ON PLANAR CAUSTICS

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to the memory of Sergei Duzhin

Abstract

We study local invariants of planar caustics, that is, invariants of Lagrangian maps from surfaces to $\mathbb{R}^2$ whose increments in generic homotopies are determined entirely by diffeomorphism types of local bifurcations of the caustics. Such invariants are dual to trivial codimension 1 cycles supported on the discriminant in the space $\mathcal{L}$ of the Lagrangian maps.

We obtain a description of the spaces of the discriminantal cycles (possibly non-trivial) for the Lagrangian maps of an arbitrary surface, both for the integer and mod2 coefficients. It is shown that all integer local invariants of caustics of Lagrangian maps without corank 2 points are essentially exhausted by the numbers of various singular points of the caustics and the Ohmoto-Aicardi linking invariant of ordinary maps. As an application, we use the discriminantal cycles to establish non-contractibility of certain loops in $\mathcal{L}$.

Keywords: Lagrangian map, caustic, local bifurcation, normal form, local invariant, discriminantal cycle.

Mathematics Subject Classification 2010: 58K65, 53D12

1. Introduction

Vassiliev's famous singularity theory approach to knot invariants [17] has been successfully applied to the study of invariants of various types of generic curves on surfaces.

The interest in this direction was initiated by Arnold's introduction in [4] of three order 1 Vassiliev-type invariants of regular planar curves. Two of Arnold's invariants, those dual to triple point and direct self-tangency bifurcations, were then generalised to the higher order settings [18,13,10].

Arnold gave also a classification of order 1 invariants of planar wave fronts [5], which was refined by Aicardi in [1]. A few years later followed Chernov's classification of similar invariants of fronts on arbitrary surfaces [16].

To add to this list, we must mention the Ohmoto-Aicardi classification of order 1 invariants of maps of surfaces to a plane done in [14] in terms of bifurcations of the critical value curves.

Our paper concerns invariants of another natural set of surficial curves, which are caustics, that is, the critical value sets of Lagrangian maps between two surfaces. Shapes of generic caustics on surfaces are similar to those of wave fronts or of the critical value sets in the Ohmoto-Aicardi case, but the set of generic 1- and 2-parameter bifurcations involved is richer, which makes the situation more interesting.

We are considering the space $\mathcal{L} = \mathcal{L}(M, E, N)$ of all Lagrangian immersions of a closed (that is, compact without boundary) surface M to the space E of a Lagrangian fibration $E \to N$, where N is another surface. In most cases we are assuming both surfaces oriented. We are looking for invariants of such Lagrangian maps whose increments along generic paths in $\mathcal{L}$ are completely determined by diffeomorphism types of the local bifurcations of the caustics in N. These are what should be called local order 1 invariants of the caustics, but we call them just *local* since no higher-order invariants will be considered. We denote by $\Xi \subset \mathcal{L}$ the set of Lagrangian maps at which the caustics bifurcate, and call this set the *discriminant.* We should remark that Ξ is not what one would consider as a complete discriminant in the space of all Lagrangian maps since it ignores bifurcations of self-intersection points in E of the immersed surfaces M. Respectively, our space of local invariants of caustics is a subset of the space of all local order 1 invariants of Lagrangian maps.

Up to a choice of an additive constant (individual for each connected component of $\mathcal{L}$), any numerical local invariant I is defined by its *derivative* $I' = \sum x_i X_i$, where the $X_i \subset \Xi$ are discriminantal strata of codimension 1 in $\mathcal{L}$, and the x_i are the increments of I across them. This linear combination is a trivial codimension 1 cycle in $\mathcal{L}$. Therefore, construction of such linear combinations (without an a priori knowledge of the invariants) splits into two parts:

i) establishing conditions on linear combinations of the codimension 1 strata to be cycles (we call them *discriminantal cycles*), and
ii) checking the triviality of the discriminantal cycles.

The first part is approachable via an appropriate development of singularity theory methods, and does not depend on the choice of M, E and N (except for the orientability) and of a particular connected component of $\mathcal{L}(M, E, N)$. The second part is either sufficiently straightforward (when an integral, that is, homotopy-free interpretation of a relevant invariant is available), or quite hard (when there is no such interpretation, and this is a general situation). In the latter case, knowledge of the fundamental group of a particular connected component of $\mathcal{L}(M, E, N)$ could be helpful, but calculation of this group is a even more complicated task. On the other hand, discriminantal cycles themselves may be used for testing non-contractibility of certain loops in $\mathcal{L}$, and we are giving examples of this in Section 5.

The main result of this paper is a complete description of the spaces of discriminantal cycles for caustics on surfaces. Translation of this description to a description of the local invariants themselves has turned out to be the most complete for the target $N = \mathbb{R}^2$ and the subset $\mathcal{L}_1$ of $\mathcal{L}(M, E, \mathbb{R}^2)$ consisting of maps without corank 2 singularities. For such a setting, up to a choice of additive constants on connected components of $\mathcal{L}_1$, all rational local invariants of caustics are linear combinations of the numbers of various singular points of the caustics and of the linking invariant of ordinary maps from [14]. In all the other cases, the question of triviality of certain discriminantal cycles is open, which allows to only bound the dimensions of the invariant spaces.

The structure of the paper is as follows.

Section 2 reminds the generalities about Lagrangian maps, describes stable singularities of planar caustics and gives examples of their local invariants. Section 3 lists discriminantal strata of codimension 1 in $\mathcal{L}$, and its Subsection 3.4 states our main results. Section 4 proves our main theorems via analysis of generic 2-parameter families of caustics. Its Subsection 4.4 considers adjustments needed if at least one of the source and target surfaces is not oriented. Finally, in Section 5, we use the discriminantal cycles corresponding to corank 2 degenerations of maps to prove non-contractibility of certain loops in the spaces of Lagrangian maps of the 2-sphere. It would be very interesting to see to what extent the results of this section could be generalised to loops in other connected components of $\mathcal{L}(S^2, E, N)$ and to the source different from S^2.

2. Lagrangian maps

2.1. *General definitions*

We start with recalling a series of standard definitions which may be found, for example, in [6] or [3].

A symplectic structure on a manifold E^{2n} is a closed non-degenerate differential 2-form ω.

A *Lagrangian submanifold* of a symplectic manifold (E^{2n}, ω) is its n-dimensional submanifold the restriction to which of the symplectic form vanishes.

A *fibration* $p : E^{2n} \to N^n$ of a symplectic manifold E over a base N is called *Lagrangian* if all its fibres are Lagrangian submanifolds. A composition $M \xrightarrow{i} E \xrightarrow{p} N$ where i is an embedding of a manifold M^n into E^{2n} as a Lagrangian submanifold is what is usually called a *Lagrangian map.* However, in this paper we allow i to be a Lagrangian immersion.

The critical value set $\mathcal{C} \subset N$ of a Lagrangian map $p \circ i$ in called the *caustic* of the map.

All Lagrangian fibrations of the same dimension are locally isomorphic. In this paper we will be mostly considering the $n = 2$ case, and in all our local normal forms, we will be using the standard local model of $T^*\mathbb{R}^2$ fibred over $\mathbb{R}^2$. The symplectic form here is $\omega = dU\wedge du + dV\wedge dv$, where u,v and U,V are coordinates respectively on the plane and along the fibres of the fibration.

A germ of a Lagrangian surface $L\subset T^*\mathbb{R}^2_{u,v}$ is defined by its *generating family of functions* $F(x,u,v)$:

$$L = \{(u,v,U,V)\,|\,\exists\, x : F_x = 0, U = F_u, V = F_v\}\,.$$

The minimal dimension of the variable x here is the corank of the derivative of the projection $L\to\mathbb{R}^2_{u,v}$ at the base point. The smoothness of L requires that the rank of the matrix $(F_x)_{x,u,v}$ of the second derivatives at the base point must be equal to the dimension of x. The caustic $\mathcal{C}\subset\mathbb{R}^2_{u,v}$ consists of those points (u,v) for which the member $F(\cdot,u,v)$ of the generating family has non-Morse critical points.

An *equivalence* of two Lagrangian maps $L_j\to E_j\to N_j$, $j=1,2$, is a commutative diagram

$$\begin{array}{ccc} L_1 \to & E_1 & \to N_1 \\ \downarrow & \varphi\downarrow & \downarrow \\ L_2 \to & E_2 & \to N_2 \end{array}$$

in which all vertical arrows are diffeomorphisms, and $\varphi^*(\omega_2)=\omega_1$ holds for the corresponding symplectic structures. In terms of the local generating families $F(x,u,v)$ of functions this corresponds to the stable $\mathcal{R}_+$-equivalence preserving the fibration $(x,u,v)\mapsto(u,v)$ (see [6]). The stability here is in the sense of addition of non-degenerate quadratic forms in extra x-variables.

In our exposition, we will be using $N=\mathbb{R}^2$ for the target surface and $T^*\mathbb{R}^2\to\mathbb{R}^2$ for the Lagrangian fibration. The differences existing with more general settings will be addressed in the remarks. All the local normal forms of maps or families of maps that we will be using will be considered near the origins of the source, target and parameter spaces.

2.2. *Generic planar caustics and their local invariants*

According to the classical result of Whitney [20], the critical point set of a generic C^∞ map (not necessarily Lagrangian) between surfaces is a smooth curve. At isolated points on this curve the map has the *pleat* singularity, for which one can choose local coordinates in the source and target so that the map is $(z_1,z_2)\mapsto(t_1,t_2)=(z_1^3+z_1z_2,z_2)$. At all other points of the critical curve, the map has the *fold* singularity, with the local normal form $(z_1,z_2)\mapsto(z_1^2,z_2)$. See Fig. 1. Fold singularities correspond to regular branches of the critical value set, while pleat points provide semi-cubical cusps of this set. If both the source and target surfaces are oriented, we distinguish two types of pleats, of local degrees $+1$ and -1. Regular branches of a generic critical value set meet transversally.

Similarly, in the case of Lagrangian maps from surfaces to the plane, a generic caustic $\mathcal{C}$ (that is, the critical value set) is a planar curve whose only singularities

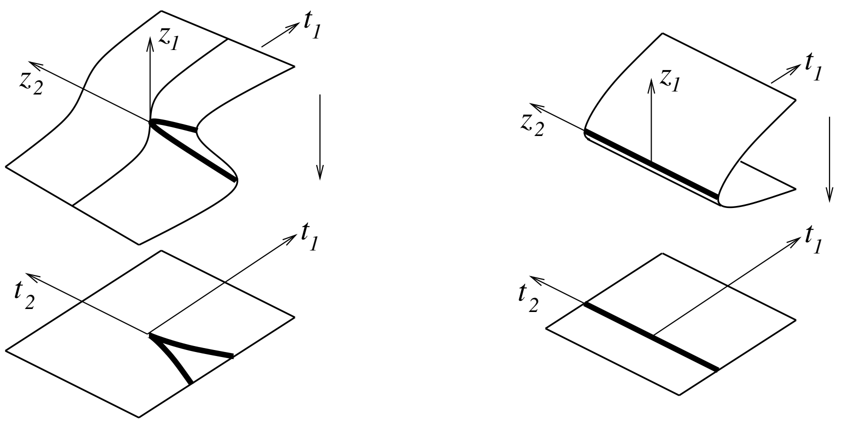

Figure 1. Pleat and fold singularities.

are points of transversal self-intersection and semi-cubical cusps. Translating the above normal forms of maps to the Lagrangian language of local generating families and using the standard notations of the corresponding function singularities (see [3]), we introduce the following notations for the points of $\mathcal{C}$:

- A_2, regular points of a caustic, with a local generating family $x^3 + ux$;
- A_2^2, points of transversal intersection of two regular local branches;
- $A_3^{s,\sigma}$, $s = \pm$, $\sigma = \pm$, semi-cubical cusps, with a local generating family $sx^4 + ux^2 + vx$, and the sign σ indicating the local degree ± 1 of the Lagrangian map.

Remark 2.1. We emphasise that the Lagrangian pleats A_3 with different choices of the sign $s = \pm$ are *not* Lagrangian equivalent, in spite of being equivalent in the oriented Whitney setting. The reason is that the function x^4 cannot be transformed to $-x^4$ by a change of the real coordinate x (see [3] for details).

Following [14], we co-orient a caustic $\mathcal{C} \subset \mathbb{R}^2$ to its side with a higher number of local pre-images. We will also show the same information by orienting $\mathcal{C}$ so that its (orientation, co-orientation) frame gives the orientation of the plane, as in Fig. 2.

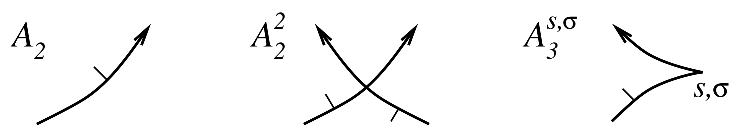

Figure 2. Singularities of generic caustics.

Consider the space $\mathcal{L} = \mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ of all Lagrangian maps $M \looparrowright T^*\mathbb{R}^2 \to \mathbb{R}^2$, where the first arrow is a Lagrangian immersion and the second the canonical projection. Maps whose caustics have more complicated singularities than those described above form the *discriminantal* hypersurface Ξ in $\mathcal{L}$.

Consider connected components of $\mathcal{L} \setminus \Xi$. A numerical *invariant* is a way to assign numbers to each of them. Along a generic path in $\mathcal{L}$, the values of an invariant change at the moments of discriminant crossings.

Definition 2.2. We say that an *invariant is local* if every increment of the invariant is completely determined by the diffeomorphism type of the local bifurcation of the caustic at the crossing.

For non-$\mathbb{Z}_2$-valued invariants, the discriminant should be locally co-oriented.

A local invariant I defines its *derivative* $I' = \sum_i x_i X_i$, where the $X_i \subset \Xi$ are the strata of codimension 1 in $\mathcal{L}$ we are able to distinguish for the needs of Definition 2.2, and the x_i are the local increments of I along generic paths in $\mathcal{L}$ crossing the X_i in the co-orienting direction. On the other hand, I is defined by I' on each connected component $\mathcal{L}_j$ of $\mathcal{L}$ up to an arbitrary choice of the value of I at a non-discriminantal base point in $\mathcal{L}_j$.

Since the total increment of I along any loop in $\mathcal{L}_j$ vanishes, the derivative $\sum_i x_i X_i$ must be a trivial codimension 1 cycle in $\mathcal{L}_j$. The vanishing of the total increment on contractible loops (that is, the derivative being a cycle, maybe non-trivial) is equivalent to its vanishing on small loops in $\mathcal{L}$ around codimension 2 strata of the discriminant. Finding the relevant cyclicity constraints on the increments x_i is the problem we are mainly concentrating on in this paper. Cycles of the form $\sum_i x_i X_i$ will be called *discriminantal*.

To establish the non-triviality of a discriminantal cycle, one should point out a loop in $\mathcal{L}$ with a non-zero index of intersection with the cycle. The loop itself is then non-contractible. We will give examples of this in Section 5.

On the other hand, one of the ways to establish the triviality of a cycle $I' = \sum_i x_i X_i$ is to find an *integral* (that is, path-independent) interpretation of its antiderivative I in terms of the geometry of individual caustics.

Example 2.3. The number of isolated singularities of $\mathcal{C}$ of a particular type is, of course, a local invariant. We have five such invariants:

I_d, *the number of double points* A_2^2*;*
$I^c_{s,\sigma}$, $s, \sigma = \pm$, *the numbers of* (s,σ)*-cusps.*

For Lagrangian maps to a plane but not to a more complicated surface, we have a sixth local invariant. It is the restriction to the set of Lagrangian maps $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ of the self-linking invariant of the critical value sets of generic smooth maps from M to the plane, as introduced by Ohmoto and Aicardi in [14]. Basically, this Bennequin-type invariant is the writhe of a ribbon defined by the critical value set in $PT^*\mathbb{R}^2$ which is then embedded into $\mathbb{R}^3$. We recall its exact definition now.

Let $\mathcal{C}$ be this time the critical value set of a generic C^∞ map from a surface M to oriented $\mathbb{R}^2$. The curve $\mathcal{C}$ is oriented in the way we oriented caustics earlier. Considering each point of $\mathcal{C}$ with its normal direction to the curve, we lift $\mathcal{C}$ to a link $\widetilde{\mathcal{C}}$ in $PT^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times S^1$. Now, for a fixed small $\varepsilon > 0$ and each point $c \in \mathcal{C}$, take the two points on the normal to $\mathcal{C}$ at c at the distance ε from c. Let $\mathcal{C}_\varepsilon \subset \mathbb{R}^2$ be the curve formed by all such points, and $\widetilde{\mathcal{C}}_\varepsilon \subset PT^*\mathbb{R}^2$ the corresponding link. Choose a small $\varepsilon_0 > 0$. The union $\widetilde{C}$ of all the $\widetilde{\mathcal{C}}_\varepsilon$ for $0 \leq \varepsilon \leq \varepsilon_0$ is an oriented multi-component ribbon in $PT^*\mathbb{R}^2$ with the core $\widetilde{\mathcal{C}}$.

We orient the solid torus $PT^*\mathbb{R}^2$ as $\mathbb{R}^2 \times S^1$, with the circular factor oriented by the positive rotational direction in the plane. We embed $PT^*\mathbb{R}^2$ unknottedly into $\mathbb{R}^3$ which we take with the orientation coming from the solid torus, and define the linking number $\ell(\mathcal{C})$ as the writhe of the ribbon $\widetilde{C} \subset \mathbb{R}^3$. For this we consider the diagram of $\widetilde{C}$ obtained from a generic projection of the $\mathbb{R}^3$ to a plane. We calculate the linking number $\ell_0(\widetilde{\mathcal{C}})$ as the usual algebraic sum of positive and negative crossings in the link diagram of $\widetilde{\mathcal{C}}$, and we also calculate the algebraic number $\ell_1(\widetilde{C})$ of signed half-twists by which the ribbon diagram differs from the blackboard one. Finally, $\ell(\mathcal{C}) := \ell_0(\widetilde{\mathcal{C}}) + \ell_1(\widetilde{C})/2$.

3. Generic codimension 1 bifurcations of planar caustics

We will now describe the strata from which we will be building up discriminantal cycles in $\mathcal{L} = \mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$. They correspond to bifurcations met in generic one-parameter families of caustics. Wherever the letters s or σ appear in the notations below, they always mean $\pm$ like in the previous section. The indices e and h distinguish between the elliptic and hyperbolic versions of similar bifurcations.

3.1. *Corank 1 multi- and uni-germs*

First of all we list all bifurcations in generic 1-parameter families which involve only corank 1 singularities of the corresponding Lagrangian maps.

We start with bifurcations of multi-germs. In each of them, one of the participating local components is a smooth A_2 branch of the caustic. We illustrate such a bifurcation in Fig. 3 only with the final curve (that is, the one to the positive side of the corresponding discriminantal stratum in $\mathcal{L}$) and indicate the shift of the smooth branch during the transition. The notations we are introducing are self-explanatory, with the letter T staying for the tangency of the caustic components.

We have (see Fig. 3):

$A_2^{3,k}$, $k = 2, 3$, triple point of a caustic. The post-bifurcational triangular region has k sides co-oriented outwards. Respectively, for the pre-bifurcational triangle, this number is $3 - k$.

$TA_2^{2,k}$, $k = 0, 1, 2$, tangency of two smooth branches. Here k is the number of sides of the new-born bi-gon co-oriented outwards.

$A_3^{s,\sigma}A_2^k$, $k = 0, 1$, an (s, σ)-cusp passes through a smooth branch of a caustic. Here the value of k distinguishes between the two co-orientation possibilities of the regular curve.

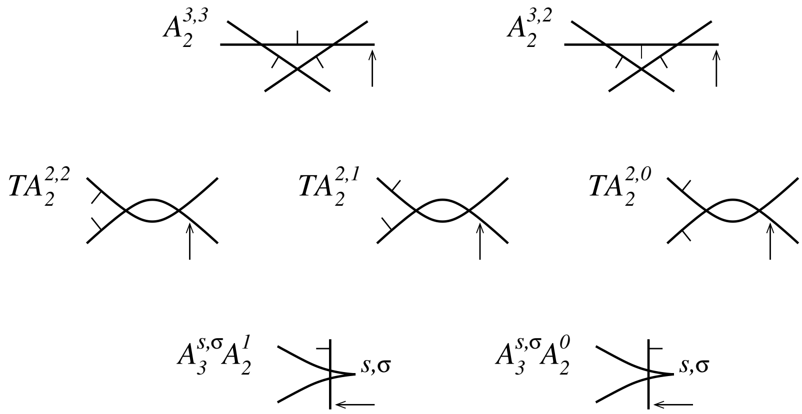

Figure 3. Generic bifurcations of multi-germ caustics.

For the uni-germs, we have the following transformations of the caustics (see Fig. 4 where the directions of the positive moves are from the left to the right):

$A_3^{s,\sigma;e}$, birth of a lips component, with two (s, σ)-cusps.

$A_3^{s,\sigma;h}$, a beaks bifurcation of the caustic, with two (s, σ)-cusps appearing.

$A_4^{s,\sigma}$, a swallowtail bifurcation. The (s, σ)-cusp is the first of the two cusps on the local post-bifurcational curve if we follow its conventional orientation.

Normal forms of the generating family bifurcations in the last three cases respectively are

$$s(x^4 + (v^2 - \lambda)x^2) + ux\,, \qquad s(x^4 + (\lambda - v^2)x^2) + ux\,, \qquad x^5 - \lambda x^3 + vx^2 + ux\,,$$

where λ is the parameter increasing in the bifurcation (see [2,21]).

3.2. *Corank 2 bifurcations in one-parameter families*

According to [21], any local bifurcation of caustics in this case may be obtained as a generic one-parameter family of planar sections of the spatial caustics of the $\mathcal{R}_+$-miniversal deformations of the $D_4^\pm$ function singularities:

$$F(x, y; u, v, w) = \pm x^2 y + \frac{1}{3}y^3 + \frac{1}{2}wy^2 + vy + ux\,.$$

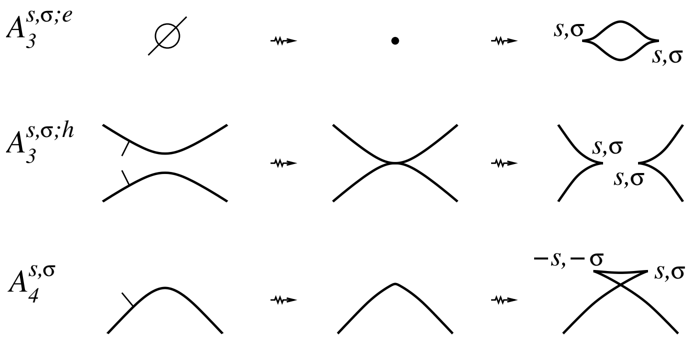

Figure 4. Generic corank 1 bifurcations of uni-germ caustics.

These two caustics are shown in Fig. 5. The parameter of such a sectional family reduces to $\lambda = w + \alpha u + \beta v$, $\alpha, \beta \in \mathbb{R}$, $\alpha^2 \mp \beta^2 \neq 0$. The inequality condition here means that the sectional surface passing through the origin should not be tangent to the self-intersection locus (real for D_4^+ and imaginary for D_4^-) of the spatial caustic. The families are actually uni-modular: if one of the coefficients α and β is not zero, it may be normalised to ± 1, the other staying arbitrary.

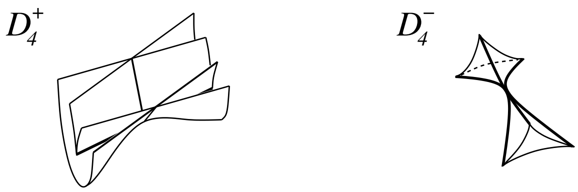

Figure 5. The $D_4^{\pm}$ caustics in $\mathbb{R}^3$.

We co-orient the corresponding discriminantal strata in $\mathcal{L}$ by the *decrease* of the above parameter λ, which means that in the sectional planar caustics the $(-, \sigma)$-cusps change to the $(+, \sigma)$-cusps. We distinguish five pairs of corank 2 bifurcations shown in Fig. 6. The subscripts in the notations of the first four of them store the information about the post-bifurcational double points: either their number, or the right/left position of the only point.

3.3. *Derivatives of the standard invariants*

In what follows, it will be convenient for us to operate with sums of the above elementary discriminantal strata differing only in certain indices in their notation. In such cases we will omit the corresponding signs or letter and assume that the

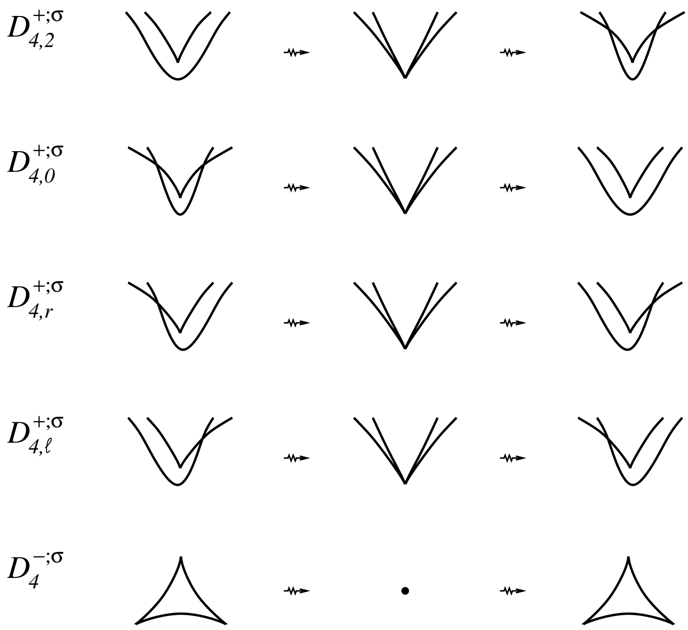

Figure 6. Generic corank 2 bifurcations of caustics. All cusps of the curves on the left are $(-,\sigma)$, and all of the curves on the right are $(+,\sigma)$.

summation is done across the whole range of the omitted symbols, for example:

$$A_2^3 = A_2^{3,3} + A_2^{3,2}\,, \quad A_3A_2 = \sum_{s,\sigma=\pm} (A_3^{s,\sigma}A_2^1 + A_3^{s,\sigma}A_2^0)\,, \quad D_{4,2}^+ = D_{4,2}^{+;+} + D_{4,2}^{+;-}\,.$$

However, to avoid notational confusion, we will use

$$A_3^{s,\sigma;e/h} = A_3^{s,\sigma;e} + A_3^{s,\sigma;h} \quad \text{and} \quad A_3^{e/h} = A_3^e + A_3^h\,.$$

Within these settings we have the following summary of the increments of the invariants introduced in Section 2.2.

Lemma 3.1. *The derivatives of the invariants counting the double points and various cusps of planar caustics, and of their linking invariant are*

$$\begin{aligned} I_d' &= 2TA_2^2 + 2A_3A_2 + A_4 + 2D_{4,2}^+ - 2D_{4,0}^+\,, \\ {I^c_{s,\sigma}}' &= 2A_3^{s,\sigma;e/h} + A_4^{s,\sigma} + A_4^{-s,-\sigma} + sD_4^{+;\sigma} + 3sD_4^{-;\sigma}\,, \\ I_\ell' &= 2TA_2^{2,2} - 2TA_2^{2,1} + 2TA_2^{2,0} + A_3^{e/h} - 2D_{4,2}^+ + 2D_{4,0}^+\,. \end{aligned}$$

Proof. The expressions for the first five derivatives are provided by a simple inspection of the bifurcation figures.

The A-part of the linking derivative is a translation to our notations of the increment count done in [14] for the linking invariant of critical value sets of maps from surfaces to the plane.

To obtain the D-part, we consider the set of all Lagrangian maps $M \hookrightarrow T^*\mathbb{R}^2 \to \mathbb{R}^2$ as a subset of the space $\Omega(M, \mathbb{R}^2)$ of all smooth maps from M to $\mathbb{R}^2$. In Ω, the bifurcations of Fig. 6 are no longer stable as one-parameter families, and we deform them into generic paths along which the planar critical value sets undergo sequences of local corank 1 transitions shown in Fig. 7 (deformed $D^+_{4,0}$ and $D^+_{4,\ell}$ paths are opposite to the two in the Figure). Now the D-part of the expression follows from its A-part. $\square$

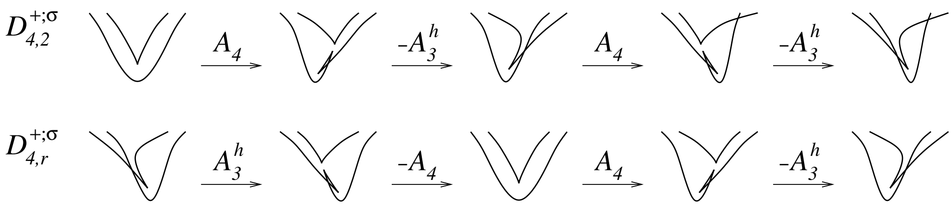

Figure 7. The D-moves of the caustics as sequences of transitions of the critical value sets of arbitrary (not necessarily Lagrangian) smooth maps. The notation of the steps is in terms of their Lagrangian analogues. Notice that the s signs of the cusps should not be used now since they are not defined in $\Omega(M, \mathbb{R}^2)$ (see Remark 2.1).

Remark 3.2. The $D^{\pm}_4$ caustics of Fig. 5 are stable as critical value sets of Lagrangian maps between 3-manifolds. However, the corresponding local singularities of maps are of codimension 1 in the space of all smooth maps between these manifolds. In particular, a small generic perturbation within smooth maps deforms the D^+_4 caustic to the left surface in Fig. 8 [7,11]. The sequences in Fig. 7 are generic 1-parameter families of planar sections of this surface.

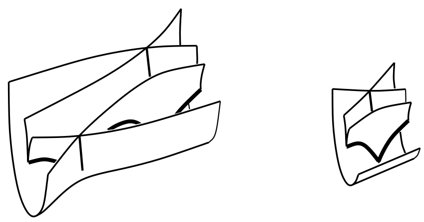

Figure 8. Stable perturbation of the D^+_4 caustic via a smooth non-Lagrangian deformation of a map between 3-manifolds. The surface has the axial symmetry which produces the whole surface from its swallowtail half shown on the right.

3.4. *Classification of the discriminantal cycles and invariants*

All statements in this section refer to any closed oriented surface M and any connected component of $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$. The target plane is oriented. All invariants are considered up to a choice of additive constants on connected components of the spaces of maps.

The main result of this paper is

Theorem 3.3. *The space of rational discriminantal cycles in $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ has rank 8. It is spanned by the derivatives of the six invariants $I_d, I^c_{s,\sigma}, I_\ell$ and the two cycles*

$$D_4^{\pm;\sigma} = D_{4,2}^{+;\sigma} + D_{4,0}^{+;\sigma} + D_{4,r}^{+;\sigma} + D_{4,\ell}^{+;\sigma} + D_4^{-;\sigma}, \quad \sigma = \pm\,.$$

For a basis of the discriminantal cycles in $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ in the integer case one can, for example, take the derivatives of

$$\begin{gathered} I^c/2 = (I^c_{+,+} + I^c_{+,-} + I^c_{-,+} + I^c_{-,-})/2,\ (I_d - I^c_{+,+} - I^c_{-,+})/2, \\ (I_\ell - I^c/2 + I^c_{+,+} + I^c_{-,+})/2, \qquad I^c_{+,+}, \qquad I^c_{+,-}, \end{gathered} \tag{3.1}$$

and the cycles

$$((I^c_{+,-} - I^c_{-,+})' - D_4^{\pm;+} - D_4^{\pm;-})/2, \qquad D_4^{\pm;+}, \qquad D_4^{\pm;-}\,. \tag{3.2}$$

Passing to the mod2 coefficients, we have

Theorem 3.4. *The space of $\mathbb{Z}_2$ discriminantal cycles in $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ has rank 9. It is spanned by the mod2 reductions of the above eight integer cycles and $A_2^3 + A_3^{\pm,+}A_2$.*

Among the cycles appearing in these two theorems, the triviality of the $D_4^{\pm;\sigma}$ and $A_2^3 + A_3^{\pm,+}A_2$ is not known. Their triviality may also depend on the choice of a particular connected component of $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$. Therefore, passing from discriminantal cycles to invariants, we have only estimates:

Corollary 3.5. *The rank of the space of integer local invariants on a particular connected component of $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ is at least 6 and at most 8. For mod2-valued invariants, the bounds are respectively 6 and 9.*

In Section 5 we will have an example when the rank in the integer case is less than 8.

The minimal invariant spaces guaranteed by the corollary are spanned, for example, by the five invariants in (3.1) and $I^c_{-,+}$.

Let $\mathcal{L}_1(M, T^*\mathbb{R}^2, \mathbb{R}^2) \subset \mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ be the set of all Lagrangian maps without corank 2 points. Discriminantal cycles in $\mathcal{L}_1$ do not contain any D-summands.

Theorem 3.6. *The space of integer discriminantal cycles in $\mathcal{L}_1(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ has rank 6. Its basis is formed, for example, by the derivatives of the invariants (3.1) and $(I^c_{+,-} - I^c_{-,+})/2$. Respectively, these six invariants form a basis of the space of all integer local invariants on $\mathcal{L}_1(M, T^*\mathbb{R}^2, \mathbb{R}^2)$.*

The mod2 setting adds here two linearly independent cycles, $A_2^3 + A_3^{\pm,+}A_2$ and $A_3^{+,+}A_2 + A_3^{-,-}A_2$, whose triviality is not known.

Remark 3.7. All statements of the above three theorems about the discriminantal cycles stay valid if we replace Lagrangian maps $M \looparrowright T^*\mathbb{R}^2 \to \mathbb{R}^2$ with Lagrangian maps $M \looparrowright E^4 \to N$, where N is an arbitrary oriented surface. The upper bounds for the ranks of the invariant spaces stated in the corollary and last theorem also stay true, but the lower bounds should be reduced by 1 since the cycle I'_ℓ may no longer be trivial.

Theorems 3.3, 3.4 and 3.6 are proved in Section 4.3, with the preparations occupying Sections 4.1 and 4.2.

4. Bifurcations in 2-parameter families of Lagrangian maps

Our proof of the classification theorems of the previous section is based on the study of bifurcations in generic 2-parameter families of caustics in the next two subsections. The bifurcation diagram of each family yields a linear equation on the increments of our local invariants across the codimension 1 strata: the equation states that the total increment along a small generic loop in $\mathcal{L}$ around the codimension 2 stratum must vanish. The whole system of these equations guarantees that the corresponding linear combination of codimension 1 strata is a discriminantal cycle in $\mathcal{L}$.

The generating families will now depend on four parameters: local coordinates u and v on the target plane, and bifurcational parameters λ_1 and λ_2. The value range for the variables s and σ is always $\{+, -\}$, tracing the (s, σ)-types of the cusps in the bifurcations.

In Section 3 we singled out 35 discriminantal strata which we will call *elementary.* A part of our strategy to choose a particular sequence of bifurcations will be to show as soon as possible that the increments across some of them must coincide and, therefore, such strata may be united into sums like those used in Lemma 3.1. We call these sums *big* strata. We denote the increment across a particular elementary or big stratum as the stratum itself, but in small characters.

Quite a few bifurcations in our analysis will be of the form $S \cdot A_2$, by which we mean a generic A_2 line passing through a generic codimension 1 bifurcation S. The co-orientation of the A_2 line will not be important.

4.1. Corank 1 maps

The bifurcations we are considering in this subsection differ from those considered in [14] for non-Lagrangian maps by the involvement of the sign σ of the cusps.

a) The simplest $S{\cdot}A_2$ bifurcations have the diagram shown in the left of Fig. 9, which gives us the equation $g = h$. In particular, this happens if the codimension 1 bifurcation S is of types **1** or **2** of the table below. Our conclusion in case **1** is that the triple point stratum A_2^3 may participate in discriminantal cycles only over $\mathbb{Z}_2$, which will be noted by the square brackets in the formulas (but not in the diagrams).

The middle bifurcation diagram in Fig. 9 is of the $A_4^{s,\sigma}\cdot A_2$ degeneration, corresponding to case **3** of the table.

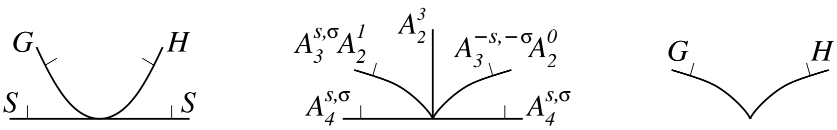

Figure 9. The diagrams of the simplest $S{\cdot}A_2$ singularities, of the $A_4^{s,\sigma}{\cdot}A_2$ bifurcations, and of the simplest cubic degenerations. They correspond to the equations **1**–**5**.

The last diagram of Fig. 9 serves the codimension 2 cubic versions of the codimension 1 'quadratic' degenerations $A_3^{s,\sigma;e}$ and $TA_2^{2,1}$. Namely, the first cubic bifurcation has generating family $sx^4 + (v^3 + \lambda_2 v + \lambda_1)x^2 + ux$, while the second is the interaction of the curves $v = 0$ and $v = u^3 + \lambda_2 u + \lambda_1$ with opposite co-orientations. The conclusions derived from these two cases are in lines **4** and **5** of the table. The superscripts *opp* and *dir* are used there for opposite and direct tangencies.

	S	equation	big stratum
1.	$TA_2^{2,2}$	$a_2^{3,3} = a_2^{3,2}$	
	$TA_2^{2,1}$	$2a_2^{3,2} = 0$	$[A_2^3] = A_2^{3.3} + A_2^{3,2}$
2.	$A_3^{s,\sigma;e}$	$a_3^{s,\sigma}a_2^1 = a_3^{s,\sigma}a_2^0$	$A_3^{s,\sigma}A_2 = A_3^{s,\sigma}A_2^1 + A_3^{s,\sigma}A_2^0$
3.	$A_4^{s,\sigma}$	$a_3^{s,\sigma}a_2 - a_3^{-s,-\sigma}a_2 = [a_2^3]$	$A_3^{\pm(+,\sigma)}A_2 = A_3^{+,\sigma}A_2 + A_3^{-,-\sigma}A_2$ over $\mathbb{Z}$
4.	$A_3^{s,\sigma;e}$	$a_3^{s,\sigma;e} = a_3^{s,\sigma;h}$	$A_3^{s,\sigma;e/h} = A_3^{s,\sigma;e} + A_3^{s,\sigma;h}$
5.	$TA_2^{2,1}$	$ta_2^{2,2} = ta_2^{2,0}$	$TA_2^{2,opp} = TA_2^{2,2} + TA_2^{2,0}$
			$TA_2^{2,dir} = TA_2^{2,1}$

b) In Figs. 10 and 11, we show bifurcation diagrams of three more codimension 2 degeneration. The corresponding incremental equations are **6**–**8** below.

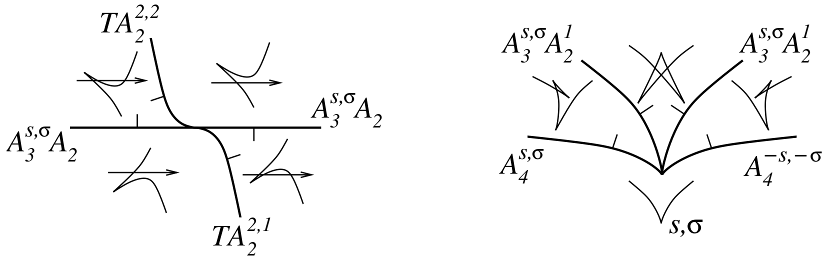

Figure 10. The line-and-cusp tangency bifurcation, and a family $F = sx^6+\lambda_1x^4+\lambda_2x^3+vx^2+ux$ of planar sections of the A_5 caustic in $\mathbb{R}^4$.

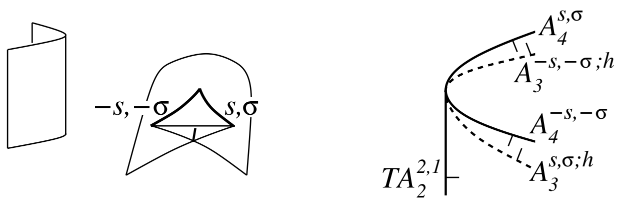

Figure 11. Bifurcations of a swallowtail section by a smooth surface tangent to the self-intersection line.

	equation	big stratum
6.	$2a_3^{s,\sigma}a_2 = ta_2^{2,opp} + ta_2^{2,dir}$	$A_3A_2 = \sum_{s,\sigma=\pm} A_3^{s,\sigma}A_2$ over $\mathbb{Z}$
7.	$a_4^{s,\sigma} = a_4^{-s,-\sigma}$	$A_4^{\pm(+,\sigma)} = A_4^{+,\sigma} + A_4^{-,-\sigma}$
8.	$2a_4^{\pm(+,\sigma)} = a_3^{+,\sigma;h} + a_3^{-,-\sigma;h} + ta_2^{2,dir}$	

4.2. *Corank 2 bifurcations*

a) *The $D_{4,r}^{+;\sigma}{\cdot}A_2$ family.* Comparing the events on the left and on the right in Fig. 12 during the motion of the additional A_2 component and recalling from equations **1** above that $2a_2^3 = 0$, we conclude that the incremental equation here reduces to

$$\mathbf{9}. \quad a_3^{-,\sigma}a_2 = a_3^{+,\sigma}a_2\,.$$

This provides us with a big stratum $A_3^{\pm,\sigma}A_2 = A_3^{+,\sigma}A_2 + A_3^{-,\sigma}A_2$ over $\mathbb{Z}_2$. (We already have A_3A_2 over the integers.) All the other versions of the $D_4^+{\cdot}A_2$ bifurcations yield the same.

b) *Degenerate sections of the D_4^+ caustic.* In Section 3.2 we quoted the normal form of a generic function on $\mathbb{R}^3$ in presence of the D_4^+ caustic. We will now denote this caustic $\mathcal{C}(D_4^+)$. A standard argument using the description from [21] of the vector fields tangent to $\mathcal{C}(D_4^+)$ shows that the functions next in the hierarchy in

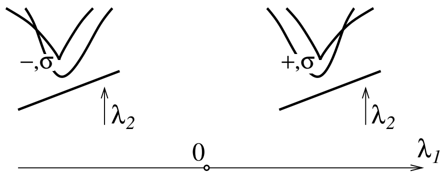

Figure 12. The $D^{+;\sigma}_{4,r}{\cdot}A_2$ bifurcation.

this case can be reduced to the normal forms

$$w \pm u \pm v + \gamma v^2\,, \quad \gamma \in \mathbb{R} \setminus \{0\}\,.$$

Here γ is a modular coefficient. The zero level of such a function is tangent at the origin to one of the two rays of the self-intersection locus of $\mathcal{C}(D_4^+)$. Making use of a natural notion of a versal deformation of a function germ on $\mathbb{R}^3$ with respect to the group of diffeomorphisms preserving the caustic, we see that the functions above give us two-parameter families of caustics defined by generating families of functions depending on two additional parameters (λ_1, λ_2):

$$x^2 y + \frac{1}{3}y^3 + \frac{1}{2}(\pm u \pm v + \gamma v^2 + \lambda_2 v + \lambda_1)y^2 + vy + ux\,. \tag{4.1}$$

Various choices of the signs in this formula (including the sign of γ) give us the bifurcation diagrams in the (λ_1, λ_2)-plane shown in Fig. 13. Comparison of the first two diagrams there implies $d^{+;\sigma}_{4,r} = d^{+;\sigma}_{4,\ell}$, which allows us to introduce a big stratum

$$D^{+;\sigma}_{4,r/\ell} = D^{+;\sigma}_{4,r} + D^{+;\sigma}_{4,\ell}\,.$$

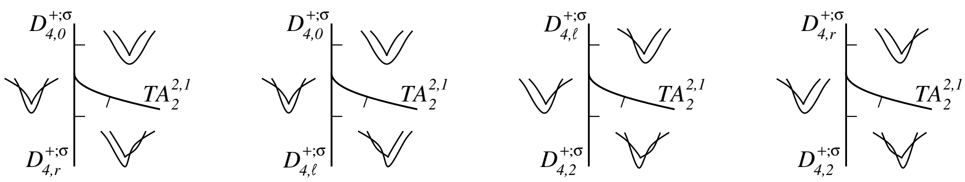

Figure 13. Bifurcation diagrams of the families (4.1). In each diagram, the cusp on the left of the vertical strata is $(-,\sigma)$ and the cusps on the right are $(+,\sigma)$. The opposite cusp option is not shown since it yields the same set of four incremental equations.

Now Fig. 13 provides two linearly independent incremental equations for each $\sigma = \pm$:

$$\mathbf{10}. \quad \begin{aligned} ta_2^{2,1} &= d^{+;\sigma}_{4,r/\ell} - d^{+;\sigma}_{4,0} \\ &= d^{+;\sigma}_{4,2} - d^{+;\sigma}_{4,r/\ell} \end{aligned}$$

We remark that we are not considering here functions on $(\mathbb{R}^3, \mathcal{C}(D_4))$ whose zero level is tangent at the origin to the cuspidal edge of the caustic since such functions would correspond to the change of topology of the source surface of our Lagrangian maps. However, such functions will appear in Section 5 where they will be used for constructing non-contractible loops in the spaces of Lagrangian maps.

c) *The D_5 family.* In Fig. 14 we are showing the bifurcation diagrams of two-parameter families of caustics coming from the deformations

$$x^2y \pm y^4 + \lambda_1 y^3 + \lambda_2 y^2 + vy + ux$$

of the $D_5^\pm$ function singularities. Since each diagram has two pairs of double strata, these two-parameter families of planar caustics are infinitely degenerate. However, these families are the principal quasi-homogeneous parts of generic two-parameter slicings of the big $D_5^\pm$ caustics in $\mathbb{R}^4$, and yield the same incremental equations as such generic slicings do, namely:

$$\mathbf{11}. \qquad \begin{aligned} d_{4,2}^{+;\sigma} - d_4^{-;\sigma} + a_3^{+,\sigma;e} + a_3^{+.-\sigma;h} - 2a_4^{+,-\sigma} &= 0 \\ -d_{4,0}^{+;\sigma} + d_4^{-;\sigma} + a_3^{-,\sigma;e} + a_3^{-.-\sigma;h} - 2a_4^{-,-\sigma} &= 0 \end{aligned}$$

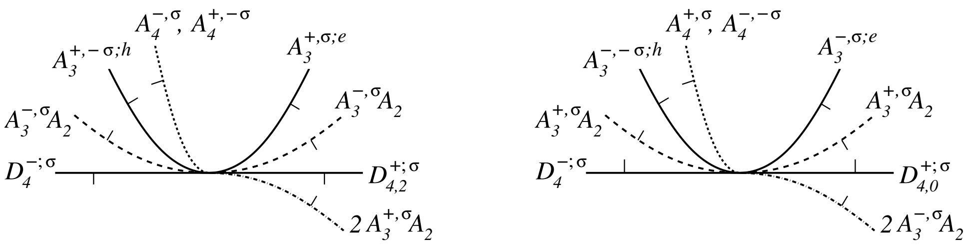

Figure 14. The D_5^+ and D_5^- bifurcation diagrams.

This finishes the process of deriving the incremental equations. It is not very difficult to show that no other stable codimension 2 bifurcation of planar caustics delivers an equation linearly independent (both mod2 and over the integers) from the equations already listed.

4.3. *Proofs of the classification theorems*

We initially had 35 elementary discriminantal strata. Over the previous two subsections we have been able to join them, both over $\mathbb{Z}$ and $\mathbb{Z}_2$, into 19 bigger. Equations on the increments of the invariants across these 19 strata obtained during the bifurcation analysis are collected in the columns of the first half of Table 1 below. Equations which are integer linear combinations of the others are not included there. That is why only one of the equations **6** and only the first pair of the equations **11** are in the table. We are using dots instead of zeros.

With 11 linearly independent equations in 19 unknowns, we have a rank 8 solution space. The 8 columns of the second half of Table 1 contain the coefficients of the discriminantal cycles mentioned in Theorem 3.3. They are easily seen to form a basis of the solution space over the rationals, which proves the theorem.

The rank of the mod2 solutions matrix in Table 1 is 5. This yields a few parity conservation laws for appropriate linear combinations of numbers of double points and various cusps of caustics in homotopies. The most obvious one is that the parity of the total number of cusps stays the same.

To produce an integer solution basis from the rational one we can, for example, consider its modification (3.1–3.2) from Section 3.4.

The mod2 reduction of the incremental equations drops the rank of the coefficient matrix by 1, due to the elimination of the equation **1**. Therefore, a mod2 solution basis may be obtained by addition to the reduced basis (3.1–3.2) of one cycle, for example, $A_2^3 + A_3^{\pm,+}A_2$ (or $A_2^3 + A_3^{\pm,-}A_2$). This proves Theorem 3.4.

Table 1	1	3	6	8	8	10	10	10	10	11	11	I'_d	$I^{c\ \prime}_{+,+}$	$I^{c\ \prime}_{+,-}$	$I^{c\ \prime}_{-,+}$	$I^{c\ \prime}_{-,-}$	I'_ℓ	$D_4^{\pm;+}$	$D_4^{\pm;-}$
A_2^3	2	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$TA_2^{2,opp}$	.	.	−1	.	.	.	.	.	.	.	.	2	.	.	.	.	2	.	.
$TA_2^{2,dir}$	.	.	−1	−1	−1	1	1	1	1	.	.	2	.	.	.	.	−2	.	.
$A_3^{\pm,+}A_2$	.	1	2	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.
$A_3^{\pm,-}A_2$	.	−1	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.
$A_3^{+,+;e/h}$	.	.	.	−1	.	.	.	.	.	1	1	.	2	.	.	.	1	.	.
$A_3^{+,-;e/h}$	.	.	.	.	−1	.	.	.	.	1	1	.	.	2	.	.	1	.	.
$A_3^{-,+;e/h}$	.	.	.	.	−1	.	.	.	.	.	.	.	.	.	2	.	1	.	.
$A_3^{-,-;e/h}$	.	.	.	−1	.	.	.	.	.	.	.	.	.	.	.	2	1	.	.
$A_4^{\pm(+,+)}$	.	.	.	2	.	.	.	.	.	.	−2	1	1	.	.	1	.	.	.
$A_4^{\pm(+,-)}$	.	.	.	.	2	.	.	.	.	−2	.	1	.	1	1	.	.	.	.
$D_{4,2}^{+;+}$	.	.	.	.	.	.	.	−1	.	1	.	2	1	.	−1	.	−2	1	.
$D_{4,2}^{+;-}$	.	.	.	.	.	.	.	.	−1	.	1	2	.	1	.	−1	−2	.	1
$D_{4,r/\ell}^{+;+}$	.	.	.	.	.	−1	.	1	.	.	.	.	1	.	−1	.	.	1	.
$D_{4,r/\ell}^{+;-}$	.	.	.	.	.	.	−1	.	1	.	.	.	.	1	.	−1	.	.	1
$D_{4,0}^{+;+}$	.	.	.	.	.	1	.	.	.	.	.	−2	1	.	−1	.	2	1	.
$D_{4,0}^{+;-}$	.	.	.	.	.	.	1	.	.	.	.	−2	.	1	.	−1	2	.	1
$D_4^{-;+}$	.	.	.	.	.	.	.	.	.	−1	.	.	3	.	−3	.	.	1	.
$D_4^{-;-}$	.	.	.	.	.	.	.	.	.	.	−1	.	.	3	.	−3	.	.	1

For Theorem 3.6, avoiding corank 2 maps, we have to restrict our attention to the A-strata and equations **1**–**8** only. To cover the integer and mod2 options simultaneously we have to consider this time all four strata $A_3^{s,\sigma}A_2$ individually. The equation-cycle table for this case is Table 2 below. We have 7 linearly independent equations in 13 unknowns. The set of the discriminantal cycles suggested for an integer basis in the theorem occupies the second half of the table, and is indeed linearly independent. In the table we set

$$\begin{aligned} \hat{I}_d &= (I_d - I^c_{+,+} - I^c_{-,+})/2, & \hat{I}^c_{-,+} &= (I^c_{-,+} - I^c_{+,-})'/2, \\ I^c/2 &= (I^c_{+,+} + I^c_{+,-} + I^c_{-,+} + I^c_{-,-})/2, & \hat{I}_\ell &= (I_\ell - I^c/2 + \hat{I}^c_{+,+} + I^c_{-,+})/2. \end{aligned}$$

The mod2 reduction reduces this time the rank of the equation matrix by 2. Therefore, two extra basic cycles should be added to obtained a $\mathbb{Z}_2$-basis, for example, $A_2^3 + A_3^{\pm,+}A_2$ and $A_3^{+,+}A_2 + A_3^{-,-}A_2$. This finishes our proofs.

Table 2 ($\mathcal{L}_1$)	1	3	3	6	6	8	8	$\hat{I}'_d$	$I^{c\ \prime}_{+,+}$	$I^{c\ \prime}_{+,-}$	$\hat{I}^{c\ \prime}_{-,+}$	$I^{c\ \prime}/2$	$\hat{I}'_\ell$
A_2^3	2	1	1	.	.	.	.	.	.	.	.	.	.
$TA_2^{2,opp}$	.	.	.	−1	−1	.	.	1	.	.	.	.	1
$TA_2^{2,dir}$	.	.	.	−1	−1	−1	−1	1	.	.	.	.	−1
$A_3^{+,+}A_2$	.	1	.	2	.	.	.	1	.	.	.	.	.
$A_3^{+,-}A_2$	.	.	1	.	2	.	.	1	.	.	.	.	.
$A_3^{-,+}A_2$	.	.	−1	.	.	.	.	1	.	.	.	.	.
$A_3^{-,-}A_2$	.	−1	.	.	.	.	.	1	.	.	.	.	.
$A_3^{+,+;e/h}$	.	.	.	.	.	−1	.	−1	2	.	.	1	1
$A_3^{+,-;e/h}$	.	.	.	.	.	.	−1	.	.	2	−1	1	.
$A_3^{-,+;e/h}$	.	.	.	.	.	.	−1	−1	.	.	1	1	1
$A_3^{-,-;e/h}$	.	.	.	.	.	−1	.	.	.	.	.	1	.
$A_4^{\pm(+,+)}$	.	.	.	.	.	2	.	.	1	.	.	1	.
$A_4^{\pm(+,-)}$	.	.	.	.	.	.	2	.	.	1	.	1	.

4.4. *Non-oriented source or target*

Assume first of all that the source surface M is not oriented while the target plane has an orientation chosen. This means gluing together discriminantal strata of codimension 1 in $\mathcal{L}(M, T^*\mathbb{R}^2, \mathbb{R}^2)$ differing only by the sign σ in their notation. The modified equations and basic cycles are collected in Table 3. The sign σ is now gone from the notations.

Table 3	1	3	6	8	10	10	11	I'_d	$I^c_+{}'$	$I^c_-{}'$	I'_ℓ	$D_4^\pm$
A_2^3	2	1	.	.	.	.	.	.	.	.	.	.
$TA_2^{2,opp}$	.	.	−1	.	.	.	.	2	.	.	2	.
$TA_2^{2,dir}$	.	.	−1	−1	1	1	.	2	.	.	−2	.
$A_3^+A_2$	.	1	2	.	.	.	.	2	.	.	.	.
$A_3^-A_2$	.	−1	.	.	.	.	.	2	.	.	.	.
$A_3^{+;e/h}$	.	.	.	−1	.	.	2	.	2	.	1	.
$A_3^{-;e/h}$	.	.	.	−1	.	.	.	.	.	2	1	.
A_4	.	.	.	2	.	.	−2	1	1	1	.	.
$D_{4,2}^+$	.	.	.	.	.	−1	1	2	1	−1	−2	1
$D_{4,r/\ell}^+$	.	.	.	.	−1	1	.	.	1	−1	.	1
$D_{4,0}^+$	.	.	.	.	1	.	.	−2	1	−1	2	1
D_4^-	.	.	.	.	.	.	−1	.	3	−3	.	1

Analysis of Table 3 and comparison with the oriented case show that in the space $\mathcal{L}(M,E,N)$ for non-oriented source M and oriented target N,

i) a rational basis of discriminantal cycles is formed by the cycles in the second half of Table 3;

ii) for a basis over the integers one can take

$$I^c{}'/2, \quad I^c_+{}', \quad (I'_d + I'_\ell - I^c{}'/2)/2, \quad (I'_d - I^c_+{}' + D_4^\pm)/2, \quad D_4^\pm\,;$$

iii) to obtain a $\mathbb{Z}_2$-basis one should add $A_2^3 + A_3^+A_2$ to the mod2 reductions of the integer basis.

We see that, depending on the triviality of the linear combinations of the I'_ℓ, $D_4^\pm$ and $A_2^3 + A_3^+A_2$ cycles, the rational or integer local invariant spaces have ranks at least 3 and at most 5, with the upper bound goes up to 6 over $\mathbb{Z}_2$.

Switching to the space $\mathcal{L}_1(M,E,N)$ of corank at most 1 maps, we need to drop every mentioning of the $D_4^\pm$ cycle in the above items. In particular, this reduces all the rank bounds by 1. In particular, we have

Proposition 4.1. *The space of integer local invariants of Lagrangian maps of a non-oriented surface M to oriented $\mathbb{R}^2$ is 4-dimensional. Its basis is formed by the invariants*

$$I^c/2,\ I^c_+,\ (I_d + I_\ell - I^c/2)/2 \quad \textit{and} \quad (I_d - I^c_+)/2.$$

Assume now the target surface N non-oriented making no assumption on orientability of the source M. In addition to the loss of the local degree index σ in the notations of the strata in Section 3 we have had so far in the current section, this condition allows for only one type of the A_4 bifurcation and also makes no difference between the $D_{4,r}^+$ and $D_{4,\ell}^+$ transitions. However, this does not imply any further amendment to Table 3. Therefore, all our observations about the spaces of discriminantal cycles stay true if at least one of the surfaces M and N is not oriented.

5. Non-trivial discriminantal cycles

In this section we are showing that some of the integer discriminantal cycles we have found are non-trivial in perhaps the simplest possible situations, namely in the space of Lagrangian mappings of a 2-sphere, in its component of contractible maps (the contractibility requirement includes the contractibility of the induced map to the Lagrangian Grassmannian). The idea is to construct a loop in a space of Lagrangian maps having a non-zero intersection number with a cycle. The loop in its turn will be non-contractible.

We start with a two-parameter family of caustics formed by the bifurcations of the section of the D_4^+ caustic in $\mathbb{R}^3$ by a smooth sheet tangent to the cuspidal edge at the D_4^+ point, as illustrated in Fig. 15. The corresponding generating family depending on two additional parameters λ is

$$F(x,y,u,v,\lambda) = x^2y + \frac{1}{3}y^3 + (v+\lambda_1)y^2 + (av^2+\lambda_2)y + ux\,,$$

where $a > 1$ is a constant. The equation $F_y = 0$ shows that the source bifurcates between a sphere in the xyv-coordinate space and the empty set. The local degrees of the Lagrangian maps at all pleat points are the same, and we are assuming them to be $+1$ at this moment.

Consider now a path γ in $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$ that may be described in appropriate coordinates as induced from the generating family F as shown in Fig. 15. A more complicated version of the same homotopy of planar caustics appeared in [15] (without any relation to the sections of the D_4^+ caustic) as a candidate for a non-trivial loop in $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$ in assumption that the orientation of S^2 is ignored. We are going to show that γ is indeed a non-trivial loop in such a setting provided the space of the Lagrangian maps is interpreted correctly.

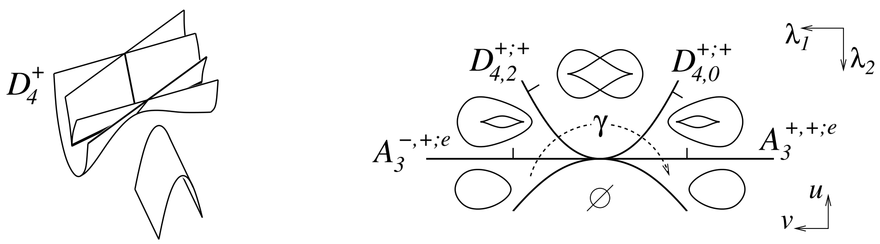

Figure 15. Sections of the D_4^+ caustic by a smooth sheet non-transversal to its cuspidal edge. In the bifurcation diagram, all the cusps in its left half are of $(-,+)$-type, while all those in the right half are $(+,+)$-cusps.

A Lagrangian map of a surface $M \to T^*\mathbb{R}^2 \to \mathbb{R}^2_{u,v}$, defined by a global generating family F of functions in the way considered for map germs in Section 2.1, lifts to a map $M \to \mathbb{R}^3_{u,v,F}$ if we use the values of the family as the third coordinate in the target. The image of such a map is called a *wave front* (see [3] or [6] for the theory of Legendrian maps and other related topics). Using this lifting, the homotopy of the Lagrangian maps in $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$ corresponding to the path γ in Fig. 15 may be understood via the homotopy of the corresponding wave fronts in $\mathbb{R}^2_{u,v,F}$. The latter is an eversion of a flying saucer front: starting γ with the saucer with the inward co-orientation, we are changing it to the outward one. See Fig. 16.

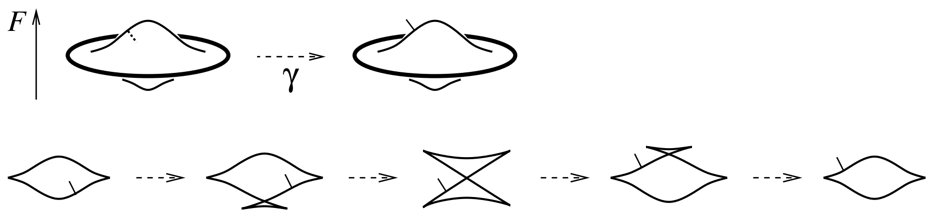

Figure 16. Eversion of a flying saucer front, and the sequence of bifurcations of its sections by the planes $v = const$ during the homotopy of the planar caustics along the path γ in Fig. 15.

From Fig. 16 we see that the final Lagrangian map γ_1 of the path γ is in the same connected component of the complement to the discriminant Ξ in $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$ as the composition $\gamma_0 \circ j$ of the initial map with an orientation-reversing involution j of the sphere, for example, with the reflection $(x, y, v) \mapsto (-x, y, v)$. In our construction, we will be interested only in how γ and Ξ meet, and therefore we can assume that γ_1 and $\gamma_0 \circ j$ coincide.

Let $\bar\gamma$ be a path in $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$ formed by all the compositions $\gamma_t \circ j$, where $\gamma = \{\gamma_t, 0 \le t \le 1\}$. The path $\Gamma = \bar\gamma\gamma$ is a loop in $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$.

Proposition 5.1. (conjectured by Ohmoto) *The loop Γ is not contractible.*

Proof. According to Fig. 15, the indices of intersection of Γ with the discriminantal cycles $D_4^{\pm,+}$ and $D_4^{\pm,-}$, contributed respectively by γ and $\widehat{\gamma}$, are both $+2$. □

Corollary 5.2. *Consider the space of integer discriminantal cycles in the connected component of $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)$ containing maps for which the induced maps of S^2 to the Lagrangian Grassmannian of 2-planes in $\mathbb{R}^4$ are contractible. In its subspace spanned by the cycles $D_4^{\pm,+}$ and $D_4^{\pm,-}$, only the difference $D_4^{\pm,+} - D_4^{\pm,-}$ may be the derivative of an integer-valued local invariant.*

The version of the above for a non-oriented sphere is as follows. Elimination of the orientation of S^2 means that we do not distinguish between the two orientation options, that is, the space $\mathcal{L}(S^2_{nonor}, T^*\mathbb{R}^2, \mathbb{R}^2)$ is the quotient $\mathcal{L}(S^2, T^*\mathbb{R}^2, \mathbb{R}^2)/\mathbb{Z}_2$

where the $\mathbb{Z}_2$-action is by composing with any fixed orientation-reversing involution of the sphere. Within such a setting, the path γ in Fig. 15 is closed in this quotient. As it was noticed in Section 4.4, the strata in Fig. 15 lose now the second superscript in their notation, and we see that the intersection index of γ with the $D_4^{\pm}$ discriminantal cycle is 2. Hence this cycle may be the derivative of a mod2 local invariant, but not of an integer or rational one. This addresses the question from [15].

Due to the local nature of all the constructions of this section, all the claims we have done here stay valid for any Legendrian fibration $E^4 \to N^2$, not just for the cotangent bundle of $\mathbb{R}^2$.

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