}x + q_m;$$
$$D_m: \quad F = x_1^2 x_2 \pm x_2^{m-1} + q_1 x_2^{m-2} + \cdots + q_{m-2}x_2 + q_{m-1}x_1 + q_m;$$
$$E_6: \quad F = x_1^3 \pm x_2^4 + q_1 x_1 x_2^2 + q_2 x_1 x_2 + q_3 x_2^2 + q_4 x_1 + q_5 x_2 + q_m.$$

Generic wave fronts germs in 3-space are diffeomorphic either to swallowtails (A_4) or to collection of mutually transversal smooth surfaces (A_2) and cuspidal ridges (A_3).

3.2. *Metamorphosis of wave front*

Consider a germ at the origin of the R-versal deformation depending on parameters q

$$F(x,q) = f(x) + \sum_{i=1}^{n} q_i\varphi_i(x) \qquad (1)$$

of the polynomial $f(x)$ having at the origin a critical point of multiplicity $\mu \leq n$.

Assume the germs $\varphi_i, \quad i = 1,\ldots,\mu$ form a basis of local gradient factor algebra

$$Q_f = C^\infty(x)/C^\infty(x)\left\{\frac{\partial f}{\partial x}\right\},$$

of the algebra $C^\infty(x)$ of germs at the origin of smooth functions in x. Assume that $\varphi_{]m+1},\ldots,\varphi_n$ are equal to zero.

Proposition.(see[[6]]) *The real-analytic vector fields which are tangent to the wave front of the Legendrian projection germ determined by the generating family F form a free module over the ring of germs of functions in q with n generators.*

In other words the wave fronts of R-versal families of functions are Saito's free divisors. A distinguished system of generators $V_1,\ldots,V_n$ are easy to describe. For any $\varphi_j, \quad j = 1,\ldots,\mu$ consider the decomposition

$$-F(x,q)\varphi_j(x) = \sum_{i=1}^{\mu} e_{i,j}\varphi_i(x)\mathrm{mod}(C^\infty(x,q)\{\frac{\partial F(x,q)}{\partial x}\}).$$

Then vector fields

$$V_j = \sum_{i=1}^{\mu} e_{ij}(q)\frac{\partial}{\partial q_i} \quad j = 1,\ldots,\mu \text{ and } V_j = \frac{\partial}{\partial q_j}, \text{ for } j = \mu+1,\ldots,n$$

form the basis of tangent (logarithmic) vector fields. An easy proof of this is bases just on the wave front property mentioned above. An one-parameter group of diffeomotphisms mapping the wave front to itself correspond to a family of V-equivalences of the family F with itself. The infinitesimal version of the latter condition provides the decomposition being the linear combination of the decompositions for V_j.

Let L_t be a family of Legendre submanifolds germs in PT^*M smoothly depending on $t \in \mathbf{R}$. We can choose generating families $F(x,q,t)$ of L_t also smoothly depending on t. Then $F(x,q,t)$ considered as a family of functions in x with parameters q,t is a generating family for a (big) Legendre submanifold in $PT^*(M \times R)$.

Hence, to reduce the family of Legendrian projections L_t to a normal form we can reduce the big Legendre projection to a normal form, and then using diffeomorphisms preserving the big front normalize the fibration of the extended configuration space q,t by level hypersurfaces of the function t.

Let the big generating family be R–stable (this holds generically in small dimensions) and be equivalent to the family (1). Take a distinguished generator φ_μ from the annihilator of the maximal ideal of the algebra Q_f. Applying logarithmic vector fields to a generic function t on the parameter space, the normal forms of a generic function are either q_j, for $j > \mu$ or $\pm q_* + \sum_{j=\mu+1}^{n} \pm q_j^2$.

These formulas describe singularities of moving wave front transformations. In variational problems the family of wave fronts are given by the level sets of the action function. The distance function $f(x,q)$ between point x from certain initial variety X_0 and a point q in the ambient space determines a family of equidistants of X_0 whose generic metamorphosis are also described by these normal forms.

3.3. *Affine generating families*

An example of wave front propagation different from the Riemannian distance function is provided by the generating families related to the systems of chords described in [[7]].

Let M, a_0 and N, b_0 be two germs at points a_0 and b_0 of smooth hypersurfaces in an affine space $\mathbf{R}^n$. Let $\mathbf{r}_i : U_i^{n-1} \to \mathbf{R}^n \quad i = 1,2$ be local regular parametrizations of M and N, where U_i are viscinities of the origin in $\mathbf{R}^{n-1}$ with local coordinates u and v respectively, $\mathbf{r}_1(0) = a_0, \ \mathbf{r}_2(0) = b_0$.

A parallel pair is a pair of points $a \in M, \ b \in N, \ a \neq b$ such that the hyperplane T_aM which is tangent to M at a is parallel to the tangent hyperplane T_bN.

Suppose the distinguished pair a_0, b_0 is a parallel one. A *chord* is the straight line $l(a,b)$ passing through a parallel pair: $l(a,b) = \{q \in \mathbf{R}^n \mid \ q = \lambda a + \mu b,\ \lambda \in \mathbf{R},\ \mu \in \mathbf{R},\ \lambda + \mu = 1\}$.

An affine (λ,μ)–equidistant E_λ of the couple (M,N) is the set of all $q \in \mathbf{R}^n$ such that $q = \lambda a + \mu b$ for given $\lambda \in \mathbf{R},\ \mu \in \mathbf{R},\ \lambda + \mu = 1$ and all parallel pairs a, b (close to a_0, b_0).

The *extended* affine space is the space $\mathbf{R}_e^{n+1} = \mathbf{R} \times \mathbf{R}^n$ with baricentric cooordinate $\lambda \in \mathbf{R},\ \mu \in \mathbf{R},\ \lambda{+}\mu = 1$ on the first factor (called affine time).

Denote by $pr : w = (\lambda, q) \mapsto q$ the projection of $\mathbf{R}_e^{n+1}$ to the second factor.

An affine extended wave front $W(M,N)$ of the couple (M,N) is the union of all affine equidistants each embedded into its own slice of the extended affine space: $W(M,N) = \{(\lambda, E_\lambda)\} \subset \mathbf{R}_e^{n+1}$.

The bifurcation set $\mathrm{Bif}(M,N)$ of a family of affine equidistants (or of the family of chords) of the couple M,N is the image under pr of the locus of the critical points of the restriction $pr_r = pr\,|_{W(M,N)}$ A point is critical if pr_r at this point fails to be a regular projection of a smooth submanifold.

In general $\mathrm{Bif}(M,N)$ consists of two components: the caustic Σ being the projection of singular locus of extended wave front $W(M,N)$ and the envelope Δ being the (closure of) the image under pr_r of the set of regular points of $W(M,N)$ which are the critical points of the projection pr restricted to the regular part of $W(M,N)$.

The caustic consists of the singular points of momentary equidistants E_λ while the envelope is the envelope of family of regular parts of momentary equidistants.

On the other hand the affine wave front is swept out by the liftings to $\mathbf{R}_e^{n+1}$ of chords. Each of them has regular projection to configuration space $\mathbf{R}^n$. Hence the bifurcation set $B(M,N)$ is essentially the envelope of the family of chords.

A germ of a family $F(x,w)$ of functions in $x \in \mathbf{R}^k$ with parameters $w = (t,q) \in \mathbf{R}_e^{n+1}$ where $t \in \mathbf{R}$ and $q \in \mathbf{R}^n$ determines the following collection of varieties:

The fiberwise critical set is the set $\mathbf{C}_F \subset \mathbf{R}^k \times \mathbf{R} \times \mathbf{R}^n$ of the solutions

(x,w) of so-callled Legendre equations:

$$F(x,w) = 0, \qquad \frac{\partial F}{\partial x} = 0.$$

The wave front (discriminant) is $W(F) = \{(t,q) \mid \exists x : (x,w) \in \mathbf{C}_F\}$.

The intersections of (big) wave front with $t = \mathrm{const}$ subspaces are called *momentary* wave fronts $W_t(F)$.

The bifurcations set $\mathrm{Bif}(F)$ is the image under the projection $pr : (t,q) \mapsto q$ of the points of $W(F)$ where the restriction $pr\,|_{W(F)}$ fails to be a regular projection of a smooth submanifold. Projections of singular points of $W(F)$ form the caustic $\Sigma(F)$, and singular projections of regular points of $W(F)$ determines the envelope or criminant $\Delta(F)$.

Family F is generating family of a Legendre subvariety $\widehat{L}(F) \subset PT^*(\mathbf{R}^{n+1})$ which is smooth provided that the Legendre equations are locally regular, i.e standard Morse conditions are fulfilled [[1]].

Two germs of families $F_i \quad i = 1,2$ are called *space-time-contact–*equivalent ("v" - for short) if there exist a non-zero function $\phi(x,t,q)$ and a diffeomorphism $\theta : \mathbf{R}^k \times \mathbf{R}^{n+1} \to \mathbf{R}^k \times \mathbf{R}^{n+1}$, of the form

$$\theta : (x,t,q) \mapsto (X(x,t,q), T(t,q), Q(q))$$

such that $\phi F_1 = F_2 \circ \theta$.

The sum of the family $F(x,t,q)$ with a non-degenerate quadratic form in extra variables $y_1, \dots, y_m$ is called a stabilization of F. Two germs of families are *v-stable* equivalent if they are v-equivalent to stabilizations of one and the same family in fewer variables.

The bifurcation diagrams of v-stable equivalent families are diffeomorphic. Theory of singularities of functions with respect to this equivalence group see in [[8,9]].

The critical points of the projection $pr\,|_{\mathbf{C}_F}$ satisfy the equation:

$$\det \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial t} \\ \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial t \partial x} \end{vmatrix} = 0.$$

Since the first k entries $\frac{\partial F}{\partial x}$ of the first row vanish, the determinant factorises. Hence the bifurcation diagram $B(F)$ splits into two components. One of them (which is the *criminant* $\Delta(F)$) is the image of the projection $(x,t,q) \mapsto q$ of the subvariety $\mathbf{C}_d \subset \mathbf{C}$ determined by the equation $\frac{\partial F}{\partial t} = 0$. The other one (which is the *caustic* $\Sigma(F)$) is the image of the projection $(x,t,q) \mapsto q$ of the subvariety $\mathbf{C}_s \subset \mathbf{C}$ determined by the equation $det\left(\frac{\partial^2 F}{\partial x^2}\right) = 0$.

The following version of Hyugens principle holds: the criminant (envelope) coincides with the wave front of F, considered as a family in variables x and t with parameters q only.

Definition. An affine generating family $\mathcal{F}$ of a pair M, N is a family of functions in $u, v, p \in U_1 \times U_2 \times ((\mathbf{R}^n)^\wedge \setminus \{0\}, 0$ with parameters $\lambda, q \in \mathbf{R} \times \mathbf{R}^n$ of the form

$$\mathcal{F}(u,v,p) = \lambda < \mathbf{r}_1(u) - q, p > + \mu < \mathbf{r}_2(v) - q, p > .$$

Here $\lambda, \mu = 1 - \lambda$ are baricentric cooordinates on $\mathbf{R}$, and $<,>$ is the standard pairing of vectors from $\mathbf{R}^n$ and covectors p from the dual space $(\mathbf{R}^n)^\wedge$.

Proposition. *The germ at a point* $q_0 = \lambda_0 a_0 + \mu_0 b_0$ *of affine equidistants generated by a pair* (M, a_0), (N, b_0) *coincides with the family of momentary wave fronts generated by the germ* $\mathcal{F}$ *at the point* $x = 0$, $y = 0$, $[p] = [d\mathbf{r}_1|_{a_0}] = [d\mathbf{r}_2|_{b_0}]$. *The wave front* $W_{\mathcal{F}}$ *coincides with the affine extended wavefront* $W(M,N)$. *Bifurcation diagram* $\mathrm{Bif}(\mathcal{F})$ *coincides with the set* $B(M,N)$.

The classification of germs of functions $f(x,t)$ with zero one-jet with respect to stable v-equivalence (without parameters) starts with the orbits [[8,9]] $(x \in \mathbf{R})$:

$$B_k: \quad \pm x^2 + t^k; \qquad C_k: \quad x^k + tz; \quad k = 2,3,4 \qquad F_4 : x^3 + t^2.$$

The complement to them has codimension 4. Their miniversal deformations in parameters $q \in \mathbf{R}^3$ are as follows:

$$B_k: \quad \pm x^2 + t^k + q_{k-1}t^{k-2} + \cdots + q_1;$$

$$C_k: \quad x^k + xt + q_{k-1}x^{k-2} + \cdots + q_3 x^2 + q_2 t + q_1;$$

$$F_4 : x^3 + t^2 + q_3 xt + q_1 x + q_2 t + q_1.$$

Introduce a non-generic singularity class (related to D.Mond classification of mappings from plane to space):

$$\tilde{C}_4 : F = q_1 + t(q_2 + t + x_1 q_3 + x_1^3 + x_2^2).$$

The following results were proven in [[7]].

Theorem.(Transversal case) *If* $n \leq 5$ *and the intial chord* (a,b) *is not parallel to* $T_a M$ *then there is an open dense subset of the space of germs of hypersurfaces* M *and* N *such that at any point the criminant is void and caustic is diffeomorphic to that of some of simple singularities* A_m, D_m, E_m *provided that* $m \leq n+1$.

Theorem. (Tangential case) *If* $n = 3$ *and the intial chord* (a,b) *is parallel to* $T_a M$ *then there is an open dense subset of the space of germs of surfaces* M *and* N *such that the criminant coinsides with the ruled surface swept by bitangent chords, and the bifurcation set* $B(M,N)$ *germs at any point of a bitangent chord is diffeomorphic to the bifurcation diagram of some of simple classes* B_k, C_k, $k = 2,3,4$, F_4 *or of the exceptional class* $\tilde{C}_4$.

References

1. V. I. Arnold, *Singularities of caustics and wave fronts*, Kluwer Academic Publ., Dordrecht-Boston-London, 1990.
2. V. I. Arnold, V. V. Goryunov, O. V. Lyashko and V. A. Vassiliev, *Singularities II. Classification and Applications*, Encyclopaedia of Mathematical Sciences, vol.39, Dynamical Systems VIII, Springer Verlag, Berlin a.o., 1993.
3. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable maps. Vol. I*, Monographs in Mathematics **82**, Birkhäuser, Boston, 1985.
4. C. Viterbo, *Generating functions, symplectic geometry and applications*, Proc. Intern. Congr. Math., Zurich, 1994.
5. A. A. Agrachev, G. Charlot, J. P. Gauthier, V. M. Zakalyukin, *On subriemannian caustics and wave fronts for contact distributions in three-space*, Journal of Dynamical and Control Systems, **6** (2000) n.3 365-395
6. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on parameters, versality of mappings*, Journal of Soviet Mathematics **27** (1984) 2785-2811.
7. P. J. Giblin, V. M. Zakalyukin, *Singularities of centre symmetry sets*, Proc. London Math. Soc. **90** 2005 n.3 132-166.

8. V. V. Goryunov, *Projections of generic surfaces with boundary*, Theory of singularities and its applications (ed. V.I.Arnold), Advances in Soviet Mathematics 1, AMS, Providence, RI, 1990 157-200.
9. V. M. Zakalyukin, *Envelopes of families of wave fronts and control theory*, Proc. Steklov Inst. Math. **209** (1995) 133-142