MODULI OF VECTOR BUNDLES ON CURVES AND GENERALIZED THETA DIVISORS

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1. Lecture II: Moduli spaces and generalized theta divisors

1.1. **The moduli space.** Back to the boundedness problem: we want to see that semistable bundles do the job. First a technical point.

Lemma 1.1. *Let E be a semistable bundle on X.*

(a) If $\mu(E) > 2g-2$, then $h^1 E = 0$.

(b) If $\mu(E) > 2g-1$, then E is globally generated.

Proof. Homework. □

Proposition 1.2. *The set $\mathcal{S}(r,d)$ of isomorphism classes of semistable bundles of rank r and degree d is bounded.*

Proof. Fix $\mathcal{O}_X(1)$ a polarization on the curve. By the Lemma, there exists a fixed $m >> 0$ such that for all F in $\mathcal{S}(r,d)$ we have $h^1F(m) = 0$ and $F(m)$ is globally generated. Let $q := h^0F(m) = \chi(F(m))$, which is constant by Riemann-Roch. The global generation of $F(m)$ means that we have a quotient

$$\mathcal{O}_X^{\oplus q}(-m) \xrightarrow{\beta} F \longrightarrow 0.$$

These all belong to the Quot scheme $\mathrm{Quot}_{r,d}(\mathcal{O}_X^{\oplus q}(-m))$, which is a bounded family. □

The quotient β can be realized in many ways: fix a vector space $V \cong k^q$, and choose an isomorphism $V \cong H^0F(m)$. On $\mathrm{Quot}_{r,d}(V \otimes \mathcal{O}_X(-m)) \cong \mathrm{Quot}_{r,d}(\mathcal{O}_X^{\oplus q}(-m))$ we have a natural $GL(V)$-action, namely each $g \in GL(V)$ induces a diagram

$$\begin{tikzcd}
V \otimes \mathcal{O}_X(-m) \arrow[r] \arrow[d, "g\otimes \mathrm{id}"'] & Q \\
V \otimes \mathcal{O}_X(-m) \arrow[ur] &
\end{tikzcd}$$

The scalar matrices act trivially on these quotients, so in fact we have a $PGL(V)$-action.

Proposition 1.3. *Let $\Omega \subset \mathrm{Quot}_{r,d}(V \otimes \mathcal{O}_X(-m))$ be the set of quotients Q such that Q is semistable and $V \cong H^0Q(m)$. Then Ω is invariant under the $PGL(V)$-action, and we have a bijection*

$$\Omega/PGL(V) \longrightarrow \mathcal{S}(r,d).$$

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Proof. The set Ω is clearly invariant, and the points in the same orbit give isomorphic quotient bundles, so we have a natural map $\Omega/PGL(V) \longrightarrow \mathcal{S}(r,d)$. Since $m >> 0$, by global generation the map is surjective. Suppose now that we have two different quotients inducing an isomorphism ϕ:

$$\begin{tikzcd} V \otimes \mathcal{O}_X(-m) \arrow[r] & Q \arrow[d, "\phi"', "\cong"] \\ V \otimes \mathcal{O}_X(-m) \arrow[r] & Q \end{tikzcd}$$

This induces an isomorphism $H^0Q(m) \overset{H^0\phi(m)}{\longrightarrow} H^0Q(m)$, which corresponds to an element $g \in GL(V)$. This is uniquely determined up to scalars, so we can in fact consider it in $PGL(V)$. $\square$

Although we will not have time to get into this during the lectures, the main point of Geometric Invariant Theory (GIT) in this context is essentially to show that in fact this quotient has the structure of a projective algebraic variety[1]. The GIT machinery constructs the space $U_X(r,d)$ – an algebraic variety replacement of $\mathcal{S}_{r,d}$ – the *moduli space of S-equivalence classes of semistable vector bundles of rank r and degree d on X*. We denote by $U_X^s(r,d)$ the open subset corresponding to isomorphism classes of stable bundles. These spaces have the following basic properties:

(1) $U_X(r,d)$ is a projective variety (i.e irreducible), of dimension $r^2(g-1)+1$.

(2) $U_X(r,d)$ is in general only a *coarse* moduli space (i.e. there is no universal family). In fact one can show that $U_X(r,d)$ is fine if and only if $(r,d)=1$.

Remark 1.4. By twisting with an arbitrary line bundle of degree $e \in \mathbf{Z}$ we see that there is an isomorphism

$$U_X(r,d) \cong U_X(r,d+re),$$

which means that in many arguments we can consider that $d >> 0$.

We consider also a variant of $U_X(r,d)$ when the determinant of the vector bundles is fixed. More precisely, for any $L \in \mathrm{Pic}^d(X)$, we denote by $SU_X(r,L)$ the moduli space of (S-equivalence classes of) semistable bundles of rank r and fixed determinant L. These are the fibers of the natural determinant map

$$\det : U_X(r,d) \longrightarrow \mathrm{Pic}^d(X).$$

Here are more consequences which are derived from the GIT construction:

(3) $U_X(r,d)$ and $SU_X(r,L)$ are normal, Gorenstein, with rational singularities.

(4) $\mathrm{Sing}(U_X(r,d)) = U_X(r,d) - U_X^s(r,d)$, unless $g=2$, $r=2$ and $d=$ even, when $U_X(r,d)$ is smooth. Same for $SU_X(r,L)$. Moreover, if $h=(r,d)$ and we denote $r_0 := r/h$ and $d_0 := d/h$, then the dimension of $\mathrm{Sing}(U_X(r,d))$ is $\frac{r^2}{2}(g-1)+2$ if h is even and $\frac{r^2+r_0^2}{2}(g-1)+2$ if h is odd (homework).

(5) If E is stable, then $T_E\, U_X(r,d) \cong H^1(E^\vee \otimes E) \cong \mathrm{Ext}^1(E,E)$.

[1]This is literally true if we consider S-equivalence classes of semistable bundles instead of isomorphism classes.

Problem 1.5. *A general good description of the tangent space (cone) to the moduli space at a singular (equivalently, semistable) point is not known, except in a few special cases – this is an important problem.*

2. Generalized theta divisors

Consider first the following situation: on $\mathrm{Pic}^d(X)$ we have, for each $L \in \mathrm{Pic}^{g-1-d}$, a theta divisor

$$\Theta_L := \{M \mid h^0(M \otimes L) \neq 0\},$$

which is a "translate" of the principal polarization Θ on the Jacobian $J(X) \cong \mathrm{Pic}^0(X)$ (or of $W_{g-1} = \{N \mid h^0 N \neq 0\} \subset \mathrm{Pic}^{g-1}(X)$). Note that the numerical choice is such that $\chi(M \otimes L) = 0$.

Now fix $E \in U_X(r,d)$[2]. When can we have a vector bundle F such that $\chi(E \otimes F) = 0$? By Riemann-Roch we need $\mu(E \otimes F) = g - 1$, in other words

$$\mu(F) = g - 1 - \mu(E).$$

Using the notation $h = (r,d)$, $r_0 = r/h$ and $d_0 = d/h$, we see that the only possibilities are $\mathrm{rk}\ F = kr_0$ and $\deg\ F = k(r_0(g-1) - d_0)$, with $k \geq 1$. Fix such an F.

Claim: If there exists $E \in U_X(r,d)$ such that $H^0(E \otimes F) = 0$, then

$$\Theta_F := \{E \mid h^0(E \otimes F) \neq 0\} \subset U_X(r,d)$$

is a divisor with a natural scheme structure (a *generalized theta divisor*). The same is true on $SU_X(r,L)$.

Proof. I will explain here only the case $(r,d) = 1$, when there exists a universal family, and discuss more the other case in the afternoon[3]. It is a general example of a determinantal construction: let $\mathcal{E}$ be the universal bundle on $X \times U_X(r,d)$ and D an effective divisor on X with $\deg D >> 0$. We can consider the following natural sequence obtained by pushing forward to $U_X(r,d)$:

$$0 \to p_{U*}(\mathcal{E} \otimes p_X^* F) \to p_{U*}(\mathcal{E} \otimes p_X^* F(D)) \to p_{U*}((\mathcal{E} \otimes p_X^* F(D))_{|D \times U_X(r,d)}) \to R^1 p_{U*}(\mathcal{E} \otimes p_X^* F) \to 0,$$

Note that the 0 on the right is obtained by base change, since for any $E \in U_X(r,d)$ we have that $h^1(E \otimes F(D)) = 0$, as D has sufficiently large degree and the family of E's is bounded. Let's redenote this sequence

$$0 \to K \to G \xrightarrow{\alpha} H \to C \to 0.$$

In fact we have $K = 0$ (exercise). In any case, by base change and Riemann-Roch, G and H are vector bundles on $U_X(r,d)$ of the same rank $r \cdot \mathrm{rk}(F) \cdot \deg(D)$. Where we can apply base change, fiberwise we have

$$H^0(E \otimes F(D)) \to H^0(E \otimes F(D)_D) \to H^1(E \otimes F).$$

We see immediately that the degeneracy locus of α is set-theoretically precisely Θ_F. This means that Θ_F has a determinantal scheme structure and since $\mathrm{rk}\ G = \mathrm{rk}\ H$ we have that

[2] I will always use somewhat abusively vector bundle notation instead of S-equivalence class notation for simplicity. For anything we are interested in, it can be checked easily using Jordan-Hölder filtrations that the statement is independent of the choice in the S-equivalence class.

[3] In any case, the bottom line is that for this problem we can in fact pretend that there's always a universal family.

codim $\Theta_F \le 1$. But since there exists E such that $h^0(E \otimes F) = 0$, i.e $E \notin \Theta_F$, we have that codim $\Theta_F \le 1$. $\square$

In fact the requirement above is satisfied for general F, so we always have enough generalized theta divisors. I will skip the proof of this, which is not hard but appeals to things that we have not discussed. Note however that, by semicontinuity, it is obvious for example on the moduli spaces that contain direct sums of line bundles, for instance $U_X(k, k(g-1))$.

Proposition 2.1. *If $F \in U_X(kr_0, k(r_0(g-1) - d_0))$ and $E \in U_X(r,d)$ are general, then*

$$H^0(E \otimes F) = H^1(E \otimes F) = 0.$$

One of the most important facts that has been proved about the moduli spaces of vector bundles, again using the GIT description, is the following:

Theorem 2.2. *(Drezet-Narasimhan) (1) $U_X(r,d)$ and $SU_X(r,L)$ are locally factorial.*

(2) For any $F \in U_X(kr_0, k(r_0(g-1) - d_0))$ such that Θ_F is a divisor, the line bundle $\mathcal{O}(\Theta_F)$ on $SU_X(r,L)$ does not depend on the choice of F. The Picard group of $SU_X(r,L)$ is isomorphic to $\mathbf{Z}$, generated by an ample line bundle $\mathcal{L}$ (called the determinant line bundle*), and $\Theta_F \in |\mathcal{L}^k|$.*

(3) The inclusions $\mathrm{Pic}(\mathrm{Pic}^d(X)) \subset \mathrm{Pic}(U_X(r,d))$ (given by the determinant morphism) and $\mathbf{Z} \cdot \mathcal{O}(\Theta_F) \subset \mathrm{Pic}(U_X(r,d))$, with $k = 1$, induce an isomorphism

$$\mathrm{Pic}(U_X(r,d)) \cong \mathrm{Pic}(\mathrm{Pic}^d(X)) \oplus \mathbf{Z}.$$

We have the following transformation formula: if $F, F' \in U_X(kr_0, k(r_0(g-1) - d_0))$, then

$$\mathcal{O}(\Theta_F) \cong \mathcal{O}(\Theta_{F'}) \otimes \det{}^*(\det(F) \otimes \det(F')^{-1}).$$

(4) $\omega_{SU_X(r,L)} \cong \mathcal{L}^{-2h}$, where $h = (r,d)$ (in particular, $SU_X(r,L)$ are Fano varieties).

2.1. **Verlinde formula.** The theorem above tells us that the only line bundles on the moduli space $SU_X(r,L)$ are the powers of the determinant line bundle $\mathcal{L}$. The Verlinde formula tells us the dimension $s_{r,k}$ of the space of global sections $H^0(SU_X(r,L), \mathcal{L}^k)$, for all $k \ge 1$. For simplicity, from now on I will restrict to the case when $L = \mathcal{O}_X$, and denote $SU_X(r) := SU_X(r, \mathcal{O}_X)$; there are analogous but technically more complicated results for any L. The Verlinde formula, in a form due to Zagier, reads

$$s_{r,k} = \left(\frac{r}{r+k}\right)^g \cdot \sum_{\substack{S \cup T = \{1,\ldots,r+k\} \\ |S|=r, |T|=k}} \prod_{\substack{s \in S \\ t \in T}} \left|2 \cdot \sin \pi \frac{s-t}{r+k}\right|^{g-1}.$$

Example 2.1: Note that $s_{1,k} = 1$, while $s_{r,1} = r^g$. This last number is the same as the dimension of the space of classical theta functions of level r, i.e. $h^0(J(X), \mathcal{O}_J(r\Theta))$! We'll see in a second that this is not a coincidence.

Remark 2.3. By Kodaira vanishing (which works on varieties with rational singularities – why?) we have

$$H^i(SU_X(r), \mathcal{L}^k) = 0, \text{ for all } i > 0 \text{ and } k > 0.$$

This implies that

$$h^0(SU_X(r), \mathcal{L}^k) = \chi(\mathcal{L}^k) = \frac{\mathcal{L}^{(r^2-1)(g-1)}}{(r^2-1)(g-1)!} \cdot k^{(r^2-1)(g-1)} + \text{l.o.t.}$$

This is a polynomial in k, with integral output, neither of which seems to be obvious from the formula!

What about $U_X(r, 0)$? Note that we have a natural morphism obtained by taking tensor product with line bundles:

$$\tau : SU_X(r) \times \mathrm{Pic}^{g-1}(X) \xrightarrow{\otimes} U_X(r, r(g-1)).$$

One can check the following facts (all in the homework). First, τ is étale Galois, with Galois group $X[r]$, the group of r-torsion points in $\mathrm{Pic}^0(X)$, which has order r^{2g}. Second, if we denote the canonical polarization

$$\Theta = \Theta_{can} := \{E \mid h^0 E \neq 0\} \subset U_X(r, r(g-1)),$$

we have that

$$\tau^* \mathcal{O}_U(\Theta) \cong \mathcal{L} \boxtimes \mathcal{O}_{\mathrm{Pic}}(r\Theta).$$

The Künneth formula gives then for any $k \geq 1$ an isomorphism

$$H^0(\tau^* \mathcal{O}_U(k\Theta)) \cong H^0(SU_X(r), \mathcal{L}^k) \otimes H^0(\mathrm{Pic}^{g-1}(X), \mathcal{O}(r\Theta)).$$

Putting these facts together, we obtain that

$$h^0(U_X(r, r(g-1), \mathcal{O}_U(k\Theta)) = s_{r,k} \cdot \frac{k^g}{r^g}.$$

Corollary 2.4.

$$h^0(U_X(k, k(g-1), \mathcal{O}_U(r\Theta)) = s_{r,k} = h^0(SU_X(r), \mathcal{L}^k).$$

Proof. Stare at the Verlinde formula until you see that the expression $s_{r,k} \cdot r^{-g}$ is symmetric in r and k. □

Corollary 2.5.

$$h^0(U_X(k, k(g-1), \mathcal{O}_U(\Theta)) = 1, \text{ for all } k \geq 1.$$

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