MODULI OF VECTOR BUNDLES ON CURVES AND GENERALIZED THETA DIVISORS

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These are rough notes for my lectures and the afternoon sessions at the Cologne Summer School, August 7-11. The main omission for now is that I haven't (at least yet) included references in the text. The bibliography however contains books and references that I rely on for the general theory.

1. Lecture I: Bounded families, semistable bundles, examples

1.1. **Arbitrary vector bundles.** Let X be a smooth projective curve of genus g over an algebrically closed field k. Later we will need to assume that the characteristic is 0, but for now we can work in full generality. We will identify freely vector bundles with locally free sheaves.

Definition 1.1. Let E be a vector bundle on X, of rank r. The *determinant* of E is the line bundle $\det E := \wedge^r E$. The *degree* of E is the degree d of $\det E$[1]. The *slope* of E is the rational number $\mu(E) = \frac{d}{r}$.

The Riemann-Roch formula for vector bundles on curves says:

$$\chi(E) = h^0 E - h^1 E = d + r(1-g).$$

An important example is the following: $\chi(E) = 0 \iff \mu(E) = g - 1$.

From the moduli point of view, the initial idea would be to construct an algebraic variety (or scheme) parametrizing the isomorphism classes of all vector bundles with fixed invariants, i.e. rank r and degree d. Note that fixing these invariants is the same as fixing the Hilbert polynomial of E.

Definition 1.2. Let $\mathcal{B}$ be a set of isomorphism classes of vector bundles. We say that $\mathcal{B}$ is *bounded* if there exists a scheme of finite type S over k and a vector bundle F on $S \times X$ such that all the elements of $\mathcal{B}$ are represented by some $F_s := F_{|\{s\}\times X}$ with $s \in S$.

We find easily that the initial idea above is too naive.

Lemma 1.3. *The set of isomorphism classes of vector bundles of rank r and degree d on X is not bounded.*

Proof. Assuming that the family is bounded, use the notation in Definition 1.2. The projection to S is flat, while F is locally free, so F is flat over S. By the semicontinuity theorem, this implies that there are only a finite number of possible $h^i F_s$ for $i = 0, 1$.

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[1]This is the same as the first Chern class $c_1(E)$.

Fix now a point $x \in X$ and define for each $k \in \mathbf{N}$ the vector bundle

$$E_k := \mathcal{O}_X(-kx) \oplus \mathcal{O}_X((k+d)x) \oplus \mathcal{O}_X^{\oplus r-2}.$$

They all clearly have rank r and degree d. On the other hand, when $k \to \infty$ we see that $h^0 E_k$ and $h^1 E_k$ also go to ∞, which gives a contradiction. $\square$

There exists however a well-known bounded moduli problem in this context, which produces the Quot scheme. Let E be a vector bundle of rank r and degree e on X, and fix integers $0 \le k \le r$ and d. We would like to parametrize all the quotients

$$E \longrightarrow Q \longrightarrow 0$$

with Q a coherent sheaf of rank k and degree d on X.[2] We consider the *Quot functor*:

$$\underline{\mathrm{Quot}}^E_{k,d} : \text{ Algebraic varieties}/k \to \text{Sets}$$

associating to each S the set of coherent quotients of $E_S := p_X^* E$ which are flat over S and have rank k and degree d over each $s \in S$. This is a contravariant functor associating to $T \xrightarrow{f} S$ the map taking a quotient $E_S \to Q$ to the quotient $E_T = (f \times id)^* E_S \to (f \times id)^* Q$.

Theorem 1.4 (Grothendieck)**.** *There exists a projective scheme $Quot_{k,d}(E)$ of finite type over k, which represents the functor $\underline{\mathrm{Quot}}^E_{k,d}$.*

This means the following: there exists a "universal quotient" $E_{\mathrm{Quot}_{k,d}(E)} \to \mathcal{Q}$ on $\mathrm{Quot}_{k,d}(E) \times X$, which induces for each variety S an isomorphism

$$\mathrm{Hom}(S, \mathrm{Quot}_{k,d}(E)) \cong \underline{\mathrm{Quot}}^E_{k,d}(S)$$

given by

$$(S \xrightarrow{f} \mathrm{Quot}_{k,d}(E)) \to (E_S \to (f \times id)^* \mathcal{Q}).$$

The terminology is: the Quot functor (scheme) is a *fine* moduli functor (space). We will discuss more about Quot schemes in the afternoon sessions. Let me just mention here the following basic fact, which is a standard consequence of formal smoothness.

Proposition 1.5. *Let E be a vector bundle of rank r and degree e, and*

$$q : \ [0 \to G \to E \to F \to 0]$$

a point in $Quot_{k,d}(E)$. Then:

(1) There is a natural isomorphism $T_q Quot_{k,d}(E) \cong \mathrm{Hom}(G,F) (\cong H^0(G^\vee \otimes F))$.

(2) If $\mathrm{Ext}^1(G,F) (\cong H^1(G^\vee \otimes F)) = 0$, then $Quot_{k,d}(E)$ is smooth at q.

(3) We have

$$h^0(G^\vee \otimes F) \ge \dim_q Quot_{k,d}(E) \ge h^0(G^\vee \otimes F) - h^1(G^\vee \otimes F).$$

The last quantity is $\chi(G^\vee \otimes F) = rd - ke - k(r-k)(g-1)$ (by Riemann-Roch).

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[2] Cf. the homework for the notions of rank and degree for an arbitrary coherent sheaf on X.

1.2. **Semistable vector bundles.** To remedy the problem explained in the previous subsection, one introduces the following notion.

Definition 1.6. Let E be a vector bundle on X of rank r and degree d. It is called *semistable* (respectively *stable*) if for any subbundle $0 \neq F \hookrightarrow E$, we have $\mu(F) \leq \mu(E)$ (respectively $\mu(F) < \mu(E)$). It can be checked that in the definition we can replace subbundles with arbitrary coherent subsheaves (cf. the homework).

Here are some basic properties which will be useful in the sequel:

Proposition 1.7. *If E and F are stable vector bundles and $\mu(E) = \mu(F)$, then every non-zero homomorphism $\phi : E \to F$ is an isomorphism. In particular* $\mathrm{Hom}(E, E) \cong k$ *(i.e. E is* simple*).*

Proof. Say $G = \mathrm{Im}(\phi)$. Then by definition we must have $\mu(E) \leq \mu(G) \leq \mu(F)$ and wherever we have equality the bundles themselves must be equal. Since $\mu(E) = \mu(F)$, we have equality everywhere, which implies easily that ϕ must be an isomorphism. Now if $\phi \in \mathrm{Hom}(E, E)$, by the above we see that $k[\phi]$ is an finite field extension of k. Since this is algebraically closed, we deduce that $\phi = \lambda \cdot \mathrm{Id}$, with $\lambda \in k^*$. □

Proposition 1.8. *Fix a slope $\mu \in \mathbf{Q}$, and let $SS(\mu)$ be the category of semistable bundles of slope μ. Then $SS(\mu)$ is an abelian category.*

Proof. Homework. □

Definition 1.9. Let $E \in SS(\mu)$. A *Jordan-Hölder* filtration of E is filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_p = E$$

such that each quotient E_{i+1}/E_i is stable of slope μ.

Proposition 1.10. *Jordan-Hölder filtrations exist. Any two have the same length and, upon reordering, isomorphic stable factors.*

Proof. Since the rank decreases, there is a $G \subset E$ stable of slope μ. This implies that E/G is semistable of slope μ and we repeat the process with E/G instead of E. The rest is a well-known general algebra argument. □

Definition 1.11. (1) For any Jordan-Hölder filtration $E_\bullet$ of E, we define

$$\mathrm{gr}(E) := \mathrm{gr}(E_\bullet) = \bigoplus_i E_{i+1}/E_i.$$

This is called the *graded object* associated to E (well-defined by the above).

(2) A vector bundle E is called *polystable* if it is a direct sum of stable bundles of the same slope. (So for E semistable $\mathrm{gr}(E)$ is polystable.)

(3) Two bundles $E, F \in SS(\mu)$ are called *S-equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(F)$.

Sometimes we can reduce the study of arbitrary bundles to that of semistable ones via the following:

Proposition 1.12. *Let E be a vector bundle on X. Then there exists an increasing filtration*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_p = E$$

such that

(1) *Each quotient E_{i+1}/E_i is semistable.*
(2) *We have $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all i.*

The filtration is unique; it is called the Harder-Narasimhan filtration *of E.*

Proof. Homework. □

Example 1.1: (1) All line bundles are stable. Any extension of vector bundles in $SS(\mu)$ is also in $SS(\mu)$.

(2) If $(r, d) = 1$, then stable is equivalent to semistable.

(3) If $X = \mathbf{P}^1$, by Grothendieck's theorem we know that every vector bundle splits as $E \cong \mathcal{O}_(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$, so it is semistable iff all a_i are equal.

(4) We will study some very interesting examples below. Until then, here's the first type of example which requires a little argument. Say L_1 and L_2 are line bundles on X, with $\deg L_1 = d$ and $\deg L_2 = d + 1$. Consider extensions of the form

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0.$$

These are parametrized by $\mathrm{Ext}^1(L_2, L_1) \cong H^1(L_1 \otimes L_2^{-1})$. By Riemann-Roch this is isomorphic to k^{g-2}, so as soon as $g \geq 3$ we can choose the extension to be non-split. For such a choice E is stable: first note that $\mu(E) = d + 1/2$. Consider any line subbundle M of E. If $\deg M \leq d$ everything is fine. If not, the induced map $M \to L_2$ must be non-zero (otherwise M would factor through L_1, of too low degree). This immediately implies that it must be an isomorphism, which is a contradiction since the extension is non-split.

For later reference, let me also mention the following important:

Theorem 1.13. *Assume that* $\mathrm{char}(k) = 0$. *If E and F are semistable bundles, then $E \otimes F$ is also semistable*[3]. *In particular, for any k, $S^k E$ and $\wedge^k E$ are also semistable.*

Proof. The second assertion follows from the first, since in characteristic 0 symmetric and exterior powers are direct summands of tensor powers. I will sketch a proof of the first statement, due to Gieseker, in the afternoon session. □

1.3. **Example: Lazarsfeld's bundles.** Here's the more interesting example promised above. It is based on a construction considered by Lazarsfeld in the study of syzygies of curves.

Consider a line bundle L on X of degree $d \geq 2g + 1$. Denote by M_L the kernel of the evaluation map:

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} L \longrightarrow 0$$

[3] It is also true that if E and F are actually stable, then $E \otimes F$ is polystable.

and let $Q_L = M_L^\vee$. Note that rk $Q_L = h^0L - 1 = d - g$ and deg $L = d$, so $\mu(Q_L) = \frac{d}{d-g}$. The main property of Q_L is the following:

Proposition 1.14. *If $x_1, \ldots, x_d$ are the points of a generic hyperplane section of $X \subset \mathbb{P}(H^0(L))$, then Q_L sits in an extension:*

$$0 \longrightarrow \bigoplus_{i=1}^{d-g-1} \mathcal{O}_X(x_i) \longrightarrow Q_L \longrightarrow \mathcal{O}_X(x_{d-g} + \ldots + x_d) \longrightarrow 0.$$

Proof. Afternoon session. □

This implies the stability of Q_L – the proof below is due to Ein-Lazarsfeld.

Proposition 1.15. *Under the assumptions above Q_L is a stable bundle.*

Proof. Let's see that the dual M_L is stable. One can actually prove a bit more: M_L is *cohomologically stable*, i.e. for any line bundle A of degree a and any $t < \mathrm{rk} M_L = d - g$:

$$H^0(\bigwedge^t M_L \otimes A^{-1}) = 0 \text{ if } a \geq t \cdot \mu(M_L) = -\frac{td}{d-g}. \tag{1}$$

This implies the stability of M_L: indeed, if $F \hookrightarrow M_L$ is a subbundle of degree a and rank t, then we have an inclusion $A := \bigwedge^t F \hookrightarrow \bigwedge^t M_L$, which implies that $H^0(\bigwedge^t M_L \otimes A^{-1}) \neq 0$. By cohomological stability we must have $\mu(F) = \frac{a}{t} < \mu(M_L)$, so M_L is stable.

Let's prove (1). Take exterior powers in the dual of the sequence in Proposition 1.14 to obtain

$$0 \to \mathcal{O}_X(-x_{d-g} - \ldots - x_d) \otimes \bigwedge^{t-1}(\bigoplus_{i=1}^{d-g-1} \mathcal{O}_X(-x_i)) \to \bigwedge^t M_L \to \bigwedge^t(\bigoplus_{i=1}^{d-g-1} \mathcal{O}_X(-x_i)) \to 0.$$

In other words we have an exact sequence

$$0 \to \bigoplus_{1 \leq i_1 < \ldots < i_{t-1} \leq d-g-1} \mathcal{O}_X(-x_{i_1} - \ldots - x_{i_{t-1}} - x_{d-g} \ldots - x_d)) \to \bigwedge^t M_L$$

$$\to \bigoplus_{1 \leq j_1 < \ldots < j_t \leq d-g-1} \mathcal{O}_X(-x_{j_1} - \ldots - x_{j_t})) \to 0.$$

We tensor this sequence by A^{-1}. It can be checked easily that on both extremes H^0 is zero, as the points x_i are general. This implies what we want. □

Corollary 1.16. *If* $\mathrm{char}(k) = 0$, *then for all p the bundle $\bigwedge^p Q_L$ is semistable.*

1.4. **Example: Raynaud's bundles.** Let $X \hookrightarrow J(X)$ be an Abel-Jacobi embedding. Denote $A = J(X)$. Then A has a principal polarization Θ, which induces an isomorphism $A \cong \widehat{A} = \mathrm{Pic}^0(A)$. Denote by $\mathcal{P}$ a Poincaré bundle on $A \times \widehat{A}$. For any $m \geq 1$, we consider what's called the *Fourier-Mukai* transform of $\mathcal{O}_{\widehat{A}}(-m\Theta)$, namely:

$$F := \widehat{\mathcal{O}_A(-m\Theta)} := R^g p_{A*}(p_{\widehat{A}}^* \mathcal{O}(-m\Theta) \otimes \mathcal{P}).$$

By base change, this is a vector bundle with fiber over $x \in A$ isomorphic to

$$H^g(\widehat{A}, \mathcal{O}_{\widehat{A}}(-m\Theta) \otimes \mathcal{P}_{|\{x\}\times \widehat{A}}) \cong H^0(J(X), \mathcal{O}_{J(X)}(m\Theta) \otimes P_x)^\vee,$$

where P_x is the line bundle in $\mathrm{Pic}^0(J(X))$ corresponding to $x \in J(X)$. (Note that as x varies with $J(X)$, P_x varies with $\mathrm{Pic}^0(J(X))$.) Hence F is a vector bundle of rank m^g. Define the vector bundle $E := F_{|X}$ on X.

The claim is that this is a semistable bundle. Indeed, consider the multiplication by m map $\phi_m : A \to A$. By a result of Mukai (cf. afternoon session) we have that

$$\phi_m^* F \cong H^0\mathcal{O}_{\widehat{A}}(m\Theta) \otimes \mathcal{O}_{\widehat{A}}(m\Theta) \cong \bigoplus_{m^g} \mathcal{O}_{\widehat{A}}(m\Theta).$$

We consider the étale base change $\psi : Y \to X$, where $Y = \phi_m^{-1}(X)$. The decomposition of F via pull-back by ϕ_m implies that $\psi^* E$ is semistable on Y. Applying the Lemma below, we deduce that E is semistable.

Let's also compute the slope of E. Note that $\deg\, \psi^* E = \deg\, \phi_m \cdot \deg\, E = m^{2g} \cdot \deg\, E$. This gives

$$\deg\, E = \frac{m^g \cdot m \cdot (\theta \cdot [Y])}{m^{2g}} = \frac{\theta \cdot [Y]}{m^{g-1}}.$$

But since $Y = \phi_m^{-1}(X)$, $\phi_m^* \theta \equiv m^2 \cdot \theta$ and $\theta \cdot [X] = g$, we have that $m^2 \cdot (\theta \cdot [Y]) = g \cdot m^{2g}$. Putting everything together we finally obtain

$$\deg\, E = g \cdot m^{g-1}, \text{ i.e. } \mu(E) = \frac{g}{m}.$$

Lemma 1.17. *Let $f : Y \to X$ be a finite morphism of smooth projective curves, and E a vector bundle on X. Then E is semistable if and only if $f^* E$ is semistable.*

Proof. Afternoon session. □

Example 1.2: I will only mention in passing one more idea, which has to do with moduli of curves and Farkas' lectures. The point is that certain types of stable bundles can exist only on special curves, just like in the usual Brill-Noether theory for line bundles. One can look for vector bundles with "many" sections, i.e. wonder whether for a given k there exist semistable bundles E on X of rank r and degree d (or fixed determinant L), which have k independent sections.

For instance, if $g(X) = 10$, one can see that the condition that there exist a semistable bundle of rank 2 and determinant ω_X with at least 7 sections is codimension 1 in $\mathcal{M}_{10}$, i.e. such bundles exist only on curves filling up a divisor in $\mathcal{M}_{10}$. The closure of this divisor in $\overline{\mathcal{M}}_{10}$ is a very interesting divisor, the first divisor shown to be of "slope" smaller than expected. However, strictly speaking these vector bundles are of the same kind as the Lazarsfeld examples above.

References

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