SINGULARITIES OF PROJECTIONS

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Projections appear naturally in problems on dependence of different mathematical objects on parameters. For example, in the bifurcation theory of equilibrium points of differential equations, it is important to study the projection of the variety of these points, situated in the product of the parameter and phase space, to the parameters. Singularities of this projection correspond to bifurcations of the equilibria.

At first glance, the set of maps that the theory of projections operates with is smaller than the one for the general theory of mappings. The equivalence relation is also more restrictive: *two projections* of varieties from the spaces of fibrations onto their bases should be considered *equivalent* if it is possible to send one of these varieties to the other by a diffeomorphism of the ambient spaces which transforms fibres of the first fibration to fibres of the second one. On the germ level this corresponds to the subgroup of the contact group respecting distinguished parameters.

But, on the other hand, one can treat a mapping of smooth manifolds $N \to P$ as a projection of its graph from the product $N \times P$ onto the base P. The classification of such projections is the same as the classification of mappings $N \to P$ up to diffeomorphisms of the source and the target, i.e. up to *the left-right equivalence* [16]. It is also not so difficult to see that even the codimensions of the germs of the corresponding objects in the corresponding function spaces coincide. If we also permit the variety we are projecting to have singularities, then we obtain a natural extension of the notion of left-right equivalence to the case of mappings of singular varieties to smooth ones. In this lecture we give a survey of the following topics of the theory of projections:

(1) Singularities of images of projections of generic space curves to a plane.
(2) Projections of a generic surface from 3-space onto a plane, including the projective classification of points of a generic surface in the projective 3-space.
(3) Some general principles of classification of projections of complete intersections.
(4) Finally, we concentrate on the case of projections of complete intersections onto a line and show that thier properties are very close to the properties of functions on smooth manifolds.

In our exposition we follow [3] and [5,§1.3]. We are trying to be independent of any external references.

1. Projections of space curves to a plane

We start with the classification of images of the objects being projected, i.e. with a coarser equivalence relation.

Let γ be a smooth curve embedded into $\mathbb{R}P^3$. Consider the projection $\mathbb{R}P^3 \setminus O \to \mathbb{R}P^2$ from a point O not on γ: to each point we associate the line through this point and the projection centre O.

Theorem 1.1 [11]. *Assume that the curve γ is generic. Then*

(1) *For a generic point O, the image of γ obtained under the projection is an immersed curve with ordinary double points.*
(2) *If the centre of the projection is situated on a certain exceptional surface in $\mathbb{R}P^3$, the image has one of the singularities shown in Fig.1.I.*
(3) *If the centre is on a certain exceptional curve in $\mathbb{R}P^3$, the image has either two singularities from (2) or one of the singularities shown in Fig.1.II.*
(4) *For certain isolated positions of the projection centre, the image has either three (2)-singularities, or one singularity from (2) and one singularity from (3), or one of the singularities shown in Fig.1.III.*

This list is complete for a generic curve γ.

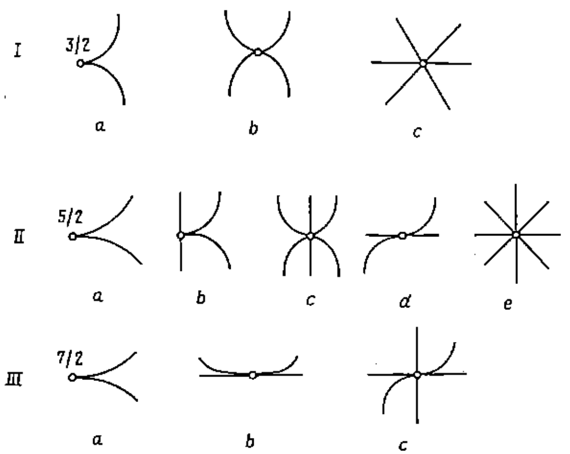

Fig.1. Local images of projections of generic curves

The genericity of the curve is understood here to be the genericity of the embedding $\gamma \to \mathbb{R}P^3$ [6].

Remarks on the drawings. In Ia and IIb cases the cusp has degree $3/2$ ($t \mapsto (t^2, t^3)$), in case IIa — $5/2$ ($t \mapsto (t^2, t^5)$), in IIIa — $7/2$ ($t \mapsto (t^2, t^7)$). The two branches in Ib and IIc have first order contact, in IId and IIIc second order contact, and in IIIb third order contact. In all the remaining cases all the intersections are maximally nondegenerate. Only the configuration IIe has a modulus (i.e. a continuous numerical invariant).

Realizations of the singularities are based on the following. We get k local branches of the image for the projection from a point on a line which intersects γ in k points. We get two tangent branches for the projection along the line AB, where A and B are points of γ with the tangents to γ situated in one plane. The ordinary cuspidal point is obtained by the projection along a line tangent to γ. In order to get a $5/2$-cusp, we need this tangent to be applied at a point with zero torsion. For the special choice of the projection centre on such a tangent we get a $7/2$-cusp.

2. Projections of surfaces onto a plane

In this section we consider a surface in three-dimensional space. According to the classical Whitney theorem [27], the only possible singularities of its generic projection onto a plane are folds (along curves) and cusps (at isolated points) (cases **2** and **3** in Fig.2 below). But, projecting along some special nongeneric directions, one can get more complicated singularities. That is what the story will be about now.

Definition. *A projection of a surface* Γ embedded in to $\mathbb{R}P^3$ from a point O not on the surface is a diagram $\Gamma \hookrightarrow \mathbb{R}P^3 \setminus O \to \mathbb{R}P^2$, where the first arrow is the embedding and the second arrow is the projection as in the previous section. Two projections are called *equivalent* if there exists a commutative 3×2-diagram whose rows are these projections and whose columns are diffeomorphisms.

The equivalence relation just introduced is the equivalence under a diffeomerphism of the ambient 3-space *fibred over the base* $\mathbb{R}P^2$ of the projection.

The definitions for germs are analogous.

According to Platonova [25] and O.Shcherbak [26,3,6], the hierarchy of projection-germs for a generic surface is as follows:

$$\begin{array}{ccccccccc}
\mathbf{1} & \leftarrow & \mathbf{2} & \leftarrow & \mathbf{3} & \leftarrow & \mathbf{6} & \leftarrow & \mathbf{8} \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & \mathbf{4} & \leftarrow & \mathbf{7} & & \mathbf{11} \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathbf{5} & \leftarrow & \mathbf{10} & & \\
 & & & & \uparrow & & & & \\
 & & & & \mathbf{9} & & & &
\end{array}$$

The numbers here correspond to the germs equivalent to the projection-germs at the origin of the surfaces $z = f(x,y)$ along the pencil of rays parallel to the x-axis, where for the nonsingular case **1** $f = x$ and for all the other cases f is given by the following table:

type	$f(x,y)$	type	$f(x,y)$
2	x^2	**7**	$x^4 + x^2y + xy^2$
3	$x^3 + xy$	**8**	$x^5 \pm x^3y + xy$
4	$x^3 \pm xy^2$	**9**	$x^3 \pm xy^4$
5	$x^3 + xy^3$	**10**	$x^4 + x^2y + xy^3$
6	$x^4 + xy$	**11**	$x^5 + xy$

Theorem 2.1 [3]. *For any generic surface and for any centre of projection in three-dimensional space the projection germ is equivalent to one of these fourteen germs at the origin.*

The singularities of the apparent contours and sets of critical values corresponding to each of the singularities listed are shown in Fig.2.

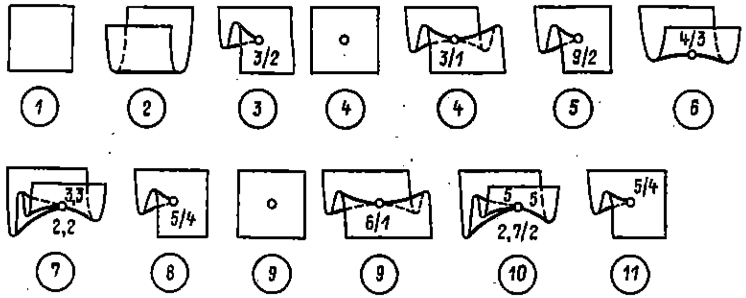

Fig.2. Generic singularities of apparent contours

For a projection from a generic centre, the projection-germ at a generic point of the surface is of type **1**, at the points of a certain curve it has type **2** (fold) and at some isolated points it has type **3** (Whitney cusp). The cusps appear for projections along asymptotic rays of the surface.

For the description of the remaining singularities, which occur only for special projection centres, we need the projective classification of points of a generic surface in projective 3-space ([21,24];Fig.3).

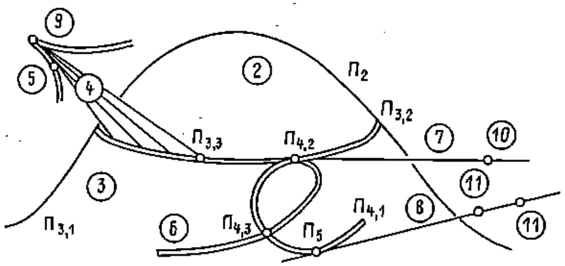

Fig.3. Centres of nongeneric projections

A smooth *curve of parabolic points* $\Pi_{3,2}$ divides the surface into a *domain of elliptic points* Π_2 (no real tangent lines of order higher than 1) and a *domain of hyperbolic points* $\Pi_{3,1}$ (there are two such tangents; they are called *asymptotic lines* and their directions at the point of application are called *asymptotic directions*).

In the domain of hyperbolic points there is aslo a smoothly immersed *curve of inflections of asymptotics* $\Pi_{4,1}$ (the order of contact of the asymptotic line with the surface is higher than 2 at points of this curve). On this curve there are isolated *points of bi-inflections* Π_5 (the fourth order of the contact) and *points of self-intersections* $\Pi_{4,3}$ (the third order contact for the both asymptotic lines). At the $\Pi_{4,2}$-points the curve of inflections of asymptotics is simply tangent to the curve of parabolic points (at such a point the unique asymptotic direction is tangent to the curve of parabolic points). Finally, there are also discrete $\Pi_{3,3}$-points on the curve of parabolic points, we will discuss these points later on.

Thus, the hierarchy of points on a generic surface is as follows:

$$\begin{tikzcd}
& & \Pi_5 \arrow[dl] \\
\Pi_{3,1} & \Pi_{4,1} \arrow[l] & \Pi_{4,3} \arrow[l] \\
\Pi_2 & \Pi_{3,2} \arrow[l] \arrow[ul] & \Pi_{4,2} \arrow[l] \arrow[ul] \\
& & \Pi_{3,3} \arrow[ul]
\end{tikzcd}$$

The projective-dual surface has a cuspidal edge at the points corresponding to $\Pi_{3,2}$ and $\Pi_{3,3}$, and a swallow tail at the $\Pi_{4,2}$-points.

We write out the normal forms of the p-jet of the surface at the points of each class with respect to projective transformations:

class	normal form	restrictions	p	codim
Π_2	x^2+y^2	–	2	0
$\Pi_{3,1}$	$xy+x^3+y^3$	–	3	0
$\Pi_{3,2}$	$y^2+x^3+xy^3+ax^4$	–	4	1
$\Pi_{4,1}$	$xy+y^3+x^4+hx^3y$	–	4	1
$\Pi_{4,2}$	$y^2+x^2y+vx^4$	$v\neq 0,1/4$	4	2
$\Pi_{4,3}$	$xy+x^4+ax^3y+bxy^3\pm y^4$	–	4	2
Π_5	$xy+y^3\pm x^3y+\phi_5$	$\phi_5(x,0)\neq 0$	5	2
$\Pi_{3,3}$	$y^2+x^3+ax^4+by^5\pm xy^4+x^2\phi_3$	–	5	2

Here $a,b,h,v\in\mathbb{R}$ and ϕ_r is a homogeneous polynomial of degree r in x and y.

Theorem 2.2. *The p-jet of a generic surface at any point can be reduced by a projective transformation to one of the normal forms of this table.*

Remark. Kulikov computed the numbers of points and the degrees of the curves of the listed classes on an algebraic surface of degree d in $\mathbb{C}P^3$. His results are as follows [20]:

	Π_5 $5d(d-4)(7d-12)$	$\Pi_{4,3}$ $5d(7d^2-28d+30)$
	$\Pi_{4,2}$ $2d(d-2)(11d-24)$	$\Pi_{3,3}$ $10d(d-2)(7d-16)$
curve degree	$\Pi_{3,2}$ $4d(d-2)$	$\Pi_{4,1}$ $d(11d-24)$

Corollary 2.3. *On a real surface of sufficiently high odd degree in $\mathbb{R}P^3$ there is at least one real point of asymptotic bi-inflection, at least one point of inflection of both asymptotics, and at least one curve of parabolic points.*

Remark. The paper [23] contains results for hypersurfaces in $\mathbb{C}P^4$ analogous to Kulikov's.

Now we return to the description of the singular projections from the nongeneric centres (Fig.3). The singularities **4** and **6** are realized by projections from almost all the points of certain surfaces, namely, the ruled surfaces formed by the asymptotic lines, drawn at the points of the parabolic curve and at the points of the curve of inflections of asymptotics respectively. The first of these surfaces is developable and has a cuspidal edge. For a generic point on this edge as the projection centre, we obtain the singularity **5**.

7 and **8** are also realized by projections from points of certain curves. We see **7** from a generic point of an asymptotic line tangent to the curve of parabolic points (i.e. passing through a $\Pi_{4,2}$-point), and we see **8** from a generic point of an asymptotic tangent of the fourth order.

Finally, we get **9**, **10** and **11** for projection from isolated points only. **9** occurs for a projection from a cusp of the cuspidal edge of the developable surface mentioned above (the corresponding asymptotic line passes through a $\Pi_{3,3}$-point). We realize **11** by projections from the two "focal" points on an asymptotic tangent of the fourth order, and we realize **10** projecting from one special point on an asymptotic line tangent to the line of parabolic points (at $\Pi_{4,2}$-point).

Remarks. a) Projections of a surface from various points of the 3-space form a 3-parameter family. Thus, it should be expected that in the problem under consideration there would appear all the remaining singularities which occur in generic 3-parameter families of projections of surfaces onto a plane. Namely, the projections of the surfaces $z=x^4\pm x^2y^2+xy^2$, $x^5+x^3y+x^2y$ and $x^6+ax^4y+x^3y+xy$, $a\in\mathbb{R}$, along the x-axis [19]. But, according to Platonova and O.Shcherbak, these projections are defined by the properties of the projecting line, not by the properties of a point on this line. For that reason these projections cannot be realized in our problem.

b) Projections by pencils of parallel lines were considered in [2] (10 types excluding **9**, **10**, **11**). Their list coincides with the list of the singularities in generic 2-parameter families [19] and with the list of the singularities of left-right codimension ≤ 2 [2,8].

Looking at the surface from one of the most singular points (**9**,**10**,**11**) and slightly moving the head one can see the surface as shown in Figs.4–7 [3]. In the middle of each Figure there is the bifurcation diagram consisting of the nongeneric projection centres with various components of the complement marked as the corresponding drawings.

3. Projections of complete intersections

We now consider the general case of a projection of a subvariety V of a space E onto a base B of a fibration: $V\hookrightarrow E\to B$. We do not assume V is smooth.

Definition. *A suspension over a projection $V\hookrightarrow E\to B$ is defined by an embedding of E as a subfibration into a space of a larger fibration with the same base. The stable equivalence of projections* is the equivalence of their suitable suspensions.

Theorem 3.1 [16]. *The left-right ($\equiv\mathcal{A}$-)classification of map-germs of smooth manifolds $V,0\to B$ is eqivalent to the stable classification of projection-germs up to fibrations with $(\dim V)$-dimensional fibres. A map-germ with an n-dimensional kernel at 0 is realized already by fibrations with n-dimensional fibres.*

In order to get the complete $\mathcal{A}$-classification of germs $V,0\to B$ for V singular it is sufficient to consider projections from spaces of fibrations with fibres of dimension equal to the embedding dimension of V.

A projection of V to B defines a family of subvarieties in the fibres of the fibration $E\to B$ (namely, the family of intersections of V with the fibres). If the dimension of V is less than the dimension of the base, then in almost every fibre the corresponding subvariety is empty, and if $\dim V\geq\dim B$, then this subvariety has the same codimension in the fibre as V has in E. It is natural to consider the projection-germ $V\hookrightarrow E\to B$ as a deformation of the subvariety V^o of the distinguished fibre.

Let V be a germ of the zero-set of some mapping f. Then the equivalence of projections up to diffeomorphisms of the space of the fibration fibred over the base is *the fibred contact ($\equiv\mathcal{K}$-)equivalence* of the corresponding mappings. Formally this means that two map-germs $f_1,f_2:(\mathbb{C}^n\times\mathbb{C}^p,0)\to(\mathbb{C}^m,0)$, $x\in\mathbb{C}^n$, $u\in\mathbb{C}^p$, are equivalent if and only if there exist germs on $(\mathbb{C}^n\times\mathbb{C}^p,0)$ of a diffeomorphism $(x_1,u_1)\mapsto(x_2(x_1,u_1),u_2(u_1))$ and of an $m\times m$-matrix M, $\det M(0)\neq 0$, such that

$$f(x_1,u_1)\equiv M(x_1,u_1)\cdot f_2(x_2(x_1,u_1),u_2(u_1)).$$

For $p=0$ this is the usual notion of *$\mathcal{K}$-equivalence*.

If we are interested in projections of finite codimension in the function space, the maximal degeneration of the variety being projected should be an isolated singularity of a complete intersection and V^o should be of finite $\mathcal{K}$-codimension.

Note that perturbing a projection of a singular variety V we consider not only the variations of its position with respect to the fibres of the fibration, but we also deform V itself, smoothing its singularities. This remark extends also to the corresponding $\mathcal{A}$-classification of mappings of singular varieties to smooth ones.

Definition. *A projection is said to be 'onto'* if the base dimension does not exceed $\dim V$.

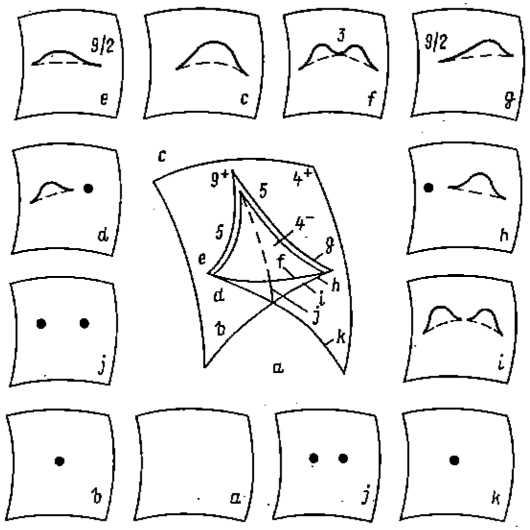

Fig.4. Bifurcations of the 9^+ singularity

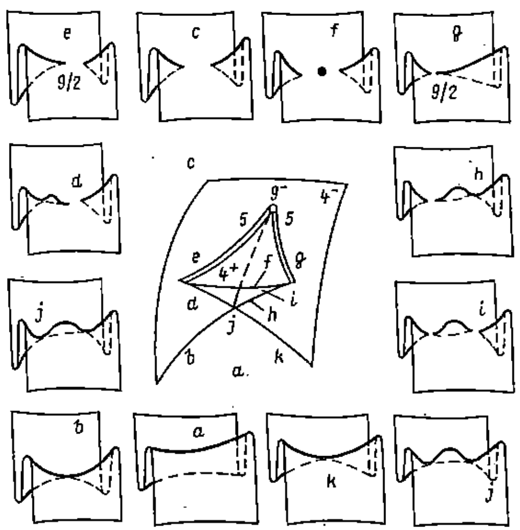

Fig.5. Bifurcations of the 9^- singularity

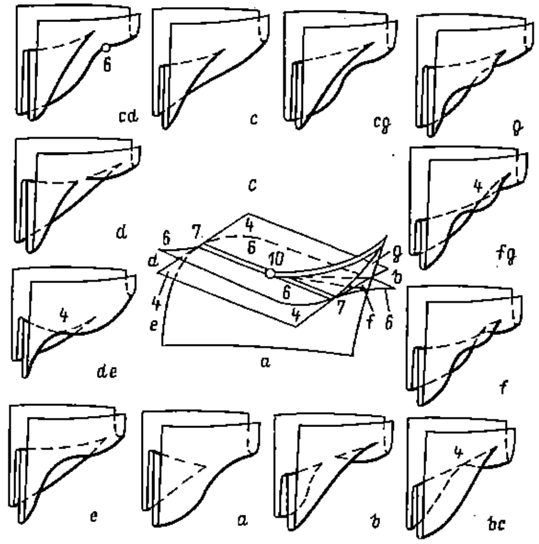

Fig.6. Bifurcations of the 10 singularity

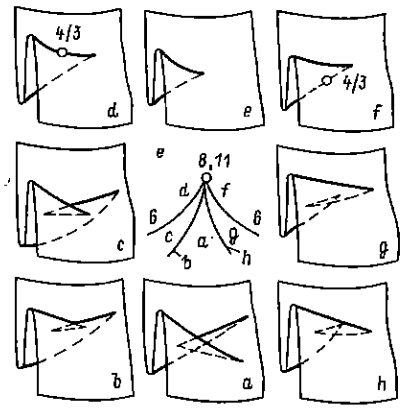

Fig.7. Bifurcations of the 8 and 11 singularities

The initial part of the projection 'onto' classification with respect to the fibred contact group (that is, the left-right classification of mappings between manifolds of nonincreasing dimension) was obtained in [18,19]. One can find there the lists of all simple and enveloping singularities, and many of their adjacencies. Both the real and the complex cases were considered. Here we will cite only the part of these results on the simple complex singularities. (Recall that a singularity is called *simple* if sufficiently small perturbations give only representatives of a finite number of equivalence classes.)

The treatment of projections as deformations of a distinguished fibre V^o and the classification of simple complete intersections V^o [12,13] provide

Theorem 3.2 [18,19,3]. *A simple germ of projection 'onto' is stably fibred equivalent to a deformation either of a hypersurface or of a curve in a three-dimensional space or of a fat point of a plane.*

The classification of simple singularities is most succinct in the second of these cases.

Theorem 3.3 [18,19,3]. *A deformation of a germ of a curve that is not equivalent to a plane curve defines a simple projection-germ if and only if it is a versal deformation of a simple singularity of a curve in $\mathbb{C}^3$.*

One can find the list of simple curve singularities in [12,13,5].

For the two other cases the list of simple projections is very extensive and we will limit ourselves to the case of projections onto a line. We note only that hypersurface projections provide quite interesting examples of singularities which are $\mathbb{R}$-simple but not $\mathbb{C}$-simple. There is a whole series of such projections: $V = \{x_1^3 + u_1^2 + u_2^2 \pm u_1^k x_1 \pm x_2^2 \pm \cdots \pm x_n^2 = 0\}$, $k \geq 2$, $(x,u) \mapsto u$ [18,19].

4. Projections onto a line

Consider a projection $V \hookrightarrow \mathbb{C}^{n+1} \to \mathbb{C}$, where V is a complete intersection of positive dimension.

A. Classification.

Theorem 4.1 [14,18,19]. (1) *Any simple projection of a complete intersection of positive dimension onto $\mathbb{C}$ is stably equivalent either to a projection of a hypersurface or to a projection of a curve from three-dimensional space.*

(2) *A simple germ of projection of a hypersurface in the space of a fibration $(x,u) \mapsto u$ can be reduced by a complex fibred diffeomorphism $(x,u) \mapsto (h(x,u), v(u))$ to the projection onto the u-axis of a germ at the origin of the variety $f(x,u) = 0$, where f is one of the functions of the following list:*

type	f	type	f
$A_\mu, \mu \geq 0$	$u + x_1^{\mu+1} + q$	E_8	$u + x_1^3 + x_2^5 + q$
$D_\mu, \mu \geq 4$	$u + x_1^2 x_2 + x_2^{\mu-1} + q$	$B_\mu, \mu \geq 2$	$u^\mu + x_1^2 + q$
E_6	$u + x_1^3 + x_2^4 + q$	$C_\mu, \mu \geq 3$	$ux_1 + x^\mu + q$
E_7	$u + x_1^3 + x_1 x_2^3 + q$	F_4	$u^2 + x_1^3 + q$

Here $q = x_r^2 + \cdots + x_n^2$, $r = 3$ for D_μ and E_μ, and $r = 2$ in all the other normal forms.

(3) *The simple projections of curve-germs from 3-space onto a line, that are not stably equivalent to projections of plane curves, form two infinite series which can be reduced to the projections onto the u-axis of the germs $f = 0$ of the following curves:*

type	f
$C_{k,\ell}, 2 \leq k \leq \ell$	$xy, x^k + y^\ell + u$
$F_{2k+1}, 2 \leq k$	$x^2 + y^3, y^k + u$
$F_{2k+4}, 1 \leq k$	$x^2 + y^3, xy^k + u$

Observe that the list (2) of the simple hypersurface projections onto a line coincides with the list of the simple functions on a manifold with the boundary $u = 0$ [1,6]. This is a consequence of the quasihomogeneity of all the simple and enveloping boundary singulatities.

The hierarchy of the simple projections is given in Fig.8. There we set for the curves $C_{1,\ell} = C_{\ell+1}$, $C_{1,1} = B_2$, $F_3 = B_3$.

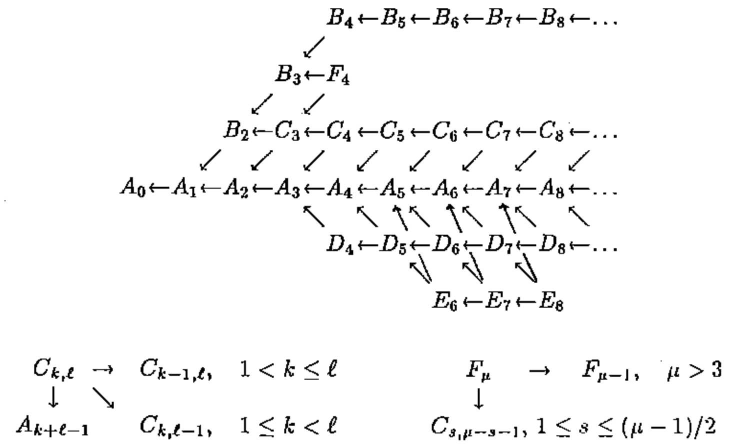

Fig.8. Hierarchy of simple projections onto a line

B. Dynkin diagrams. The coincidence of the lists of the simple boundary functions and simple projections suggests the following definition of the intersection form for projections.

Consider a projection $V_0 \hookrightarrow E \to \mathbb{C}$. Fix a small ball of radius ρ centered at $0 \in E$. Along with the complete intersection V_0 defined by the equation $f = 0$ in this ball, consider a neighbouring smooth complete intersection $V_\varepsilon = f^{-1}(\varepsilon)$, $|\varepsilon| \ll \rho$. For generic ε the manifold V_ε intersects transversally the fibre of the fibration $E \to \mathbb{C}$, passing through 0, by a submanifold V_ε^o of complex codimension

1 in V_ε. The coordinate function on the base $\mathbb{C}$ of the fibration induces a function on V_0. If the critical point 0 of this function on V_0 is isolated (a singular point of the manifold is counted as critical), the quotient-space $V_\varepsilon/V_\varepsilon^o$ is homotopy equivalent to a wedge of finite a number μ of spheres of the middle dimension. We call μ *the Milnor number*.

Consider a two-sheeted covering $\mathcal{V}_\varepsilon \to V_\varepsilon$ that is branched along V_ε^o. Let H be the middle-dimensional integer homology of $\mathcal{V}_\varepsilon$. The interchange of the sheets of the covering induces an involution on H. Let H^- be the anti-invariant part of H. μ is the rank of H^-. The intersection form on H provides a bilinear form on H^-. We call this form *the intersection form of the projection*.

Given a basis of H^- one can build up a Dynkin diagram corresponding to the intersection form. For an even-dimensional V_0 the intersection form is symmetric and the diagram is defined in the usual way [1,7,5]. For $\dim V_0$ odd we need to be more careful with the definition of short and long cycles [3].

Let H^+ be the involution-invariant part of H. H^- projects naturally to the quotient H/H^+. We say that an anti-invariant element $h \in H^-$ is *short* if its image in H/H^+ is not divisible by 2, and *long* if it is.

We construct the Dynkin diagram for the skew-symmetric form in the following way. Its vertices i correspond to the basic elements e_i of H^-. The multiplicity of the edge between two vertices is equal to the index of intersection of the corresponding cycles if at least one of these cycles is short, and to half this index if the both are long. The orientation $i \to j$ of an edge shows that the intersection number (e_i, e_j) is positive. An edge that joins vertices corresponding to cycles of different length is equipped with a sign ">" that is open towards the long cycle. When the graph is a tree the orientations of the edges are omitted since they might be done arbitrarily by a choice of the orientations of the basic cycles.

With the described conventions the Dynkin diagrams of the projections $A_\mu, \ldots, F_4$ are ordinary Dynkin diagrams of the corresponding Lie algebras [9,1,7,4,5] shown in Fig.9.

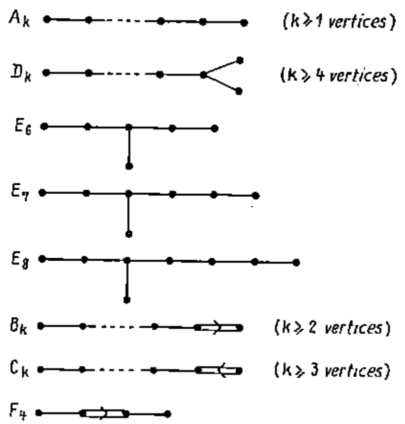

Fig.9. Dynkin diagrams of simple projections of hypersurfaces onto a line

Theorem 4.2 [14]. *The intersection forms of the singulerities F_μ and $C_{k,\ell}$ are given in suitable bases by the Dynkin diagrams of Fig.10.*

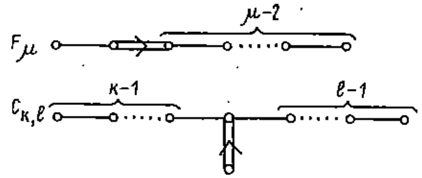

Fig.10. Dynkin diagrams of simple projections of space curves onto a line

C. Distinguished bases. The bases mentioned in the above theorem can be chosen to be special in the following sense [14].

On the same ball in the space E of the fibration on which the mapping f that defines the complete intersection V_0 is given, we consider a generic small perturbation $g : E \to \mathbb{C}^p$ of f. We may assume the common zero-set Y of the first $p-1$ coordinate functions of g to be smooth as well as its intersection Y^o with the fibre of $E \to \mathbb{C}$ passing through $0 \in E$. Assume also the last coordinate function g_p to have only Morse critical points on Y and Y^o.

The pair of nonsingular levels W and W^o of the function g_p on Y and Y^o is isomorphic to the pair $(V_\varepsilon, V_\varepsilon^o)$ of smooth manifolds we considered earlier. The corresponding two-sheeted coverings, the anti-invariant homologies and the intersection forms on these homologies are also isomorphic. Consequently, as in the case of boundary singularities [1,7,5], the critical values of g_p on Y and Y^o define vanishing cycles and semicycles in the homology of the pair $(V_\varepsilon, V_\varepsilon^o)$. Lifting to the two-sheeted covering, we get long and short cycles that generate the anti-invariant homology H^-. Chose a basis for H^- in the set of these cycles. We call this basis *special*. The diagrams of Theorem 4.2 are diagrams in special bases.

The suggested algorithm has the evident imperfection that it depends on the choice made: for a projection onto a line, which is not stably equivalent to a projection of a hypersurface, the number of critical points of g_p exceeds the rank of H^-. The following definition of a distinguished (short) basis for H^- avoids such arbitrariness.

Let W and W^o be the same as before. Consider the projection $E \to \mathbb{C}$ as a function. The genericity of the mapping g maens that the restriction of this function u on W is a Morse function, with nondegenerate critical points and distinct critical values. The number of these critical values is equal to the number of spheres in the wedge to which the quotient $W/W^o \simeq V_\varepsilon/V_\varepsilon^o$ is homotopic (this is the Milnor number μ).

Cosider a system of μ paths on the u-axis going from a noncritical value $u = 0$ to the μ critical values. Approaching the end of one of these paths we contract a cycle on W/W^o. If the paths have no intersections and self-intersections, μ relative cycles obtained in this way form a basis for the relative homology. The corresponding μ anti-invariant cycles in the homology of the double covering of W which is branched along W^0 form a *distinguished basis* of μ short cycles in H^-.

For the ADE singularities this construction provides the ordinary Dynkin diagrams. But already for B_μ we get unusual ones: in the symmetric case the diagram is a disjoint set of its vertices, and in the skew-case it is the complete graph with double edges. For the projections $C_{k,\ell}$ and F_μ there exist distinguished short bases with the Dynkin diagrams shown in Fig.11.

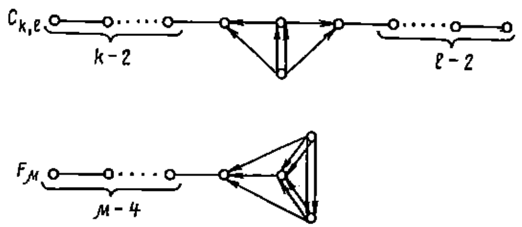

Fig.11. Dynkin diagrams in distinguished short bases

D. Milnor numbers. Consider a projection $(x,u) \mapsto u$ onto a line of a germ at the origin of an isolated singularity of a complete intersection $f(x,u) = 0$, $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C}^p,0)$, $n \geq p$. For brevity we will call such a projection '*the projection f*'.

The Milnor number of the projection f is the number of the Morse critical points of the function u on the variety being projected that stick together at the origin. So, it can be expressed by the formula

$$\mu = \mu_{alg} = \dim_{\mathbb{C}} \mathcal{O}_{x,u}/I,$$

where $\mathcal{O}_{x,u}$ is the ring of holomorphic function-germs on $(\mathbb{C}^{n+1},0)$, and I is the ideal generated by the coordinate functions of the mapping f and all the p-minors of the matrix $(\partial f/\partial x)$ [15].

Another description of this number is as follows.

There are different groups acting on the functional space of all the projections and corresponding to different transformations of the base $\mathbb{C}$, namely to: the identity, a shift by a constant, an arbitrary diffeomorphism. If the variety V being projected is smooth, these groups provide respectively the $\mathcal{R}$-, $\mathcal{R}_+$- and $\mathcal{A}$-equivalence [6,4] of the composed mappings $V \to \mathbb{C}$ onto the base of the fibration. So, we shall call by the same names the corresponding equivalences of projections of arbitrary complete intersections.

The groups of the $\mathcal{R}$-, $\mathcal{R}_+$- and $\mathcal{A}$-equivalences of projections onto a line are good geometrical subgroups of the group of contact equivalence of maps from $\mathbb{C}^{n+1}$ to

$\mathbb{C}^p$ [10]. Therefore the finite determinacy and versality theorems are true for them. For example, the deformation

$$f(x,u) + \lambda_1 e_1(x,u) + \cdots + \lambda_\tau e_\tau(x,u),$$

where $e_1, \ldots, e_\tau$ is a basis of the linear space

$$\mathcal{O}^p_{x,u}/\{I_f \mathcal{O}^p_{x,u} + \mathcal{O}_{x,u}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle\},$$

is an $\mathcal{R}$-miniversal deformation of the projection f. (Here $\mathcal{O}^p_{x,u}$ is the space of holomorphic map-germs from $(\mathbb{C}^{n+1},0)$ to $\mathbb{C}^p$.) One can write out $\mathcal{R}_+$- and $\mathcal{A}$-miniversal deformations in the similar way with the only difference that in the denominator of the quotient space one should add the term $\mathbb{C}\partial f/\partial u$ or $\mathcal{O}_u \partial f/\partial u$ respectively.

Let τ, τ_+ and $\tau_{\mathcal{A}}$ be the dimensions of the bases of the corresponding deformations of the same projection f. Obviously, $\tau - 1 = \tau_+ \geq \tau_{\mathcal{A}}$. τ is the index in the notations of the singularities in the classification Theorem 4.1, $\tau(C_{k,\ell}) = k+\ell$. Since all the simple singularities are quasihomogeneous, we get for them $\tau - 1 = \tau_+ = \tau_{\mathcal{A}}$.

Theorem 4.3 [15]. *For a germ of a projection of a complete intersection of positive dimension onto a line* $\mu = \tau$.

This property shows that a projection onto a line is a natural generalization of the notion of a function on a smooth manifold (the projection onto the line defines the height function).

Remark. For a projection of a fat point, τ and μ_{alg} do not necessary coincide. E.g., for the projection of a fat point $x^k + u = x^\ell = 0$ from $\mathbb{C}^2$, we get $\tau = k + \ell - 1 \neq \mu_{alg} = \ell$ [17].

E. Bifurcation diagrams. Let $F(x,u,\lambda)$ be an $\mathcal{R}_+$-miniversal deformation of a projection f, $\lambda \in \mathbb{C}^{\tau-1}$ being the parameter of the deformation. $F = 0$ is a smooth complete intersection in $\mathbb{C}^{n+\tau}$.

Definition. *The discriminant* $\Delta \subset \mathbb{C}^\tau$ *of a projection* f is the germ at the origin of the set of the critical values of the mapping $(x,u,\lambda) \mapsto (u,\lambda)$ restricted to the germ of the manifold $F = 0$.

An $\mathcal{R}_+$-miniversal deformation of the projection $(x,u) \mapsto u$ onto the line of a complete intersection $f(x,u) = 0$ of positive dimension is a contact versal deformation of the complete intersection $f(x,0) = 0$ (u is the additional parameter). Therefore the discriminant of the projection f is the discriminant of the complete intersection $f(x,0) = 0$ (possibly, multiplied by a complex linear space).

Consider the projection $\pi : (u,\lambda) \mapsto \lambda$ of the discriminant Δ (see Fig.12 where $f \in C_3$ and $F = x^3 + ux + \lambda_1 x^2 + \lambda_2$).

Definition. *The bifurcation diagram of projections* $\Sigma \subset \mathbb{C}^{\tau-1}$ of a projection f is the union of the π-image of the singular set of Δ with the set of the critical values of π on the smooth part of Δ.

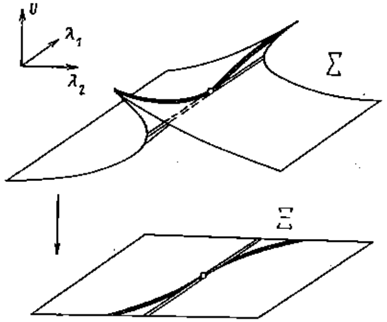

Fig.12. The discriminant Δ and the bifurcation diagram Σ of the projection C_3

The diagram Σ is the set of the values of the parameter λ of the $\mathcal{R}_+$-miniversal deformation for which either the variety $V_\lambda = \{F(\cdot,\cdot,\lambda) = 0\} \subset \mathbb{C}^{n+1}$ is singular or the restriction of the function u on this variety is not a Morse function.

Generally the bifurcation diagram of a projection consists of the three hypersurfaces:

(1) the projection of the cuspidal edge of the discriminant (corresponding to a degeneration of a critical point of the function u on a smooth variety V_λ);
(2) the projection of the selfintersection of the discriminant (corresponding to equal critical values of u on a smooth V_λ);
(3) the set of the critical values of the projection of the regular part of Δ (V_λ is singular).

For a projection of a smooth hypersurface we can write its equation f as $f_0(x)+u=0$. In this case Δ and Σ are the discriminant and the bifurcation diagram of functions of the function f_0 (see, e.g.[6,7,4]).

In Fig.13 there are shown the bifurcation diagrams of all the real projections onto a line with $\tau = 3,4$ [5]. There are two real forms for the complex singularity D_4 and three real forms for $C_{2,2}$.

Note that the pair $(\mathbb{C}^{\mu-1}, \Sigma(D_\mu))$ is a double covering of the pair $(\mathbb{C}^{\mu-1}, \Sigma(C_\mu))$ branched along the stratum B_2.

F. Normal forms of vector fields. The bifurcation diagram of projections Σ of a projection onto a line is obtained by the projection π of the (u,λ)-space $\mathbb{C}^\tau$, containing the discriminant, along the phase lines of the vector field $\partial/\partial u$. This vector field (and, consequently, the projection π itself) has the following stability property provided by the versality theorem for the group of $\mathcal{R}_+$-equivalence of projections.

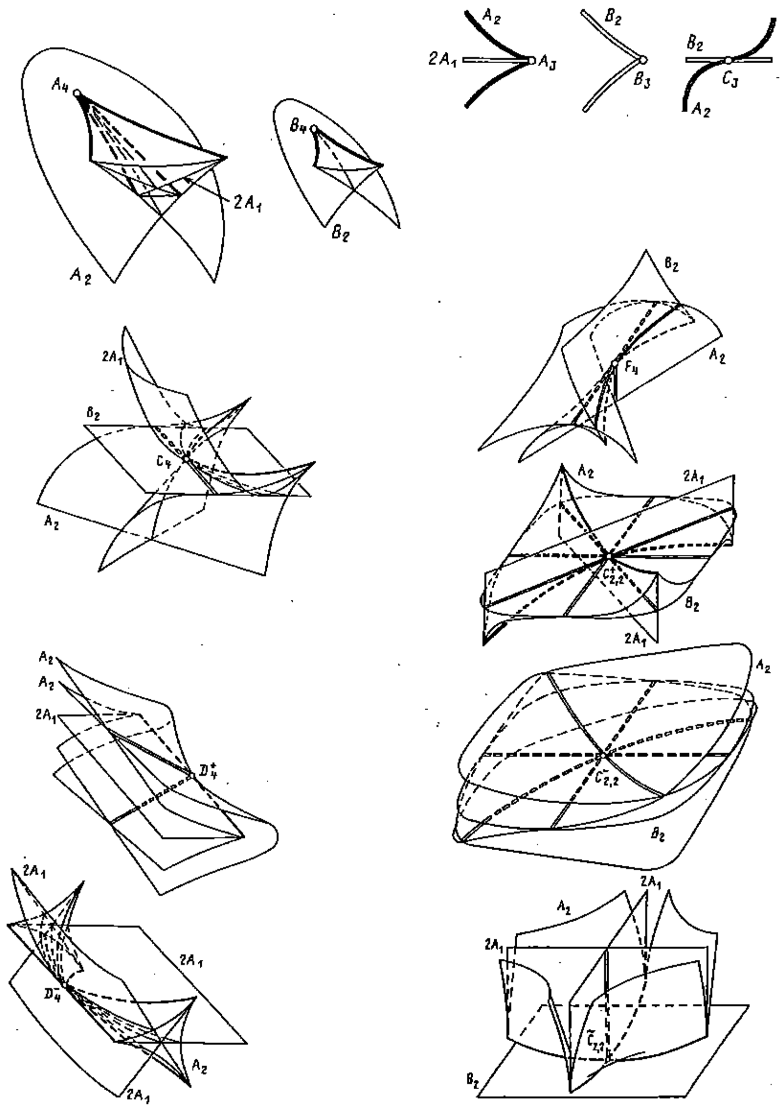

Fig.13. Bifurcation diagrams of projections of low codimension

Theorem 4.4 [14,19,3]. *The germ of the vector field $\partial/\partial u$ is stable in a neighbourhood of the origin with respect to the discriminant of the projection: a germ of every nearby holomorphic vector field at a suitable nearby point can be transformed to the germ of $\partial/\partial u$ at the origin by a biholomorphism sending the discriminant into itself.*

This assertion is a generalization of the theorem on the stability of a vector field transversal to the hyperplane tangent to the discriminant of an isolaed function singularity [2,22].

Example 4.5. Consider the discriminant of the projection C_3 (Fig.12). Consider a germ at the origin of a vector field v on $\mathbb{C}^3$ satisfying the two conditions:

(1) the vector $v(0)$ lies in the plane $\lambda_2 = 0$, tangent to the discriminant, but does not lie on the line $\lambda_2 = u = 0$ tangent to the cuspidal edge;

(2) for a vector field ν on $\mathbb{C}^3$ which is tangent to the discriminant and does not vanish at the origin, the vector $[v, \nu](0)$ is not tangent to Δ.

Then v reduces to the normal form $\partial/\partial u$ by a diffeomorphism of $\mathbb{C}^3$ preserving the discriminant.

G. The $k(\pi, 1)$-theorem.

Theorem 4.6 [14,19]. *The germ at the origin of the space $\mathbb{C}^{\tau-1} \setminus \Sigma$ of the complement to the bifurcation diagram of a simple projection onto a line is a $k(\pi,1)$-space, where π is a subgroup of finite index in the Artin braid group on τ threads.*

All the simple projection are quasihomogeneous. Let $\alpha_0, \alpha_1, \dots, \alpha_{\tau-1}$ be the weights of the variables $u, \lambda_1, \dots, \lambda_{\tau-1}$. Then the index mentioned is

$$\tau! \alpha_0^{\tau-1} / \alpha_1 \cdot \ldots \cdot \alpha_{\tau-1}.$$

Note that the bifurcation diagrams of the projections of singular hypersurfaces onto a line are distinct from the bifurcation diagrams of functions of the boundary singularities with the same names [1].

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