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Local invariants of maps between 3-manifolds

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Abstract

We classify order 1 invariants of maps between 3-manifolds whose increments in generic homotopies are defined entirely by diffeomorphism types of local bifurcations.

We show that in the oriented case the space of integer invariants has rank 7 for any source and target, and give a geometric interpretation of its basis. The mod2 setting, with $\mathbb{R}^3$ as the target, adds another 4 linearly independent invariants, one of which combines the self-linking of the cuspidal edge of the critical value set with the number of connected components of the edge.

We also analyse general non-oriented settings and obtain either exact descriptions or rank estimates for the spaces of integer- and mod2-valued invariants.

The proofs are based on the study of bifurcations in generic 1- and 2-parameter families of maps.

Introduction

After Vassilev's introduction of finite-type invariants of knots [**10**], a similar approach has been developed to invariants of smooth maps in a range of other low dimensions. The first to be mentioned is Arnold's study of order 1 plane curve invariants [**1**, **2**]. Among numerous consequences of this study was construction of theory of higher-order invariants for certain classes of curves in the plane [**6**, **11**]. Order 1 local invariants of maps of surfaces to $\mathbb{R}^2$ and $\mathbb{R}^3$ have also been classified [**5**, **8**], followed by a higher-order construction in a special case (see [**7**]). Finally, the last order 1 setting considered so far has been that for maps of 3-manifolds to the plane [**12**].

This paper makes the next natural step in the direction of increasing dimensions, and is devoted to the classification of the invariants of maps between 3-manifolds. Throughout the paper, we assume that our manifolds have no boundary, and the source is compact. We are trying not to restrict our attention to only $\mathbb{R}^3$ as a target. We are considering all four possible source–target orientation combinations. The invariants we study are order 1 local, that is, those whose increments in generic homotopies of maps are determined entirely by the diffeomorphism types of local bifurcations. Since no higher-order invariants will be involved, we call such invariants just *local*. All invariants are considered modulo choices of additive constants.

For given source and target manifolds, M and N, each particular connected component of the space $\Omega(M, N)$ of all smooth maps between the two manifolds gives rise to its own space of local invariants. Different components may have different topology, which may imply difference between the invariant spaces. However, none of the results of this paper depends on the component choice. Therefore, we will frequently refer to local invariants of maps between a pair of manifolds without even mentioning particular components of $\Omega(M, N)$.

The invariants will be integer- and mod2-valued. We describe most of them in terms of the geometry of the critical value set $\mathcal{C}$ of a map.

Our first main result, Theorem 2.1, states that the space of integer local invariants has rank 7, provided both M and N are oriented. In such a case all the invariants are coming from the

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numbers of points of $\mathcal{C}$ of various isolated singular types (for example, numbers of positive and negative swallowtails), Euler characteristic of the critical point set, and the linking number in $J^1(M,N)$ of the 1-jet extension of a map with the set of all 1-jets of corank at least 2.

Our second main result, Theorem 4.1, concerns mod2 local invariants of maps of an oriented 3-manifold to oriented $\mathbb{R}^3$. This time the invariant space turns out to be of rank 11. We express four new generators, not coming from the integer setting, in terms of their increments across codimension 1 strata in $\Omega(M,\mathbb{R}^3)$. However, we have been able to find a homotopy-independent interpretation for only one new generator: it combines a self-linking of the cuspidal edge of $\mathcal{C}$ with the number of connected components of the edge (Theorem 4.3). A problem of integrating three other mod2 invariants stays open.

Theorem 4.1 allows us to estimate the mod2 invariant spaces for arbitrary oriented source and target manifolds (Corollary 4.2).

Our proofs of Theorems 2.1 and 4.1 are based on the analysis of generic 1- and 2-parameter families of maps. Straightforward simplifications of the analysis yield descriptions of the integer and mod2 invariant spaces in the cases when at least one of the two participating 3-manifolds is not oriented (Theorems 6.1 and 6.2). In particular, we find that, for non-oriented source and arbitrary target manifolds, the integer invariant space has rank 4 (certain attempts at this result with non-oriented $\mathbb{R}^3$ as the target were taken in [**9**]), while for maps from an oriented 3-manifold to a non-oriented one the rank of the integer space is either 4 or 5.

All our analysis is local, and thus gives exact ranks of the invariant spaces for connected components of $\Omega(M,N)$ with trivial fundamental groups. In the majority of the integer settings, passing to non-trivial fundamental groups does not reduce the rank since all the invariants possess homotopy-independent interpretations. However, in the remaining integer setting (oriented source, and non-oriented target) and for mod2 invariants, it would be very interesting to understand when such reduction is possible, that is, when the linear combinations of the strata in $\Omega(M,\mathbb{R}^3)$, dual to the local invariants of maps from M to $\mathbb{R}^3$, fail to stay trivial codimension 1 cycle if the target is changed.

The structure of the paper is as follows.

In Sections 1–5, we consider maps between oriented 3-manifolds.

Section 1 describes a stratification of the critical value set $\mathcal{C}$ of a generic map in terms of stable multi-singularities of maps, with an emphasis on the contribution of the orientations.

In Section 2, we introduce seven natural local invariants, and formulate in Theorem 2.1 that certain simple linear combinations of the seven form a basis of the space of all integer local invariants.

Section 3 contains a complete list of codimension 1 strata in the space of all maps and provides expressions for the basic invariants in dual terms, as linear combinations of such strata.

In Section 4, we consider mod2 invariants. Theorem 4.1 on the rank and basis of the mod2 invariant space for $N=\mathbb{R}^3$ is stated there, along with Theorem 4.3 on a way to derive a local invariant from the geometry of the framed link defined by the cuspidal edge of $\mathcal{C}$.

Section 5 contains proofs of Theorems 2.1 and 4.1. Here, we study bifurcations of codimension 2 singularities of the maps.

Finally, Section 6 carries out invariant classification, both over $\mathbb{Z}$ and $\mathbb{Z}_2$, in the assumption that either source or target manifold, or both of them are not oriented.

1. *Stratification of a generic critical value set*

The main object of our study is the critical value set $\mathcal{C}(f)$ of a generic map f between two 3-manifolds. A regular point of $\mathcal{C}$ is the value of f at its fold point, an A_1 singularity, near which we can locally write f as $(f_1,f_2,f_3):(x,y,z)\mapsto(x^2,y,z)$. In the first five sections of the paper, we are considering maps between oriented manifolds, and so, in local normal forms within these sections, we assume that $dx\wedge dy\wedge dz$ and $df_1\wedge df_2\wedge df_3$ are our fixed orientations

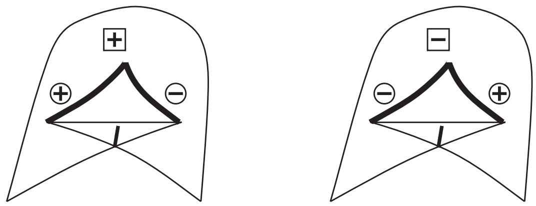

FIGURE 1. *Positive and negative swallowtails.*

of the source and target. Many of our normal forms will have ambiguities in sign choices which cannot be eliminated if the source and target orientations stay fixed.

Following [8], we fix a natural *co-orientation* of the A_1 stratum of $\mathcal{C}$, to its side on which the number of local preimages of a point is *greater*.

Irregular points of $\mathcal{C}$ are coming from various multi- and uni-germ singularities of generic maps between 3-manifolds. They are:

- A_1^2, transversal intersections of two smooth sheets;
- A_1^3, same for three sheets;
- $A_2^{\pm}$, cuspidal edges, consisting of values of f at its pleat points, that is, points near which f has normal form $(\pm x^3 + yx, y, z)$;
- $A_2^{\pm}A_1$, transversal intersections of edges with regular sheets;
- $A_3^{\pm}$, swallowtail points, at which f locally is $(\pm x^4 + yx^2 + zx, y, z)$.

The sign $\pm$ in the edge notation stays for the local degree ± 1 of the map. Two edges of opposite signs meet at a swallowtail. A swallowtail is positive if the positive edge rotates to the negative edge anti-clockwise as seen from the pyramidal side of the surface (see Figure 1).

Near edges, the co-orientation of the smooth part of $\mathcal{C}$ is inside the smaller of the two local connected regions.

Let $\Omega = \Omega(M, N)$ be the space of all C^∞-maps between two oriented three-dimensional manifolds, M and N. As has already been mentioned, we assume throughout the paper that both manifolds are without boundaries, and that M is compact. Mappings with more complicated singularities of their critical value sets than those described above form the *discriminantal* hypersurface Ξ in Ω.

Consider connected components of $\Omega \setminus \Xi$. A numerical *invariant* is a way to assign numbers to each of them. Along a generic path in Ω, the values of an invariant change at the moments of discriminant crossings.

DEFINITION 1.1. We say that an *invariant is local* if every increment of the invariant is completely determined by the diffeomorphism type of the local bifurcation of the map at the crossing.

For integer invariants, this assumes that the discriminant should be locally co-oriented.

2. *Integer invariants in the oriented setting*

2.1. *Examples*

It is clear that the number of isolated singularities of $\mathcal{C}$ of a particular type is a local invariant. We introduce notations for five such invariants:

I_t, *the number of triple points* A_1^3;

$I_{s_\pm}$, *the numbers of positive and negative swallowtails*;
$I_{c_\pm}$, *the numbers of* $A_2^\pm A_1$ *points.*

Another obvious local invariant is

I_χ, *half of the Euler characteristic of the critical locus* $\mathcal{K}$.

The Euler characteristic of $\mathcal{K}$ is even since $\mathcal{K}$ is a smooth surface with a natural co-orientation in the oriented source M, for example, to the side where the local degree of the map is $+1$.

2.2. *Linking with* Σ^2

To define our next invariant, consider the subset Σ^2 of the jet space $J^1(M,N)$. The subset consists of all 1-jets with the linear parts of corank at least 2, and is of codimension 4 in the jet space. It has a canonical co-orientation (see, for example, Section 3.3) which does not depend on the orientability of M and N.

Fix a map $f_0 \in \Omega(M,N) \setminus \Xi$. Consider a generic homotopy $\{f_t\}_{0\leqslant t\leqslant 1}$ in Ω from f_0 to a map $f_1 \notin \Xi$ in the same connected component of Ω. The images of the extensions $j^1 f_t$ define a four-dimensional film $\varphi \subset J^1(M,N)$ which meets Σ^2 at isolated points. We orient φ as $[0,1]\times M$, and, having Σ^2 co-oriented, obtain the intersection index $\langle \varphi, \Sigma^2\rangle$. We assume the orientation of $J^1(M,N)$ being the co-orientation of Σ^2 followed by its orientation, hence the index compares the orientation of φ with the co-orientation of Σ^2 at the meeting points. We now introduce

I_{Σ^2}, *the linking number of the image of the* 1*-jet extension of a map with* Σ^2, *as*

$$I_{\Sigma^2}(f_1) = \langle \varphi, \Sigma^2\rangle + I_{\Sigma^2}(f_0),$$

where the last term is an arbitrary fixed number.

Owing to the parallelizability of oriented 3-manifolds, the invariant is *integrable*, that is, may be expressed in a homotopy-independent way. Namely, the parallelizability allows us to consider $J^1(M,N)$ as $\mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^3)\times M\times N$, and Σ^2 in it as $\Sigma_0^2\times M\times N$, where $\Sigma_0^2 \subset \mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^3)$ is the set of operators of corank at least 2. Let $\psi \subset \mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^3)$ be a codimension 3 closed film with boundary Σ_0^2. We orient the codimension 3 closed film $\Psi = \psi\times M\times N \subset J^1(M,N)$ so that Σ^2 is its boundary. Up to an additive constant, we have

$$I_{\Sigma^2}(f) = \langle \mathrm{Image}(j^1 f), \Psi\rangle.$$

All 6 invariants from the previous section were introduced in a homotopy-independent, we call it *integral*, form.

2.3. *Classification*

All through the paper, local invariants are considered up to additive constants which may be chosen arbitrarily for each connected component of Ω.

THEOREM 2.1. (a) *The space of integer local invariants of maps between two oriented three-dimensional manifolds has rank* 7.

(b) This space is generated by

$$(I_{s_+}\pm I_{s_-})/2,\quad (I_{c_+}+I_{c_-})/2,\quad I_t,\quad (I_t+I_{c_+})/2,\quad I_\chi,\quad I_{\Sigma^2}.$$

We prove part (b) of the theorem in Section 3.3 modulo part (a). Part (a) itself will be derived in Section 5.4 from the study of codimension 2 singularities undertaken in Section 5.

3. *Codimension 1 bifurcations*

A local invariant is dual to a trivial codimension 1 cycle in $\Omega(M,N)$. Such cycle is a linear combination of strata of codimension 1 singularities of our maps. Its coefficients are the increments of the invariant at crossings of the strata (in the positive direction if we are working over $\mathbb{Z}$). We are now listing all codimension 1 singularities up to orientation-preserving local diffeomorphisms of the source and target.

3.1. *Multi-germs*

First of all, we have interactions of generic singularities of $\mathcal{C}$ which involve certain tangencies and further intersections. They differ by the co-orientations of the participating smooth sheets and by the signs of the edges and swallowtails. It is usually possible to co-orient the corresponding codimension 1 strata in Ω. In such cases, our figures may show the critical sets only for the 'positive resolutions' and indicate by arrows how one of the local components has been moved during the bifurcations.

The notations of the bifurcations below are self-explanatory, with T used for the tangency of the participating strata. Letters e and h distinguish between elliptic and hyperbolic versions of similar bifurcations. Letter r stays for the number of faces of the bounded region appearing *after* the bifurcation, which are co-oriented outward the region. If there is no such bounded region, we make a special comment on the meaning of r. The figures illustrate only one particular value of r from the range. Signs of the edges and swallowtails are usually omitted from the figures.

So, we have (see Figure 2):

$A_1^{4,r}$, $r=2,3,4$, intersection of four smooth sheets. The pre-bifurcation tetrahedral region has $4-r$ faces co-oriented outwards. Therefore, the $r=2$ stratum $A_1^{4,2}$ is not co-orientable in Ω by local means.

$TA_1^{3,r}, r=0,1,2,3$, three smooth sheets are pairwise transversal to each other, but the line of intersection of any two of them is tangent to the third sheet at the moment of bifurcation.

$TA_1^{2,e,r}, r=0,1,2$, elliptic tangency of two smooth sheets.

$TA_1^{2,h,r}, r=0,1$, same, but hyperbolic. We write $r=1$ if the sheets have the same co-orientation, and $r=0$ if the co-orientations are opposite. For $r=1$, we fail to locally co-orient the stratum in Ω.

$A_2^{\pm}A_1^{2,r}, r=0,1,2$, cuspidal edge meets the intersection of two smooth sheets.

$A_2^{2,e,\pm,\pm}$, two edges of given signs meet face to face. We will use $A_2^{2,e,+,-}$, not $A_2^{2,e,-,+}$.

$A_2^{2,h,\pm,\pm}$, one of the edges is overtaking the other. If the signs of the edges coincide, we fail to co-orient the stratum in Ω by local means. For $A_2^{2,h,+,-}$, we set the positive side of the bifurcation to be that with two $A_2^+A_1$ points.

$A_3^{\pm}A_1^r, r=0,1$, a smooth sheet passes through a swallowtail.

$TA_2^{\pm}A_1^{e,r}, r=0,1$, cuspidal edge becomes tangent to a smooth sheet so that the two local components of $\mathcal{C}$ do not intersect before the bifurcation.

$TA_2^{\pm}A_1^{h,r}, r=0,1$, the hyperbolic version of the previous. For $r=1$, the co-orientation of the A_1 sheet is towards the cuspidal edge before the bifurcation. For $r=0$, it is opposite.

3.2. *Uni-germs*

Figure 3 shows codimension 1 degenerations of corank 1 uni-germs. The normal forms of the degenerate maps are taken below from [4]. The sign of the real parameter λ in the local formulas for the families co-orients the strata in Ω. Since all families have normal forms $(h(x,y,z,\lambda),y,z)$, we are listing the functions h only.

$A_2^{\pm,+,+}: \pm(x^3+(y^2+z^2-\lambda)x)$, birth of a flying saucer. Here $\pm$ is the sign of the edge, and the two pluses in the notation are the signs of the squares in the coefficient of x.

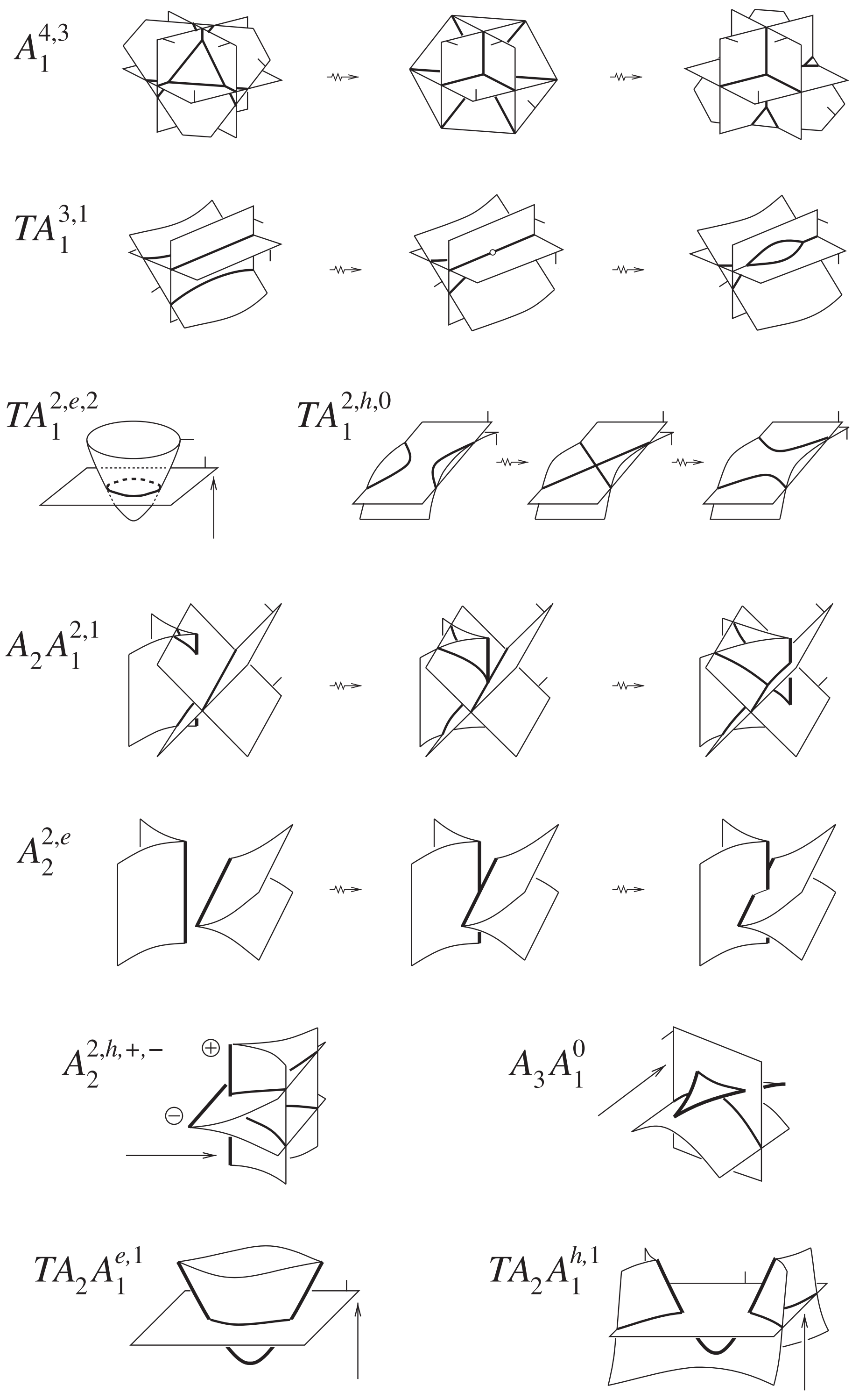

Figure 2. *Codimension 1 multi-germs.*

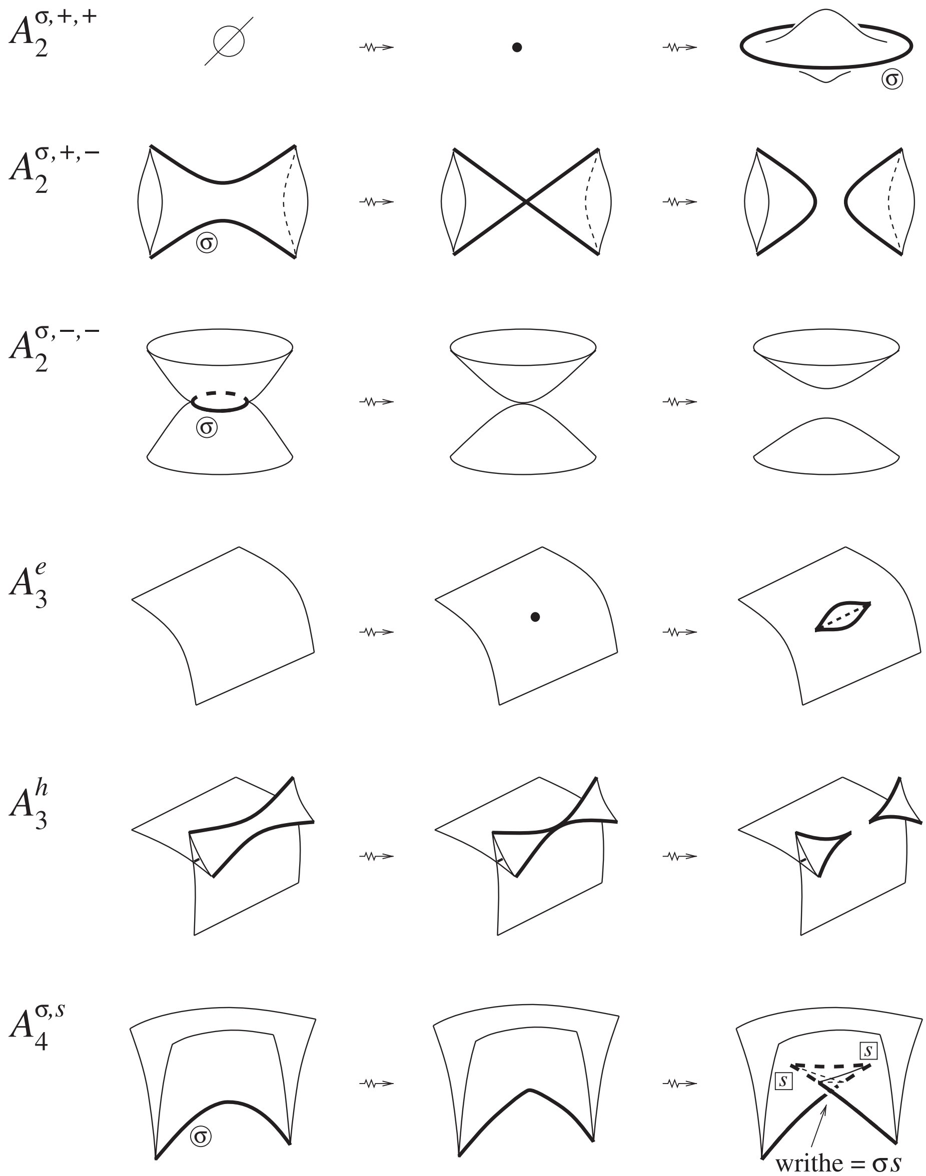

FIGURE 3. *Codimension 1 corank 1 uni-germs.*

$A_2^{\pm,+,-}$: $\pm(x^3 + (y^2 - z^2 + \lambda)x)$, hyperbolic transformation of an edge.
$A_2^{\pm,-,-}$: $\pm(x^3 - (y^2 + z^2 + \lambda)x)$, death of a compact component of an edge.
A_3^e, birth of cuspidal lips.
A_3^h, beaks bifurcation on an edge.
$A_4^{\sigma,s}, \sigma, s = \pm : \sigma(x^5 - \lambda x^3 + syx^2 + zx)$. Here σ is the local degree of the whole map, while s is the sign of the two swallowtails born in the bifurcation.

Finally, we have four corank 2 degenerations:

$D_4^{+,\pm}$: $(x^2 + y^2 + zy + \lambda x, \pm xy, z)$, where $\pm$ is the sign of the edge (see Figure 4, $\lambda = 0$).
$D_4^{-,\pm}$: $(\pm(x^2 - y^2) + zx - \lambda y, xy, z)$, of the local degree ± 2.

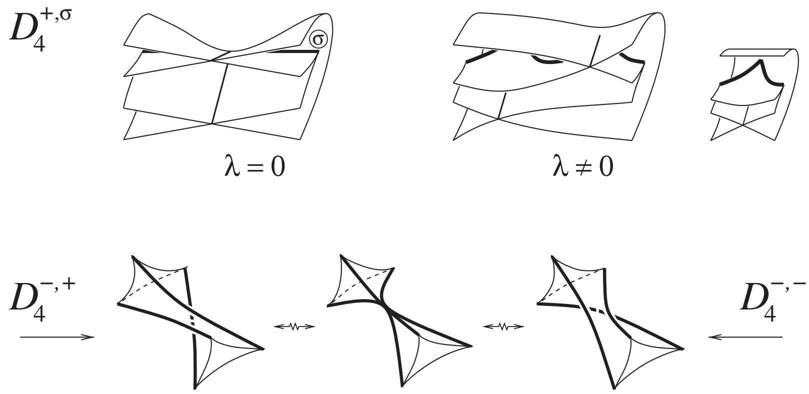

Figure 4. *Codimension 1 corank 2 uni-germs.*

The critical value sets of these families have been depicted in [**3**]. However, their features responsible for the co-orientability of the strata in Ω have not been emphasized.

What happens geometrically during the D_4 transitions is presented in Figure 4. For $\lambda = 0$, we are getting the purse and pyramid, the well-known stable singularities D_4^+ and D_4^- of caustics in three dimensions. The critical value sets for $\lambda = -1$ are mirror images of those for $\lambda = 1$. This provides the co-orientation of the strata in Ω.

Namely, we co-orient the $D_4^{+,\pm}$ strata, so that to their positive side (corresponding to $\lambda > 0$) we have a pair of positive swallowtails (hence for $\lambda < 0$ we have a negative pair). Note that the surface $D_4^{+,\pm}, \lambda \neq 0$, in Figure 4 is the result of gluing its half shown on the right with a copy of this half obtained by the rotation through $180°$.

In the $D_4^{-,\pm}$ cases, the edges for $\lambda \neq 0$ are not as straight as they were for $\lambda = 0$. They form screws with the rotation angle $2\pi/3$. We choose the positive resolution of $D_4^{-,\sigma}$ to be that with the $\sigma 2\pi/3$ rotation.

3.3. *The integer invariants in terms of linear combinations of the strata*

Lemma 3.1. *The basic integer-valued invariants of Theorem* 2.1 *are dual to the following linear combinations of the codimension* 1 *strata*:

$$
\begin{aligned}
I_s/2 = (I_{s_+} + I_{s_-})/2 \quad &: \quad A_3^{e/h} + A_4,\\
(I_{s_+} - I_{s_-})/2 \quad &: \quad A_4^{\pm,+} - A_4^{\pm,-} + 2D_4^{+,\pm},\\
I_c/2 = (I_{c_+} + I_{c_-})/2 \quad &: \quad 2A_2^{2,e} + A_3A_1 + TA_2A_1 + A_4,\\
I_t \quad &: \quad 2TA_1^3 + 2A_2A_1^2 + A_3A_1,\\
(I_t + I_{c_+})/2 \quad &: \quad TA_1^3 + A_2A_1^2 + 2A_2^{2,e,+,+} + A_2^{2,e/h,+,-} + A_3A_1 + TA_2^+A_1 + A_4^{+,\pm},\\
I_\chi \quad &: \quad A_2^{\pm,+,+} + A_2^{\pm,+,-} + A_2^{\pm,-,-},\\
I_{\Sigma^2} \quad &: \quad D_4.
\end{aligned}
$$

The last relation holds for an appropriate co-orientation of Σ^2.

Here a missing index $\pm$ or r or e/h in the notation of a stratum means the sum of the elementary strata along the whole range of the index, for example,

$$A_3A_1 = A_3^+A_1^1 + A_3^+A_1^0 + A_3^-A_1^1 + A_3^-A_1^0, \quad \text{or} \quad A_2^{2,e} = A_2^{2,e,+,+} + A_2^{2,e,+,-} + A_2^{2,e,-,-},$$

$$\text{or} \quad TA_2A_1 = TA_2^+A_1^e + TA_2^+A_1^h + TA_2^-A_1^e + TA_2^-A_1^h.$$

We keep an index if its omission may cause confusion, for example, $A_4^{\pm,+} = A_4^{+,+} + A_4^{-,+}$. We also use notation like $A_3^{e/h} = A_3^e + A_3^h$.

Proof of Lemma 3.1. The expressions for the first six invariants easily follow from inspection of Figures 2–4. We do the seventh invariant in detail.

Let j be a regular point of the variety $\Sigma^2 \subset J^1(M^3, N^3)$. The linear part of j is a rank 1 linear operator $\psi : \mathbb{R}^3 \to \mathbb{R}^3$. Let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ be bases of, respectively, the kernel and cokernel of ψ. We take $(a_1 \otimes b_1) \wedge (a_1 \otimes b_2) \wedge (a_2 \otimes b_1) \wedge (a_2 \otimes b_2)$ as the co-orientation of Σ^2 at j (it does not depend on the order in the initial bases).

The Jacobi matrix of the $D_4^{+,+}$ family is

$$\begin{pmatrix} 2x+\lambda & 2y+z & y \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It confirms that the 1-jet extension of the family meets Σ^2 just at the origin. It also provides the linear map $(\lambda, x, y, z) \mapsto (2x+\lambda, y, 2y+z, x)$ from the domain of the family to the oriented four-dimensional transversal to Σ^2. The determinant of the map is 1. Since, following Section 2.1, the domain of $D_4^{+,+}$ is oriented by $d\lambda \wedge dx \wedge dy \wedge dz$, the intersection index of the 1-jet extension of the family with Σ^2 is $+1$. Similar calculations for the other three D_4 families give the same result. □

Proof of part (b) of Theorem 2.1. The claim now follows from part (a) of Theorem 2.1 and an observation that the minor of the coefficients of the strata $A_3^{e/h}, A_4^{+,+}, TA_2^-A_1, A_3A_1, TA_2^+A_1, A_2^{\pm,+,+}$ and $D_4^{-,-}$ in the linear combinations of Lemma 3.1 is 1. □

4. *mod2 invariants*

4.1. *Lists of invariants*

In the case of the simplest target we have the following result.

THEOREM 4.1. *The space of mod*2 *local invariants of maps from an oriented three-dimensional manifold to oriented* $\mathbb{R}^3$ *has rank* 11. *Its basis is formed by seven invariants of Theorem* 2.1 *reduced modulo* 2, *and four further invariants dual to the following linear combinations of the strata*:

$$\begin{aligned} I_8 &: TA_1^2 + A_2^{+,+,+} + A_2^{+,+,-} + A_2^{+,-,-} + D_4^{\pm,+}, \\ I_9 &: A_1^4 + A_2^+A_1^2 + A_2^{2,e/h,+,+}, \\ I_{10} &: A_2A_1^2 + A_2^{2,e/h,+,+} + A_2^{2,e/h,-,-}, \\ I_{11} &: A_2^{2,e/h} + A_4^{-,\pm}. \end{aligned}$$

We prove the theorem in Section 5.4 after analysing codimension 2 degenerations in the rest of Section 5.

As for a geometric sense of the four new invariants, I_8 may be interpreted as the parity of a generalized number of self-tangencies of the regular part of the critical value set in a generic homotopy, and I_9 as a similarly generalized parity for quadruple points of $\mathcal{C}$. The generalizations are taking account of the behaviour of cuspidal edges of fixed sign. The I_{10} invariant must be somehow related to the linking of the self-intersection of $\mathcal{C}$ and its cuspidal edge. An interpretation of I_{11} is given in the next subsection.

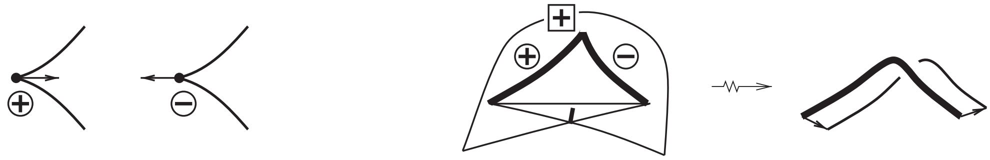

Figure 5. *Making a framed link from the edges.*

In the general setting, we have the following estimate.

Corollary 4.2. *The rank of the space of* mod2 *local invariants of maps between arbitrary oriented* 3*-manifolds is at least* 7 *and at most* 11*. The upper bound is achieved for connected components of* $\Omega(M,N)$ *with trivial fundamental groups.*

The lower bound here is provided by the modulo 2 reduction of the integer invariants. A reason for the upper bound will be given in Section 5.4.

4.2. *The number of components and self-linking of the cuspidal edge*

Among the four new invariants introduced in Theorem 4.1, the best understanding in terms of the geometry of the critical value set we have at the moment is that of the I_{11} invariant, or rather of its linear combination with the reduced integer invariants. To formulate the statement, we start with a construction of a framed link from cuspidal edges of $\mathcal{C}$.

The way we introduce the framing at a regular point of an edge depends on the sign of the edge and is illustrated by the transversal sections shown in Figure 5, left. Near swallowtails, we smoothen the edge and join the framings along the two branches by adding a half-twist of the sign coinciding with that of the swallowtail.

Let us arbitrarily orient our framed link, and calculate its *writhe* w as the algebraic number of crossings of the cores of the components in the link diagram obtained plus the sum of the algebraic numbers of full rotations done by the framing of each of the components around its own core. Since the number of crossings of two different components in a link diagram is even, the quantity $w \bmod 4$ does not depend on the orientations of the components.

We denote by n the number of components of our link.

Theorem 4.3. *The* mod2 *invariant* $I_{fe} = n + w/2$ *is local. It is dual to the cycle*

$$A_2^{2,e/h} + A_2^{\pm,+,+} + A_2^{\pm,+,-} + A_2^{\pm,-,-} + A_3^{e/h} + A_4^{+,\pm} + D_4.$$

We do not touch here the question of $(w/2) \bmod 2$ being either integer or half-integer. The point is that its increments are integer.

Recalling the expressions for our earlier invariants, we see that the linear combination in the theorem states that, up to an additive constant,

$$I_{fe} = I_{11} + I_\chi + I_s/2 + I_{\Sigma^2}.$$

Proof of Theorem 4.3. The only increments of $(n + w/2) \bmod 2$ which need detailed consideration are those across the $A_2^{\pm,+,-}$, A_3^h and $D_4^{-,\pm}$ strata. Coincidence of the increments across all other strata with the coefficients in the linear combination given in the theorem follows easily from Figures 2–4.

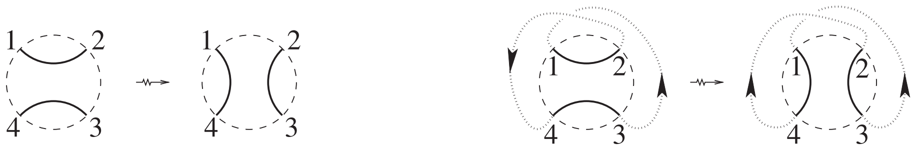

FIGURE 6. *The $A_2^{\pm,+,-}$ link transformations.*

FIGURE 7. *Edges in the $D_4^{-,\pm}$ transformations.*

$A_2^{\pm,+,-}$. The move makes a local modification of the link as shown in Figure 6 (left), with the fragments equipped with the blackboard framing. The rest of the link diagram is outside the disc and connects points 1, 2, 3, 4 by arcs which we will call *external*.

Assume the external arcs are 14 and 23. Then the transformation produces two components from one. We choose an orientation of the initial component, and orient the two final so that the external arcs keep their orientations. Then the writhe is kept unchanged and the increment of I_{fe} is 1 due to the change in n.

If the external arcs are 12 and 34, then the move reverses the one just considered.

In the remaining case of external arcs 13 and 24, the number of components is preserved. Assume the orientations of the only components involved are as in Figure 6, right. The change in I_{fe} is due to the reorientation of the external arc 24. The change in $(w/2)\,\mathrm{mod}2$ due to the crossings of this arc with the other components of the link is 0: indeed we can close the arc with the diameter 24 to obtain a separate link component, and reorientation of a link component does not affect $(w/2)\,\mathrm{mod}2$. Also, no change to I_{fe} comes from the self-linking of the arc 24 after its reorientation. The only change may come from the mutual crossings of the external arcs 24 and 13. This increment is $2 \cdot \frac{1}{2} \cdot \langle \overrightarrow{42}, \overrightarrow{31} \rangle = \langle \overrightarrow{42}, \overrightarrow{31} \rangle \equiv 1\,\mathrm{mod}2$, since the endpoints of the arcs alternate on the boundary of the disc. We have used here the angular brackets for the total algebraic number of positive and negative crossings of the arcs. This completes the $A_2^{\pm,+,-}$ considerations.

The A_3^h case is basically the same.

$D_4^{-,\pm}$. A move of this type replaces three local edge fragments having the rotation angle of the framing $2\pi/3$ along each of them with similar fragments having the framing rotating in the opposite direction. This provides the increment of 1 in I_{fe}. Therefore, the theorem follows from the following statement.

LEMMA 4.4. *Consider a local modification of a framed link shown in Figure 7, left. Assume that the framing of all participating fragments is blackboard. Then the move preserves the quantity* $(n + w/2)\,\mathrm{mod}2$.

The sense of the numbers n and w is the same as earlier, in the particular case of a link constructed from the edges.

Proof. Like for the $A_2^{\pm,+,-}$ transformation, we inspect all possible ways to connect the endpoints of the fragments by external arcs. Combinatorially, there are 15 different configurations, but the symmetries involved reduce their number to at most 8. We consider one of them as an illustration. All others are done similarly.

Assume that the external arcs are 13, 26 and 45, as shown in Figure 7, right. Similar to the $A_2^{\pm,+,-}$ case, we choose here orientations of the components we have before and after the transformation. The increment of $(w/2)\,\mathrm{mod}2$ is seen to be

$$\langle\overrightarrow{31},\overrightarrow{62}\rangle+\langle\overrightarrow{31},\overrightarrow{45}\rangle+(0-0)/2.$$

The last summand here reflects the change in the signs of the crossings inside the ellipse. The first summand is odd since the endpoints of the arcs alternate along the ellipse, while the second is even since the endpoints do not alternate. Therefore, the total increment of $(w/2)\,\mathrm{mod}2$ is 1. Since n is reduced by 1 in the transformation, the quantity $(n+w/2)\,\mathrm{mod}2$ stays unchanged. □

This completes our proof of Theorem 4.3. □

5. *Bifurcations in 2-parameter families*

Our main tool to prove part (a) of Theorem 2.1, and Theorem 4.1 will be a study of bifurcations of codimension 2 singularities of maps between 3-manifolds carried out in this section. We will consider generic 2-parameter families of either maps or just of the critical value sets, and construct the bifurcation diagrams of the families. Each diagram yields a linear equation on the increments of a local invariant across the codimension 1 strata. The equation represents the fact that the total increment along a generic loop in Ω must be zero (cf. [**5**]): in particular, it should be so for a small loop around the origin in the parameter space of a codimension 2 map singularity. We denote the increment across a particular stratum as for the stratum itself, but in small characters. Increments which may be non-zero only in the mod2 case will be kept in square brackets.

The normal forms of all simple uni-germs of maps we are using are taken from [**4**]. The majority of the other normal forms are easy to obtain. The relevant calculations are rather routine even in the most involved cases. Therefore, we omit such calculations as well as details of calculations of the bifurcation diagrams.

5.1. *Gluing codimension 1 strata together*

We start with all 52 strata, four of which ($A_1^{4,2}, TA_1^{2,h,1}, A_2^{2,h,+,+}$ and $A_2^{2,h,-,-}$) we have failed to co-orient in Ω. Our initial goal is to reduce the number of unknown increments.

5.1.1. *Extra A_1 component.* First of all, we consider the easiest kind of codimension 2 bifurcations when an extra generic A_1 sheet of $\mathcal{C}$ passes through a point of a codimension

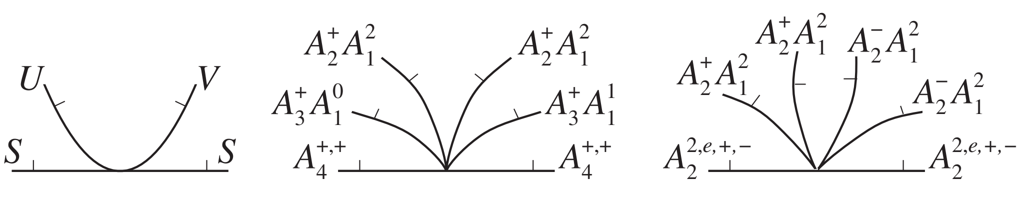

FIGURE 8. *Discriminants of the families obtained from interaction of a generic smooth sheet with a codimension 1 bifurcation.*

1 bifurcation S. For cases **1**–**5** of the table below, the planar discriminants are of the form shown in Figure 8, left. Such a discriminant gives the equation $u = v$ for the increments. The discriminants for cases **6** and **7** are in the same figure in its centre and right. The equations obtained at this stage allow us to consider in what follows sums of the strata differing only by certain indices in their notation. Such bigger strata will be denoted as introduced in Section 3.3 and the notation of the increments will follow the pattern. If one of the summands in a big stratum is non-co-orientable, then the increment of any integer invariant across the big stratum is zero.

	S	r	Equation	Big stratum
1.	$TA_1^{3,r}$	$2,3$	$[a_1^{4,2}] = a_1^{4,3} = a_1^{4,4}$	A_1^4
2.	$TA_1^{2,e,r}$	$0,1,2$	$ta_1^{3,r+1} = ta_1^{3,r}$	TA_1^3
3.	$TA_2^{\sigma}A_1^{e,r}$	$0,1$	$a_2^{\sigma}a_1^{2,r+1} = a_2^{\sigma}a_1^{2,r}$	$A_2^{\sigma}A_1^2$
4.	$A_2^{\sigma,+,+}$		$ta_2^{\sigma}a_1^{e,0} = ta_2^{\sigma}a_1^{e,1}$	$TA_2^{\sigma}A_1^e$
	$A_2^{\sigma,-,-}$		$ta_2^{\sigma}a_1^{h,0} = ta_2^{\sigma}a_1^{h,1}$	$TA_2^{\sigma}A_1^h$
5.	A_3^e		$a_3^+a_1^0 = a_3^-a_1^1$	
			$a_3^+a_1^1 = a_3^-a_1^0$	
6.	$A_4^{+,+}$		$a_3^+a_1^0 = a_3^+a_1^1$	A_3A_1
7.	$A_2^{2,e,+,-}$		$2a_2^+a_1^2 = 2a_2^-a_1^2$	$A_2A_1^2$ over $\mathbb{Z}$

As usual, here σ is either plus or minus.

5.1.2. *Cubic bifurcations.* The $A_3^{e/h}$ singularities have normal forms $(x, y, z) \mapsto (x^4 \pm y^2x^2 + zx, y, z)$. Writing y^3 instead of $\pm y^2$, we obtain a codimension 2 uni-germ, with a versal deformation given by $(\cdots + (y^3 + \lambda_1 y + \lambda_2)x^2 + \cdots)$. Its discriminant is a semi-cubical parabola $4\lambda_1^3 + 27\lambda_2^2 = 0$, and yields coincidence of the increments across its half-branches. Similarly replacing quadratic configurations by cubic in some other codimension 1 bifurcations S, we obtain a list like in the previous subsection:

	S	Equations	Big stratum
8.	A_3^e	$a_3^e = a_3^h$	$A_3^{e/h} = A_3^e + A_3^h$
9.	$A_2^{\sigma,+,\pm}$	$a_2^{\sigma,+,+} = a_2^{\sigma,+,-} = a_2^{\sigma,-,-}$	$A_2^{\sigma,q} = A_2^{\sigma,+,+} + A_2^{\sigma,+,-} + A_2^{\sigma,-,-}$
10.	$TA_2^{\sigma}A_1^e$	$ta_2^{\sigma}a_1^e = ta_2^{\sigma}a_1^h$	$TA_2^{\sigma}A_1 = TA_2^{\sigma}A_1^e + TA_2^{\sigma}A_1^h$
11.	$TA_1^{2,e,r}, TA_1^{2,h,r}$	$ta_1^{2,e,2} = -ta_1^{2,e,0} = ta_1^{2,h,0}$	$TA_1^{2,\text{opp}} = TA_1^{2,e,2} - TA_1^{2,e,0} + TA_1^{2,h,0}$
		$ta_1^{2,e,1} = [ta_1^{2,h,1}]$	$TA_1^{2,\text{dir}} = TA_1^{2,e,1} + TA_1^{2,h,1}$

To avoid confusion with other singularities, the notation of the majority of big strata here has not been obtained by simple omission of the indices. We are using indices q for quadratic, dir = direct for tangency of two smooth sheets with coinciding co-orientations, and opp for similar tangency with opposite co-orientations.

5.2. *Particular multi-germ families*

So far we have halved the number of unknown increments to 26, out of which four may be non-trivial only in the mod2 case.

5.2.1. *Non-transversal interaction with a cuspidal edge.* Figure 9 illustrates three codimension 2 events when the plane tangent to the critical point set at its edge point is in a special position with other local components of $\mathcal{C}$:

the plane coincides with the plane tangent to a smooth A_1 sheet;
the plane contains the tangent direction of the line of intersection of two A_1 sheets;
the plane contains the tangent direction of another cuspidal edge.

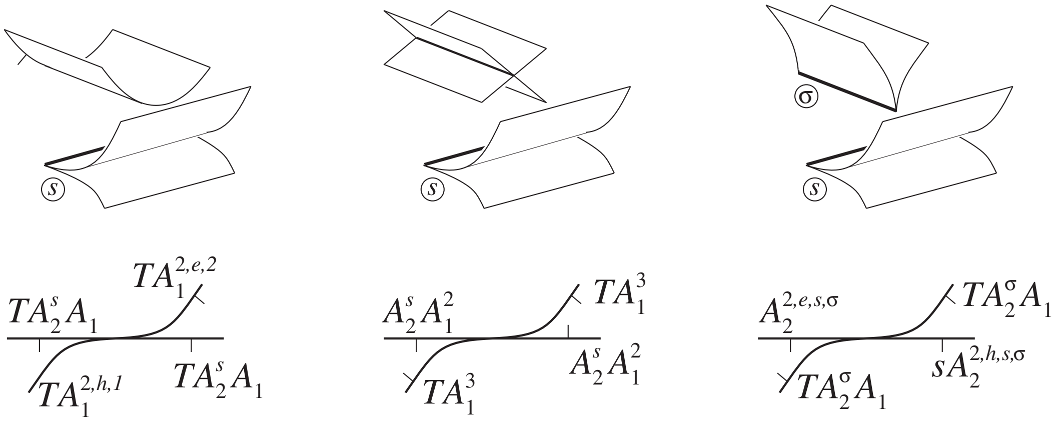

FIGURE 9. *Codimension 2 degenerations due to special positions with respect to the tangent plane at an edge point.*

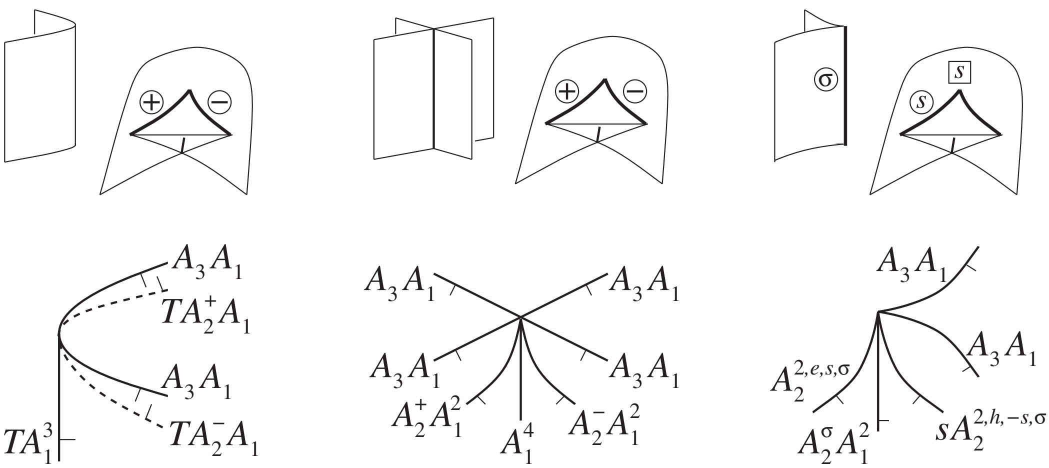

FIGURE 10. *Codimension 2 degenerations involving swallowtails.*

Respectively, we obtain new equations for the increments:

$$\begin{array}{lrcl}
\mathbf{12}. & ta_1^{2,\mathrm{opp}} & = & [ta_1^{2,\mathrm{dir}}] \\
\mathbf{13}. & 2ta_1^3 & = & 2a_2^+a_1^2 \\
 & & = & 2a_2^-a_1^+ \\
\mathbf{14}. & 2ta_2^+a_1 & = & a_2^{2,e,+,+} + [a_2^{2,h,+,+}] \\
 & & = & a_2^{2,e,+,-} + a_2^{2,h,+,-} \\
 & 2ta_2^-a_1 & = & a_2^{2,e,+,-} - a_2^{2,h,+,-} \\
 & & = & a_2^{2,e,-,-} + [a_2^{2,h,-,-}].
\end{array}$$

Equation **12** allows us to unite all codimension 1 strata corresponding to self-tangencies of regular sheets and introduce a big stratum $TA_1^2 = TA_1^{2,\mathrm{dir}} + TA_1^{2,\mathrm{opp}}$, which may appear in mod2 invariants only.

5.2.2. *Interaction with a swallowtail.* Figure 10 shows the events involving swallowtails. In the first case, at the most degenerate moment, the direction of the self-intersection curve at the swallowtail point is tangent to the incoming smooth sheet. The two other cases are clear. The bifurcations provide the following equations:

$$
\begin{array}{lrcl}
\mathbf{15}. & 2a_3a_1 & = & ta_2^+a_1 + ta_2^-a_1 + ta_1^3 \\
\mathbf{16}. & a_2^+a_1^2 & = & a_2^-a_1^2 + [a_1^4] \\
\mathbf{17}. & 2a_3a_1 & = & a_2^+a_1^2 + a_2^{2,e,+,+} - a_2^{2,h,+,-} \\
 & & = & a_2^+a_1^2 + a_2^{2,e,+,-} + [a_2^{2,h,+,+}] \\
 & & = & a_2^-a_1^2 + a_2^{2,e,+,-} + [a_2^{2,h,-,-}] \\
 & & = & a_2^-a_1^2 + a_2^{2,e,-,-} + a_2^{2,h,+,-}.
\end{array}
$$

5.3. *Uni-germs of codimension 2*

A map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ may be treated as induced by a map from the target $(\mathbb{R}^3, 0)$ to the base B_{f_0} of a contact versal deformation of the complete intersection $f_0 : \{f = 0\}$. For example, the degenerate maps in the $A_4^{\pm,\pm}$ families are induced by generic maps from $(\mathbb{R}^3, 0)$ to $(B_{A_4}, 0)$.

Figure 11 shows normal forms and discriminants of map germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ induced by:

a generic map $(\mathbb{R}^3, 0) \to (B_{A_5}, 0)$;
a generic codimension 1 map in the set of all maps $(\mathbb{R}^3, 0) \to (B_{A_4}, 0)$;
a generic codimension 2 map $(\mathbb{R}^3, 0) \to (B_{A_3}, 0)$;
a generic codimension 1 map $(\mathbb{R}^3, 0) \to (B_{(x^2,y^2)}, 0)$;
a generic map $(\mathbb{R}^3, 0) \to (B_{(x^2+y^3,xy)}, 0)$.

In the first and fifth cases, the maps are uni-modular. The other three normal forms are taken from [**4**]. The order of the signs in a diagram label is the same in which they appear in the corresponding normal form.

Figure 11 gives the following equations:

$$
\begin{array}{lrcl}
\mathbf{18}. & a_2^{2,h,+-} & = & a_4^{+,+} - a_4^{-,+} \\
 & & = & a_4^{+,-} - a_4^{-,-} \\
\mathbf{19}. & 2a_3^{e/h} & = & a_4^{+,+} + a_4^{+,-} - 2ta_2^+a_1 \\
 & & = & a_4^{-,+} + a_4^{-,-} - 2ta_2^-a_1 \\
\mathbf{20}. & a_2^{+,q} - a_2^{-,q} & = & [ta_1^2] \\
\mathbf{21}. & a_2^{+,q} - a_2^{-,q} & = & d_4^{+,+} - d_4^{+,-} \\
\mathbf{22}. & d_4^{+,+} - d_4^{-,+} & = & a_4^{+,+} - a_4^{+,-} \\
 & d_4^{+,-} - d_4^{-,-} & = & a_4^{-,+} - a_4^{-,-}.
\end{array}
$$

Equation **20** makes up a big stratum $A_2^q = A_2^{+,q} + A_2^{-,q}$ over $\mathbb{Z}$.

5.4. *Proofs of the classification results in the oriented case*

We have now obtained enough relations between the increments to prove the classification claims made in Sections 2.3 (apart from what has already been proved in Section 3.3) and 4.1.

The equations obtained earlier in this section reflect the fact that the intersection index with the codimension 1 trivial cycle in $\Omega(M, N)$, dual to a local invariant, vanishes on contractible loops. Therefore, the equations may be sufficient if, for example, $N = \mathbb{R}^3$. Let ν be the rank of the invariant space for $\Omega(M, \mathbb{R}^3)$. For general N, the fundamental group of the space of maps (or of its particular connected component) may be non-trivial, which may add further vanishing conditions to our system of equations and thus force the rank of the invariant space to drop below ν.

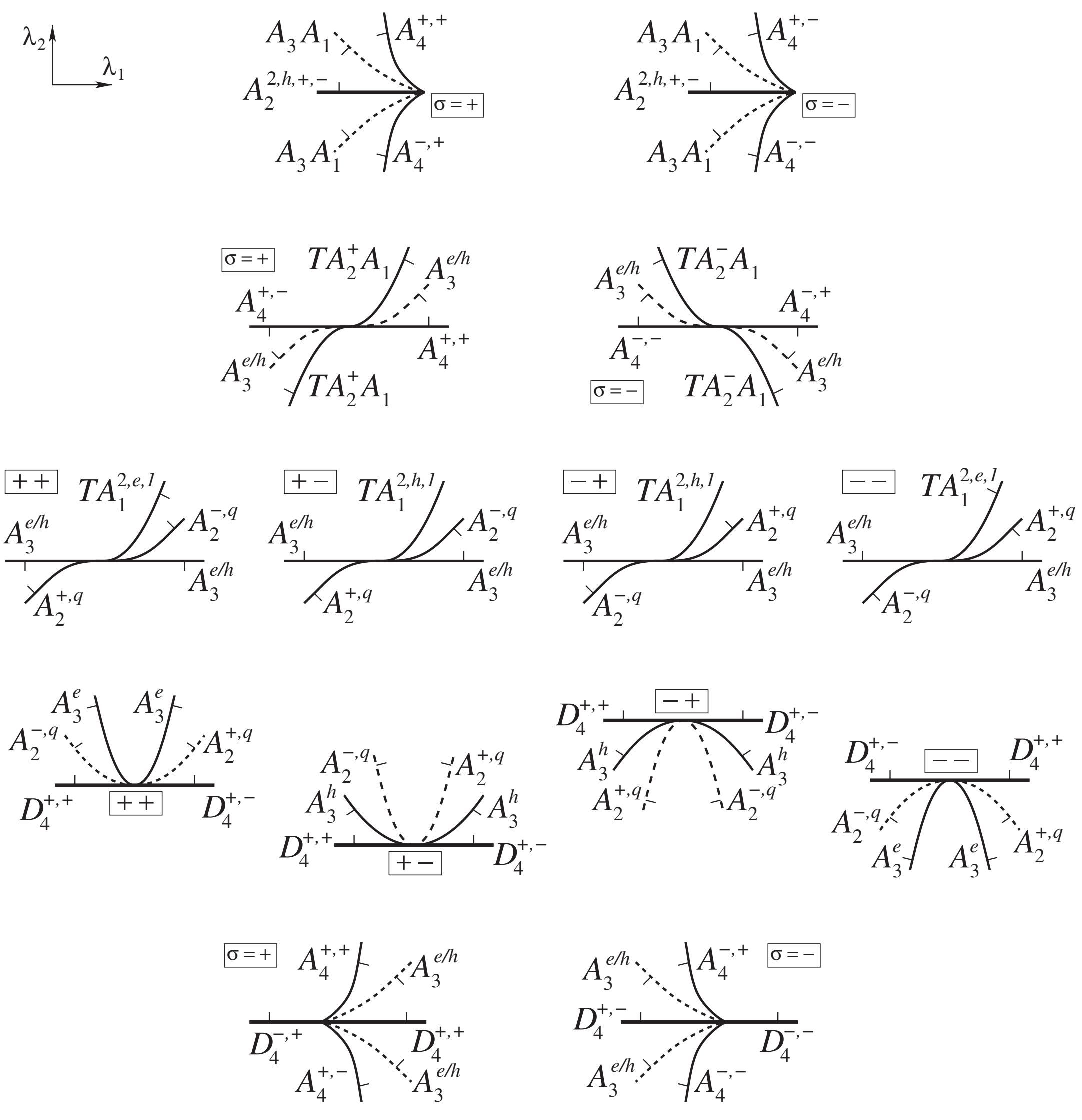

Figure 11. *Discriminants of the families.*

$$(x^6 + (\lambda_1 \pm y + \alpha z)x^4 + \lambda_2 x^3 + \sigma y x^2 + zx, y, z), \quad \sigma = \pm, \quad \alpha \in \mathbb{R};$$
$$(\sigma x^5 + yx^3 + (\pm y^2 + \lambda_1 y + \lambda_2)x^2 + zx, y, z), \quad \sigma = \pm;$$
$$(x^4 + yx^2 + (\pm y^2 + \lambda_1 y + \lambda_2 \pm z^2)x, y, z);$$
$$(x^2 + yz, y^2 + (\pm z^2 + \lambda_1 z + \lambda_2 \pm (x^2 + yz))x, z);$$
$$(\sigma x^2 + y^3 + (\pm z^2 + \alpha xy + \lambda_1)y^2 + \lambda_2 x + zy, xy, z), \quad \sigma = \pm, \quad \alpha \in \mathbb{R}.$$

On the other hand, the solution space of the complete system of equations on the increments must contain the duals of all integral invariants of maps from M to N. Let μ be the rank of the space spanned by such duals.

Thus, we have the following obvious estimates.

Proposition 5.1. *Within the notation introduced, the rank of the space of local invariants of maps between manifolds M and N is at least μ, and at most ν.*

Of course, we assume here that the orientation settings for the source and target manifolds are fixed, along with the value ring of the invariants. Also, the whole space of maps from M to N may be replaced in the context by its connected component.

Proof of Theorem 2.1, *part (a).* In the integer case, after joining elementary strata into the bigger ones, we now have 11 strata

$$TA_1^3, \quad A_2A_1^2, \quad A_2^{2,e,+,+}, \quad A_2^{2,e,+,-}, \quad A_2^{2,e,-,-}, \quad A_2^{2,h,+,-},$$
$$A_3A_1, \quad TA_2^+A_1, \quad TA_2^-A_1, \quad A_2^q, \quad A_3^{e/h},$$

along with a further 8 corresponding to the variety of sign choices in $A_4^{\pm,\pm}$ and $D_4^{\pm,\pm}$.

On the other hand, 12 equations

$$\mathbf{13}, \mathbf{14}(1), \mathbf{14}(2), \mathbf{17}(1), \mathbf{17}(2), \mathbf{18}(1), \mathbf{18}(2), \mathbf{19}(1), \mathbf{19}(2), \mathbf{21}, \mathbf{22}(1), \mathbf{22}(2)$$

are linearly independent. Therefore, the rank of the space of all integer local invariants is at most seven. However, we have already observed by the end of Section 3.3 that the duals of the seven invariants mentioned in part (b) of the theorem are linearly independent. Moreover, all these seven invariants are integral. Hence, in the notation of Proposition 5.1 we must have $\nu = \mu = 7$, and the proposition implies part (a) of Theorem 2.1. □

Proof of Theorem 4.1. Over $\mathbb{Z}_2$, equations **14**, **19** and **22** join the elementary strata in pairs to provide 7 big strata

$$A_2^{2,e/h,+,+}, \quad A_2^{2,e/h,+,-}, \quad A_2^{2,e/h,-,-}, \quad A_4^{+,\pm}, \quad A_4^{-,\pm}, \quad D_4^{\pm,+}, \quad D_4^{\pm,-}.$$

Besides these, we have another 11 strata:

$$A_1^4, \quad TA_1^3, \quad TA_1^2, \quad A_2^+A_1^2, \quad A_2^-A_1^2, \quad A_3A_1, \quad TA_2^+A_1, \quad TA_2^-A_1, \quad A_2^{+,q}, \quad A_2^{-,q}, \quad A_3^{e/h}.$$

Seven equations

$$\mathbf{15},\ \mathbf{16},\ \mathbf{17}(1),\ \mathbf{17}(3),\ \mathbf{18}(1),\ \mathbf{20},\ \mathbf{21} \qquad (1)$$

are independent, and a basis of their solution space is formed by the 11 invariants given in Theorem 4.1. It is not so difficult to show that none of the codimension 2 singularities we have not considered so far (for example, quintuple points of $\mathcal{C}$, or a smooth sheet passing through a $D_4^{\pm,\pm}$ point) adds to the rank of the mod2 system (1). □

Proof of Corollary 4.2. Owing to Proposition 5.1, the upper bound for the rank of the space of mod2 invariants for an arbitrary target manifold N is provided by the similar rank $\nu = 11$ which was obtained in Theorem 4.1 for $N = \mathbb{R}^3$. □

REMARKS. (1) At the moment, we are not in a position to raise the lower bound $\mu = 7$ stated in Corollary 4.2 since we are lacking integral interpretations for the invariants I_8, I_9 and I_{10}, and the way we integrated I_{11} in Section 4.2 is based on the target being $\mathbb{R}^3$ (or, for example, S^3).

(2) There is no prohibition for the rank in the corollary to depend on a choice of a particular connected component of $\Omega(M, N)$.

6. *Non-oriented settings*

As an easy corollary of our codimension 2 study in the oriented case, we are going to obtain similar classification results, both over $\mathbb{Z}$ and $\mathbb{Z}_2$, assuming that at least one of the 3-manifolds M and N is either non-oriented or non-orientable. The notation I_s and I_c will now be used for the total numbers of swallowtail and A_2A_1 points of the critical value set. There will also appear a new mod2 invariant $I_{\Sigma^{1,1,1,1}}$, the *linking number* of the image of the k-jet extension of a map with the stratum $\Sigma^{1,1,1,1}$ in $J^k(M,N)$, $k \geqslant 5$. Its definition is absolutely similar to the pathwise definition of I_{Σ^2} in Section 2.1.

6.1. *Non-oriented source*

THEOREM 6.1. *Assume that the source manifold M is not oriented. Then the rank of the space of local invariants is* 4 *over the integers, while it is at least* 4 *and at most* 6 *over* $\mathbb{Z}_2$.

An integer basis is formed by the invariants $I_s/2$, $I_c/2$, I_t and I_χ.

The mod2 upper bound is achieved for connected components of $\Omega(M,N)$ with trivial fundamental groups. In such a case, I_{Σ^2} and $I_{\Sigma^{1,1,1,1}}$ should be added to the above four invariants to complete a mod2 basis.

Like earlier, we are not considering here the question of whether some of our invariants may be half-integers, but we emphasize that their increments are integer.

Proof. (1) We assume for the start that the target is oriented $\mathbb{R}^3$, and the invariants are integer.

Loss of the source orientation erases the signs of cuspidal edges and swallowtails of the critical value set $\mathcal{C}$. However, $\mathcal{C}$ stays co-oriented as before.

The list of codimension 1 bifurcations becomes shorter. All multi-germs lose all signs in their notation, and the corresponding strata are now the sums of the former strata along the whole range of the lost signs. As for the uni-germs, we now have to operate from the start with the stratum $A_2^{\pm,+,+} = A_2^{+,+,+} + A_2^{-,+,+}$ (we keep the $\pm$ for the moment for convenience), and similar strata $A_2^{\pm,+,-}$ and $A_2^{\pm,-,-}$. The target orientation allows us to pair up the elementary strata we had before and obtain

$$A_4^{+,+} + A_4^{-,-}, \quad A_4^{+,-} + A_4^{-,+}, \quad D_4^{+,+} - D_4^{+,-}, \quad D_4^{-,+} - D_4^{-,-}. \tag{2}$$

The A_4 sums here are distinguished by the local writhe of the edge; see Figure 3. The reason for taking the last difference is clear from the lower half of Figure 4. The first difference has a geometric co-orientation: when looking into the purse, so that its plane of symmetry is horizontal, we see the upper swallowtail moving left during a positive crossing of $D_4^{+,+} - D_4^{+,-}$ while the lower one is moving right.

The appearance of the D_4 differences indicates that I_{Σ^2} is no longer an integer invariant, which is consistent with the fact that the 4-film we used to define this linking number (see Sections 2.1 and 3.3) is now either non-orientable or non-oriented.

A straightforward adjustment, to the case of a non-oriented source, of the study of codimesion 2 singularities done in Section 5 reduces us to integer linear combinations of 11 strata:

$$TA_1^3, \quad A_2A_1^2, \quad A_2^{2,e}, \quad A_3A_1, \quad TA_2A_1, \quad A_2^q, \quad A_3^{e/h}, \quad \text{and} \quad (2).$$

The modified versions of 7 increment equations

$$\mathbf{13}, \quad \mathbf{14}(1), \quad \mathbf{15}, \quad \mathbf{18}(1), \quad \mathbf{19}(1), \quad \mathbf{21}, \quad \mathbf{22}(1) \tag{3}$$

are independent. An integer basis of their solution space is provided by the linear combinations of the codimension 1 strata dual to the invariants $I_s/2$, $I_c/2$, I_t and I_χ (see Lemma 3.1). This proves Theorem 6.1 over the integers in the case of the trivial fundamental groups.

(2) For an arbitrary oriented target N, the integer version of the theorem follows from Proposition 5.1. According to what has just been shown, we should now take in it the upper bound $\nu = 4$, while the lower bound μ is also 4, since all four integer invariants mentioned in the theorem are integral.

(3) We are now left with the mod2 setting.

We first take $\mathbb{R}^3$ as the target. Over $\mathbb{Z}_2$, the consideration reduces to linear combinations of 11 strata:

$$A_1^4, \quad TA_1^3, \quad TA_2^2, \quad A_2A_1^2, \quad A_2^{2,e/h}, \quad A_3A_1, \quad TA_2A_1, \quad A_2^q, \quad A_3^{e/h}, \quad A_4^{\pm,\pm}, \quad D_4^{\pm,\pm}.$$

Equations **15**, **16**(1), **17**(1), **18**(1) and **20**(1) show that the increments across the first 5 strata vanish. No other equations impose additional constraints. Therefore, an arbitrary linear combination of the other 6 strata corresponds to a mod2 local invariant. The invariants $I_s/2$, $I_c/2$, I_t, I_χ, I_{Σ^2} and $I_{\Sigma^{1,1,1,1}}$ are dual to, respectively,

$$A_3^{e/h} + A_4^{\pm,\pm}, \quad A_3A_1 + TA_2A_1 + A_4^{\pm,\pm}, \quad A_3A_1, \quad A_2^q, \quad D_4^{\pm,\pm} \quad \text{and} \quad A_4^{\pm,\pm},$$

which are linearly independent over $\mathbb{Z}_2$. The result does not depend on whether an orientation of $\mathbb{R}^3$ is fixed or not. This proves the theorem for connected components of $\Omega(M, N)$ with trivial fundamental groups.

(4) In the general setting over $\mathbb{Z}_2$, the estimates of the theorem are once again coming from Proposition 5.1. The actual rank of the mod2 invariant space for arbitrary N depends on vanishing or non-vanishing of the total increments of linear combinations of the invariants I_{Σ^2} and $I_{\Sigma^{1,1,1,1}}$ on non-trivial loops in a particular connected component of $\Omega(M, N)$. □

6.2. *Non-oriented target*

THEOREM 6.2. *Assume that the source manifold M is oriented and the target N is not. Then the rank of the space of local invariants is either 4 or 5 over the integers, while it is at least 4 and at most 6 over $\mathbb{Z}_2$.*

Both upper bounds are achieved for connected components of $\Omega(M, N)$ with trivial fundamental groups. In such a case an integer basis is formed by $I_s/2$, $I_c/2$, I_t, I_χ, I_{Σ^2}, to which $I_{\Sigma^{1,1,1,1}}$ should be added to complete a mod2 basis.

Proof. Consider the integer situation. The only difference with the orientation setting considered in part (1) of the proof in the previous subsection is that we now have sums

$$A_4^{\pm,+} = A_4^{+,+} + A_4^{-,+}, \quad A_4^{\pm,-} = A_4^{+,-} + A_4^{-,-},$$
$$D_4^{+,\pm} = D_4^{+,+} + D_4^{+,-}, \quad D_4^{-,\pm} = D_4^{-,+} + D_4^{-,-}$$

instead of (2). This takes equation **21** off the list (3), implying that the invariant I_{Σ^2} survives over $\mathbb{Z}$ if its total increment along any generic loop in a chosen connected component of $\Omega(M, N)$ vanishes. (This is again consistent with the way the integer invariant I_{Σ^2} was defined in Sections 2.1 and 3.3, which needed an orientation of M.) However, the lack of an integral interpretation for I_{Σ^2} when at least one of the participating 3-manifolds is non-orientable disallows us to state that I_{Σ^2} indeed survives in the general setting of the theorem. Hence we are left with the two options for the rank of the invariant space over the integers.

Over $\mathbb{Z}_2$, the result follows from Proposition 5.1. □

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