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THE POINCARÉ POLYNOMIAL OF THE SPACE OF FORM-RESIDUES ON A QUASI-HOMOGENEOUS COMPLETE INTERSECTION

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Suppose that quasihomogeneous functions $f_1, \ldots, f_n \in \mathbf{C}[z_1, \ldots, z_m]$, $n \leqslant m$, give in $\mathbf{C}^m$ a complete intersection X_0 with an isolated singularity at the origin, and let $\deg z_i = A_i$, $\deg f_j = D_j (i = 1, \ldots, m;$ $j = 1, \ldots, n)$. Then (see [1]), the non-singular fibre X_ε of the map $f = (f_1, \ldots, f_n)$: $(\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ has the homotopy type of a bouquet of μ spheres of dimension $m - n$, and it was proved in [2] that $\mu = \dim_{\mathbf{C}} \mathcal{O}_m/I$ where $\mathcal{O}_m = \mathbf{C}\{z_1, \ldots, z_m\}$, and I is the ideal generated by the coordinate functions and all n-minors of the Jacobian matrix of f. We represent an arbitrary element in $\widetilde{H}^{m-n}(X_\varepsilon; \mathbf{C})$ as a form-residue

$$q\, dz_1 \wedge \ldots \wedge dz_m / df_1 \wedge \ldots \wedge df_n, \quad q \in \mathcal{O}_m/I.$$

We set $Q = \mathcal{O}_m/I$, $\mathcal{O}_n = \mathbf{C}\{y_1, \ldots, y_n\}$, $B = \sum_{j=1}^{n} D_j - \sum_{i=1}^{m} A_i$ and denote by $p_R(t)$ the Poincaré polynomial of the graded ring R.

THEOREM.

$$p_Q(t) = \operatorname{res}_{s=0} \frac{s^{n-m}}{1+s} \left[s^{m-n-1} \prod_{j=1}^{n} (1 - t^{D_j}) \Big/ \prod_{i=1}^{m} (1 - t^{A_i}) - \right.$$
$$\left. - t^B \prod_{i=1}^{m} \frac{1 + s t^{A_i}}{1 - t^{A_i}} \prod_{j=1}^{n} \frac{1 - t^{D_j}}{1 + s t^{D_j}} + s^{-2} t^B \right].$$

PROOF. For $n = m$ the formula is obvious. We consider the case $n < m$. Let $\{\Omega^p, d\}$, $p \geqslant 0$, be the complex of germs at zero of holomorphic differential forms on $\mathbf{C}^m$. We write Ω_f for the factor-complex $\Omega_f^p = \Omega^p / df_1 \wedge \Omega^{p-1} + \ldots + df_n \wedge \Omega^{p-1}$.

It is proved in [2] that the sequence

$$0 \to f^* \mathcal{O}_n \hookrightarrow \Omega_f^0 \xrightarrow{d} \Omega_f^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega_f^{m-n-1} \xrightarrow{d} d\Omega_f^{m-n-1} \to 0$$

is exact. Considering the grading $\deg z_i = \deg dz_i = A_i (i = 1, \ldots, m)$ and using the fact that

$$p_{\Omega_f^p}(t) = \operatorname{res}_{s=0} s^{-p-1} \prod_{i=1}^{m} \frac{1 + s t^{A_i}}{1 - t^{A_i}} \prod_{j=1}^{n} (1 + s t^{D_j})^{-1},$$

$$p_{f^* \mathcal{O}_n}(t) = \prod_{j=1}^{n} (1 - t^{D_j})^{-1} \quad \text{(see [3])},$$

we obtain

$$p_{d\Omega_f^{m-n-1}}(t) = \operatorname{res}_{s=0} s^{-m+n} \sum_{k=0}^{m-n-1} (-s)^k \prod_{i=1}^{m} \frac{1 + s t^{A_i}}{1 - t^{A_i}} \prod_{j=1}^{n} (1 + s t^{D_j})^{-1} +$$
$$+ (-1)^{m-n} \prod_{j=1}^{n} (1 - t^{D_j})^{-1} = \operatorname{res}_{s=0} \frac{s^{n-m}}{1+s} \left[\prod_{i=1}^{m} \frac{1 + s t^{A_i}}{1 - t^{A_i}} \prod_{j=1}^{n} (1 + s t^{D_j})^{-1} - \right.$$
$$\left. - s^{-2} \prod_{j=1}^{n} (1 - t^{D_j})^{-1} \right],$$

since
$$\sum_{k=0}^{m-n-1} (-s)^k \equiv \frac{1}{1+s} \bmod (s^{m-n}) \text{ and } (-1)^{m-n} = -\operatorname{res}_{s=0} \frac{s^{n-m-2}}{1+s}.$$

We consider now the map from $d\Omega_f^{m-n-1}$ to Ω^m of multiplication by $df_1 \wedge \ldots \wedge df_n$ and, on dividing the holomorphic form of highest degree in $\mathbf{C}^m$ by $dz_1 \wedge \ldots \wedge dz_m$, we identify Ω^m with $\mathcal{O}_m$. Then we obtain a map $df\colon d\Omega_f^{m-n-1} \to \mathcal{O}_m$ of degree B. It is proved in [2] that df is an embedding and that $\mathcal{O}_m/\operatorname{Im} df$ is a free $f^*\mathcal{O}_n$-module of rank μ. Thus, if $M = \mathcal{O}_m/\left(\operatorname{Im} df + \sum_{j=1}^{n} f_j \mathcal{O}_m\right)$, then

$$p_M(t) = (p_{\mathcal{O}_m}(t) - t^B p_{d\Omega_f^{m-n-1}}(t)) \prod_{j=1}^{n} (1 - t^{D_j}),$$

which coincides exactly with the expression for $p_Q(t)$ in the statement of the theorem. But since it is clear that

$$I \supset \operatorname{Im} df + \sum_{j=1}^{n} f_j \mathcal{O}_m, \qquad \dim M = \dim Q = \mu,$$

we have

$$I = \operatorname{Im} df + \sum_{j=1}^{n} f_j \mathcal{O}_m \text{ and } M = Q.$$

COROLLARY. *The polynomial $p_Q(t)$ is reciprocal for $n = 1$ and $n = m - 1$.*

In the remaining cases the polynomial is not reciprocal, in general.

We remark also that if $n = m - 1$,

$$J_i = \det \frac{\partial (f_1, \ldots, f_{m-1})}{\partial (z_1, \ldots, \hat{z}_i, \ldots, z_m)} \quad \text{and} \quad J_{ij} = \frac{\partial}{\partial z_j} J_i,$$

then the element of highest weight in Q has the form

$$\sum_{i,\, j=1}^{m} (-1)^{i+j} (J_{ii}J_{jj} - J_{ij}J_{ji}).$$

References

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