Projection of 0-dimensional complete intersections onto a line and the $k(\pi,1)$-conjecture

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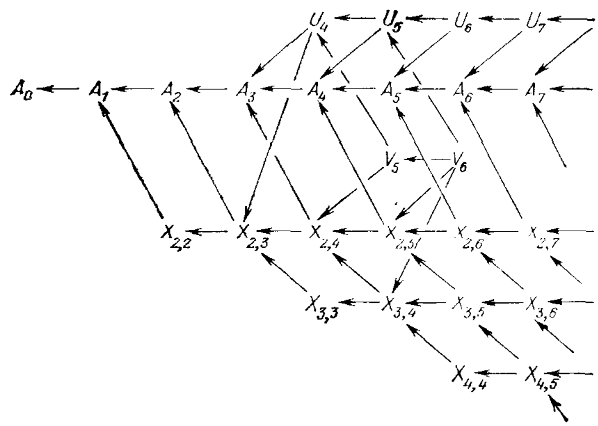
The concept of the projection of a surface from a fibre space to its base was introduced in [3], and the equivalence of projections was defined in the natural manner. In this note we provide a list of simple projections of 0-dimensional complete intersections from $\mathbf{C}^2$ to $\mathbf{C}^1$, and we prove that the germs of complements to the bifurcation diagrams of some of these are Eilenberg-MacLane spaces.

1. ***Theorem*** **1.** *The germ of a projection of a 0-dimensional complete intersection from* $(\mathbf{C}^2, 0)$ *to* $(\mathbf{C}^1, 0)$ *is simple if and only if it is equivalent to the germ at zero of the projection* $(x, u) \mapsto u$ *of one of the complete intersections* $f = (f_1, f_2) = 0$ *indicated in the table.*

In the table, ν is the dimension of the base of the miniversal deformation of the projection (see [3]), and N is the multiplicity of the point $\{0\} = \{f^{-1}(0)\}$.

Notation	f	ν	N
A_μ, $\mu \geqslant 0$	$(x, u^{\mu+1})$	μ	$\mu+1$
$X_{k,l}$, $2 \leqslant k \leqslant l$	$(x^k + u, x^l)$	$k+l-2$	l
U_{n+2}, $n \geqslant 2$	$(x^2 + u^n, ux)$	$n+2$	$n+2$
V_5	(x^2, u^2)	5	4
V_6	$(x^3 + u^2, ux)$	6	5

The full contiguity diagram of the projections in Theorem 1 is as follows:



2. Let F: $(\mathbf{C}^2 \times \mathbf{C}^\nu, 0) \to (\mathbf{C}^2, 0)$ be a miniversal deformation of the germ of the projection $(x, u) \mapsto u$ of the complete intersection $f = 0$, and let $\lambda \in \mathbf{C}^\nu$ be the deformation parameter. We denote by $\tilde{F}$ a representative of F defined for $|x| < \varepsilon$, $|u| < \varepsilon$, $|\lambda| < \rho$, $\varepsilon \ll \rho \ll 1$. We set $M_\lambda = (\tilde{F}|_{\lambda=\mathrm{const}})^{-1}(0)$.

Definition. The germ at zero of the set of those values of the parameter λ for which $\Pi(M_\lambda)$ consists of fewer than N points is called the *bifurcation diagram* $\Sigma \subset \mathbf{C}^\nu$ of the germ *of the projection* of the zero-dimensional complete intersection $f^{-1}(0)$ from $(\mathbf{C}^2, 0)$ to $(\mathbf{C}^1, 0)$, $(x, u) \overset{\Pi}{\mapsto} u$.

A bifurcation diagram has two components: $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 corresponds to the appearance of multiple points in M_λ, and Σ_2 to the fact that distinct points on M_λ have the same u-coordinate (this component is empty for A_μ).

Conjecture. *Let Σ be the bifurcation diagram of the germ of a simple 0-dimensional complete intersection from $(\mathbf{C}^2, 0)$ to $(\mathbf{C}^1, 0)$. Then the germ at zero of the space $C^\nu \setminus \Sigma$ is a $k(\pi, 1)$-space.*

3. ***Theorem* 2.** *The conjecture is true for the projections A_μ, $\mu \geqslant 0$; $X_{k,k}$, $k \geqslant 2$; $X_{2,l}$, $l \geqslant 3$; U_{n+2}, $n \geqslant 2$.*

Proof. All the projections in Theorem 1 have quasihomogeneous miniversal deformations with positive weights in the deformation parameters $\lambda_1, \ldots, \lambda_\nu$. For such deformations the diagram $\Sigma^\nu \setminus \mathbf{C}^\nu$ is defined globally, and the assertion of the conjecture is equivalent to that for the whole space $\mathbf{C}^\nu \setminus \Sigma$.

A_μ.

The theorem follows from the fact that the definitions of the bifurcation diagram of the projection of A_μ and of that at zero of a function with a critical point A_μ coincide [1].

$X_{k,k}$, $\nu = 2k - 2$.

Miniversal deformation:

$$(u + \lambda_1 x + \ldots + \lambda_{k-1}x^{k-1}, \quad \lambda_k + \lambda_{k+1}x + \ldots + \lambda_{2k-2}x^{k-2} + x^k).$$

We set $\lambda' = (\lambda_1, \ldots, \lambda_{k-1})$, $\lambda'' = (\lambda_k, \ldots, \lambda_{2k-2})$; let $W \subset \mathbf{C}^{k-1}$ be the complexification of the system of mirrors of the Weyl group A_{k-1}, $\Delta \subset \mathbf{C}^{k-1}$ the bifurcation diagram at zero of the function with critical point A_{k-1}. It is not difficult to show that the map $\mathbf{C}^{2k-2} \setminus \Sigma \to \mathbf{C}^{k-1} \setminus \Delta$, $(\lambda', \lambda'') \mapsto \lambda''$ gives a locally trivial fibration with fibre $\mathbf{C}^{k-1} \setminus W$. Thus, $\mathbf{C}^{2k-2} \setminus \Sigma$ is a $k(\pi, 1)$-space.

$X_{2,l}$, $\nu = l$.

Miniversal deformation:

$$(x^2 + u, \; x^l + \lambda_l x^{l-1} + \ldots + \lambda_2 x + \lambda_1).$$

Let $\mathbf{C}^l$ be the space with coordinates $x_1, \ldots, x_l$, $Y \subset \mathbf{C}^l$ the system of mirrors of the group D_l; let $A_{l-1} \subset D_l$ be the subgroup generated by the reflections in the diagonals $x_i = x_j$, $1 \leqslant i < j \leqslant l$. For the singularity $X_{2,l}$, $\mathbf{C}^\nu \setminus \Sigma$ is the space $\mathbf{C}^l \setminus Y$, factored by the action of A_{l-1}. Hence, $\mathbf{C}^l \setminus \Sigma$ is a $k(\pi, 1)$-space, where π is a subgroup of index 2^{l-1} in the group BD_l of generalized braids [2].

U_{n+2}, $\nu = n + 2$.

Miniversal deformation:

$$(u^n + \lambda_1 u^{n-1} + \ldots + \lambda_{n-1}u + \lambda_n + \lambda_{n+1}x + x^2, \; ux - \lambda_{n+2}).$$

$\Sigma = \{\lambda \subset \mathbf{C}^{n+2} \mid$ the polynomial $u^{n+2} + \lambda_1 u^{n+1} + \ldots + \lambda_{n-1}u^3 + \lambda_n u^2 + \lambda_{n+1}\lambda_{n+2}u + \lambda_{n+2}^2$ has a multiple root$\}$. It is easy to see that $\mathbf{C}^{n+2} \setminus \Sigma$ is a regular double covering of $\mathbf{C}^{n+}$ … where $Z = \{(\Lambda_1, \ldots, \Lambda_{n+2}) \in \mathbf{C}^{n+2} \mid$ the polynomial $v^{n+2} + \Lambda_1 v^{n+1} + \ldots + \Lambda_{n-1}v^3 + \Lambda_n v^2 + \Lambda_{n+1}v + \Lambda_{n+2}$ has a multiple or a zero root$\}$. Hence, $\mathbf{C}^{n+2} \setminus \Sigma$ is a $k(\pi, 1)$-space, where π is a subgroup of index 2 in the group BC_{n+2} of generalized braids [2].

Remarks. a) For the singularity U_{n+2}, Σ_1 is the bifurcation diagram at zero of the complete intersection $(x^2 + u^n, ux)$, $\Sigma_2 = \{\lambda_{n+2} = 0\}$. As Knörrer has shown in [4], for $n = 2$ the s… $\mathbf{C}^4 \setminus \Sigma_1$ has a non-trivial group π_3.

b) It is not known whether the $k(\pi, 1)$-conjecture is true for the projections $X_{k,l}$, $3 \leqslant k <$ and V_6.

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