VECTOR FIELDS ON BIFURCATION VARIETIES

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Quite often we need to classify functions on a space containing a distinguished hypersurface. This happens, for example, when we consider functions on varieties with boundaries, equivariant singularities, perestroikas, section singularities, [1,22,23,4,5,7,11,9,10]. On the infinitesimal level problems of this kind require the description of the Lie algebra of vector fields tangent to the distinguished hypersurface. In many cases this hypersurface arises as a discriminant or a bifurcation diagram of a certain object. In this lecture, following [3,§1.5], we give a survey of some results concerning holomorphic vector fields tangent to such singular varieties, namely, to

(1) discriminants and bifurcation diagrams of functions,
(2) discriminants and bifurcation diagrams of projections onto a line,
(3) discriminants of complete intersections.

In all these cases the module of vector fields preserving the hypersurface is free over the ring of functions on the ambient space. Such a hypersurface is called a *free divisor* in the sense of Saito [19,8]. Knowledge of the generators of the module of these vector fields is also useful calculating generators of the dual object, which is the module of differential forms with a logarithmic pole along the discriminant. This leads to expressions for the coefficients of the Gauss-Manin connection of the corresponding singularity [18].

1. Functions on smooth varieties

Consider a germ at the origin in $\mathbb{C}^n$ of a holomorphic function $f(x)$ with an isolated critical point. Recall:

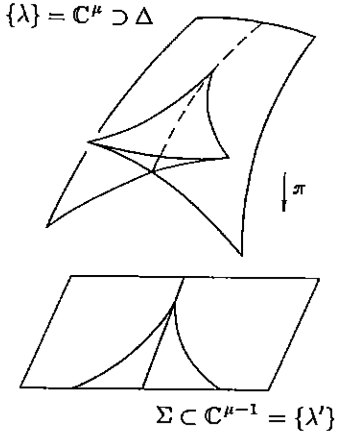
(1) a deformation $F(x,\lambda)$, $\lambda = (\lambda_0, \ldots, \lambda_{\mu-1}) \in \mathbb{C}^\mu$, of this function is *$\mathcal{R}$-miniversal* if its initial velocities $\partial F/\partial\lambda_i|_{\lambda=0}$ represent a $\mathbb{C}$-linear basis of the space $\mathcal{O}_x/\mathcal{O}_x\langle\partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle$;
(2) a deformation $\Phi(x,\lambda')$, $\lambda' = (\lambda_1, \ldots, \lambda_{\mu-1}) \in \mathbb{C}^{\mu-1}$, of f is *truncated $\mathcal{R}$-miniversal* (or *$\mathcal{R}_+$-miniversal*) if the similar condition holds for the space $\mathfrak{m}_x/\mathcal{O}_x\langle\partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle$.

Here $\mathcal{O}_x$ denotes the space of germs at the origin of holomorphic functions on $\mathbb{C}^n$ and $\mathfrak{m}_x$ is the ideal in $\mathcal{O}_x$ of germs vanishing at the origin.

The parameter spaces $\mathbb{C}^\mu$ and $\mathbb{C}^{\mu-1}$ of these deformations contain respectively:

(1) *the discriminant* $\Delta = \{\lambda|$ the surface $F(x,\lambda) = 0$ in the x-space is not smooth$\}$;

(2) *the bifurcation diagram of functions* $\Sigma = \{\lambda' |$ the function $\Phi(x,\lambda')$ on the x-space has either a non-Morse critical point or equal critical values$\}$.



We say that *a vector field* v *is tangent to a hypersurface* $\phi = 0$ if the derivative of ϕ along v belongs to the ideal generated by ϕ. We are going to describe the modules Θ_Δ and Θ_Σ of the germs at the origins of holomorphic vector fields tangent to the discriminant and to the bifurcation diagram of functions.

The versality conditions on F and Φ imply existence of the following decompositions:

$$F \cdot \partial F/\partial\lambda_i \equiv$$
$$v_{0,i}\partial F/\partial\lambda_0 + \cdots + v_{\mu-1,i}\partial F/\partial\lambda_{\mu-1} \mod \mathcal{O}_{x,\lambda}\langle \partial F/\partial x_1, \ldots, \partial F/\partial x_n\rangle,$$
$$i = 0, \ldots, \mu-1 \quad [22,23],$$

and

$$\Phi^j \equiv w_{0,j} +$$
$$w_{1,j}\partial\Phi/\partial\lambda_1 + \cdots + w_{\mu-1,j}\partial\Phi/\partial\lambda_{\mu-1} \mod \mathcal{O}_{x,\lambda}\langle \partial\Phi/\partial x_1, \ldots, \partial\Phi/\partial x_n\rangle,$$
$$j = 1, \ldots, \mu-1 \quad [6],$$

where $v_{s,i}(\lambda)$ and $w_{s,j}(\lambda')$ are germs of holomorphic functions.

Theorem 1.1.

(1) [22,23] $\Theta_\Delta = \mathcal{O}_\lambda\langle v_0, \ldots, v_{\mu-1}\rangle$, $\quad v_i = v_{0,i}\partial_{\lambda_0} + \cdots + v_{\mu-1,i}\partial_{\lambda_{\mu-1}}$.

(2) [6] $\Theta_\Sigma = \mathcal{O}_{\lambda'}\langle w_1, \ldots, w_{\mu-1}\rangle$, $\quad w_j = w_{1,j}\partial_{\lambda_1} + \cdots + w_{\mu-1,j}\partial_{\lambda_{\mu-1}}$.

The both modules are free. For the simple functions this procedure gives the basis of Θ_Δ different to the basis obtained by the convolution of invariants of Coxeter groups [2,s.2.5.7].

In practice it is more convenient to take the deformations as

$$F(x,\lambda) = \Phi(x,\lambda') + \lambda_0 = f(x) + \lambda_{\mu-1}e_{\mu-1}(x) + \cdots + \lambda_1 e_1(x) + \lambda_0,$$

where $e_1, \ldots, e_{\mu-1}$ is a $\mathbb{C}$-linear basis of $\mathfrak{m}_x/\mathcal{O}_x\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle$.

For $F(x,\lambda) = \Phi(x,\lambda') + \lambda_0$ it is also convenient to use for calculations of the Θ_Σ generators the matrix $\mathcal{V} = (v_{s,i}|_{\lambda_0=0})$, $s,i = 0, \ldots, \mu-1$, of the components of the Θ_Δ generators. Set $\pi : (\lambda_0, \lambda') \mapsto \lambda'$ and let us write a vector field as a column of its components.

Corollary 1.2 [21]. $w_j = \pi_*(\mathcal{V}^{j-1}\nu_0)$, $j = 1, \ldots, \mu-1$, *where* $\nu_0 = v_0|_{\lambda_0=0}$.

Example 1.3. A_3. $F(x,\lambda) = x^4 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$.

$$\partial F/\partial\lambda_0 = 1, \qquad \partial F/\partial\lambda_1 = x, \qquad \partial F/\partial\lambda_2 = x^2.$$

$$4F \cdot \partial F/\partial\lambda_0 = x \cdot \partial F/\partial x + 2\lambda_2 \cdot x^2 + 3\lambda_1 \cdot x + 4\lambda_0 \cdot 1,$$
$$4F \cdot \partial F/\partial\lambda_1 = (x^2 + \lambda_2/2) \cdot \partial F/\partial x + 3\lambda_1 \cdot x^2 + (4\lambda_0 - \lambda_2^2) \cdot x + (-\lambda_1\lambda_2/2) \cdot 1,$$
$$4F \cdot \partial F/\partial\lambda_2 = (x^3 + \lambda_2 x/2 + 3\lambda_1/4) \cdot \partial F/\partial x + (4\lambda_0 - \lambda_2^2) \cdot x^2 +$$
$$(-2\lambda_1\lambda_2) \cdot x + (-3\lambda_1^2/4) \cdot 1.$$

$$\begin{aligned} v_0 &= (4\lambda_0, 3\lambda_1, 2\lambda_2) \text{ — Euler field,} \\ v_1 &= (-\lambda_1\lambda_2/2, 4\lambda_0 - \lambda_2^2, 3\lambda_1), \\ v_2 &= (-3\lambda_1^2/4, -2\lambda_1\lambda_2, 4\lambda_0 - \lambda_2^2). \end{aligned}$$

$$\Phi(x,\lambda) = x^4 + \lambda_2 x^2 + \lambda_1 x. \qquad w_1 = (3\lambda_1, 2\lambda_2), \quad w_2 = (-7\lambda_1\lambda_2^2, 9\lambda_1^2 - 2\lambda_2^3).$$

Consider the projection $p : (x,\lambda) \mapsto \lambda$. The decompositions preceding the theorem show that a field v preserves the discriminant if and only if it is p-liftable to a vector field $\tilde{v}$ (i.e. $p_*\tilde{v} = v$) on the (x,λ)-space tangent to the smooth variety $F(x,\lambda) = 0$. We can also see that, for $F(x,\lambda) = \Phi(x,\lambda') + \lambda_0$, a vector field preserves the bifurcation diagram Σ if and only if it is π-liftable to a field on $\mathbb{C}^\mu$ tangent to the discriminant.

These remarks are corollaries of the following general property. Consider a germ of a reduced hypersurface V in $\mathbb{C}^\mu$. Suppose the restriction to V of the fibration $\pi : \mathbb{C}^\mu \to \mathbb{C}^{\mu-1}$ is proper. Let d be the multiplicity of the intersection of V with $\pi^{-1}(0)$. Then V is given by an equation which is polynomial of degree d along the fibre of the projection. Let $D : \mathbb{C}^{\mu-1} \to \mathbb{C}$ be the discriminant of this polynomial. We call the zero-variety of D the bifurcation variety.

Theorem 1.4 [16,17]. *Let* $\Xi \subset \mathbb{C}^{\mu-1}$ *be the set of the base points over which more than two points of the hypersurface glue together (i.e. the number of different points of the hypersurface over such base point does not exceed* $d-2$*). Suppose the dimension of* Ξ *is strictly less than the dimension of the bifurcation variety. Then every germ of a holomorphic vector field on the base, preserving the bifurcation variety, is liftable to a germ of a holomorphic field tangent to* V *in the space of the fibration.*

Let us return to the discriminant Δ and the diagram Σ.

The dimension of the linear space spanned by evaluating vector fields of Θ_Δ at $0 \in \mathbb{C}^\mu$ (this is the rank of the system of vectors at the origin of the basic fields) is equal to the difference between the Milnor number μ and the Tjurina number τ of the singularity f. At an arbitrary point $\lambda \in \mathbb{C}^\mu$ the corank of the system of basic vector fields coincides with the contact codimension of the corresponding multi-germ. The corank of the basic system of Θ_Σ at $\lambda' \in \mathbb{C}^{\mu-1}$ is measured by the left-right codimension of the multi-germ $\Phi(\cdot, \lambda')$ at the critical points.

Corollary 1.5. (e.g.[4,6]) *The following statements are equivalent:*

(1) *the finction f is right-equivalent to a quasihomogeneous one;*
(2) *every holomorphic vector field preserving the discriminant $\Delta \subset \mathbb{C}^\mu$ vanishes at the origin;*
(3) *every holomorphic vector field preserving the bifurcation diagram of functions $\Sigma \subset \mathbb{C}^{\mu-1}$ vanishes at the origin.*

Remark 1.6. All the assertions of this section can be extended to the case of functions on a variety with a smooth boundary $x_1 = 0$. To do this we only need to substitute $x_1\partial F/\partial x_1$ and $x_1\partial\Phi/\partial x_1$ for $\partial F/\partial x_1$ and $\partial\Phi/\partial x_1$ in the decompositions. The adaptation for functions with linear singularities [20] is also easy [13].

2. Projections onto a line

Consider a fibration $\mathbb{C}^{n+1} \to \mathbb{C}$, $(x,u) \mapsto u$. Let $f(x,u) = 0$ be a germ at the origin of a complete intersection of positive dimension with an isolated singularity, $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^p, 0)$, $n \geq p$. We call such an object in the space of the fibration *'a projection f'*.

There is a notion of *$\mathcal{R}_+$-equivalence* on the space $\mathcal{O}^p_{x,u}$ of all projections [3, s.1.3.4]: we say that two projections f' and f'' are $\mathcal{R}_+$-equivalent iff we can find germs at $0 \in \mathbb{C}^{n+1}$ of a $p \times p$-matrix M, $\det M(0) \neq 0$, and of a diffeomorphism $(x', u) \mapsto (x''(x',u), u + c)$, $c =$const, such that

$$f'(x',u) \equiv M(x',u) \cdot f''(x''(x',u), u + c).$$

Actually, this is an equivalence of height functions on complete intersections. This generalizes the notion of the $\mathcal{R}_+$-equivalence of functions on smooth manifolds. (If we do not require a diffeomorphism to be fibred, we get the usual *contact equivalence*.)

An $\mathcal{R}_+$-miniversal deformation of a projection f is a deformation $F(x,u,\lambda')$, $\lambda' = (\lambda_1, \ldots, \lambda_{\mu-1}) \in \mathbb{C}^{\mu-1}$, of a map-germ $f(x,u)$ such that its initial velocities $\partial F/\partial\lambda_i|_{\lambda=0}$ represent a $\mathbb{C}$-linear basis of the space

$$\mathcal{O}^p_{x,u}/\{I_f\mathcal{O}^p_{x,u} + \mathcal{O}_{x,u}\langle\partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle + \ \mathbb{C}\,\partial f/\partial u\}.$$

Here $I_f \subset \mathcal{O}_{x,u}$ is the ideal generated by the coordinate functions of the mapping f.

Let $\mathbb{C}^\mu$ be the (u,λ)-space.

The spaces $\mathbb{C}^\mu$ and $\mathbb{C}^{\mu-1}$ contain respectively:

(1) *the discriminant* $\Delta = \{(u,\lambda')|$ the variety $F(x,u,\lambda') = 0$ in the x-space is not smooth$\}$;
(2) *the bifurcation diagram of projections* $\Sigma = \{\lambda'|$ either the variety $F(x,u,\lambda') = 0$ in the (x,u)-space is not smooth or the function u on this variety has degenerate critical points or equal critical values$\}$.

We are going to describe the modules Θ_Δ and Θ_Σ of the tangent vector fields. The description will be quite similar to the previous section.

In what follows it is convenient to set $u = \lambda_0$ and $\lambda = (\lambda_0, \lambda')$.

The versality condition implies existence of the decompositions in the space $\mathcal{O}^p_{x,\lambda}/\{I_F\mathcal{O}^p_{x,\lambda} + \mathcal{O}_{x,\lambda}\langle\partial F/\partial x_1, \ldots, \partial F/\partial x_n\rangle\}$ [12]:

$$u\partial F/\partial\lambda_i \equiv v_{0,i}\partial F/\partial\lambda_0 + \ldots + v_{\mu-1,i}\partial F/\partial\lambda_{\mu-1}, \qquad i = 0, \ldots, \mu-1;$$

$$u^j\partial F/\partial u \equiv w_{0,j}\partial F/\partial\lambda_0 + \ldots + w_{\mu-1,j}\partial F/\partial\lambda_{\mu-1}, \qquad j = 1, \ldots, \mu-1,$$

where $v_{s,i}(\lambda')$ and $w_{s,j}(\lambda')$ are holomorphic functions.

Let $\delta_{s,i}$ be the Kronecker symbol.

Theorem 2.1 [12].

(1) $\Theta_\Delta = \mathcal{O}_\lambda\langle v_0, \ldots, v_{\mu-1}\rangle$,
$$v_i = (v_{0,i} - \delta_{0,i}u)\partial_{\lambda_0} + \cdots + (v_{\mu-1,i} - \delta_{\mu-1,i}u)\partial_{\lambda_{\mu-1}}.$$
(2) $\Theta_\Sigma = \mathcal{O}_{\lambda'}\langle w_1, \ldots, w_{\mu-1}\rangle$,
$$w_j = w_{1,j}\partial_{\lambda_1} + \cdots + w_{\mu-1,j}\partial_{\lambda_{\mu-1}}.$$

Both modules are free.

The liftability properties of the elements of Θ_Δ and Θ_Σ from the previous section are hold again. The corank at (u,λ') of the system of Θ_Δ-basic fields is equal to the contact codimension of the multi-germ of the variety $F(\cdot,u,\lambda') = 0$ at its singular points. The corank of the system $w_1, \ldots, w_{\mu-1}$ at $\lambda' \in \mathbb{C}^{\mu-1}$ is the left-right codimension of the germ of the projection $F(\cdot,\cdot,\lambda')$. Consequently, if the projection f is quasihomogenious, all the basic vector fields vanish at the origin.

Corollary 1.2 expressing the fields w_j in terms of the matrix of the fields v_i extends to the case of projections word to word.

Example 2.2 [13]. Consider a miniversal deformation of the simple projection $C_{2,2}$:

$$F(x,u,\lambda') = (x_1^2 + \lambda_1 x_1 + u + \lambda_2 x_2 + x_2^2, x_1x_2 + \lambda_3).$$

The calculations provide the following matrices of the components of the basic fields (we write a field out in a line):

$$\Theta_\Delta : \quad \begin{array}{llll} 2u & \lambda_1 & \lambda_2 & 2\lambda_3 \\ -6\lambda_2\lambda_3 & 4u - \lambda_1^2 & -8\lambda_3 & \lambda_1\lambda_3 \\ -6\lambda_1\lambda_3 & -8\lambda_3 & 4u - \lambda_2^2 & \lambda_2\lambda_3 \\ 4\lambda_3 - 2\lambda_1\lambda_2 & -3\lambda_2 & -3\lambda_1 & u \end{array}$$

$$\Theta_\Sigma : \quad \begin{array}{lll} \lambda_1 & \lambda_2 & 2\lambda_3 \\ \lambda_1^3 + 32\lambda_2\lambda_3 & \lambda_2^3 + 32\lambda_1\lambda_3 & -\lambda_1^2\lambda_3 - \lambda_2^2\lambda_3 \\ \lambda_1^5 - 36\lambda_1^2\lambda_2\lambda_3 + & \lambda_1^5 - 36\lambda_1\lambda_2^2\lambda_3 + & 128\lambda_3^3 - 176\lambda_1\lambda_2\lambda_3^2 - \\ \quad +320\lambda_1\lambda_3^2 - 4\lambda_2^3\lambda_3 & \quad +320\lambda_2\lambda_3^2 - 4\lambda_1^3\lambda_3 & \quad -\lambda_1^4\lambda_3 - \lambda_2^4\lambda_3 \end{array}$$

Let us remark that in this example the matrix for Θ_Δ also provides basic vector fields tangent to the discriminant of the fat point $(x_1^2 + x_2^2, x_1x_2)$. This is so because our deformation gives a 4-parameter miniversal deformation of this fat point (u is the additional parameter).

In [13] there are given examples of reduction to normal forms of functions on spaces containing the discriminant or the bifurcation diagram of a projection of low codimension.

3. Isolated singularities of complete intersections

Recall that the discriminant Δ of an isolated singularity of complete intersection is the subset in the base of a contact versal deformation of this singularity consisting of the values of the deformation parameters corresponding to non-smooth perturbed complete intersections.

Theorem 3.1 [15]. *The discriminant of an isolated complete intersection singularity is a free divisor.*

Let us show how to construct free generators of Θ_Σ.

Let $F(x,\lambda)$ be a contact versal deformation of a complete intersection $f_0(x) = 0$, $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n \geq p$. $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ is the parameter of the deformation. Suppose the λ_0-axis has a finite index of intersection μ with the discriminant $\Delta \subset \mathbb{C}^{r+1}$. Then μ is the dimension of the base of $\mathcal{R}$-miniversal deformation of the projection $(x, \lambda_0) \mapsto \lambda_0$ onto the line of the complete intersection $f = F|_{\lambda_1 = \cdots = \lambda_r = 0} = 0$. Thus the space

$$\mathcal{O}^p_{x,\lambda_0}/\{I_f\mathcal{O}^p_{x,\lambda_0} + \mathcal{O}_{x,\lambda_0}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle\}$$

is μ-dimensional and for its linear basis one can take the restrictions to the $\lambda_1 = \cdots = \lambda_r = 0$-plane of the elements

$$\begin{aligned}
&\partial F/\partial\lambda_0, \ldots, \lambda_0^{\mu_0-1}\partial F/\partial\lambda_0,\\
&\qquad\qquad \partial F/\partial\lambda_1, \ldots, \lambda_0^{\mu_1-1}\partial F/\partial\lambda_1,\\
&\qquad\qquad\qquad\qquad \ldots, \partial F/\partial\lambda_r, \ldots, \lambda_0^{\mu_r-1}\partial F/\partial\lambda_r,
\end{aligned}$$

where all $\mu_i \geq 0$ and $\mu_0 + \mu_1 + \cdots + \mu_r = \mu$.

In the space $\mathcal{O}^p_{x,\lambda}/\{I_F\mathcal{O}^p_{x,\lambda} + \mathcal{O}_{x,\lambda}\langle \partial F/\partial x_1, \ldots, \partial F/\partial x_n\rangle\}$ there exist decompositions:

$$\lambda_0^{\mu_i} \cdot \partial F/\partial\lambda_i \equiv v_{0,i}\partial F/\partial\lambda_0 + \cdots + v_{r,i}\partial F/\partial\lambda_r, \qquad i = 0, \ldots, r,$$

where $v_{si}(\lambda)$ are polynomials in λ_0 of degree strictly less than μ_s ($v_{si} = 0$ for $\mu_s = 0$).

Theorem 3.2 [12].

$$\Theta_\Delta = \mathcal{O}_\lambda\langle v_0, \ldots, v_r\rangle, \qquad v_i = (v_{0i} - \delta_{0i}\lambda_0^{\mu_i})\partial_{\lambda_0} + \cdots + (v_{ri} - \delta_{ri}\lambda_0^{\mu_i})\partial_{\lambda_r}.$$

For a quasihomogeneous complete intersection of positive dimension there is another algorithm for constructing generators of the module Θ_Δ.

Let $F(x,\lambda)$ be this time a quasihomogeneous τ-parameter miniversal deformation of a complete intersection $f_0(x_0) = 0$, $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n > p$. Let $\alpha_1, \ldots, \alpha_\tau$

be the weights of the parameters $\lambda_1, \ldots, \lambda_\tau$. The Euler vector field $\mathfrak{e} = \alpha_1\lambda_1\partial_{\lambda_1} + \cdots + \alpha_\tau\lambda_\tau\partial_{\lambda_\tau}$ is tangent to the discriminant $\Delta \subset \mathbb{C}^\tau$. Let us express the other generators of Θ_Δ in terms of $\mathfrak{e}$.

Let $\Psi = (\Psi_{ij}(\lambda))$ be the matrix of multiplication by the function $\psi(x) \in \mathcal{O}_x$ in the $\mathcal{O}_\lambda$-module $\mathcal{O}^p_{x,\lambda}/\{I_F\mathcal{O}^p_{x,\lambda} + \mathcal{O}_{x,\lambda}\langle \partial F/\partial x_1, \ldots, \partial F/\partial x_n\rangle\}$ with respect to the generators $\partial F/\partial\lambda_1, \ldots, \partial F/\partial\lambda_\tau$:

$$\psi \cdot \partial F/\partial\lambda_j \equiv \Psi_{1j}\partial F/\partial\lambda_1 + \cdots + \Psi_{\tau j}\partial F/\partial\lambda_\tau, \qquad j = 1, \ldots, \tau.$$

The matrix Ψ is determined up to addition to its columns of columns of components of any fields from Θ_Δ.

We identify a vector field on $\mathbb{C}^\tau$ with the τ-colunm of its components. One can easily see that the field $\Psi\mathfrak{e}$ preserves the discriminant.

Now, consider the ideal $I \subset \mathcal{O}_x$ generated by the coordinate functions of the mapping f_0 and by all the p-minors of the Jacobi matrix $(\partial f_0/\partial x)$. Its codimension coincides with the Tjurina number τ [14]. Let $\psi_1, \ldots, \psi_\tau \in \mathcal{O}_x$ represent a basis of $\mathcal{O}_x/I$ and $\Psi_1, \ldots, \Psi_\tau$ be the corresponding multiplication matrices.

Theorem 3.3 [13]. *The vector fields $\Psi_1\mathfrak{e}, \ldots, \Psi_\tau\mathfrak{e}$ are free generators of the $\mathcal{O}_\lambda$-module of vector fields tangent to the discriminant of the quasihomogeneous isolated singularity of complete intersection $f_0 = 0$ of positive dimension.*

4. Equation of a free divisor

Let $\Xi \subset \mathbb{C}^r$ be a free divisor. The module of vector fields on $\mathbb{C}^r$ tangent to Ξ is free. Let $v_i = v_{1i}\partial_{\lambda_1} + \cdots + v_{ri}\partial_{\lambda_r}$, $i = 1, \ldots, r$, be its generators.

Theorem 4.1 [19]. $\quad \Xi = \{\det(v_{ij}) = 0\}$.

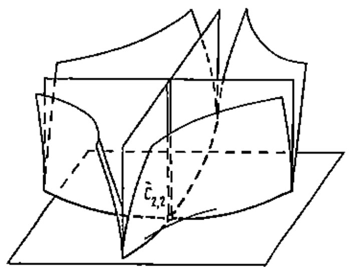
Examples 4.2.

a) Example 1.3 provides the equation $27\lambda_1^3 + 8\lambda_1\lambda_2^3 = 0$ for the bifurcation diagram of functions A_3.

b) According to Example 2.2, the bifurcation diagram Σ of projection $C_{2,2}$ is given in $\mathbb{C}^3$ by the equation

$$\begin{aligned}
&\lambda_3(\lambda_1^2 - \lambda_2^2)(409\lambda_3^3 + 768\lambda_1\lambda_2\lambda_3^2 +\\
&27\lambda_1^4\lambda_3 - 6\lambda_1^2\lambda_2^2\lambda_3 + 27\lambda_2^4\lambda_3 + \lambda_1^3\lambda_2^3)\\
&\qquad = 0.
\end{aligned}$$

This surface is shown on the right.



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