HOMEWORK: MODULI OF VECTOR BUNDLES ON CURVES AND GENERALIZED THETA DIVISORS

MIHNEA POPA

1. Semistable bundles

(1) (Riemann-Roch for vector bundles on curves.) If E is a vector bundle of rank r and degree d on a smooth projective curve X of genus g, then

$$\chi(E) = d + r(1-g).$$

Prove this assuming Serre duality and anything you want about line bundles, but not Hirzebruch-Riemann-Roch.

(2) Let $\mathcal{F}$ be a coherent sheaf on a smooth projective curve. Then:

(1) If $\mathcal{F}$ is torsion-free, then it is locally free.
(2) $\mathcal{F}$ is the direct sum of a locally free sheaf and a torsion sheaf (i.e. a sheaf supported on a finite set of points).
(3) If $\mathcal{F} = G \oplus \tau$ is a direct sum as above, we define $\text{rk } \mathcal{F} = \text{rk } G$ and $\deg \mathcal{F} := \deg G + \text{length } \tau$. Show that we still have the Riemann-Roch formula

$$\chi(\mathcal{F}) = \deg \mathcal{F} + \text{rk } \mathcal{F} \cdot (1-g).$$

(3) Let E be a vector bundle on a smooth projective curve X. Then:

(1) The degrees of all coherent subsheaves of E are bounded from above. (In particular we can talk about subsheaves of *maximal degree*, or quotients of *minimal degree*.)
(2) If $F \subset E$ is a subsheaf of maximal degree (or $E \to Q \to 0$ is a quotient of minimal degree), then F is in fact a subbundle (resp. Q is a quotient vector bundle).

(4) Let F be a stable bundle on the curve X. For any exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

we have $\text{Hom}(G, E) = 0$.

(5) Let E and F be vector bundles such that $\chi(E \otimes F) = 0$. If $H^0(E \otimes F) = 0$ (or, equivalently by Riemann-Roch, $H^1(E \otimes F) = 0$), then E and F are semistable.

(6) The category $SS(\mu)$ of semistable vector bundles on a smooth projective curve is an abelian category.

Date: August 6, 2006.

(7) Harder-Narasimhan filtrations (cf. the lectures) exist and are unique.

(8) Let E be a semistable vector bundle on a smooth projective curve X. Then:

(1) If $\mu(E) > 2g-2$, then $h^1E = 0$.
(2) If $\mu(E) > 2g-1$, then E is globally generated.

2. Quotients

(9) Let E be a vector bundle of rank r and degree e on a smooth projective curve X.

(1) Show that all the vector bundle quotients of E of rank k can be identified with sections of the Grassmann bundle (of quotients) projection $\pi : \mathbb{G}(k,E) \to X$.
(2) Let $E \xrightarrow{q} F \to 0$ be a vector bundle quotient of E of rank k. By the above this corresponds to a section $s : X \to \mathbb{G}(k,E)$. If $Y = s(X)$, then show that

$$\deg F = d \iff \mathcal{O}_{\mathbb{G}}(1) \cdot Y = rd - ke$$

where $\mathcal{O}_{\mathbb{G}}(1)$ on $\mathbb{G}(k,E)$ is the line bundle inducing the Plücker line bundle on the fibers. Deduce that d determines and is determined by the cohomology class of Y.

(10) Using the exercise above, show that if d_k is the minimal degree of a rank k quotient of a vector bundle E of rank r, then

$$\dim \operatorname{Quot}_{k,d_k}(E) \leq k(r-k).$$

(Hint: use the Rigidity (naive bend-and-break) Lemma.)

3. Moduli spaces

(11) Let $r > 0$ and d be two integers, $h := (r,d)$ and $r_0 := r/h$, $d_0 := d/h$. The dimension of the singular locus Sing $U_X(r,d)$ is equal to $\frac{r^2}{2}(g-1)+2$ if h is even, and $\frac{r^2+r_0^2}{2}(g-1)+2$ if h is odd.

(12) (Moduli of vector bundles on elliptic curves.) Let X be an elliptic curve (so that $X \cong \operatorname{Pic}^0(X)$.) The purpose of this exercise is to give a description of the moduli space of stable bundles on X in the case the rank and degree are coprime.

(1) If E and F are semistable bundles on X and $\mu(E) < \mu(F)$, then $H^1(E \vee \otimes F) = 0$.
(2) Every unstable bundle on X is a direct sum of semistable bundles.
(3) A simple vector bundle E on X (i.e. $\operatorname{Hom}(E,E) \cong k$) is semistable.

Assume for the next items in addition that $(r,d) = 1$ and $d > 0$.

(4) Let E be a stable bundle of rank r and degree d. Show that $h^0E = d$ and that a generator of $H^0\omega_X \cong k$ gives an isomorphism $H^1(E^\vee) \cong H^0(E)^\vee$.
(5) Fix an isomorphism as above. Show that we can construct an extension

$$0 \longrightarrow H^0E \otimes \mathcal{O}_X \xrightarrow{\alpha} F \longrightarrow E \longrightarrow 0$$

where the boundary map is $\mathrm{Id} : H^0E \to H^0E$. Moreover, show that α can be identified with the evaluation map $H^0F \otimes \mathcal{O}_X \to F$.
(6) Show that F is stable.
(7) Show that via the mapping $E \to F$ we obtain an isomorphism

$$U_X(r, d) \cong U_X(r + d, d).$$

(8) Conclude that the determinant map $\det : U_X(r, d) \to \mathrm{Pic}^d(X)$ is an isomorphism.

Remark 3.1. The case when $(r, d) > 1$ is more difficult. It can be shown that there are no stable bundles of rank r and degree d, and that $U_X(r, d)$ is isomorphic to the symmetric product S^hX, where $h = (r, d)$. (Cf. Atiyah's original paper.)

(13) Consider the tensor product map

$$\tau : SU_X(r) \times \mathrm{Pic}^0(X) \longrightarrow U_X(r, 0).$$

(1) Show that τ is étale Galois, with Galois group $X[r] \subset \mathrm{Pic}^0(X)$, the group of r-torsion points in the Jacobian of X. (This means that $X[r]$ has a transitive action on the fibers of τ.)
(2) For any line bundle $N \in \mathrm{Pic}^{g-1}(X)$, show that

$$\tau^*\mathcal{O}_U(\Theta_N) \cong \mathcal{L} \boxtimes \mathcal{O}_{\mathrm{Pic}}(r\Theta_N).$$

4. Generalized theta divisors

(14) Fix r and d, and $E \in U_X(r, d)$. Let F_1 and F_2 be vector bundles on X such that

$$\mu(F_1 \otimes E) = \mu(F_2 \otimes E) = g - 1$$

and such that Θ_{F_1} and Θ_{F_2} are defined. Say $\mathrm{rk}\ F_1 = m \cdot \mathrm{rk}\ F_2$. Then

$$\mathcal{O}(\Theta_{F_1}) \cong \mathcal{O}(\Theta_{F_2})^{\otimes m} \otimes \det{}^*(\det F_1 \otimes (\det F_2)^{-m}).$$

(15) Consider the tensor product map

$$\tau : U_X(r, 0) \times U_X(k, 0) \longrightarrow U_X(kr, 0)$$

and the map

$$\phi := \det \times \det : U_X(r, 0) \times U_X(k, 0) \longrightarrow J(X) \times J(X).$$

Then, for any $N \in \mathrm{Pic}^{g-1}(X)$,

$$\tau^*\mathcal{O}(\Theta_N) \cong p_1^*\mathcal{O}_{J(X)}(k\Theta_N) \otimes p_2^*\mathcal{O}_{J(X)}(r\Theta_N) \otimes \phi^*\mathcal{P},$$

where $\mathcal{P}$ is a Poincaré line bundle on $J(X) \times J(X)$, normalized such that $\mathcal{P}_{|\{0\}\times J(X)} \cong \mathcal{O}_J$.