Numerisch triviale Faserungen und Blätterungen

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0.1. Die Klassifikation algebraischer Varietäten

Historisch begann die Algebraische Geometrie mit dem Studium von algebraischen Kurven, zuerst in der Antike die Kegelschnitte, dann in der Neuzeit, beginnend mit Newton, ebene algebraische Kurven von höherem Grad (speziell ihrer Singularitäten), und im 19. Jahrhundert auch Raumkurven. Riemann schließlich gelang mit analytischen und topologischen Methoden eine befriedigende Strukturtheorie algebraischer Kurven.
Die italienische Schule um Castelnuovo, Enriques und Severi konnte dann Anfang des 20. Jahrhunderts algebraische Flächen zufriedenstellend klassifizieren. Da ihrer Arbeit aber die Hilbertsche Strenge fehlte und ihre Schüler oftmals falsche Ergebnisse über algebraische Dreifaltigkeiten präsentierten, geriet das ganze Gebiet in Verruf.
Van der Waerden und danach Zariski und Weil stellten die Algebraische Geometrie mit den Methoden der kommutativen Algebra wieder auf eine solide Grundlage, und Grothendieck vereinigte in seinem unvollendeten opus magnum “ *Éléments de géométrie algébrique*” die Kommutative Algebra und die Algebraische Zahlentheorie mit der Algebraischen Geometrie.
Nachdem diese Grundlagenarbeit Ende der 1960er (fast) zum Abschluß gebracht worden war, wandte man sich wieder den klassischen Problemen zu und erneuerte und vervollständigte zuerst die Theorie der Kurven und Flächen. Iitaka stellte 1972 gewagte, aber hochinteressante Vermutungen über höherdimensionale Varietäten auf, und Ueno bewies 1977 das erste Strukturtheorem für Dreifaltigkeiten. Trotzdem war klar, daß ihr Ansatz für eine umfassende Strukturtheorie von höherdimensionale Varietäten nicht ausreichte — vor allem fehlte ein Analogon zu den minimalen Modellen algebraischer Flächen.
Um 1980 kam dann ein entscheidender Durchbruch: Mori bewies mit Hilfe verschiedener neuer Ideen den ersten großen Schritt für die Existenz von minimalen Modellen höherdimensionaler Varietäten. Gleichzeitig definierte Reid, was minimale

Modelle von höherdimensionalen Varietäten überhaupt sein sollen und untersuchte, wozu man sie benutzen könnte, wenn sie denn existieren. Für Dreifaltigkeiten wurde das so sich abzeichnende Programm Ende der 80er von Mori und Kollár erfolgreich abgeschlossen und damit verschiedene tiefe Strukturtheoreme für Dreifaltigkeiten bewiesen, darunter die Abundance-Vermutung und die endliche Erzeugtheit des kanonischen Rings.

Beide Vermutungen (in Dimension ≥ 4) bzw. Theoreme (in Dimension 3) beschreiben Eigenschaften des kanonischen Geradenbündels K_X, also der höchsten äußeren Potenz des Kotangentialbündels $\Omega^1_X = T^*_X$ einer (glatten) algebraischen Varietät X. Die Wichtigkeit des Bündels K_X rührt von zwei Beobachtungen:

Erstens definieren globale Schnitte von (holomorphen) Geradenbündeln rationale Abbildungen in andere algebraische Varietäten (z.B. projektive Räume $\mathbb{CP}^n$), und das kanonische Bündel existiert auf jeder algebraischen Varietät.

Zweitens hat das kanonische Bündel gute funktorielle Eigenschaften unter birationalen Abbildungen, also solchen rationalen Abbildungen $f : X \dashrightarrow Y$, die auf einer offenen Teilmenge $U \subset X$ Isomorphismen sind: f induziert einen Isomorphismen zwischen den globalen Schnitten von K_X und K_Y. Das trifft auch für alle Potenzen $K_X^{\otimes m}$, aber nicht für das zum kanonischen Bündel duale antikanonische Bündel K_X^{-1} (und seine Potenzen) zu.

Die Funktorialität bzgl. birationaler Abbildungen ist wichtig bei einer sehr allgemeinen Strategie, Strukturtheoreme algebraischer Varietäten zu erhalten. Diese Strategie wurde schon von der italienischen Schule zur Klassifikation der Flächen benutzt:

Schritt 1: Man definiert zuerst eine Äquivalenzrelation auf der Menge aller algebraischen Varietäten, bei der zwei algebraische Varietäten als äquivalent gelten, wenn sie zueinander birational sind. Man versucht außerdem zu verstehen, wie verschiedene Varietäten in einer Äquivalenzklasse zusammenhängen. Z.B. hätte man gerne einige einfache Typen von Operationen, so dass man durch die Ausführung einer endlichen Anzahl dieser Operationen von einer Varietät zur anderen gelangt.

Schritt 2: Dann sucht man sich in jeder Äquivalenzklasse einen ausgezeichneten Repräsentanten, ein *minimales Modell*.

Schritt 3: Die Eigenschaften dieser minimalen Modelle werden dann verwendet, um einen Überblick über alle Äquivalenzklassen zu bekommen.

Die “einfachen Operationen”, die von einer algebraischen Fläche zu einer birational äquivalenten führen, sind Auf- und Niederblasungen von rationalen (-1)-Kurven, also Kurven $\cong \mathbb{CP}^1$ mit Selbstschnitt -1. Die minimalen Modelle sind Flächen, auf denen sich keine (-1)-Kurve zum Niederblasen findet. Die Äquivalenzklassen werden dann seit Kodaira zuerst grob nach einer Invarianten sortiert, die durch die von den globalen Schnitte von $K_X^{\otimes m}$ erzeugten rationalen Abbildungen $f_m : X \dashrightarrow \mathbb{CP}$ bestimmt wird:

Die *Kodaira-Dimension* ist definiert als

$$\kappa(X) := \max_m \dim f_m(X)$$

bzw. $-\infty$, falls $H^0(X, K_X^{\otimes m}) = 0$ für alle $m \in \mathbb{N}$. Wegen der Funktorialität von K_X ist $\kappa(X)$ konstant in einer Äquivalenzklasse, und sie ist immer $\leq$ der Dimension von X. Die minimalen Modelle in den Äquivalenzklassen mit $\kappa(X) = -\infty$ sind $\mathbb{CP}^2$ und Regelflächen, also Flächen X mit einem Morphismus $f : X \to E$ auf eine Kurve E, dessen Fasern alle $\cong \mathbb{CP}^1$ sind. Für Flächen mit $\kappa(X) = 0, 1$ gibt es ebenfalls vollständige Listen von Äquivalenzklassen, während für $\kappa(X) = 2$ (sogenannte *Flächen vom allgemeinen Typ*) wenigstens einige Beziehungen zwischen weiteren Invarianten bekannt sind.

Iitakas Arbeiten nutzen weitere Eigenschaften der plurikanonischen Bündel $K_X^{\otimes m}$, die auch in Dimension ≥ 3 gelten (falls $\kappa(X) \geq 0$): Die Bildvarietäten $X^{(m)} := f_m(X)$ werden für genügend große und teilbare m birational, also wächst $\dim H^0(X, K_X^{\otimes m})$ wie $m^{\kappa(X)}$, und der *kanonische Ring*

$$R(X) := \oplus_{m=0}^{\infty} H^0(X, K_X^{\otimes m})$$

hat Transzendenzgrad $\kappa(X) + 1$ über $\mathbb{C}$. Die birationale Äquivalenzklasse der $X^{(m)}$ heißt leicht mißbräuchlich *Iitaka-Varietät* $I(X)$ von X. Iitaka zeigte, dass es zu jedem X einen birationalen Morphismus $\pi : \widetilde{X} \to X$ und einen Morphismus $f : \widetilde{X} \to I(X)$ gibt, so dass die Fasern von f über einer dichten offenen Teilmenge von $I(X)$ Kodaira-Dimension 0 haben. f heißt Kodaira-Iitaka-Faserung, und nach Konstruktion gilt $\dim I(X) = \kappa(X)$, aber nicht unbedingt

$$\kappa(I(X)) = \dim I(X) = \kappa(X).$$

Dies zeigen einfache Gegenbeispiele von elliptischen Faserungen über $\mathbb{P}^1$ mit multiplen Fasern. Campana konnte aber 2001 $I(X)$ zusammen mit der Kodaira-Iitaka-Faserung als "Orbifold" vom allgemeinen Typ (also mit maximal möglicher Kodaira-Dimension) interpretieren.

Iitaka stellte außerdem die Frage, ob die $X^{(m)}$ irgendwann (für genügend große und teilbare m) isomorph werden. Dies ist richtig, falls der kanonische Ring endlich erzeugt ist.

Auf Dreifaltigkeiten wurde die endliche Erzeugtheit des kanonischen Rings mit den Mitteln der Mori-Theorie gezeigt. Moris entscheidende Idee war, dass die Geometrie *projektiver* algebraischer Varietäten von den auf ihnen liegenden Kurven, besonders den rationalen, kontrolliert wird. Dazu führte er den Kegel der Kurven auf X,

$$NE(X) \subset H_2(X, \mathbb{R})$$

ein, der aus den positiven Linearkombinationen der Homologieklassen von algebraischen Kurven auf X besteht. Weiter definierte er zu einem Morphismus $f : X \to Y$ (mit zusammenhängenden Fasern) zwischen *projektiven normalen* Varietäten einen Unterkegel $NE(f)$, der von allen Kurven $C \subset X$ erzeugt wird mit $f(C) = \mathrm{Punkt}$. Die

erste fundamentale, aber triviale Beobachtung des Mori-Programms ist, dass $NE(f)$ den Morphismus f eindeutig bestimmt. Außerdem ist $NE(f)$ ein extremaler Unterkegel von $NE(X)$, also eine Facette des Randes.
Das erste große Resultat von Mori war nun, dass umgekehrt jeder extremale Strahl von $NE(X)$, dessen Homologieklassen $c_1(K_X)$ negativ schneiden, zu einem Morphismus $f : X \to Y$ gehört, der entweder

- eine *Fano-Kontraktion* auf eine Varietät Y mit $\dim Y < \dim X$ ist,
- eine *divisorielle Kontraktion* eines *exzeptionellen Ortes* E von $\operatorname{codim} E = 1$ ist mit $\dim f(E) < \dim E$ und f ein Isomorphismus auf $X - E$, oder
- eine *kleine Kontraktion* ist, bei der f immer noch ismorph auf $X - E$ ist, aber $\operatorname{codim} E \geq 2$.

Zum Beispiel können bei Flächen die Morphismen $f : X \to C$ von einer Regelfläche auf die Basiskurve C als Fano-Kontraktionen und die Niederblasung einer (-1)-Kurve E als divisorielle Kontraktion von E gesehen werden, da die Einschränkung von K_X auf die Fasern von f bzw. auf E negativ ist. Kleine Kontraktionen kommen auf Flächen nicht vor. Wenn man mit einer glatten projektiven Fläche startet, kommt man nach endlich vielen divisoriellen Kontraktionen entweder zu einer Fläche X mit einer Fano-Kontraktion, oder es gibt keinen extremalen Strahl mehr. K_X heißt in letzterem Fall *nef*. Solche Flächen sind genau die oben beschriebenen minimalen Modelle.
Wenn man minimale Modelle in höheren Dimensionen genauso definieren will, stößt man auf große technische Schwierigkeiten: Sowohl divisorielle als auch kleine Kontraktionen können zu *singulären* Varietäten führen. Während sich die möglichen Singularitäten bei divisoriellen Kontraktionen noch gut kontrollieren lassen, existiert auf dem Bild einer kleinen Kontraktion nicht einmal mehr ein kanonisches Bündel - man kann also keine geeigneten Extremalstrahlen definieren. Stattdessen wurden von Reid und anderen Flips und Flops eingeführt, die den exzeptionellen Ort E einer kleinen Kontraktion mit Hilfe einer Chirurgie durch einen anderen exzeptionellen Ort E' ersetzen. Auf Dreifaltigkeiten konnten dann Kollár und Mori (und viele andere) zeigen, dass

- eine endliche Anzahl von divisoriellen Kontraktionen und Flips von einer (glatten) projektiven Dreifaltigkeit zu einem (möglicherweise singulären) minimalen Modell im obigen Sinne führt und
- zwei dieser minimalen Modelle, falls sie birational zueinander sind, durch endlich viele Flops miteinander verbunden werden können.

Damit ist für Dreifaltigkeiten eine befriedigende Definition von minimalen Modellen gefunden.
In beliebigen Dimensionen verschafft das Studium rationaler Kurven auf einer projektiven Varietät X einen ersten Überblick über die birationalen Äquivalenzklassen. Zunächst folgt aus der Mori-Theorie die Verallgemeinerung der Beobachtung, dass die

bei Flächen vorkommenden Kontraktionen rationale Kurven kontrahieren: Der exzeptionelle Ort einer divisoriellen Kontraktion bzw. die Fasern einer Fano-Kontraktion werden auch in beliebigen Dimensionen von rationalen Kurven überdeckt. Falls X eine Fano-Kontraktion zuläßt, wird damit ganz X von rationalen Kurven überdeckt — eine solche Varietät heißt *unigeregelt*. Umgekehrt wird jede unigeregelte Varietät von rationalen Kurven überdeckt, deren (gemeinsame) Homologieklasse die Chern-Klasse $c_1(K_X)$ negativ schneidet. Insbesondere können deshalb K_X und auch jede Potenz $K_X^{\otimes m}$ keinen globalen Schnitt haben, da dieser Schnitt eine der überdeckenden rationalen Kurven nichtnegativ schneiden müsste. Die Kodaira-Dimension von X ist also $-\infty$.

Eine wichtige Vermutung besagt, dass dies eine vollständige geometrische Charakterisierung ist:

$$\kappa(X) = -\infty \;\Leftrightarrow\; X \text{ unigeregelt.}$$

Für Dreifaltigkeiten zeigte Miyaoka 1988 diese Vermutung, indem er zuerst von einer nicht unigeregelten Varietät zu einem minimalen Modell überging, auf dem K_X nef ist. Die Behauptung erhielt er dann aus Verschwindungssätzen für nef Geradenbündel und einigen Klassifizierungsresultaten für Dreifaltigkeiten.

Die Vermutung kann als nullter Fall einer allgemeineren Vermutung gelten: Einem nef Geradenbündel L kann eine *numerische Dimension*

$$\nu(X, L) := \max\{k : c_1(L)^k > 0\}$$

zugeordnet werden. Die *Abundance-Vermutung* stellt dann fest: Für nicht unigeregelte Varietäten mit nef kanonischem Bündel (etwa auf minimalen Modellen) ist

$$\kappa(X) = \nu(X, K_X) = \nu(X).$$

Die Wichtigkeit der Abundance-Vermutung liegt in dem Zusammenhang, den sie zwischen der komplexen Geometrie des kanonischen Bündels und seiner Potenzen und der topologischen Invariante $\nu(X)$ herstellt.

Ein großer Fortschritt in der Klassifikationstheorie kam durch die Einführung der rational zusammenhängenden Varietäten durch Kollár, Miyaoka und Mori 1992. Dabei geht es um Varietäten, bei denen zwei allgemeine Punkte durch (eine Kette von) rationale(n) Kurven verbunden werden können. Graber, Harris und Starr zeigten Ende der 90er, dass jede unigeregelte Varietät eine rationale Abbildung $f : X \dashrightarrow Y$ auf eine *nicht unigeregelte* Varietät Y besitzt, so dass f über einer offenen Teilmenge von Y ein Morphismus ist und die Fasern von f dort maximale rational zusammenhängende Untervarietäten von X sind. f heißt dann der maximale rational zusammenhängende Quotient oder kurz MRC-Quotient von X.

Alle diese Theoreme und Vermutungen zusammengenommen, ergibt sich folgendes Bild einer ersten groben Klassifikation von birationalen Äquivalenzklassen projektiver Varietäten:

- Falls X eine unigeregelte Varietät ist, also von rationalen Kurven überdeckt wird, gibt es einen MRC-Quotienten
$$f : X \dashrightarrow Y$$
auf eine nicht unigeregelte Varietät Y.
- Falls X nicht unigeregelt ist, ist $\kappa(X) \geq 0$, und die Kodaira-Faserung
$$f : X \dashrightarrow Y$$
bildet X auf eine Varietät Y von allgemeinem Typ ab (zumindest in Campanas "Orbifold"-Sinn). Die (allgemeinen) Fasern von f haben Kodaira-Dimension $\kappa(F) = 0$. Die Dimension der Basis Y ist dabei schon durch die topologisch invariante numerische Dimension $\nu(X)$ gegeben, falls K_X nef ist.

Die "Bausteine" von projektiven Varietäten wären demnach (falls alle Vermutungen richtig sind) rational-zusammenhängende Varietäten, Varietäten mit Kodaira-Dimension 0 und solche vom allgemeinen Typ.

0.2. Numerisch triviale Faserungen und Blätterungen

In dieser Habilitationsschrift werden nun Arbeiten zusammengefasst, die sich mit neuen Ansätzen zur Lösung der Abundance-Vermutung befassen. Einerseits werden Ergebnisse und Methoden von Boucksom, Demailly, Paun und Peternell ausgebaut, die 2004 zeigten, wie man die Abundance-Vermutung auch ohne die bis jetzt nur vermutete Existenz von minimalen Modellen formulieren kann. Dazu betrachten sie pseudo-effektive Geradenbündel, die als Limes von effektiven Geradenbündeln einer schwachen Positivität genügen und die nef und effektiven Geradenbündel umfassen.
Da z.B. das zum exzeptionellen Divisor E der Aufblasung von $\mathbb{P}^2$ in einem Punkt gehörige Geradenbündel negativen Selbstschnitt hat, lässt sich die Definition der numerischen Dimension nicht so einfach auf pseudo-effektive Geradenbündel übertragen. Boucksom, Demailly, Paun und Peternell führen dazu ein neues Schnittprodukt auf den Kohomologieklassen der pseudo-effektiven Geradenbündel ein, das duch Weglassen der für die Geradenbündel exzeptionellen Orte im klassischen Schnittprodukt immer ≥ 0 bleibt. Die technische Schwierigkeit liegt darin, dass der exzeptionelle Ort nicht immer wie E in der Aufblasung von $\mathbb{P}^2$ Kodimension 1 haben muss. Boucksom, Demailly, Paun und Peternell beheben dieses Problem auf analytische Weise durch Betrachten beliebiger (fast) positiver Metriken auf dem Geradenbündel; algebraisch entspricht diesem Ansatz der Rückzug in beliebige Aufblasungen. Das neue Schnittprodukt stimmt für nef Geradenbündel mit dem klassischen Schnittprodukt überein.
Boucksom, Demailly, Paun und Peternell definieren dann die numerische Dimension eines pseudo-effektiven Geradenbündels L als
$$\nu(X, L) := \max\{k : (c_1(L)^k)_{\geq 0} > 0\},$$

wobei $(c_1(L)^k)_{\geq 0}$ das neue Schnittprodukt bezeichne. Schließlich beweisen sie, dass K_X auf nicht-unigeregelten glatten Varietäten immer pseudo-effektiv ist und können dann die Abundance-Vermutung auch für solche Varietäten formulieren.
Der zweite Ansatz, der von den in dieser Schrift zusammengefaßten Arbeiten entwickelt wird, wurde zunächst von Tsuji 1999 studiert und lässt sich am besten differentialgeometrisch motivieren: Falls es eine hermitesche Metrik mit semipositiver Krümmung auf einem pseudo-effektiven Geradenbündel L gibt, sind die Fasern der Iitaka-Faserung tangential an die Nullrichtungen der Krümmung. Umgekehrt kann man unabhängig von der Iitaka-Faserung fragen, ob sich alle oder wenigstens einige der Nullrichtungen der Krümmung einer Metrik auf einem pseudo-effektiven Geradenbündel als Tangentialrichtungen an eine Faserung auffassen lassen.
Diese Idee kann algebraisch oder analytisch entfaltet werden:

- Im algebraischen Sinn werden Nullrichtungen der Krümmung zu Kurven, auf denen das Geradenbündel numerisch trivial wird. Dies ermöglicht für nef Geradenbündel die Konstruktion der sogenannten Nef-Faserung, die in Kapitel 1 beschrieben wird.
- Für beliebige pseudo-effektive Geradenbündel ergeben sich technische Schwierigkeiten aus der Tatsache, dass positive hermitesche Metriken auf L singulär sein können. In Kapitel 2 wird geklärt, was numerische Trivialität von Geradenbündeln auf Kurven oder auch Untervarietäten höherer Dimension bzgl. solche singulären Metriken bedeutet. Die eleganteste Charakterisierung in Theorem 3.8 sagt, dass der Rückzug des Krümmungsstroms auf die Untervarietät eine Summe von Integrationsströmen von Divisoren sein muß. In Kapitel 3 wird dann eine Faserung mit maximal-dimensionalen numerisch trivialen Fasern konstruiert, die pseudo-effektive Reduktionsabbildung.

Es stellt sich heraus, dass die Iitaka-Faserung eines Geradenbündels L die pseudo-effektive Reduktionsabbildung zu der Metrik h_m ist, die von den Schnitten einer genügend hohen Potenz $L^{\otimes m}$ erzeugt wird. Da diese Metrik aber so eng mit der Existenz von Schnitten verknüpft ist, sagt diese Identität von Faserungen wenig über das Abundance-Problem für das Geradenbündel L aus.
Die Nef-Faserung hingegen, die unabhängig von speziellen Metriken auf L definiert ist, ist manchmal feiner als die Iitaka-Faserung von L. Insbesondere Beispiel 7.1 legt nahe, wie dieser Unterschied erklärt werden kann: Die numerisch trivialen Richtungen sind hier nicht tangential an eine Faserung, sondern an eine Blätterung.
Der Begriff einer numerisch trivialen Blätterung für pseudo-effektive Klassen auf einer kompakten Kähler-Mannigfaltigkeit wird in Abschnitt 6.1 definiert und eine maximale numerisch triviale Blätterung konstruiert. Die Notwendigkeit der technisch anspruchsvollen Definition wird am Beispiel 7.2 klar: Anstelle der einzig möglichen positiven Metrik auf dem dort betrachteten Geradenbündel muß man eine approximierende Sequenz von fast positiven Metriken heranziehen, da die positive Metrik zu wenig Informationen trägt. Diese Berücksichtigung von vielen fast positiven Metriken

auf einmal steht im Einklang mit den Methoden, die Boucksom et al. bei der Einführung des positiven Schnittprodukts entwickelt haben. Sie werden im Kapitel 4 ausführlich erläutert.
In den Abschnitten 6.3 und 6.4 wird gezeigt, dass die Iitaka-Faserung die numerisch triviale Blätterung und diese wiederum die Nef-Faserung eines nef Geradenbündels L enthält. Wir erhalten so ein hinreichendes Kriterium für das Auseinanderfallen von Iitaka-Faserung und Nef-Faserung: Die maximale numerisch L-triviale Blätterung ist **keine** Faserung.
Die Abundance-Vermutung sagt nun, dass dies für das kanonische Bündel auf nicht-unigeregelten Varietäten nicht passiert. Außerdem behauptet sie, dass die Kodimension der Blätter einer numerisch K_X-trivialen Blätterung gleich der numerischen Kodimension $\dim(X) - \nu(X)$ der Varietät X ist. Die oben geschilderte Intuition hinter dem Begriff der numerisch trivialen Blätterung legt nahe, dass die numerische Kodimension zumindest größer oder gleich der Blätterdimension sein sollte: Die Potenzen $c_1(L)^k$ errechnen sich als äußeres Produkt der Krümmungsform einer Metrik auf L. Wenn also diese Metrik semipositiv ist und ihre Nullrichtung die Blätter enthalten, ergibt sich die gewünschte Schranke.
Diese Schranke wird in Abschnitt 6.2 auch bewiesen, allerdings unter einer Zusatzannahme an die Singularitäten der Blätterung: sie sollen isoliert sein. Die Beseitigung dieser Annahme erfordert ein genaues Studium des Verhaltens der fast positiven Metriken auf L in der Umgebung von Blätterungssingularitäten. Dies ist bis jetzt nicht geschehen und scheint wegen der ungeklärten Beschreibung von Blätterungssingularitäten auch sehr schwierig zu sein.
Beispiel 7.3 zeigt schließlich, dass die Ungleichung strikt sein kann. Dies muss als weiterer Beleg für die sehr komplexe Geometrie auf $\mathbb{P}^2$ aufgeblasen in 9 Punkten gesehen werden. Die Abundance-Vermutung sagt wiederum, dass derartige Phänomene bei kanonischen Bündeln auf nicht-unigeregelten Varietäten nicht auftreten können.

0.3. Frühere Veröffentlichungen

Die meisten Ergebnisse dieser Schrift sind schon in früheren Arbeiten publiziert worden. Die Nef-Faserung aus Kapitel 1 wurde in [**1**] konstruiert, während die Überlegungen zu Tsujis numerisch trivialen Faserungen in den Kapiteln 2 und 3 aus [**22**] stammen. Die Definition, Konstruktion und Eigenschaften numerisch-trivialer Blätterungen in Kapitel 6 finden sich in [**21**], ebenso wie die Einführung in die Resultate von Boucksom et al. in Kapitel 4 und die Beispiele in Kapitel 7.
Die Resultate aus Kapitel 5 sind neu; sie entstanden als genauere Ausführungen einiger Argumente in [**21**].

0.4. Danksagungen

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CHAPTER 0

Introduction

Algebraic curves were created by God,
algebraic surfaces by the devil.
Max Noether

This left little room for algebraic threefolds.
János Kollár

0.1. Classification of algebraic varieties

Classic algebraic geometry started with studying algebraic curves, first in the ancient world the conic sections, then in modern times, starting with Newton, plane algebraic curves of higher degree (especially their singularities), and in the 19th century also space curves. Finally Riemann gave a quite satisfactory structure theory of algebraic curves, using analytic and topological methods.

At the turn of the 20th century the Italian school of Castelnuovo, Enriques and Severi achieved a satisfactory structure theory for algebraic surfaces. Their work, however, lacked the Hilbertian rigor, and after their students presented frequently false results on algebraic threefolds the whole subject started to fall into disrepute.

Systematically using methods from commutative algebra Van der Waerden and afterwards Zariski and Weil placed algebraic geometry on solid foundations again, and Grothendieck united in his unfinished magnum opus "Éléments de géométrie algébrique" commutative algebra and algebraic number theory with algebraic geometry.

By the end of the sixties, the foundational work was mostly done and attention turned towards the classical problems. First the theory of curves and surfaces was redone and completed. In 1972 Iitaka proposed some bold and interesting conjectures concerning higher-dimensional varieties, and Ueno proved in 1977 the first structure theorem about threefolds along this path. It was clear, however, that the scope of their approach was limited. Above all an analog of the minimal models of surfaces was missing.

The major breakthrough came in 1980: Using several new ideas Mori accomplished the first major step towards proving the existence of minimal models for higher-dimensional varieties. At the same time Reid defined what such minimal models should be after all and pointed out several ways to use them if they exist. For threefolds, the emerging program was successfully finished by Mori and Kollár by the end of the eighties, and

several deep structure theorems for threefolds were proven, among them the abundance conjecture and the finite generatedness of the canonical ring.
Both conjectures (in dimension ≥ 4) resp. theorems (in dimension 3) describe properties of the canonical line bundle K_X that is the top exterior power of the cotangent bundle $\Omega^1_X = T^*_X$ on a smooth algebraic variety. The bundle K_X is so important because of the following two observations:
First, global sections of holomorphic line bundles define rational maps into other algebraic varieties, e.g. the projective spaces $\mathbb{CP}^n$, and the canonical bundle exists on every smooth algebraic variety.
Second, the canonical bundle has good functorial properties under birational maps that are rational maps $f : X \dashrightarrow Y$ being isomorphisms on an open subset $U \subset X$: the map f induces an isomorphism between the global sections of K_X and K_Y. This is also true for all powers $K_X^{\otimes m}$, but does not hold e.g. for the anticanonical bundle K_X^{-1} and its powers.
The functoriality w.r.t. birational maps is essential in a very general strategy for obtaining structure theorems of algebraic varieties. This strategy was already used by the Italian school to classify algebraic surfaces:

Step 1: First define an equivalence relation on all algebraic varieties by declaring two algebraic varieties as equivalent if they are birational. Try to understand how two equivalent varieties are connected. In particular one would like to have some "simple" types of operations leading in finitely many steps from one variety to the other.

Step 2: Next look for a distinguished representative, the *minimal model*, in every equivalence class.

Step 3: Finally use the properties of these minimal models to get a survey over all equivalence classes.

The "simple operations" leading from an algebraic surface to birationally equivalent surfaces are blow ups and blow downs of rational (-1)-curves that are curves $\cong \mathbb{CP}^1$ with self-intersection -1. The minimal models are surfaces on which there are no (-1)-curves for blowing down. Following Kodaira the equivalence classes will be roughly classified according to an invariant given by the rational maps $f_m : X \dashrightarrow \mathbb{CP}^N$ induced by the global sections of $K_X^{\otimes m}$, the *Kodaira dimension*. It is defined as

$$\kappa(X) := \max_m \dim f_m$$

resp. $-\infty$ if $H^0(X, K_X^{\otimes m}) = 0$ for all $m \in \mathbb{N}$. The Kodaira dimension $\kappa(X)$ is constant in an equivalence class because of the functoriality of K_X , and it always is $\leq \dim X$.
The minimal models in the equivalence classes with $\kappa(X) = -\infty$ are $\mathbb{CP}^2$ and ruled surfaces that are surfaces X with a morphism $f : X \to E$ onto a curve E whose fibers are all $\cong \mathbb{CP}^1$. There are also complete lists for equivalence classes of surfaces with $\kappa(X) = 0, 1$, whereas for surfaces X with $\kappa(X) = 2$ (so-called *surfaces of general type*), at least some relations between other invariants are known.

Iitaka's works use further properties of the pluricanonical bundles $K_X^{\otimes m}$, which also hold in dimension ≥ 3: If $\kappa(X) \geq 0$ the image varieties $X^{(m)} := f_m(X)$ become birational to each other for m sufficiently large and divisible. Consequently $\dim H^0(X, K_X^{\otimes m})$ grows like $m^{\kappa(X)}$ and the *canonical ring*

$$R(X) := \oplus_{m=0}^{\infty} H^0(X, K_X^{\otimes m})$$

has transcendental degree $\kappa(X) + 1$ over $\mathbb{C}$. By abuse of notation the birational equivalence class of the $X^{(m)}$ is called *Iitaka variety* $I(X)$ of X. Iitaka proved that for every X there exists a birational morphism $\pi : \widetilde{X} \to X$ and a morphism $f : \widetilde{X} \to I(X)$ such that the fibers of f have Kodaira dimension 0 over a dense open subset of $I(X)$. Then f is called *Kodaira-Iitaka fibration*, and by construction $\dim I(X) = \kappa(X)$, but not necessarily

$$\kappa(I(X)) = \dim I(X) = \kappa(X).$$

Simple counter examples are elliptic fibrations over $\mathbb{P}^1$ with multiple fibers. In 2001 Campana was able to interpret $I(X)$ together with the Kodaira-Iitaka fibration as an "orbifold" of general type, i.e. having maximal Kodaira dimension.
In addition Iitaka asked if the $X^{(m)}$ get at some point isomorphic, for m sufficiently large and divisible. This is true if the canonical ring is finitely generated.
On threefolds the finite generatedness of the canonical ring was shown by using Mori theory. Mori's landmark idea was that the geometry of projective algebraic varieties is controlled by the curves lying on them, especially the rational curves. To demonstrate this he introduced the cone of curves on X,

$$NE(X) \subset H_2(X, \mathbb{R})$$

that consists of all positive linear combinations of the homology classes of algebraic curves on X. To every morphism $f : X \to Y$ with connected fibers between projective normal varieties he associated a subcone $NE(f)$ generated by all curves with $f(C) = \text{point}$. The first fundamental but trivial observation of the Mori program is that $NE(f)$ determines f uniquely. Furthermore, $NE(f)$ is an extremal subcone of $NE(X)$ hence a face on the boundary.
The first fundamental result of Mori was the converse of this statement: every extremal ray of $NE(X)$ whose homology class intersects $c_1(K_X)$ negatively is associated to a morphism $f : X \to Y$ of one of the following three types:

- a *Fano contraction* onto a variety Y with $\dim Y < \dim X$,
- a *divisorial contraction* of an *exceptional locus* E of $\operatorname{codim} E = 1$ with $\dim f(E) < \dim E$, and f is an isomorphism on $X - E$, or
- a *small contraction* for which f is still an isomorphism on $X - E$, but $\operatorname{codim} E \geq 2$.

The morphisms $f : X \to C$ from a ruled surface X to the base curve C, for example, are Fano contractions, and the blow down of a (-1)-curve E on a surface is the divisorial contraction of E because the restriction of K_X to a fiber of f resp. to E are

negative. Small contractions do not occur on surfaces. Starting with a smooth projective surface a finite number of divisorial contractions either leads to a surface X with a Fano contraction, or there will be no extremal ray in $NE(X)$ intersecting K_X negatively. In this case K_X is called *nef*, and such surfaces are exactly the minimal models described above.
Defining minimal models in the same way in higher dimensions causes major technical problems: Both divisorial and small contractions can lead to *singular* varieties. Whereas possible singularities of divisorial contractions may be controlled rather well, there even does not exist a canonical bundle on the image of a small contraction – hence it is not possible to define suitable extremal rays to continue the contractions. Instead Reid and others introduced flips and flops replacing the exceptional locus E of a small contraction with another exceptional locus E', via a surgery. On threefolds Kollár and Mori (and many others) were able to show that

- a finite number of divisorial contractions and flips leads from a smooth projective threefold to a possibly singular minimal model in the sense described above and
- two of these minimal models are birational to each other iff they can be connected by a finite number of flops.

Thus a satisfactory definition of minimal models was found.
In arbitrary dimensions the study of rational curves on a projective variety X allows a first survey over the birational equivalence classes. First, Mori theory implies the generalization of the observation that on surfaces contractions contract rational curves: In arbitrary dimensions the exceptional locus of a divisorial or Fano contraction is covered by rational curves, too. Consequently, if there exists a Fano contraction on X, the variety is covered by rational curves – such varieties are called *uniruled*. Conversely, every uniruled variety is covered by rational curves whose common homology class intersects the Chern class $c_1(K_X)$ negatively. In particular, K_X and any power $K_X^{\otimes m}$ cannot have global sections, since such a section would positively intersect one of the covering rational curves. Hence the Kodaira dimension of X is $-\infty$.
An important conjecture predicts that this is a complete geometric characterization:

$$\kappa(X) = -\infty \;\Leftrightarrow\; X \text{ uniruled.}$$

For threefolds Miyaoka proved this conjecture by passing from a general non-uniruled variety to a minimal model with nef K_X. Then he obtained the claim from vanishing theorems for nef line bundles and results classifying threefolds.
The conjecture can be interpreted as case zero of a more general conjecture: A nef line bundle L has a *numerical dimension*

$$\nu(X, L) := \max\{k : c_1(L)^k > 0\},$$

and the *abundance conjecture* claims that for non-uniruled varieties with nef canonical bundle (e.g. on minmal models)

$$\kappa(X) = \nu(X, K_X) = \nu(X).$$

The abundance conjecture is important because it establishes a connection between the complex geometry of the canonical bundle and its powers and the topological invariant $\nu(X)$.
Substantial progress in classification theory was achieved when in 1992 Kollár, Miyaoka and Mori introduced the notion of rationally connected varieties. They are defined as varieties where two general points can be connected by a (chain of) rational curve(s). Around 2000 Graber, Harris and Starr proved that every uniruled variety allows a rational map $f : X \dashrightarrow Y$ on a non-uniruled variety Y such that f is a morphism over an open subset of Y and the fibers of f are maximal rationally connected subvarieties of X. Then f is called the maximal rationally connected quotient, for short MRC-quotient of X.

From all these theorems and conjectures emerges a first picture of a rough classification of birational equivalence classes of projective varieties:

- If X is a uniruled varieties i.e. covered by rational curves, there exists an MRC quotient
$$f : X \dashrightarrow Y$$
on a non-uniruled variety Y.
- If X is non-uniruled then $\kappa(X) \geq 0$, and the Kodaira fibration
$$f : X \dashrightarrow Y$$
maps X onto a variety of general type (at least in Campana's "orbifold" sense), and the (general) fibers F of f have Kodaira dimension $\kappa(F) = 0$. The dimension of the basis Y is given by the topological invariant numerical dimension $\nu(X)$ if K_X is nef.

All conjectures holding true the "building blocks" of projective varieties would be rationally connected varieties, varieties with Kodaira dimension 0 and varieties of general type.

0.2. Numerically trivial fibrations and foliations

In this Habilitationschrift several works are bundled dealing with new approaches to the abundance conjecture. On the one hand, these works further develop results and methods of Boucksom, Demailly, Paun and Peternell. They showed in 2004 how to state the abundance conjecture without using the existence of minimal models, which is only conjectural up to now. To this purpose they considered pseudo-effective line bundles, which satisfy a weak positivity property as the limit of effective line bundles and which contain nef and effective line bundles.

Since e.g. the line bundle associated to the exceptional divisor E of the blow up of $\mathbb{P}^2$ in a point has negative self intersection the definition of the numerical dimension cannot be immediately transfered to pseudo-effective line bundles. Instead, Boucksom et al. introduced a new intersection product on the cohomology classes of the pseudo-effective line bundles, which always stays ≥ 0 by removing the exceptional loci of the line bundle in the classical intersection product. The technical difficulty lies in the fact that these exceptional loci need not always be of codimension 1, as E in the blow up of $\mathbb{P}^2$. Boucksom et al. solved this problem in an analytic manner via considering arbitrary (almost) positive metrics on line bundles; algebraically this approach corresponds to pulling back to arbitrary blow ups. The new intersection product is the same as the classical one on nef line bundles.
Next, Boucksom et al. defined the numerical dimension of a pseudo-effective line bundle as

$$\nu(X, L) := \max\{k : (c_1(L)^k)_{\geq 0} > 0\},$$

where $(c_1(L)^k)_{\geq 0}$ denotes the new intersection product. Finally they prove that K_X is always pseudo-effective on a non-uniruled smooth variety and are able to state the abundance conjecture for such varieties.
The second approach developped by the papers bundled in this Schrift was first studied by Tsuji in 1999 and is most suitably motivated in a differential-geometric way: If there exists a hermitian metric with semi-positive curvature on a pseudo-effective line bundle L then the fibers of the Iitaka fibration are tangential to the null directions of the metric curvature. Vice versa one may ask independently from the Iitaka fibration whether all or at least some of the null directions belonging to the semi-positive curvature of a metric on a pseudo-effective line bundle can be interpreted as the tangent directions of a fibration. This idea can be unfolded algebraically or analytically:

- Algebraically null directions of the curvature are translated as curves on which the line bundle is numerically trivial. For nef line bundles this allows to construct the so called nef fibration which will be described in Chapter 1.
- For arbitrary pseudo-effective line bundles technical difficulties result from the fact that positive hermitian metrics on L may be singular. In Chapter 2 the notion of numerical triviality of line bundles on curves or also higher-dimensional subvarieties w.r.t. such singular metrics will be clarified. The most elegant characterization in Theorem 3.8 states that the pull back of the curvature current onto the subvariety must be the sum of integration currents of divisors. In Chapter 3 a fibration with maximal-dimensional numerically trivial fibers is constructed, the pseudo-effective reduction map.

It turns out that the Iitaka fibration of a line bundle L is the pseudo-effective reduction map w.r.t. a metric h_m generated by the sections of a sufficiently high power $L^{\otimes m}$. Since this metric is so closely related to the existence of sections the identity of fibrations doesn't tell too much on the abundance problem for the line bundle L.

The nef fibration, however, is defined independently of special metrics on L, but sometimes it is finer than the Iitaka fibration of L. Especially Example 7.1 suggests how to explain this difference: here, the numerically trivial directions are not tangential to a fibration but to a foliation.
The notion of a numerically trivial foliation for pseudo-effective line bundles on a compact Kähler manifold will be defined in Section 6.1, and a maximal numerically trivial foliation will be constructed. The necessity of the technically demanding definition will be illustrated by Example 7.2: Instead of the unique positive but very singular metric on the considered line bundle one has to look at approximating sequences of almost positive metrics, since the positive metric carries not enough information. Taking into account many almost positive metrics at the same time fits with the methods introduced by Boucksom et al. for defining the positive intersection product. They will be outlined in detail in Chapter 4.
In Sections 6.3 and 6.4 it will be proven that the Iitaka fibration contains the numerically trivial foliation, which in turn contains the nef fibration of a nef line bundle L. Thus we obtain a sufficient criterion for the split between Iitaka and nef fibration: The maximal numerically L-trivial foliation is **not** a fibration.
Now, the abundance conjecture states that this cannot happen for the canonical bundle on a non-uniruled variety. Furthermore it claims that the codimension of the leaves of a numerically K_X-trivial foliation equals the numerical dimension $\nu(X)$. The intuition behind the notion of numerical triviality described above suggests that the numerical dimension is at least a lower bound for the codimension of the foliation leaves: The powers $c_1(L)^k$ are computed as the exterior power of the curvature form of a metric on L. Hence if this metric is semi-positive and its null directions contain the leaves, the desired bound is correct.
This bound will be proven in Section 6.2, but under an additional assumption on the singularities of the foliation: they must be isolated. To remove this assumption one carefully has to study the behaviour of almost positive metrics around singularities of foliations. This is not done up to now and seems to be very difficult because of the unknown description of foliation singularities.
Finally, Example 7.3 shows that the inequality can be strict. This should be seen as another hint to the extremely intricate geometry of $\mathbb{P}^2$ blown up in 9 points. On the other hand, the abundance conjecture tells once more that such phenomenons cannot occur on non-uniruled varieties.

0.3. Previous publications

Most of the results of this Schrift have been published earlier.
The nef fibration in Chapter 1 was constructed in [**1**] whereas the investigations on Tsuji's numerically trivial fibrations in Chapters 2 und 3 originally come from [**22**]. The definition, construction and properties of numerically trivial foliations in Chapter 6

will be found in [**21**] as will be the introduction to the results of Boucksom et al. in Chapter 4 and the examples in Chapter 7.
The results from Chapter 5 are new. They emerged as a more detailed explanation of some arguments in [**21**].

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CHAPTER 1

The nef fibration

From the algebraic-geometric point of view, the most natural way of defining numerical triviality for subvarieties with respect to a given line bundle is to use the standard intersection numbers of the line bundle with curves on the subvariety. Relevant definitions are given now.

DEFINITION 1.1. *Let X be an irreducible reduced projective complex space (projective variety, for short). A line bundle L on X is numerically trivial, iff $L.C = 0$ for all irreducible curves $C \subset X$. The line bundle is nef iff $L.C \geq 0$ for all curves C.*

Let $f : Y \to X$ be a surjective map from a projective variety Y. Then clearly L is numerically trivial (nef) if and only if f^*L is.

DEFINITION 1.2. *Let X and Y be normal projective varieties and $f : X \dashrightarrow Y$ a rational map and let $X^\circ \subset X$ be the maximal open subset where f is holomorphic. The map f is said to be almost holomorphic if some fibers of the restriction $f|_{X^\circ}$ are compact.*

In this chapter we want to prove the following structure theorem for nef line bundles on a projective variety.

THEOREM 1.3. *Let L be a nef line bundle on a normal projective variety X. Then there exists an almost holomorphic, dominant rational map $f : X \dashrightarrow Y$ with connected fibers, called a "reduction map" such that*

(i) *L is numerically trivial on all compact fibers F of f with $\dim F = \dim X - \dim Y$*
(ii) *for every general point $x \in X$ and every irreducible curve C passing through x with $\dim f(C) > 0$, we have $L.C > 0$.*

The map f is unique up to birational equivalence of Y.

This theorem was stated without complete proof in Tsuji's paper [**37**].

1.1. Construction of the reduction map

1.1.1. A criterion for numerical triviality. In order to prove Theorem 1.3 and construct the reduction map, we will employ the following criterion for a line bundle to be numerically trivial:

THEOREM 1.4. *Let X be an irreducible projective variety which is not necessarily normal. Let L be a nef line bundles on X. Then L is numerically trivial if and only if any two points in X can be joined by a connected chain C of curves such that $L.C = 0$.*

In the remaining part of the present section we will prove Theorem 1.4. The proof will be performed by a reduction to the surface case. The argumentation is then based on the following statement which in the smooth case is a simple corollary to the Hodge Index Theorem.

PROPOSITION 1.5. *Let S be an irreducible projective surface which is not necessarily normal and let $q : S \to T$ be a morphism with connected fibers onto a curve. Assume that $L \in \operatorname{Pic}(S)$ is a nef line bundle and there exists a curve $C \subset S$ such that $q(C) = T$ and*

$$L.F = L.C = 0$$

holds where F is a general q-fiber. Then L is numerically trivial.

PROOF. If S is smooth set $D = C + nF$ where n is a large positive integer. Then we have $D^2 > 0$. By the Hodge Index Theorem it follows that

$$(L.D)^2 \geq L^2.D^2$$

hence $L^2 = 0$ since by our assumptions $L.D = 0$. So equality holds in the Index Theorem and therefore L and D are proportional: $L \equiv kD$ for some rational number k. Since $0 = L^2 = k^2 D^2$ and $D^2 > 0$ we conclude that $k = 0$. That ends the proof in the smooth case.
If S is singular let $\delta : \widetilde{S} \to S$ be a desingularization of S and $\widetilde{C} \subset \widetilde{S}$ a component of $\delta^{-1}(C)$ which maps surjectively onto C. Note that the fiber of $q \circ \delta$ need no longer be connected and consider the Stein factorization

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\delta", "\text{desing.}"'] \arrow[d, "\widetilde{q}"'] & S \arrow[d, "q"] \\ \widetilde{T} \arrow[r, "\text{finite}"'] & T. \end{tikzcd}$$

It follows immediately from the construction that $\widetilde{q}(\widetilde{C}) = \widetilde{T}$, that $\delta^*(L)$ has degree 0 on $\widetilde{C}$ and on the general fiber of $\widetilde{q}$. The argumentation above therefore yields that $\delta^*(L)$ is trivial on $\widetilde{S}$. The claim follows. $\square$

1.1.2. Proof of Theorem 1.4. If L is numerically trivial the assertion of Theorem 1.4 is clear. We will therefore assume that any two points can be connected by a not necessarily irreducible curve which intersects L with multiplicity 0 and we will show that L is numerically trivial. To this end choose an arbitrary irreducible curve $B \subset X$. We are finished if we show that $L.B = 0$.

Let $a \in X$ be an arbitrary point which is not contained in B. For any $b \in B$ we can find by assumption a connected, not necessarily irreducible curve Z_b containing a and b such that $L.Z_b = 0$. Since the Chow variety has compact components and only a countable number of components we find a family $(Z_t)_{t \in T}$ of curves parametrized by a compact irreducible curve $T \subset \mathrm{Chow}(X)$ such that for every point $b \in B$ there exists a point $t \in T$ such that the curve Z_t contains both a and b. We consider the universal family $S \subset X \times T$ over T together with the projection morphisms

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d, "q"'] & X \\ T & \end{tikzcd}$$

CLAIM 1. *There exists an irreducible component $S_0 \subset S$ such that $p^*(L)$ is numerically trivial on S_0.*

PROOF OF CLAIM 1. As all curves Z_t contain the point a the surface S contains the curve $\{a\} \times T$. Let $S_0 \subset S$ be a component which contains the curve $\{a\} \times T$. Since $\{a\} \times T$ intersects all fibers of the natural projection morphism q and since $p^*(L)$ is trivial on $\{a\} \times T$ an application of Proposition 1.5 yields the claim. □

CLAIM 2. *The bundle $p^*(L)$ is numerically trivial on S.*

PROOF OF CLAIM 2. We argue by contradiction and assume that there are components $S_j \subset S$ where $p^*(L)$ is not numerically trivial. We can therefore subdivide the set I of irreducible components S_i of S into two subvarieties as follows:

$$\begin{aligned} I_{\mathrm{triv}} &:= \{i \in I : p^*(L)_{|S_i} \text{ is numerically trivial}\}, \\ I_{\mathrm{ntriv}} &:= \{i \in I : p^*(L)_{|S_i} \text{ is not numerically trivial}\}. \end{aligned}$$

Set $S_{\mathrm{triv}} := \bigcup_{i \in I_{\mathrm{triv}}} S_i$ and $S_{\mathrm{ntriv}} := \bigcup_{i \in I_{\mathrm{ntriv}}} S_i$. By assumption and Claim 1 both varieties are not empty. Since S is the universal family over a curve in $\mathrm{Chow}(X)$ the morphism q is equidimensional. In particular, since all components $S_i \subset S$ are two-dimensional every irreducible component S_i maps surjectively onto T. Thus if $t \in T$ is a general point the connected fiber $q^{-1}(t)$ intersects both S_{triv} and S_{ntriv}. It is therefore possible to find a curve $D \subset S_{\mathrm{triv}} \cap S_{\mathrm{ntriv}}$ which dominates T.
That however contradicts Proposition 1.5: On one hand since $D \subset S_{\mathrm{triv}}$, the degree of $p^*(L)_{|D}$ is 0. On the other hand we can find an irreducible component $S_j \subset S_{\mathrm{ntriv}} \subset S$ which contains D. But because $p^*(L)_{|D}$ has degree 0 on the fibers of $q_{|S_j}$ Proposition 1.5 asserts that $p^*(L)$ is numerically trivial on S_j contrary to our assumption. This ends the proof of Claim 2. □

We apply Claim 2 as follows: If $B' \subset S$ is any component of the preimage $p^{-1}(B)$ then $p^*(L).B' = 0$. That shows that $L.B = 0$ and the proof of Theorem 1.4 is done. □

1.1.3. Proof of Theorem 1.3. In order to derive Theorem 1.3 from Theorem 1.4 we introduce an equivalence relation on X with setting $x \sim y$ if x and y can be joined by a connected connected curve C such that $L.C = 0$. Then by [**11**] or [**12**, Appendix] there exists an almost holomorphic map $f : X \dashrightarrow Y$ with connected fibers to a normal projective variety Y such that two general points x and y satisfy $x \sim y$ if and only if $f(x) = f(y)$. This map f gives the fibration we are looking for.
If F is a general fiber then $L_{|F} \equiv 0$ by Theorem 1.4.
We still need to verify that $L.C = 0$ for all curves C contained in an *arbitrary* compact fiber F_0 of dimension $\dim F_0 = \dim X - \dim Y$. To do that let H be an ample line bundle on X and pick

$$D_1, \ldots, D_k \in |mH|$$

for m large such that

$$D_1. \ldots .D_k.F_0 = C + C'$$

with an effective curve C'. Then

$$L.(C + C') = L.D_1. \ldots .D_k.F$$

with a general fiber F of f hence $L.(C+C') = 0$. Since L is nef we conclude $L.C = 0$. $\square$

1.2. Nef cohomology classes

In Theorems 1.3 and 1.4 we never really used the fact that L is a line bundle; only the property that $c_1(L)$ is a nef class is important and even rationality of the class does not play any role. Hence our results directly generalize to nef cohomology classes of type $(1, 1)$. To be precise we fix a projective manifold (we stick to the smooth case for sakes of simplicity) and we say that a class $\alpha \in H^{1,1}(X, \mathbb{R})$ is nef if it is in the closure of the cone generated by the Kähler classes. Moreover α is numerically trivial if $\alpha.C = 0$ for all curves $C \subset X$.
If $Z \subset X$ is a possibly singular subspace then we say that α is numerically trivial on Z if for some (and hence for all, see [**34**]) desingularisation $\widehat{Z} \to Z$ the induced form $f^*(\alpha)$ is numerically trivial on $\widehat{Z}$ i.e. $f^*(\alpha).C = 0$ for all curves $C \subset \widehat{Z}$. Here $f : \widehat{Z} \to X$ denotes the canonical map. Similarly we define α to be nef on Z. If Z is smooth this is the same as to say that $\alpha_{|Z}$ is a nef cohomology class in the sense that $\alpha_{|Z}$ is in the closure of the Kähler cone of Z.

THEOREM 1.6. *Let α be a nef cohomology class on a smooth projective variety X. Then there exists an almost holomorphic dominant rational map $f : X \dashrightarrow Y$ with connected fibers such that*

(i) *α is numerically trivial on all compact fibers F of f with $\dim F = \dim X - \dim Y$.*
(ii) *for every general point $x \in X$ and every irreducible curve C passing through x with $\dim f(C) > 0$ we have $\alpha.C > 0$.*

The map f is unique up to birational equivalence of Y.
In particular if two general points of X can be joined by a chain C of curves such that $\alpha.C = 0$ then $\alpha \equiv 0$.

1.3. The nef dimension

Since Y is unique up to a birational map its dimension $\dim Y$ is an invariant of L which we compare to the other known invariants.

DEFINITION 1.7. *The dimension $\dim Y$ is called the nef dimension of L. We write*

$$n(L) := \dim Y.$$

As usual we let $\nu(L)$ be the numerical dimension of L, i.e. the maximal number k such that $L^k.H^{n-k} \neq 0$.

PROPOSITION 1.8. *The nef dimension is never smaller than the numerical dimension:*

$$\nu(L) \leq n(L).$$

PROOF. Fix a very ample line bundle $H \in \mathrm{Pic}(X)$ and set $\nu := \nu(L)$. Let Z be a general member cut out by $n - \nu$ elements of $|H|$. The dimension of Z will thus be $\dim Z = \nu$ and since $L^\nu.H^{n-\nu} > 0$ the restriction $L_{|Z}$ is big (and nef). Consequently $\dim f(Z) = \nu$ since otherwise Z would be covered by curves C which are contained in general fibers of f so that $L.C = 0$ contradicting the bigness of $L_{|Z}$. In particular we have $\dim Y \geq \dim f(Z) = \nu$ and our claim is shown. □

COROLLARY 1.9. *The nef dimension is never smaller than the Kodaira dimension:*

$$\kappa(L) \leq n(L).$$

PROOF. This follows from $\kappa(L) \leq \nu(L)$ (see [**14**, (6.10)]). □

REMARK 1.10. As mentioned in the Introduction the abundance conjecture predicts

$$\kappa(L) = \nu(L)$$

which implies $\kappa(L) = n(L)$.
On the other hand there exist varieties X and nef line bundles $L \neq K_X$ such that

$$\kappa(L) < \nu(L) < n(L).$$

Such examples will be discussed in Chapter 7. Their thorough analysis was the starting point of the construction of numerically L-trivial foliations.

CHAPTER 2

Tsuji's intersection numbers

In [**37**], H. Tsuji stated assertions on the structure of pseudo-effective line bundles L on a projective manifold X similar to the nef reduction of the last chapter. In particular he postulated the existence of a meromorphic "reduction map", which essentially says that through the general point of X there is a maximal irreducible subvariety which is flat w.r.t. a positive possibly singular hermitian metric h on L. The purpose of this chapter is to clarify and define this meaning of "numerical trivial" via the introduction of intersection numbers of L with curves depending on h.
We study three such definitions which are contained in Tsuji's arguments and solve the subtle question when they are equivalent. We need all three to construct the reduction map in the next chapter. The first is

DEFINITION 2.1. *Let X be a smooth projective complex manifold, let L be a holomorphic line bundle on X with positive singular hermitian metric h. If $C \subset X$ is an irreducible curve with normalization $\pi : \tilde{C} \to C$ such that h is well defined on C, i.e. $h_{|C} \not\equiv +\infty$, then define the intersection number*

$$(L,h).C := \limsup_{m\to\infty} \frac{1}{m} h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(m\pi^* L) \otimes \mathcal{I}((\pi^* h)^m)).$$

Here, $\mathcal{I}((\pi^* h)^m)$ denotes the multiplier ideal sheaf of the pulled back metric $(\pi^* h)^m$ on $\tilde{C}$.

This definition leads directly to the birational invariance of the intersection numbers, i.e. for a birational morphism we have $f : \widetilde{X} \to X$

$$(f^* L, f^* h).\bar{C} = (L,h).C$$

where $\bar{C}$ is the strict transform (s. section 2.3).
The next definition is contained in

PROPOSITION 2.2. *If C is smooth,*

$$(L,h).C = L.C - \sum_{x\in C} \nu(\Theta_{h|C}, x),$$

where $\nu(\Theta_{h|C}, x)$ is the Lelong number of the curvature current Θ_h restricted to C in $x \in C$.

This equality gives a more geometric interpretation of the intersection numbers (especially in the case of analytic singularities, s. Proposition 2.17) and is an important step towards a last equality. This is the most subtle one, and to formulate it properly, one has to remember two facts:

- The sets where plurisubharmonic functions are equal to $-\infty$ are **pluripolar** sets, whose structure is difficult to describe. They are more complicated than countable unions of algebraic sets, but at least they are still of Lebesgue measure 0, cf. [**25**].
- For a positive current Θ, the level sets of the Lelong numbers $E_c(\Theta) = \{x \in X | \nu(\Theta, x) \geq c\}$ are analytic subsets of X ([**35**],[**14**, (2.10)]).

So it is useful to introduce the following notion:

DEFINITION 2.3. *Let X, L, h be as in the previous definition. A smooth curve $C \subset X$ will be called $(L, h)-$general iff $h_{|C}$ is a well defined singular metric on C and*

(i) *C intersects no codim-2-component in any of the $E_c(h)$,*
(ii) *C intersects every prime divisor $D \subset E_c(h)$ in the regular locus D_{reg} of this divisor, C does not intersect the intersection of two such prime divisors, and every intersection point x has the minimal Lelong number $\nu(h, x) = \nu(h, D) := \min_{z \in D} \nu(h, z)$,*
(iii) *for all $x \in C$, the Lelong numbers*
$$\nu(h_{|C}, x) = \nu(h, x).$$

Using methods of [**31**] it is possible to show that in families of curves covering X (e.g. appropriate components of the Chow variety) every curve outside a pluripolar set is $(L, h)-$general, see Theorem 2.5. We can even prove the stability of this notion under certain blow ups, see Lemma 2.7. The main reason for introducing this notion lies in the equality
$$\mathcal{I}(h^m) \cdot \mathcal{O}_C = \mathcal{I}(h^m)_{|C} = \mathcal{I}(h^m_{|C}),$$
which is true for $(L, h)-$general curves. From this one easily gets the announced last equality

THEOREM 2.4. *For $(L, h)-$general smooth curves $C \subset X$,*
$$(L, h).C = \limsup_{m \to \infty} \frac{1}{m} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) \cdot \mathcal{O}_C),$$
where $\mathcal{I}(h^m) \cdot \mathcal{O}_C$ is the image of $\mathcal{I}(h^m) \otimes \mathcal{O}_C$ in $\mathcal{O}_C$.

This equality is needed in order to be able to interchange restriction (to curves C) with taking global sections (of the sheaf $\mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)$) as in the proof of the Key Lemma 3.4 in the next chapter. There are explicit counterexamples for arbitrary curves, s. section 2.1.3. On the other hand the equality is true in general in case of analytic singularities, s. Proposition 2.15.

2.1. (L, h)-general curves

We start with properties of (L, h)-general curves.

2.1.1. Slices of positive currents. The aim is to prove the following

THEOREM 2.5. *Let $\pi : X \to B$ be a smooth family X of smooth projective curves over a smooth quasiprojective base B. Let L be a pseudo-effective line bundle on X and h a positive singular hermitian metric on L. Then there is a pluripolar set $N_B \subset B$ such that for $a \in B - N_B$, every fibre $\pi^{-1}(a)$ is an $(L, h)-$general curve.*

This Theorem is essentially a consequence of Ben Messaoud's

THEOREM 2.6. *Let M, G be two complex varieties of dimension n and k, let ϕ be a plurisubharmonic function on M and let $f : M \to G$ be a submersion admitting a holomorphic section s. Then there exists a pluripolar set $E \subset G$ such that for all $a \in G \setminus E$, the restricted plurisubharmonic function $\phi_{|f^{-1}(a)} \not\equiv -\infty$ and*

$$\nu(\phi, s(a)) = \nu(\phi_{|f^{-1}(a)}, s(a)).$$

PROOF. S. [**31**, Cor. 5.4]. □

PROOF OF THEOREM 2.5. Take an open subset $U \subset X$ such that $\pi : U \cong \Delta^k \times \Delta \to \Delta^k$ with $\Delta \subset \mathbb{C}$ the unit disk. Apply Theorem 2.6 to the family $U \times \Delta \overset{\pi \times \mathrm{id}_\Delta}{\longrightarrow} \Delta^k \times \Delta$, the pulled back plurisubharmonic function and the section

$$s : \Delta^k \times \Delta \to U \times \Delta,\ (b, t) \mapsto (b, t, t).$$

Since the projection of $s(\Delta^k \times \Delta)$ on U is an isomorphism there is pluripolar set $E_U \subset U$ such that for all $x \in U \setminus E_U$

$$\nu(\phi, x) = \nu(\phi_{|\pi^{-1}(\pi(x))}, x),$$

for any plurisubharmonic function ϕ on $U \times \Delta$. Setting

$$\phi := \mathrm{pr}_U^*(\phi_h),$$

the pull back of the plurisubharmonic weight ϕ_h of h in U, it is also true that

$$\nu(\phi_h, x) = \nu(\phi_{|\pi^{-1}(\pi(x))}, x).$$

Since the countable union of pluripolar sets is again pluripolar the same is true for a pluripolar set $E \subset X$. The other two requirements of Definition 2.3 for an $(L, h)-$general curve also show that these curves must be fibres outside the countable union of analytic subsets, which is a pluripolar set. This shows the Theorem. □

2.1.2. Birational invariance of $(L,h)-$generality.

LEMMA 2.7. *Let C be a smooth $(L,h)-$general curve on X, and let $Z \subset X$ be a smooth subvariety with $C \not\subset Z$, let $\pi : \widehat{X} \to X$ be the blowup of X with centre Z. Then the strict transform $\widehat{C}$ of C is still $(\pi^* L, \pi^* h)-$general.*

PROOF. The assertion is clear as long as $Z \cap C = \emptyset$. Otherwise, let $x \in Z \cap C$ be a point such that $\nu(h_{|C}, x) = 0$. Then for y the unique preimage of x in $\widehat{C}$,

$$0 = \nu(\pi^* h, y) \leq \nu(\pi^* h_{|\widehat{C}}, y) = \nu(h_{|C}, x) = 0.$$

If $x \in Z \cap C$ is a point such that $\nu(h_{|C}, x) > 0$ then C will intersect transversally a prime divisor D of some $E_c(h)$. Consider two cases:

(a) Z is a point. Then the intersection of the strict transforms $\widehat{C} \cap \widehat{D} = \emptyset$, and C intersects the smooth exceptional divisor E transversally in a unique point $y \in E$ with $\pi(y) = x$. Furthermore,

$$\nu(\pi^* h, E) \geq \nu(h, D) = \nu(h, x) = \nu(h_{|C}, x) = \nu(\pi^* h_{|\widehat{C}}, y),$$

hence $\nu(\pi^* h, E) = \nu(\pi^* h_{|\widehat{C}}, y)$.

(b) $\dim Z \geq 1$. Then the preimage of $\widehat{C} \cap \widehat{D}$ over x consists of one point y, and by the same argument as in (a), replacing E by $\widehat{D}$, it follows

$$\nu(\pi^* h, \widehat{D}) = \nu(\pi^* h_{|\widehat{C}}, y).$$

$\widehat{D}$ cannot be singular in y since then $\nu(\pi^* h_{|\widehat{C}}, y) \geq \nu(\pi^* h, y) > \nu(\pi^* h, \widehat{D})$.

$\square$

2.1.3. A counterexample for non-$(L,h)-$general curves. First, one constructs a convex function $\chi : \mathbb{R} \to \mathbb{R}$ with slow growth at $-\infty$ (i.e. the derivation tends to 0) such that $\chi(-\infty) = -\infty$. For example, take

$$\chi(x) = \begin{cases} x & \text{for } x \geq -1 \\ -\sum_{k=1}^{n} \frac{1}{k} + (x+n)\frac{1}{n+1} & \text{for } -n-1 \leq x \leq -n \end{cases}$$

Then one considers the plurisubharmonic function $\psi = \max(\log|z_1|, \chi(\log|z_2|))$ on $\mathbb{C}^2$. The Lelong numbers $\nu(\psi, x)$ are 0 everywhere because of the slow growth of χ at $-\infty$, but the restriction of ψ onto $C = \{z_2 = 0\}$ has Lelong number $\nu(\psi_{|C}, x) = 1$ for all points $x \in C$.

The induced metric h may be extended to a metric of the relatively ample line bundle $\mathcal{O}(1)$ on the $\mathbb{P}^1-$bundle $\mathbb{C} \times \mathbb{P}^1$ which yields the counterexample.

2.2. Intersection numbers

The aim of this section is to prove Proposition 2.2 and Theorem 2.4.

2.2.1. Proof of Proposition 2.2. The first step is to compare the sum of the restricted Lelong numbers on arbitrary smooth curves $C \in X$ with $h_{|C} \not\equiv \infty$ to the ordinary intersection number of C with L:

LEMMA 2.8.

$$\sum_{x \in C} \nu(h_{|C}, x) \leq L.C.$$

PROOF. Since $h_{|C}$ is positive, the curvature current $i\Theta_{h_{|C}} \geq 0$, too. By a theorem of Siu, the Lelong level sets $E_c(\phi) = \{x \in X : \nu(\phi, x) \geq c\}$ are analytic [**14**, (2.10)]. But then there are only countably many points $(x_i)_{i \in \mathbb{N}}$ on C with $\nu(h_{|C}, x_i) \neq 0$. By Siu's decomposition formula [**14**, (2.18)] the current $i\Theta_{h_{|C}} - \sum_{i=1}^{N} \nu(h_{|C}, x_i)[x_i]$ is still positive for arbitrary N (where $[x_i]$ is the integration current of the divisor x_i). Consequently the first Chern class of the ($\mathbb{R}-$) divisor $L_{|C} - \sum_{i=1}^{N} \nu(h_{|C}, x_i)x_i$ is ≥ 0, hence $L.C - \sum_{i=1}^{N} \nu(h_{|C}, x_i) \geq 0$, and the claim follows. □

LEMMA 2.9. *Let C be a smooth curve and h a positive singular hermitian metric on C. Then:*

$$\limsup_{m \to \infty} \frac{1}{m} \deg_C \mathcal{I}(h^m) = \sum_{x \in C} \nu(h, x).$$

PROOF. $\mathcal{I}(h^m)$ is a torsion free subsheaf of $\mathcal{O}_C$, hence it corresponds to a divisor on C, say $\mathcal{I}(h^m) = \mathcal{O}(-D_m)$, where D_m is an effective divisor on C. We show that

$$\text{mult}_x D_m \leq \nu(h^m, x) < \text{mult}_x D_m + 1 \quad . \tag{2.2.1}$$

This is true for arbitrary positive metrics h: Choose a sufficiently small neighborhood U of x such that $\nu(h, y) < 1$ for all $y \in U \setminus \{x\}$. Let ϕ_h and Θ_h be the plurisubharmonic function and $(1,1)-$current corresponding to h in U. As explained in the proof of the previous lemma the current $\Theta = \Theta_h - \nu(h, x)[x]$ is still positive, with $\nu(\Theta, x) = 0$, $\nu(\Theta, y) < 1$ for all $y \in U \setminus \{x\}$. Let ψ be a plurisubharmonic function with $dd^c\psi = \Theta$. Then $\phi_h = \psi + \nu(h, x) \log |z - x|$, hence $\text{mult}_x \mathcal{I}(h) \geq \lfloor \nu(h, x) \rfloor$.
On the other hand $e^{-2(\psi + (\nu(h,x) - \lfloor \nu(h,x) \rfloor) \log |z - x|)}$ is locally integrable around x since the Lelong number in x is < 1, by Skoda's lemma [**14**, (5.6)]. This proves $\mathcal{I}(h)_x = ((z - x)^{\lfloor \nu(h,x) \rfloor})$, hence (2.2.1).
Now we conclude:

$$\begin{aligned} \limsup_{m \to \infty} \frac{1}{m} \deg_C D_m \stackrel{(2.2.1)}{=} \limsup_{m \to \infty} \frac{1}{m} \sum_{x \in C} \lfloor \nu(h^m, x) \rfloor \quad &\leq \quad \limsup_{m \to \infty} \frac{1}{m} \sum_{x \in C} \nu(h^m, x) \\ &= \quad \sum_{x \in C} \nu(h, x) < \infty. \end{aligned}$$

On the other hand, (2.2.1) implies

$$\lim_{m \to \infty} \frac{1}{m} \text{mult}_x D_m = \nu(h, x).$$

Hence for every $m_0 \in \mathbb{N}$:

$$\begin{aligned}
\limsup_{m\to\infty} \frac{1}{m} \deg_C D_m &= \limsup_{m\to\infty} \frac{1}{m} \sum_{x\in C} \mathrm{mult}_x D_m \\
&= \limsup_{m\to\infty} \frac{1}{m} \left(\sum_{\nu(h,x)\geq \frac{1}{m_0}} \mathrm{mult}_x D_m + \sum_{\nu(h,x)< \frac{1}{m_0}} \mathrm{mult}_x D_m \right) \\
&\overset{(*)}{=} \lim_{m\to\infty} \sum_{\nu(h,x)\geq \frac{1}{m_0}} \mathrm{mult}_x D_m + \limsup_{m\to\infty} \frac{1}{m} \sum_{\nu(h,x)< \frac{1}{m_0}} \mathrm{mult}_x D_m \\
&\geq \sum_{\nu(h,x)\geq \frac{1}{m_0}} \nu(h,x).
\end{aligned}$$

where the equality $(*)$ follows from the fact that $\{x \in C : \nu(h,x) \geq \frac{1}{m_0}\}$ is a finite set. Since $\sum_{x\in C} \nu(h,x) < \infty$ implies $\sum_{\nu(h,x)<\frac{1}{m}} \stackrel{m\to\infty}{\longrightarrow} 0$ the lemma follows. ☐

Proposition 2.2 follows from

$$\text{(2.2.2)} \quad \limsup_{m\to\infty} \frac{1}{m} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{O}_C(-D_m)) = \limsup_{m\to\infty} \frac{1}{m} \deg_C(mL - D_m)$$

PROOF. By Lemma 2.8, $\deg_C D_m \leq \sum_{x\in C} \nu(h^m_{|C}, x) \leq mL.C$. Consequently,

$$\deg_C(mL - D_m) \geq 0.$$

Let $g(C)$ be the genus of the curve C. If $\deg_C(mL - D_m) \leq 2g(C) - 2$ and $mL - D_m$ is not effective, $H^0(C, \mathcal{O}(mL - D_m)) = 0$. If $\deg_C(mL - D_m) \leq 2g(C) - 2$ and $mL - D_m$ is effective,

$$H^0(C, \mathcal{O}(mL - D_m)) \leq \deg_C(mL - D_m) + 1 \leq 2g(C) - 1.$$

If $\deg_C(mL - D_m) > 2g(C) - 2$, then $H^1(C, \mathcal{O}(mL - D_m)) = 0$, and (2.2.2) will follow by Riemann-Roch. ☐

2.2.2. Proof of Theorem 2.4. One main ingredient of the proof, which is useful in many circumstances, is

EXTENSION THEOREM 2.10 (Ohsawa-Takegoshi). *Let $\Omega \subset \mathbb{C}^n$ be a bounded open pseudoconvex set, $L = \{z_i = \dots z_n = 0\}$, $1 \leq i \leq n$, a linear subspace, and $\psi \in \mathrm{Psh}(\Omega)$ with $\psi_{|L} \neq -\infty$.*
Then there is a constant $C > 0$, only depending on n, such that for all holomorphic functions f on L with $\int_{L\cap\Omega} |f|^2 e^{-2\psi} d\lambda_L < \infty$, there is an $F \in \mathcal{O}(\Omega)$ such that $F_{|L} = f$ and

$$\int_\Omega |F|^2 e^{-2\psi} d\lambda_\Omega \leq C \cdot \int_{L\cap\Omega} |f|^2 e^{-2\psi} d\lambda_L$$

PROOF. S. [**14**, (12.9)]. ☐

Now let C be a smooth $(L,h)-$ general curve in the smooth projective variety X. Let D_m, D'_m be the effective divisors corresponding to the ideal sheaves $\mathcal{I}(h^m_{|C})$ and $\mathcal{I}(h^m)_{|C}$, as explained in subsection 2.2.1. The Extension Theorem implies a natural inclusion

$$\mathcal{I}(h^m_{|C}) \subset \mathcal{I}(h^m)_{|C},$$

hence $\deg_C D'_m \leq \deg_C D_m$, and we can prove

$$\limsup_{m\to\infty} \frac{1}{m} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{O}_C(-D'_m)) \quad = \quad \limsup_{m\to\infty} \frac{1}{m} \deg_C(mL - D'_m) \tag{2.2.3}$$

similarly to (2.2.2).
The $(L,h)-$generality implies

$$\mathcal{I}(h^m_{|C}) = \mathcal{I}(h^m)_{|C}.$$

PROOF. By Skoda's Lemma [**14**, (5.6)], $\mathcal{I}(h^m_{|C})_x = \mathcal{I}(h^m)_{|C,x} = \mathcal{O}_{C,x}$ for all points $x \in C$ with $\nu(h,x) = \nu(h_{|C},x) = 0$.
Let $x \in C$ be a point with $\nu = \nu(h,x) = \nu(h_{|C},x) > 0$. By definition of (L,h)-generality there exists a divisor D through x locally defined by $g \in \mathcal{O}_{X,x}$, $x \in D_{\mathrm{reg}}$ and $\nu = \nu(h,x) = \nu(h,D)$. As explained before it follows that

$$\mathcal{I}(h^m)_x = (g^{\lfloor m\nu \rfloor}) \subset \mathcal{O}_{X,x}.$$

Similarly we show

$$\mathcal{I}(h^m_{|C})_x = (g_{|C}^{\lfloor m\nu \rfloor}) \subset \mathcal{O}_{C,x}.$$

□

2.3. Birational invariance

Since the intersection numbers $(L,h).C$ are computed by pulling back to the normalization $\widehat{C}$ it is obvious that the intersection number $(\pi^*L, \pi^*h).\bar{C}$ where $\bar{C}$ is the strict transform of C via the birational map π does not change. The aim of this section is to generalize this observation. In the next section we apply it in the case of analytic singularities, thus obtaining a more algebraic definition of the intersection numbers.

LEMMA 2.11. *Let $\mu : C' \to C$ be a finite morphism between smooth curves. Let (L,h) be a pseudo-effective line bundle on C with $i\Theta_h \geq 0$. Then*

$$(\mu^*L, \mu^*h).C' = \deg\mu \cdot (L,h).C.$$

PROOF. It is enough to consider the following situation: Let $\mu : \Delta \to \Delta,\ z \mapsto z^n$ be a finite morphism on the unit disc Δ and let $\psi \in \mathrm{Psh}(\Delta)$ be a plurisubharmonic function on Δ. Then

$$\nu(\psi,0) = \liminf_{|z|\to 0} \frac{\psi(z)}{\log|z|} = \liminf_{|z|\to 0} \frac{\psi(z^n)}{\log|z^n|} = \frac{1}{n} \liminf_{|z|\to 0} \frac{\psi(z^n)}{\log|z|} = \frac{1}{n}\nu(\psi\circ\mu, 0).$$

Now the lemma follows by Proposition 2.2. □

PROPOSITION 2.12. *Let $f : Y \to X$ be a surjective morphism between smooth and projective varieties X and Y. Let (L, h) be a pseudo-effective line bundle on X with $i\Theta_h \geq 0$. Then:*

$$(L, h) \text{ numerically trivial on } X \iff (f^*L, f^*h) \text{ numerically trivial on } Y.$$

PROOF. Assume first that (L, h) is numerically trivial on X. Let $C \subset Y$ be an irreducible curve on Y with $f^*h_{|C} \not\equiv \infty$. When $f(C)$ is a point, this point will lie in the smooth part of h, and there won't be any singularity of h on C. Consequently,

$$(f^*L, f^*h).C = f^*L.C = L.f_*C = 0.$$

When $f(C)$ is another irreducible curve C' then one can lift the morphism $f_{|C}$ to the smooth normalizations $\widehat{C}, \widehat{C}'$, and the above equality follows by the lemma.
Similarly, assume that (f^*L, f^*h) is numerically trivial on Y. Let C be an irreducible curve on X with $h_{|C} \not\equiv \infty$. Then there exists an irreducible curve $C' \subset Y$ not lying in the singularity locus of f^*h such that $f(C') = C$, and the argument is as above. □

2.4. Metrics with analytic singularities

The $(L, h)-$ intersection numbers are much easier to handle if the plurisubharmonic weight of the metric h has only analytic singularities:

DEFINITION 2.13. *$\phi \in \mathrm{Psh}(\Omega)$, $\Omega \subset \mathbb{C}^n$ open, is said to have* ***analytic singularities****, if locally, ϕ can be written as*

$$\phi = \frac{\alpha}{2} \log\Big(\sum |f_i|^2\Big) + v,\ \alpha \in \mathbb{R}^+,$$

where v is locally bounded, and the f_i are (germs of) holomorphic functions.

For example, in this case Theorem 2.4 is true for arbitrary smooth curves. Furthermore it is easier to compute $(L, h)-$intersection numbers on log resolutions.

2.4.1. Properties of metrics with analytic singularities. By definition, the corresponding plurisubharmonic weight may locally be written as $\phi_h = \frac{\alpha}{2} \log(\sum |f_i|^2) + O(1)$. Define $\mathcal{J}(h/\alpha)$ as the ideal sheaf of germs of holomorphic functions f such that

$$|f| \leq C \cdot \Big(\sum |f_i|\Big).$$

One can easily prove that $\mathcal{J}(h/\alpha)_x$ is the integral closure of the ideal generated by the germs f_i (cf. [**14**, (1.11)]). Consequently, $\mathcal{J}(h/\alpha)$ is coherent. Furthermore,

$$\mathcal{J}(h/\alpha)_x = (g_1, \ldots, g_M) \Longrightarrow \phi = \frac{\alpha}{2} \log\Big(\sum |g_i|^2\Big) + O(1).$$

There exists log resolutions $\mu : X' \to X$ of $\mathcal{J}(h/\alpha)$ with X' non-singular, i.e.

(a) μ is proper birational,

(b) $\mu^{-1}\mathcal{J}(h/\alpha) = \mathcal{J}(h/\alpha)\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ where F is an effective divisor on X' such that $F + \mathrm{Exc}(\mu)$ has simple normal crossing support.

An existence proof is contained in the Hironaka package, cf. [**3**].
The main tool when dealing with metrics with analytic singularities is the following theorem which may be seen as an algebraic definition of multiplier ideals:

THEOREM 2.14. $\mathcal{I}(h) = \mu_*(K_{\bar{\Omega}/\Omega} - [\alpha F])$.

PROOF. See [**14**, (5.9)]. □

2.4.2. Intersection numbers of metrics with analytic singularities.

PROPOSITION 2.15. *Let X be a projective manifold, L a pseudo-effective line bundle and h a positive hermitian metric on L having only analytic singularities. Then for every smooth curve $C \subset X$,*

$$(L,h).C = \limsup_{m\to\infty} \frac{1}{m} h^0(C, \mathcal{O}_C(mL)\otimes\mathcal{I}(h^m)\cdot\mathcal{O}_C),$$

where $\mathcal{I}(h^m)\cdot\mathcal{O}_C$ is the image of $\mathcal{I}(h^m)\otimes\mathcal{O}_C$ in $\mathcal{O}_C$.

PROOF. Let D_m, D'_m be effective divisors corresponding to the torsion free ideal sheaves $\mathcal{I}(h^m_{|C}), \mathcal{I}(h^m)\cdot\mathcal{O}_C = \mathcal{I}(h^m)_{|C}$. By (2.2.2),(2.2.3) it is enough to show that

$$\liminf_{m\to\infty} \frac{1}{m}\deg_C D_m = \liminf_{m\to\infty} \frac{1}{m}\deg_C D'_m \tag{2.4.1}$$

Let $(x_i)_{i\in\mathbb{N}}$ be the countably many points on C such that $\mathrm{mult}_x D_m \neq 0$ or $\mathrm{mult}_x D'_m \neq 0$ for some $m \in \mathbb{N}$. Since C is smooth there is an open subset $U \subset X$ containing all the x_i such that $C = H_2 \cap \ldots \cap H_n$ is a complete intersection of very ample smooth hypersurfaces $H_i \subset U$. It is enough to prove (2.4.1) on U.
Locally, let the weight ϕ of h be of the form

$$\frac{\alpha}{2}\log(\sum |f_i|^2) + O(1).$$

Construct a log resolution $\mu : U' \to U$ for $\mathcal{J}(h/\alpha)_{|U}$ as above such that furthermore,

(c) the support of F contains the support of $\mathrm{Exc}(\mu)$,
(d) the strict transforms H'_i of the H_i are smooth, $\sum H_i + F$ has simple normal crossing support and $\mu^* H_n = H'_n + \sum b_j E_j$ where the E_j are prime components of $\mathrm{Exc}(\mu)$.

One has

THEOREM 2.16 (Local vanishing). *Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal sheaf on a smooth quasiprojective complex variety X, and let $\mu : X' \to X$ be a log resolution of $\mathfrak{a}$ with $\mathfrak{a}\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. Then for any rational $c > 0$:*

$$R^j\mu_*\mathcal{O}_{X'}(K_{X'/X} - [c\cdot F]) = 0 \text{ for } j > 0.$$

PROOF. See [**29**, Thm.9.4.1]. □

This theorem is used to prove the following inclusions of ideal sheaves on H_n: There is a $c \in \mathbb{N}$ independent of m such that

$$\mathcal{I}(h^{m+c})_{|H_n} \subset \mathcal{I}(h^m_{|H_n}) \subset \mathcal{I}(h^m)_{|H_n}.$$

Equation (2.4.1) follows by induction and $\limsup_{m\to\infty} \frac{1}{m} d_{m+c} = \limsup_{m\to\infty} \frac{1}{m} d_m$ for sequences $(d_m)_{m\in\mathbb{N}} \subset \mathbb{R}^+$.
The proof of the inclusions is modelled on the proof of the Restriction Theorem [**29**, Thm.9.5.1]. First of all, $\mu_{|H_n} : H'_n \to H_n$ is a log resolution of $\mathcal{J}(h/\alpha)_{|H_n} = \mathcal{J}(h/\alpha) \cdot \mathcal{O}_{H_n}$ by property (d) of μ. Property (c) implies that there exists a $c \in \mathbb{N}$ independent of m such that

$$K_{U'/U} - [m\alpha F] - c\alpha F \subset K_{U'/U} - [m\alpha F] - \sum b_j E_j =: B,$$

and consequently

$$\mathcal{I}(h^{m+c})_{|H_n} = \mu_*(K_{U'/X} - [(m+c)\alpha F])_{|H_n} \subset \mu_*\mathcal{O}_{U'}(B)_{|H_n}.$$

Now, $B - H'_n = K_{U'/U} - [m\alpha F] - \mu^* H_n$. Local vanishing applied on $\mathcal{J}(h/m\alpha) \cdot \mathcal{O}(-H_n)$ implies

$$R^1\mu_*\mathcal{O}_{U'}(B - H'_n) = 0.$$

Then

$$\mu_*\mathcal{O}_{U'}(B)_{|H_n} = (\mu_{|H_n})_*(\mathcal{O}_{H'_n}(B_{|H'_n}))$$

follows by taking direct images in the exact sequence

$$0 \to \mathcal{O}_{U'}(B - H'_n) \xrightarrow{\cdot H'_n} \mathcal{O}_{U'}(B) \to \mathcal{O}_{H'_n}(B_{|H'_n}) \to 0.$$

Since $K_{H'_n/H_n} = (K_{U'/U} - \sum b_j E_j)_{|H'_n}$, it follows

$$(\mu_{|H_n})_*(\mathcal{O}_{H'_n}(B_{|H'_n})) = (\mu_{|H_n})_*(K_{H'_n/H_n} - [m\alpha F_{|H'_n}]) = \mathcal{I}(h^m_{|H_n}),$$

hence the first inclusion.
The second inclusion follows by the Ohsawa-Takegoshi Extension Theorem. □

2.4.3. Computation of intersection numbers. This subsection shows how to compute the $(L, h)-$ intersection numbers for metrics with analytic singularities on a log resolution of the ideal sheaf of the singularities:

PROPOSITION 2.17. *Let X be a smooth projective variety, let (L, h) be a pseudo-effective line bundle L on X with a singular hermitian metric h such that $i\Theta_h \geq 0$ and h has analytic singularities. Let $\mathcal{J}(h/\alpha)$ be the ideal sheaf ot these singularities, let $\mu : \tilde{X} \to X$ be a log resolution of X with $\mu^*\mathcal{J}(h/\alpha) = \mathcal{O}(-F)$. Let $C \subset X$ be an irreducible curve. Then*

$$(L, h).C = \mu^* L.\bar{C} - F.\bar{C},$$

where $\bar{C}$ is the strict transform of C.

PROOF. By birational invariance,

$$(L, h).C = (\mu^* L, \mu^* h).\bar{C}.$$

But the pull back of h is just the metric given by F by definition of analytic singularities and log resolutions (s. [**14**, (3.13)]). This implies the proposition. $\square$

CHAPTER 3

Tsuji's numerically trivial fibrations

In this chapter we prove the existence of a reduction map with respect to the pair (L, h) of a pseudo-effective line bundle L and a possibly singular positive hermitian metric h on L. The aim is to get a reduction map with numerically trivial fibers where numerical triviality is defined by the intersection numbers which we thoroughly discussed in the last chapter:

DEFINITION 3.1. *Let X be a smooth projective complex manifold, let L be a pseudo-effective holomorphic line bundle on X with positive singular hermitian metric h. Then a subvariety $Y \subset X$ is called numerically trivial (with respect to (L, h)) if each curve $C \subset Y$ such that $h_{|C} \not\equiv \infty$ has intersection number $(L, h).C = 0$.*

Now we adjust Tsuji's assertions about the reduction map:

REDUCTION MAP THEOREM 3.2. *Let X be a smooth projective complex manifold, let L be a pseudo-effective holomorphic line bundle on X with positive singular hermitian metric h. Then there exists a dominant rational map $f : X \dashrightarrow Y$ with connected fibres such that:*

- (i) *(L, h) is numerically trivial on fibres over points in Y lying in the complement of a pluripolar set.*
- (ii) *For all $x \in X$ outside a pluripolar set, every curve C through x with $\dim f(C) > 0$ has intersection number $(L, h).C > 0$.*

Here, fibres of f are fibres of the graph $\Gamma_f \subset X \times Y \to Y$ interpreted as subschemes of X.
Finally, f is uniquely determined up to birational equivalence of Y.

There are two main differences to [**37**]. First, the reduction map need not be almost holomorphic. A counter example will be given in section 3.4. Second, Tsuji completely ignores the fact that the singularities of arbitrary positive singular hermitian metrics lie in pluripolar sets. This means for example, that the restriction of the singular metric may be well defined only on fibres over points lying in the complement of a pluripolar set. But this is not so bad: for example, the Zariski closure of the union of these fibres is always the whole variety.
After these adjustments it is possible to apply Tsuji's ideas in proving the Reduction Map Theorem:

(a) For each ample divisor H and each pair (L,h) of a line bundle with a positive singular hermitian metric one can define a **volume**

$$\mu_h(X, H+mL) := (\dim X)! \limsup_{l\to\infty} l^{-\dim X} h^0(X, \mathcal{O}_X(l(H+mL)) \otimes \mathcal{I}(h^{ml}))$$

and we have the following

LEMMA 3.3.

$$(L,h) \text{ numerically not trivial} \quad \Rightarrow \limsup_{m\to\infty} \mu_h(X, H+mL) = \infty.$$

(b) The lemma implies that for all N there exists an m_0 such that for arbitrarily large $l \gg 0$ there exist sections

$$0 \not\equiv \sigma_l \in H^0(X, \mathcal{O}_X(l(H+m_0L)) \otimes \mathcal{I}(h^{m_0 l}) \otimes \mathfrak{m}_x^{Nl})$$

for a sufficiently general point $x \in X$.

(c) This is used for

KEY LEMMA 3.4. *Let $f : M \to B$ be a projective surjective morphism from a smooth variety M to a smooth curve B. Let (L,h) be a pseudo-effective line bundle L with positive singular hermitian metric h. Suppose that (L,h) is numerically trivial on all fibres F of f over a set $B' \subset B$ not of Lebesgue measure 0. If furthermore there is an $(L,h)-$general curve W with $f(W) = B$, (L,h) is numerically trivial on W, then (L,h) will be numerically trivial on M.*

The proof is done by contradiction: Any σ_l as above must be 0.

(d) Finally the theorem is derived from the Key Lemma with methods similar to those in [**1**].

The intersection number equality in Theorem 2.4 is needed essentially in proving the Key Lemma 3.4, while the definition of the intersection number is used several times for switching to birationally equivalent varieties.
In the last section we prove a criterion for numerical triviality of a variety w.r.t. a pseudo-effective line bundle and a hermitian metric.

3.1. The volume μ_h and numerical triviality

The aim of this section is to prove Lemma 3.3 and the existence of a section σ_λ as in step (b) of the introduction.
The proof is by induction on $\dim X$. If $X = C$ is a smooth curve, the volume will be

$$\begin{aligned}
\mu_h(C, H+mL) &= \limsup_{l\to\infty} \frac{1}{l} H^0(C, \mathcal{O}_C(l(H+mL)) \otimes \mathcal{I}(h^{ml})) = \\
&= \limsup_{l\to\infty} \frac{1}{l} \deg_C(\mathcal{O}_C(l(H+mL)) \otimes \mathcal{I}(h^{ml})) = \\
&= \deg_C H + (L^{\otimes m}, h^{\otimes m}).C = \deg_C H + m \cdot (L,h).C,
\end{aligned}$$

where the second and the third equality follow by equation (2.2.2), while the fourth is a consequence of the homogenity of Lelong numbers.
If $\dim X = n$, then for every $n_1 \gg 0$ there will be a hyperplane pencil in $|n_1 H|$ with smooth center $Z \subset X$ such that the general element F of the pencil is smooth, and for sufficiently general F, the restricted metric $h_{|F} \not\equiv \infty$.
Step 1. (L, h) is not numerically trivial on a sufficiently general F.
Let $C \in X$ be an irreducible, not necessarily smooth curve such that $(L, h).C > 0$.

CLAIM 1. *For arbitrary $n_i \gg 0$, there exists a complete intersection*

$$H_1 \cap \ldots \cap H_{n-1} = C \cup \bigcup_k C_k, \ \ H_i \in |n_i H|,$$

such that the C_i are irreducible smooth curves with $h_{|C_i} \not\equiv \infty$.

PROOF. If $n = 2$, the curve C is a divisor, and for $m \gg 0$, the linear system $|mH - C|$ is very ample. Hence a general element $C' \in |mH - C|$ is irreducible, and $h_{|C'} \not\equiv \infty$.
For $n > 2$, the curve C is contained in an irreducible hypersurface H' with $h_{|H'} \not\equiv \infty$. For some $m \gg 0$ the linear system $|mH - H'|$ is very ample. Hence a general element $H'' \in |mH - H'|$ is irreducible, and $h_{|H''} \not\equiv \infty$. Use induction on $H_1 = H' \cup H''$. □

CLAIM 2. *For every irreducible curve $C \subset X$, the following inequality is true:*

$$(L, h).C \leq L.C - \sum_j \nu(\Theta_h, D_j) C.D_j. \tag{3.1.1}$$

where the sum is taken over all irreducible divisors D_j of X.

PROOF. Let $\pi : \widehat{C} \to C$ be the normalization of C. By the decomposition theorem of Siu [**14**, (2.18)],

$$i\Theta_h = \sum_j \nu(\Theta_h, D_j)[D_j] + R, \ R \geq 0,$$

where R is a positive residual $(1,1)-$current. Let ϕ_j, ϕ_R be plurisubharmonic functions such that (locally) $dd^c \phi_j = [D_j]$ and $dd^c \phi_R = R$. Then

$$\Theta_{\pi^* h} = \sum_j \nu(\Theta_h, D_j) dd^c(\phi_j \circ \pi) + dd^c(\phi_R \circ \pi) \geq \sum_j \nu(\Theta_h, D_j)[\pi^* D_j].$$

Since $\liminf_{z \to x} \sum f_j(z) \geq \sum_j \liminf_{z \to x} f_j(z)$ for arbitrary functions f_j, it follows

$$\nu(\Theta_{\pi^* h}, x) \geq \sum_j \nu(\Theta_h, D_j) \nu([\pi^* D_j], x) = \sum_j \nu(\Theta_h, D_j) \cdot \mathrm{mult}_x \pi^* D_j \ \ \forall x \in \widehat{C}.$$

But then

$$\begin{aligned}(L,h).C = L.C - \sum_{x\in\widehat{C}} \nu(\Theta_{\pi^*h},x) &\leq L.C - \sum_{x\in\widehat{C}}(\sum_j \nu(\Theta_h,D_j)\cdot \mathrm{mult}_x\pi^*D_j) = \\ &= L.C - \sum_j \nu(\Theta_h,D_j)C.D_j.\end{aligned}$$

□

By Theorem 2.5, on a sufficiently general fibre F, there is a smooth irreducible curve

$$C_F = H_1' \cap \ldots H_{n-1}', \quad H_i' \in |n_iH|,$$

which is $(L,h)-$general. Consequently,

$$(L,h).C_F = L.C_F - \sum_j \nu(\Theta_h,D_j)C_F.D_j.$$

But this implies together with (3.1.1)

$$(L,h).C_F \geq (L,h).(C + \sum C_i) > 0,$$

because $L.C_F = L.(C+\sum C_i)$. Hence F is not numerically trivial.

Step 2. Let $\mu : \widetilde{X} \to X$ be the blow up of X in the center Z of the pencil, with exceptional divisor E. Then

$$\limsup_{m\to\infty} \mu_{\mu^*h}(\widetilde{X},\mu^*(H+mL)) = \infty \Longrightarrow \limsup_{m\to\infty} \mu_h(X,H+mL) = \infty.$$

PROOF. First, $K_{\widetilde{X}} = \mu^*K_X + E$. By the functorial property of multiplier ideal sheaves [**14**, (5.8)] this implies $\mathcal{I}(h) = \mu_*(\mathcal{O}(E)\otimes\mathcal{I}(\mu^*h))$. Since $\mathcal{I}(\mu^*h) \subset \mathcal{O}(E)\otimes\mathcal{I}(\mu^*h) \subset \mathcal{K}_{\widetilde{X}}$ (the sheaf of total quotient rings), it follows

$$\mu_*\mathcal{I}((\mu^*h)^{ml}) \subset \mathcal{I}(h^{ml}).$$

By the projection formula,

$$\mu_*(\mu^*\mathcal{O}_X(l(H+mL))\otimes\mathcal{I}((\mu^*h)^{ml})) = \mathcal{O}_X(l(H+mL))\otimes\mu_*\mathcal{I}((\mu^*h)^{ml}).$$

Consequently,

$$h^0(\widetilde{X},\mu^*\mathcal{O}_X(l(H+mL))\otimes\mathcal{I}((\mu^*h)^{ml})) \leq h^0(X,\mathcal{O}_X(l(H+mL))\otimes\mathcal{I}(h^{ml})),$$

which implies the claim. □

Step 3. $\limsup_{m\to\infty}\mu_{\mu^*h}(\widetilde{X},\mu^*(H+mL)) = \infty$.
For $l_0 \gg 0$ the line bundle $l_0\mu^*H - E$ is ample on $\widetilde{X}$. It is enough to show that

$$\limsup_{m\to\infty}\mu_{\mu^*h}(\widetilde{X}, l_0\mu^*(H+mL)-E) = \infty.$$

Let $p : \widetilde{X} \to \mathbb{P}^1$ be the projection on $\mathbb{P}^1$. Now, the sheaf $\mathcal{O}_{\widetilde{X}}(ll_0\mu^*(H+mL)-lE)\otimes\mathcal{I}((\mu^*h)^{mll_0})$ is torsion free. Since p is flat,

$$p_*(\mathcal{O}_{\widetilde{X}}(ll_0\mu^*(H+mL)-lE)\otimes\mathcal{I}((\mu^*h)^{mll_0}))\cong\mathcal{E}_{m,l}\cong\bigoplus_{i=1}^{r}\mathcal{O}(a_i)$$

is also torsion free, hence a locally free sheaf on $\mathbb{P}^1$. Here, the $a_i = a_i(m,l)$ and $r = r(m,l)$ depend on m,l.
By upper semicontinuity and the Ohsawa-Takegoshi Extension Theorem, for a general fibre F

$$\begin{aligned} r(m,l) &= h^0(F,\mathcal{O}_F(ll_0\mu^*(H+mL)-lE)\otimes\mathcal{I}((\mu^*h)^{mll_0})_{|F}) \\ &\geq h^0(F,\mathcal{O}_F(ll_0\mu^*(H+mL)-lE)\otimes\mathcal{I}((\mu^*h)^{mll_0}_{|F})). \end{aligned}$$

Since $(l_0\mu^*(H+mL)-E)_{|F}$ is ample and $(L^{l_0}_{|F},h^{l_0}_{|F})$ is not numerically trivial by step 2, the induction hypothesis on F implies

$$\limsup_{m\to\infty}(\limsup_{l\to\infty}l^{-(n-1)}r(m,l))=\infty. \tag{3.1.2}$$

Let h_0 be a $\mathcal{C}^\infty$ hermitian metric on the ample line bundle $\mathcal{O}_{\widetilde{X}}(l_0\mu^*H-E)$ with $\Theta_{h_0}>0$, let h_1 be any $\mathcal{C}^\infty$ metric on $\mathcal{O}_{\mathbb{P}^1}(1)$. Then there exists a $c\in\mathbb{Q}_{>0}$ such that $\Theta_{h_0}-cp^*\Theta_{h_1}$ is a positive Kähler form on $\widetilde{X}$.

Claim. $\mathcal{E}_{m,l}\otimes\mathcal{O}_{\mathbb{P}^1}(-lc+1)$ is globally generated for all $l\in\mathbb{N}$ with $lc\in\mathbb{N}$, $l\gg 0$.

PROOF. By looking at the short exact sequence

$$0\to\mathcal{E}_{m,l}\otimes\mathcal{O}_{\mathbb{P}^1}(-cl)\to\mathcal{E}_{m,l}\otimes\mathcal{O}_{\mathbb{P}^1}(-cl+1)\to\mathcal{E}_{m,l}\otimes\mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_x\to 0,$$

one sees that the vector bundle $\mathcal{E}_{m,l}\otimes\mathcal{O}_{\mathbb{P}^1}(-cl+1)$ is globally generated if $H^1(\mathbb{P}^1,\mathcal{E}_{m,l}\otimes\mathcal{O}_{\mathbb{P}^1}(-cl))=0$. But this cohomology group is contained in

$$H^0(\mathbb{P}^1,R^1p_*(\mathcal{O}_{\widetilde{X}}(ll_0\mu^*(H+mL)-lE)\otimes\mathcal{I}((\mu^*h)^{mll_0}))\otimes p^*\mathcal{O}_{\mathbb{P}^1}(-cl)).$$

This higher direct image sheaf is 0 by Nadel vanishing [**14**, (5.11)], applied on preimages in $\widetilde{X}$ of open affine subsets of $\mathbb{P}^1$ and the big line bundle $ll_0\mu^*(H+mL)-lE+p^*\mathcal{O}_{\mathbb{P}^1}(-cl)$ equipped with the positive singular hermitian metric $h_0^l\otimes(p^*h_1)^{cl}\otimes(\mu^*h)^{mll_0}$. □

The claim implies $\limsup_{l\to\infty}l^{-1}(\min_i a_i)\geq c$, hence

$$\begin{aligned} & l^{-n}\cdot h^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(ll_0\mu^*(H+mL)-lE)\otimes\mathcal{I}((\mu^*h)^{mll_0}))= \\ &= l^{-n}\cdot h^0(\mathbb{P}^1,\mathcal{E}_{m,l}) \geq l^{-(n-1)}r(m,l)l^{-1}(\min_i a_i) \geq c\cdot l^{-(n-1)}r(m,l) \end{aligned}$$

(3.1.2) implies step 3, and Lemma 3.3 is proven.

LEMMA 3.5. *Let X be a complex projective variety, let (L,h) be a pseudo-effective line bundle with positive singular hermitian metric h. Assume that (L,h) is not numerically trivial. Let $x \in X$ be a sufficiently general point such that $\mathcal{I}(h^m)_x \cong \mathcal{O}_{X,x}$. Then, for any ample line bundle H, for all $N \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$ such that for $l \gg 0$ arbitrarily large there is a section*

$$0 \not\equiv \sigma_l \in H^0(X, \mathcal{O}_X(l(H+m_0L)) \otimes \mathcal{I}(h^{m_0 l}) \otimes \mathfrak{m}_{X,x}^{Nl}).$$

PROOF. By Lemma 3.3 there exists an m_0 such that the volume $\mu_h(X, H+m_0L) > N^{\dim X}+1$. Consequently, for $l \gg 0$ arbitrarily large,

$$h^0(X, \mathcal{O}_X(l(H+m_0L)) \otimes \mathcal{I}(h^{m_0 l})) \geq \frac{N^{\dim X}+1}{(\dim X)!} l^{\dim X} + o(l^{\dim X}).$$

Set $\mathcal{F} := \mathcal{O}_X(l(H+m_0L)) \otimes \mathcal{I}(h^{m_0 l})$. Since $\mathcal{I}(h^m)_x \cong \mathcal{O}_{X,x}$, it is true that $h^0(X, \mathcal{F} \otimes \mathcal{O}_X/\mathfrak{m}_x^{Nl}) = \frac{N^{\dim X}}{(\dim X)!} l^{\dim X} + o(l^{\dim X})$. Using the sequence

$$0 \to H^0(X, \mathcal{F} \otimes \mathfrak{m}_x^{Nl}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F} \otimes \mathcal{O}_X/\mathfrak{m}_x^{Nl})$$

one gets the lemma. $\square$

3.2. The Key Lemma

The proof of the Key Lemma 3.4 starts with the blow up $\pi : \widehat{M} \to M$ in W. Then a very general curve R in the smooth exceptional divisor $\widehat{W} \subset \widehat{M}$ is $(\pi^*L, \pi^*h)-$general: If D is a prime divisor in some $E_c(h)$ with $D \cap W \neq \emptyset$, then the strict transform $\widehat{D}$ of D will have minimal Lelong number $\nu(\pi^*h, \widehat{D}) \geq \nu(h, D)$. Now choose a very general curve $R \subset \widehat{W}$ such that the branching locus $\pi_{|R} : R \to W$ does not contain any of the countably many points $y \in W$ with $\nu(h_{|W}, y) > 0$. Then for $x \in \widehat{D}$,

$$\nu(\pi^*h_{|R}, x) = \nu(h_{|W}, \pi(x)) = \nu(h, y) = \nu(h, D) \leq \nu(\pi^*h, \widehat{D}) \leq \nu(\pi^*h, x),$$

hence $\nu(\pi^*h_{|R}, x) = \nu(\pi^*h, x)$. For all $x \in R$ not lying on the strict transform $\widehat{D}$ of one of the countably many divisors D as above, the Lelong number $\nu(\pi^*h_{|R}, x)$ is 0, hence

$$0 = \nu(\pi^*h, x) \leq \nu(\pi^*h_{|R}, x) = 0.$$

Now assume that (L,h) is not numerically trivial on M. By birational invariance, (π^*L, π^*h) is not numerically trivial on $\widehat{M}$ (this is an application of the definition of $(L,h)-$intersection numbers). For an ample line bundle H on $\widehat{M}$, it follows

$$\limsup_{m\to\infty} \mu_h(\widehat{M}, H + m\pi^*L) = \infty$$

by Lemma 3.3. Let $x_0 \in \widehat{W}$ be a sufficiently general point such that $\mathcal{I}(\pi^*h^m)_{x_0} \cong \mathcal{O}_{\widehat{M},x_0}$ for all integers m. By Lemma 3.5, for all N there exists an

m_0 such that for arbitrarily large $l \gg 0$ there is a non-vanishing section

$$\sigma_l \in H^0(\widehat{M}, \mathcal{O}_{\widehat{M}}(l(H + m_0\pi^*L)) \otimes \mathcal{I}(\pi^*h^{m_0 l}) \otimes \mathfrak{m}_{x_0}^{Nl}) - \{0\}.$$

Let $\mathcal{R}$ be a family of smooth intersection curves of $n-2$ divisors in $|kH_{|\widehat{W}}|$ (with k sufficiently large) through x_0 which cover $\widehat{W}$. Choose $d_0 \gg 0$ such that for general fibres F of $\widehat{f}$,

$$H^{n-2} \cdot F \cdot (H - d_0\widehat{W}) < 0.$$

Claim. There exists an $A_0 > 0$ independent of m_0 such that for very general curves $R \in \mathcal{R}$ with $h_{|R} \not\equiv \infty$ and all $0 \leq s \leq d_0 l$

$$\dim H^0(R, \mathcal{O}_R(l(H + m_0\pi^*L) - s\widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R}) \quad \leq \quad A_0 \cdot l + o(l). \tag{3.2.1}$$

PROOF. Since (π^*L, π^*h) is numerically trivial on $\widehat{W}$,

$$(\pi^*L, \pi^*h).R = \limsup_{m\to\infty} \frac{1}{m} \deg_R(\mathcal{O}_R(m\pi^*L) \otimes \mathcal{I}(\pi^*h^m)_{|R}) = 0.$$

(This is the application of Theorem 2.4, i.e. the $(L,h)-$generality of R.) Consequently,

$$\begin{aligned}
&\deg_R(\mathcal{O}_R(l(H + m_0\pi^*L) - s\widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R}) = \\
&= \quad \deg_R(\mathcal{O}_R(lH - s\widehat{W})) + \deg_R(\mathcal{O}_R(lm_0\pi^*L) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R}) \\
&\leq \quad A_0 \cdot l + o(l)
\end{aligned}$$

for some $A_0 > 0$, if $0 \leq s \leq d_0 l$.
If $\widehat{W}.R \leq 0$ the ampleness of H will imply that

$$H^1(R, \mathcal{O}_R(l(H + m_0\pi^*L) - s\widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R}) = 0$$

and hence the claim follows from Riemann-Roch .
If $\widehat{W}.R > 0$ there will exist an a_0 such that for all $s \geq a_0$ the cohomology group $H^0(R, \mathcal{O}_R(s\widehat{W})) \neq 0$. Therefore,

$$\begin{aligned}
h^0(R, \mathcal{O}_R(l(H + m_0\pi^*L) - s\widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R}) \leq \\
h^0(R, \mathcal{O}_R(l(H + m_0\pi^*L)) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R}),
\end{aligned}$$

and this gives the claim for $a_0 \leq s \leq d_0 l$. For $s \leq a_0$ one can argue as above with H ample and Riemann-Roch. $\square$

Now choose $N > A_0 + d_0$. Then

$$\deg_R \sigma_{l|R} \geq N \cdot l$$

for the corresponding section

$$\sigma_{l|R} \in H^0(R, \mathcal{O}_R(l(H + m_0\pi^*L)) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|R} \otimes \mathfrak{m}_{R,x_0}^{Nl}).$$

Because $s = 0$ the estimate (3.2.1) implies that $\sigma_{l|R} \equiv 0$ for $l \gg 0$ depending on R. But since σ_l vanishes on a Zariski closed subset and the curves in $\mathcal{R}$ cover $\widehat{W}$, there exists arbitrarily large $l \gg 0$ such that $\sigma_{l|\widehat{W}} \equiv 0$ and

$$\sigma_l \in H^0(\widehat{M}, \mathcal{O}_{\widehat{M}}(l(H + m_0\pi^*L) - \widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l}) \otimes \mathfrak{m}_{x_0}^{Nl-1}).$$

Repeating this argument for $0 < s \leq d_0 l$ one finally gets

$$\sigma_l \in H^0(\widehat{M}, \mathcal{O}_{\widehat{M}}(l(H + m_0\pi^*L) - d_0 l\widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l})).$$

Let F be sufficiently general, $\pi^*h_{|F} \not\equiv \infty$ and (π^*L, π^*h) numerically trivial on F. Let $\mathcal{S}_F$ be a family of smooth intersection curves of $n-2$ divisors in $|H_{|F}|$ covering F. Let $S \in \mathcal{S}_F$ be such a curve, with $\pi^*h_{|S} \not\equiv \infty$. Since (π^*L, π^*h) is numerically trivial on F,

$$(\pi^*L, \pi^*h).S = \limsup_{m\to\infty} \frac{1}{m} \deg_S(\mathcal{O}_S(m\pi^*L) \otimes \mathcal{I}(\pi^*h^m)_{|S}) = 0.$$

Furthermore, by assumption

$$S.(H - d_0\widehat{W}) = H^{\dim F - 1}.F.(H - d_0\widehat{W}) < 0,$$

hence

$$\deg_S(\mathcal{O}_S(l(H + m\pi^*L) - d_0 l\widehat{W}) \otimes \mathcal{I}(\pi^*h^{m_0 l})_{|S}) < 0$$

for some $l \gg 0$, and as above one concludes $\sigma_{l|F} \equiv 0$, $\sigma_l \equiv 0$ which is a contradiction.

3.3. Proof of the pseudo-effective Reduction Map Theorem

The main construction used in this proof is described by the following

LEMMA 3.6. *Let X be a complex projective variety, let M be a set of subvarieties $F_m \subset X$, $m \in M$, such that the union $\bigcup_{m\in M} F_m \subset X$ is not contained in a pluripolar set in X. Then there is a family $\mathfrak{F} \subset X \times B$ of subschemes of X, covering the whole of X, and a set $B' \subset B$ not contained in a pluripolar set of B, parametrizing subvarieties F_m, $m \in M$.*

PROOF. M may be interpreted as a subset of $\mathrm{Chow(X)}$. There are only countably many components of $\mathrm{Chow(X)}$. Hence there must be at least one component $\mathcal{C} \subset \mathrm{Chow(X)}$ such that the subschemes parametrized by the Zariski closure $Z = \overline{\mathcal{C} \cap M}$ cover the whole of X, and $\mathcal{C} \cap M$ is not a pluripolar set in Z. Otherwise, the subvarieties F_m, $m \in M$, are contained in a pluripolar set of X, contradiction. $\square$

Consider families $f : \mathfrak{X} \to \mathcal{N}$ with the following properties:

(i) $\mathfrak{X} \subset X \times \mathcal{N}$, where $\mathfrak{X}, \mathcal{N}$ are quasi-projective and irreducible, and the general fibres of f are subvarieties of X;

(ii) the projection $p : \mathfrak{X} \to X$ is generically finite;

(iii) (L, h) is defined and numerically trivial on sufficiently general fibres of f, that is on a set of fibres $\mathcal{M} \subset \mathcal{N}$ which is not contained in a pluripolar set;

(iv) the fibres are generically unique, i.e. if $U \subset \mathcal{N}$ is an open subset such that $f_{|U}$ is flat then the induced map $U \to \mathrm{Hilb}(X)$ will be generically injective.

The identity map $\mathrm{id} : X \to X$ is such a family, hence there is one with minimal base dimension $\dim \mathcal{N}$.

Claim. The projection $p : \mathfrak{X} \to X$ is birational on such a minimal family $f : \mathfrak{X} \to \mathcal{N}$.

PROOF. Assume that p is not birational.
Then, for a general fibre F of f and a general point $x \in F$ there is another fibre F' containing x, hence a curve C' with $x \in C' \subset F'$ and $C' \not\subset F$. Consequently, one gets a family of curves $g : \mathfrak{C} \to \mathcal{M}$ with $\mathfrak{C} \subset \mathfrak{X} \times \mathcal{M}$ giving a generically finite covering of $\mathfrak{X}$ such that the $f-$projection of the general $g-$fibre curve C is also a curve in $\mathcal{N}$. By blowing up and base change one can assume the following situation:

$$\begin{tikzcd} X & \mathfrak{X} \arrow[l, "p"'] \arrow[d, "f"] & \widetilde{\mathfrak{C}} \arrow[l, "\pi"'] \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{f}"] & \widetilde{\mathcal{M}} \\ & \mathcal{N} & \widetilde{\mathcal{N}} & \end{tikzcd}$$

where $\widetilde{\mathfrak{C}}$, $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ are smooth, the general fibres of $\widetilde{g}$ are smooth curves and the fibres of $\widetilde{f}$ map onto fibres of f in X. Furthermore, the maps p and π are generically finite.
Let $(\widetilde{L}, \widetilde{h})$ be the pulled back (L, h) on $\widetilde{\mathfrak{C}}$. Take an $(\widetilde{L}, \widetilde{h})-$general $\widetilde{g}-$fibre curve C in $\widetilde{\mathfrak{C}}$ such that the general $\widetilde{f}-$fibre through points of C is smooth. Look at the subvariety $G_C = \widetilde{f}^{-1}(\widetilde{f}(C)) \subset \widetilde{\mathfrak{C}}$. It may be not smooth, but by the smoothness of the general $\widetilde{f}-$fibre, the singular locus does not contain C. Hence using Lemma 2.7, an embedded resolution of G_C in $\widetilde{\mathfrak{C}}$ gives a smooth subvariety $\widehat{G_C}$ in the blow up $\widehat{\mathfrak{C}}$ such that the strict transform $\widehat{C}$ of C is still $(\mu^*\widetilde{L}, \mu^*\widetilde{h})-$general in $\widehat{\mathfrak{C}}$. By the following lemma, $\widehat{C}$ is also $(\mu^*\widetilde{L}, \mu^*\widetilde{h})-$general in $\widehat{G_C}$, and one can apply the Key Lemma: $\widehat{G_C}$ is $(\mu^*\widetilde{L}, \mu^*\widetilde{h})-$numerically trivial. By birational invariance this is true for the image of $\widehat{G_C}$ in X, too. But $\dim \widehat{G_C} = \dim F + 1$. Since all curves in a family are $(L, h)-$general outside a pluripolar set, the construction in Lemma 3.6 gives a new family

$$g : \mathfrak{Y} \to \mathcal{N}'$$

satisfying conditions (i) - (iv), and

$$\dim \mathcal{N}' = \dim \mathcal{N} - 1.$$

This is a contradiction to the minimality of $\dim \mathcal{N}$. $\square$

LEMMA 3.7. *Let $Y \subset X$ be a smooth subvariety in a projective complex variety X with a pseudo-effective line bundle L and a positive singular hermitian metric h on L such that $h_{|Y} \not\equiv \infty$. Then an $(L, h)-$general curve on X is also an $(L_{|Y}, h_{|Y})-$general curve on Y.*

PROOF. $\nu(h_{|C}, x) = 0$ implies $0 = \nu(h_{|Y}, x) \leq \nu(h_{|C}, x) = 0$, hence $\nu(h_{|Y}, x) = \nu(h_{|C}, x)$.
$\nu(h_{|C}, x) > 0$ implies that $x \in D$ for some prime divisor D on some $E_c(h)$. The restricted divisor $D_{|Y}$ may be singular but not in x: In that case,

$$\nu(h_{|Y}, x) > \nu(h_{|Y}, D_{|Y}) \geq \nu(h, D) = \nu(h_{|C}, x),$$

contradiction. $\square$

In the same way we show
Claim. Let $\widetilde{g} : \widetilde{\mathfrak{X}} \to \widetilde{\mathcal{N}}$ be another family satisfying the conditions (i) - (iv). Then there is a commutative diagram of rational maps

$$\begin{array}{ccc} X & \mathfrak{X} & \widetilde{\mathfrak{X}} \\ & \widetilde{f} & \widetilde{g} \\ & \mathcal{N} & \widetilde{\mathcal{N}} \end{array}$$

such that the general fibre of $\widetilde{g}$ is contained in a fibre of $\widetilde{f}$. $\square$
On the one hand, this claim implies the birational uniqueness of $\widetilde{f}$. On the other hand one can prove claim (ii) in the pseudo-effective Reduction Map Theorem 3.2: If (ii) is not satisfied there will be a set of points $N \subset X$ which is not contained in a pluripolar set such that

$$\forall\, x \in N\ \exists\, C_x \ni x \text{ irreducible curve},\ \dim \widetilde{f}(C_x) = 1 : (L, h).C_x = 0.$$

By Lemma 3.6 one gets a family of curves satisfying conditions (i)-(iv). The claim implies that the general fibre of this family is contained in a fibre of $\widetilde{f}$, hence also some of the curves C_x: contradiction.
Finally it is possible to prove that in part (i) of the Reduction Map Theorem, all fibres outside a pluripolar set are $(L, h)-$numerically trivial: This pluripolar set is just the set of fibres F such that $h_{|F} \equiv \infty$. Because assume to the contrary that $C \subset F$ is a curve on a fibre F such that $h_{|C} \not\equiv \infty$, hence $h_{|F} \not\equiv \infty$, and C is not $(L, h)-$numerically trivial. Then, as in step 1 of subsection 3.1, (L, h) is not numerically trivial on sufficiently general fibres F, contradiction !

3.4. Characterization of numerically trivial varieties

If X itself is numerically trivial one can prove the following consequence for the curvature current:

THEOREM 3.8. *Let X be a smooth projective complex manifold, let L be a pseudo-effective line bundle on X with positive singular hermitian metric h such that X is $(L, h)-$numerically trivial. Then the curvature current Θ_h may be decomposed as*

$$\Theta_h = \sum_i a_i [D_i]$$

where the D_i form a countable set of prime divisors on X and the $a_i > 0$.

PROOF. We start with the Siu decomposition of the curvature current [**14**, (2.18)]

$$\Theta_h = \sum_i a_i[D_i] + R$$

where the D_i are the (countably many) prime divisors in the Lelong number level sets $E_c(h)$ and the $a_i = \min_{x \in D_i} \nu(\Theta_h, x)$.
Next, take a very ample divisor H. By Theorem 2.5 there is a smooth complete intersection curve $C = H_1 \cap \ldots \cap H_{n-1}$, $H_i \in |H|$ which is $(L, h)-$ general. Now by Proposition 2.2

$$0 = (L, h).C = L.C - \sum_{x \in C} \nu(\sum_i a_i[D_i]_{|C}, x) - \sum_{x \in C} \nu(R_{|C}, x).$$

Since C is (L, h)- general the only points $x \in C$ where $\nu(\Theta_h, x) > 0$ are the intersection points with the regular part of the D_i's where furthermore $\nu(\Theta_h, x) = \nu(\Theta_h, D_i) = a_i$. Consequently

$$0 = (L, h).C = L.C - \sum_i a_i D_i.C.$$

But this implies

$$0 = R.C = \int_X R \wedge (\omega_H)^{n-1}$$

where ω_H is the strictly positive $\mathcal{C}^\infty-$ metric belonging to the very ample divisor H. Since R is a positive current it follows

$$R = 0.$$

$\square$

We can use this characterization to give a counterexample to almost holomorphy of the reduction map: Take $X = \mathbb{P}^2$ with homogeneous coordinates $(Z_0 : Z_1 : Z_2)$, $H = \mathcal{O}(1)$ and let the metric h on H be induced by the incomplete linear system of lines passing through $(1 : 0 : 0)$. The weight of h around $(1 : 0 : 0)$ is then

$$\phi_h = \frac{1}{2} \log(|z_1|^2 + |z_2|^2)$$

where $z_1 = \frac{Z_1}{Z_0}, z_2 = \frac{Z_2}{Z_0}$ are local coordinates around $(1 : 0 : 0)$. The weight ϕ_h and hence the curvature current Θ_h of h have an isolated pole in $(1 : 0 : 0)$.
Let $L = \{aZ_1 + bZ_2 = 0\}$ be a line through $(1 : 0 : 0)$. If w.l.o.g. $b \neq 0$ the weight of the metric h restricted to L is

$$\phi_{h|L} = \frac{1}{2} \log(|z_1|^2 + |-\frac{a}{b} z_1|^2) = \log|z_1| + \frac{1}{2} \log(1 + \frac{a^2}{b^2}).$$

Hence

$$\mathcal{J}(h^m_{|L}) = \mathcal{J}(m\phi_{h|L}) = \mathcal{I}^m_P$$

where $\mathcal{I}_P \subset \mathcal{O}_L$ is the ideal sheaf of $P = (1:0:0) \in L$ and

$$\begin{aligned}(H,h).L &= \limsup_{m\to\infty} \frac{h^0(L, \mathcal{O}_L(mH) \otimes \mathcal{J}(h^m_{|L}))}{m} \\ &= \limsup_{m\to\infty} \frac{h^0(L, \mathcal{O}_L)}{m} = 0.\end{aligned}$$

Consequently the lines through $(1:0:0)$ must be contained in fibers of the numerically trivial fibration w.r.t. (L,h). On the other hand, by the characterization of numerically trivial varieties above, this fibration cannot be the projection to a point since Θ_h is not the integration current of a divisor. Hence the numerically trivial fibration w.r.t. (L,h) is the composition of the blow up of $\mathbb{P}^2$ in $(1:0:0)$ and the natural projection from this blow up to $\mathbb{P}^1$ which certainly is not almost holomorphic.

3.5. The Iitaka fibration

Let X be a projective complex manifold and L a line bundle with non-negative Kodaira-Iitaka dimension $\kappa(X,L) \geq 0$. In this section we construct a metric h for L on X such that the (L,h)-numerical trivial fibration is the Iitaka fibration $f : X \dashrightarrow Y$ of L on X. To this purpose we use a result of Takayama [**36**]:
The asymptotic multiplier ideal $\mathcal{J}(||mL||)$ is defined to be the unique maximal element among all multiplier ideals $\mathcal{J}(\frac{1}{pm_0} \cdot |pm_0 mL|)$ where m_0 is chosen such that $|m_0 mL| \neq \emptyset$ ([**27**],[**28**]). With this ideal Takayama defined intersection numbers reflecting properties of the linear sytems $|mL|$:

$$||L,C|| := \lim_{m\to\infty} m^{-1} \deg_C \left(mL \otimes \mathcal{J}(||mL||)\right),$$

where C is an irreducible curve not contained in the stable base locus

$$\mathrm{SBs}(L) := \bigcap_{m\in\mathbb{N}} \mathrm{Bs}(|mL|)$$

of L. Then he showed that such a curve C is mapped to a point by f if and only if $||L,C|| = 0$.
Now we consider the set $N(L)$ of all $m \in \mathbb{N}$ such that the linear systems $|mL| \neq \emptyset$. Let m_0 be the greatest common divisor of the numbers in $N(L)$. Then there is a positive integer $m(L)$ such that $|mm_0L| \neq \emptyset$ for all positive integers $m \geq m(L)$. Choose generating sets $f_1, \ldots, f_{k_m}$ for the linear systems $|mm_0L| \neq \emptyset$ and let h_m be the (possibly singular) hermitian metric on L with plurisubharmonic weight (on the base $\Omega \subset \mathbb{C}^n$ of a local trivialization $L \cong \Omega \times \mathbb{C}$)

$$\phi_m = \frac{1}{2mm_0} \log(\sum_{i=1}^{k_m} |f_i|^2)$$

and curvature current $\Theta_m = i\partial\overline{\partial}\phi_m$ (on Ω). Let h_L be a smooth hermitian metric on L with weight ϕ_L on Ω and smooth curvature form Θ_L. Write $\Theta_m = \Theta_L + i\partial\overline{\partial}\phi'_m$

and normalize the ϕ'_m by subtracting (if necessary) a positive constant C_m such that $\sup \phi'_m \leq 0$ (this is possible because ϕ'_m is defined on the compact manifold X hence bounded from above). Then take the upper semicontinuous upper envelope ϕ' of the ϕ'_m and call h the (singular) hermitian metric on L given by the plurisubharmonic weight $\phi = \phi_L + \phi'$. By construction, ϕ' has the singularities exactly at the stable base locus $\mathrm{SBs}(L)$ of L.
To prove that the Iitaka fibration is (up to birational equivalence) the same as the numerically trivial fibration w.r.t. (L, h) first compare Tsuji's and Takayama's intersection numbers:

LEMMA 3.9. *With L, h as above,*

$$(L, h).C \leq ||L, C||$$

for smooth irreducible curves C not contained in a set of Lebesgue measure zero.

PROOF. To begin with, one has to relate the multiplier ideals $\mathcal{J}(c \cdot |mm_0L|)$ of the linear system $|mm_0L|$ and the positive rational number c with the (analytic) multiplier ideals $\mathcal{J}(\phi_m)$. The ideal $\mathcal{J}(c \cdot |mm_0L|)$ is defined via a log resolution, but since ϕ_m is a plurisubharmonic function with analytic singularities defined by generating elements of $|mm_0L|$, it follows that

$$\mathcal{J}(c \cdot mm_0\phi_m) = \mathcal{J}(c \cdot |mm_0L|)$$

by [**14**, (5.9)]. Consequently,

$$\begin{aligned} (3.5.1) \qquad ||L, C|| &= L.C + \lim_{m\to\infty} m^{-1} \max_{p\in\mathbb{N}} \deg_C \mathcal{J}(\frac{1}{m_0p}|m_0pmL|) = \\ &= L.C + \lim_{m\to\infty} m^{-1} \lim_{p\to\infty} \deg_C \mathcal{J}(m\phi_{mp}) \\ &= L.C + \lim_{m\to\infty} m^{-1} \lim_{n\to\infty} \deg_C \mathcal{J}(m\phi_n). \end{aligned}$$

The last equality is true because $\mathcal{J}(m\phi_n) \subset \mathcal{J}(m\phi_{n+1})$ for all n: The multiplier ideals do not depend on the generating set used to define ϕ_n. By multiplying the generators defining ϕ_n with a section in $H^0(X, m_0L)$ and completing this set to a generating set of $H^0(X, (n+1)m_0L)$ it is possible to choose $\phi_n \leq \phi_{n+1}$, hence the inclusion.
Next, Tsuji's intersection number may be expressed as

$$(L, h_n).C = L.C + \limsup_{m\to\infty} m^{-1} \deg_C \mathcal{J}(m\phi_n),$$

by (2.2.3) and the fact that h_n is a metric with analytic singularities, hence restriction to C and taking the multiplier ideal in the $\limsup$ above may be interchanged on smooth curves (Prop. 2.15). The inclusion $\mathcal{J}(m\phi_n) \subset \mathcal{J}(m\phi_{n+1})$ shows that

$$\begin{aligned} \lim_{n\to\infty} (L, h_n).C &= L.C + \lim_{n\to\infty} \limsup_{m\to\infty} m^{-1} \deg_C \mathcal{J}(m\phi_n) \\ &\leq L.C + \lim_{m\to\infty} m^{-1} \lim_{n\to\infty} \deg_C \mathcal{J}(m\phi_n) = ||L; C||. \end{aligned}$$

On the other hand, $(L, h_n).C = L.C - \sum_{x \in C} \nu(h_{n|C}, x)$ by 2.2. Since the upper semicontinuous upper envelope ϕ' of the ϕ'_m equals $\sup_m \phi'_m$ outside a set of Lebesgue measure zero ([**30**]), the envelope of the restrictions $\phi'_{m|C}$ equals almost everywhere the restriction $(\phi'_m)_{|C}$ on all curves outside a Lebesgue zero set. For these curves the lemma follows from the next statement, using the definition of Lelong numbers via integrals ([**14**, (2.7)]). □

LEMMA 3.10. *Let $C \subset X$ be a smooth curve not contained in $\{x \in X : \sup_m \phi'_m(x) < \phi'(x)\}$. Then for all $x \in C$*

$$\lim_{n \to \infty} \nu(h_{n|C}, x) \geq \nu(h_{|C}, x).$$

PROOF. By definition of Lelong numbers, $\nu(\phi, x) \geq \nu(\psi, x)$ if $\phi \leq \psi$. Consequently, by the same construction as for the inclusion $\mathcal{J}(m\phi_n) \subset \mathcal{J}(m\phi_{n+1})$, the Lelong numbers $\nu(h_{n|C}, x)$ of the ϕ_n form a decreasing sequence of non-negative numbers in every point $x \in C$ whose limit is $\geq \nu(h_{|C}, x)$. It remains to show the equality:
If z is a local parameter of C centered in x, the function ϕ'_n may locally on C be written as

$$\phi'_n(z) = \phi_n(z) - \phi_L(z) - C_n = \nu(h_{n|C}, 0) \log|z| + d_n \log(1 + \sum_{i=0}^{\infty} a_i |z|^i) - \phi_L(z) - C_n$$

for some real number d_n. For every $\epsilon > 0$ and a sufficiently small neighborhood of 0 it is true that

$$d_n \log(1 + \sum_{i=0}^{\infty} a_i |z|^i) - \phi_L(z) - C_n \leq -\epsilon \log|z|,$$

hence $\phi'_n(z) \leq (\nu(h_{n|C}, 0) - \epsilon) \log|z|$, which implies

$$\phi'(z) \leq (\lim_{n \to \infty} \nu(h_{n|C}, 0) - \epsilon) \log|z|$$

for almost all z around 0. Consequently, $\nu(\phi', 0) \geq \lim_{n \to \infty} \nu(h_{n|C}, 0) - \epsilon$ for all $\epsilon > 0$, and the equality follows. □

This already implies that the Iitaka fibration of L is contained in Tsuji's numerically trivial fibration for (L, h): Take a birational morphism $\mu : X' \to X$ from a smooth projective variety X' such that the Iitaka fibration induced by the linear system $|m\mu^* L|$ is a morphism $f : X \to Y$ on another smooth variety Y. The general fiber of this fibration is smooth. Smooth varieties are numerically trivial w.r.t. (L, h) iff $(L, h).C = 0$ for all sufficiently general smooth curves in this variety, by the Reduction map 3.2. Hence by the above inequality the numerically trivial fibration w.r.t. $(\mu^* L, \mu^* h)$ contains the Iitaka fibration. By birational equivalence of intersection numbers (Prop. 2.12), the numerically trivial fibration w.r.t. $(\mu^* L, \mu^* h)$ is birationally equivalent to that on X w.r.t. (L, h).

Next note that there is a positive integer m such that the Iitaka fibration of L is induced by the linear system $|mL|$ [**26**, 10.3]. This map $\phi_{|mL|}$ induced by $|mL|$ is also the numerically trivial fibration of the metric h_m: The linear system $|mL|$ has no base points on a curve C not contained in the indefinite locus of $\phi_{|mL|}$ hence the restriction of h_m on C is smooth and strictly positive. By the Lelong number inequalities in Lemma 3.10 we conclude that numerical triviality of a sufficiently general curve w.r.t. h implies numerical triviality w.r.t. h_m. In particular the numerically trivial fibration w.r.t. h is contained in that w.r.t. h_m and we are finished.

CHAPTER 4

Moving intersection numbers

Tsuji's numerically trivial fibrations depend not only on the pseudo-effective line bundle L but also on a positive possibly singular hermitian metric h on L. But since positive metrics on L might be rather rare this sometimes leads to undesirable effects. For example there are nef line bundles L such that for no choice of a positive h the (L, h)-numerically trivial fibration is equal to the nef fibration (see section 7.2).
In this chapter we describe Boucksom's concept of moving intersection numbers which lead to a solution of this problem. His idea was to look at many metrics at the same time and even allow small negative curvature which should tend to 0 in a limit process. In this manner he was able to define non-negative intersection numbers for pseudo-effective line bundles and also a notion of volume for big line bundles having nice properties. Furthermore Boucksom introduced a useful divisorial Zariski decomposition.
We give a short survey on this circle of ideas in its natural setting of Kähler manifolds and $(1, 1)$-classes, without claiming any originality or completeness. Our main source will be Boucksom's thesis [**8**] where also most of the proofs may be found. The second section proves an approximation theorem for currents of minimal singularities which will be useful later on.

4.1. Moving intersection numbers of pseudo-effective classes

Starting with Fujita's approximate Zariski decomposition ([**24**],[**15**]) Boucksom developped a notion of volume for arbitrary pseudo-effective classes ([**7**]) on compact Kähler manifolds. The construction leading to the volume was then generalized (with small modifications) to a "moving intersection product" of pseudo-effective classes. This in turn allows the definition of a numerical dimension for pseudo-effective classes.
Logically one has to start with defining the "moving intersection numbers":

DEFINITION 4.1. *Let X be a compact Kähler manifold with Kähler form ω. Let $\alpha_1, \ldots, \alpha_p \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes and let Θ be a closed positive current of bidimension (p, p). Then the moving intersection number $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \Theta)_{\geq 0}$ of the α_i and Θ is defined to be the limit when $\epsilon > 0$ goes to 0 of*

$$\sup \int_{X-F} (T_1 + \epsilon\omega) \wedge \ldots \wedge (T_p + \epsilon\omega) \wedge \Theta$$

where the T_i's run through all currents with analytic singularities in $\alpha_i[-\epsilon\omega]$, and F is the union of the $Sing(T_i)$.

It is not difficult to justify the existence of the limit above: First, on $X - F$ the currents $T_i + \epsilon\omega$ may locally be written as $T_i + \epsilon\omega = dd^c u_i$ for some bounded plurisubharmonic function u_i. By results of Bedford-Taylor [**2**] this implies the existence of the integral. In addition Boucksom [**7**] showed that these integrals are bounded by a constant only depending on the cohomological classes $\{T_i\}$ and $\{\Theta\}$ (this is where the Kähler assumption comes in). Hence the supremum always exists, and is increasing with increasing ϵ. This implies the existence of the limit. Finally it is easy to see that this limit does not depend on the choice of the Kähler form ω.
The $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \Theta)_{\geq 0}$ are symmetric in the α_i and concave and homogeneous in every variable separately. For nef classes $\alpha_i \in H^{1,1}(X, \mathbb{R})$ the moving intersection number equals the normal cohomological intersection number $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \{\Theta\})$ [**8**]. If some of the pseudo-effective classes coincide one has

LEMMA 4.2. *For pseudo-effective classes $\alpha, \alpha_{p+1}, \ldots, \alpha_n$ the moving intersection number $(\alpha^p \cdot \alpha_{p+1} \cdot \ldots \cdot \alpha_n)_{\geq 0}$ is the limit for $\epsilon \to 0$ of*

$$\sup \int_{X-F} (T + \epsilon\omega)^p \wedge (T_{p+1} + \epsilon\omega) \wedge \ldots \wedge (T_n + \epsilon\omega)$$

where $T \in \alpha[-\epsilon\omega]$ and $T_i \in \alpha_i[-\epsilon\omega]$ have analytic singularities.

PROOF. See Lemma 3.2.7 in [**8**]. □

DEFINITION 4.3. *Let X be a compact Kähler manifold. Then the numerical dimension $\nu(\alpha)$ of a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ is defined as*

$$\max\{k \in \{0, \ldots, n\} : (\alpha^k \cdot \omega^{n-k})_{\geq 0} > 0\}$$

for some (and hence all) Kähler classes ω.

Now the volume of a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ on a compact Kähler manifold may be defined as a special case of the moving intersection product:

$$\mathrm{vol}(\alpha) = (\alpha^n)_{\geq 0}.$$

But there are other useful possibilities to define it: First remember that Fujita considered projective $n-$dimensional algebraic varieties X and line bundles L over X, and defined the volume of L by

$$\mathrm{vol}(L) := \limsup_{k \to +\infty} \frac{n!}{k^n} h^0(X, kL).$$

If L is nef, the volume of L is the self-intersection L^n, by Riemann-Roch and $h^q(X, kL) \sim O(k^{n-q})$ ([**14**, (6.7)]). For arbitrary pseudo-effective classes

$\alpha \in H^{1,1}(X, \mathbb{R})$ on compact Kähler manifolds X Boucksom generalized this volume by defining

$$\mathrm{vol}(\alpha) = \sup \int_X T_{ac}^n$$

where the supremum is taken over all closed positive $(1,1)-$currents T with $\{T\} = \alpha$ and T_{ac} is the absolute continuous part of the Lebesgue decomposition $T = T_{ac} + T_{sg}$. Again, the Kähler assumption is necessary to guarantee that T_{ac}^n is locally integrable. By using singular Morse inequalities and the Calabi-Yau theorem Boucksom proved that $\mathrm{vol}(L) = \mathrm{vol}(c_1(L))$ and that $\mathrm{vol}(L) > 0$ iff L is a big line bundle, i.e. iff there is a closed strictly positive current representing $c_1(L)$.
Note that it is not necessary to look at all closed positive $(1,1)-$currents for taking the supremum. This is a consequence of an approximation theorem of Demailly:

THEOREM 4.4 ([**19**]). *Let $T = \theta + dd^c\phi$ be a closed almost positive $(1,1)-$current on a complex manifold X with hermitian metric ω such that θ is a smooth form. Suppose that $T \geq \gamma$ for some real $\mathcal{C}^\infty-$form γ. Then there exists a decreasing sequence ϕ_k of almost plurisubharmonic functions with analytic singularities such that the $T_k := \theta + dd^c\phi_k$ verify*

(i) *The ϕ_k converge pointwise and L^1_{loc} against ϕ, hence the T_k converge weakly against T.*
(ii) *$T_k \geq \gamma - \epsilon_k\omega$ for some sequence of positive numbers $\epsilon_k \to 0$.*
(iii) *The Lelong numbers $\nu(T_k, x)$ converge uniformly against $\nu(T, x)$ w.r.t. $x \in X$.*

Using another approximation theorem ([**13**]) Boucksom slightly modified this statement ([**7**]):

THEOREM 4.5. *Let the assumptions and notations be the same as in the theorem before. Then there exists a decreasing sequence ϕ_k of almost plurisubharmonic functions with analytic singularities such that the $T_k := \theta + dd^c\phi_k$ verify*

(i) *The T_k converge weakly against T, and $T_{k,ac} \to T_{ac}$ almost everywhere.*
(ii) *$T_k \geq \gamma - \epsilon_k\omega$ for some sequence of positive numbers $\epsilon_k \to 0$.*
(iii) *The Lelong numbers $\nu(T_k, x)$ converge uniformly against $\nu(T, x)$ w.r.t. $x \in X$.*

So one may define instead

$$\mathrm{vol}(\alpha) = \lim_{\epsilon \to 0^+} \sup \int_X T_{ac}^n$$

where the T's run through all closed $(1,1)-$ currents with analytic singularities in $\alpha[-\epsilon\omega]$, that is $\{T\} = \alpha$ and $T \geq -\epsilon\omega$ for some hermitian metric ω on X.

Here, closed $(1,1)-$ currents with analytic singularities are currents whose almost plurisubharmonic potentials locally look like

$$\frac{\alpha}{2}\log(|f_1|^2+\ldots+|f_p|^2)$$

with $f_1,\ldots,f_n$ holomorphic, up to a bounded $\mathcal{C}^\infty-$function. Such currents T are particularly useful because their absolut continuous part is the same as the residual part R in the Siu-decomposition $T=\sum_i a_i[D_i]+R$. Consequently, one may compute $\int_X T_{ac}^n$ by blowing up the (integral closure) of the ideal of singularities locally generated by the f_i and integrating the smooth form given by the pull back of T minus the integration currents of the exceptional divisors as they occur in the inverse image of the singularity ideal. In Fujita's setting this corresponds to blowing up the base locus of the multiples mL and decomposing the pull back of L into an effective part E_m and a free part D_m, and Fujita's theorem [**14**, (14.6)] tells us that

$$\mathrm{vol}(L)=\lim_{m\to\infty} D_m^n.$$

Finally, the last definition of $\mathrm{vol}(\alpha)$ is equivalent to the first one, with moving intersection numbers, by Lemma 4.2.

4.2. Currents with minimal singularities

In the notions of moving intersection numbers etc. introduced above it is necessary to take limits over all currents $\geq -\epsilon\omega$ in a pseudo-effective class α. Often it is enough to take limits over currents with minimal singularities or sequences of currents approximating them.

DEFINITION 4.6. *Let ϕ_1 and ϕ_2 be two almost plurisubharmonic functions on a complex manifold X. Then ϕ_1 is said to be less singular than ϕ_2 in $x\in X$ iff*

$$\phi_2\leq\phi_1+O(1)$$

in a neighborhood of X. The fact that ϕ_1 is less singular than ϕ_2 in every point is denoted by $\phi_1\preceq\phi_2$.

Now let X be compact Kähler and $\alpha\in H^{1,1}(X,\mathbb{R})$. Let θ be a smooth $(1,1)-$form representing α. Then every current in α may be written as $T=\theta+dd^c\phi$ for some almost plurisubharmonic function ϕ and

$$T_1\preceq T_2$$

shall denote the fact that $\phi_1\preceq\phi_2$.

PROPOSITION 4.7. *Let γ be a smooth $(1,1)-$form on X. Every non-empty subset of $\alpha[\gamma]$ admits a lower bound in $\alpha[\gamma]$ w.r.t. $\preceq$.*

PROOF. The proof is almost trivial and of course contained in [**18**] but is repeated for emphasizing a certain uniqueness property.
Let $(T_i)_{i\in I}$ be the given subset of $\alpha[\gamma]$. Write $T_i = \theta + dd^c\phi_i$ where ϕ_i is almost plurisubharmonic and $dd^c\phi_i \geq \gamma - \theta$. Since X is compact, all almost plurisubharmonic functions are bounded from above hence one may suppose that $\phi_i \leq 0$ by subtracting a constant. If one choose this constant such that $\sup_{x\in X}\phi_i(x) = 0$ the ϕ_i will be unique: An almost plurisubharmonic function ϕ with $dd^c\phi = 0$ is a holomorphic function.
The ϕ_i have an almost plurisubharmonic upper envelope ϕ such that $\theta + dd^c\phi \in \alpha[\gamma]$. The current $T = \theta + dd^c\phi$ is obviously a lower bound for the $(T_i)_{i\in I}$, with the following property: If $S \preceq T_i$ for all I, then $S \preceq T$. $\square$

Remark. The construction above shows that this lower bound $T = T_{min}$ is unique only up to L^∞. On the other hand, given the smooth $(1,1)-$ form θ in α, the construction leads to a well defined current $T_{min} = \theta + dd^c\phi_{min}$ via the upper envelope. Here, the almost plurisubharmonic function ϕ_{min} satisfies $\phi_i \leq \phi_{min}$ where the ϕ_i are chosen as above.
This current will be used in the following.

The currents with minimal singularities may be used to define minimal multiplicities of pseudo-effective classes, having a look at Boucksom's construction of higher dimensional Zariski decompositions [**9**]. In this paper, he interpreted the Lelong numbers of a current $T_{min,\epsilon}$ with minimal singularities in $\alpha[-\epsilon\omega]$ as the obstructions to reach smooth currents in $\alpha[-\epsilon\omega]$. This led him to

DEFINITION 4.8. *The minimal multiplicity of a pseudo-effective class $\alpha \in H^{1,1}(X,\mathbb{R})$ in $x \in X$ is defined as*

$$\nu(\alpha, x) := \sup_{\epsilon>0} \nu(T_{min,\epsilon}, x).$$

The generic minimal multiplicity on a prime divisor $D \subset X$ is defined as

$$\nu(\alpha, D) := \inf_{x\in D} \nu(\alpha, x).$$

Denoting by T_{min} a current with minimal singularities in $\alpha[0]$ one has always

$$\nu(\alpha, x) \leq \nu(T_{min}, x),\ \nu(\alpha, D) \leq \nu(T_{min}, D).$$

There are examples where $\nu(\alpha, D) < \nu(T_{min}, D)$, see section 7.2.
The following approximation of $T_{\min}$ will be useful later on:

THEOREM 4.9. *Let X be a compact Kähler manifold with Kähler form ω, let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a pseudo-effective class. Then there exists a sequence of closed $(1,1)-$currents T_k with analytic singularities in $\alpha[-\epsilon_k\omega]$ for some sequence $(\epsilon_k) \to 0$ of positive real numbers such that*

(i) *the T_k converge weakly against a closed positive $(1,1)-$current T which has minimal singularities in $\alpha[0]$,*

(ii) $\nu(T_k, x) \to \nu(\alpha, x)$ *for every point* $x \in X$,
(iii) *for all* i

$$\int_{X-Sing(T_k)} (T_k + \epsilon_k \omega)^p \wedge \omega^{n-p} \to (\alpha^p.\omega^{n-p})_{\geq 0}.$$

PROOF. To compute $(\alpha^p.\omega^{n-p})_{\geq 0}$ it is enough to determine the limit of the

$$s_\epsilon := \sup_T \int_{X-Sing(T)} (T + \epsilon\omega)^p \wedge \omega^{n-p}$$

where $T \in \alpha[-\epsilon\omega]$ has analytic singularities, by Lemma 4.2. Consequently, for each p there are two sequences $\epsilon_k \downarrow 0$, $\delta_k \to 0$ of real numbers and a sequence of closed $(1,1)-$ currents $(T_k^{(p)})_{k\in\mathbb{N}}$ with analytic singularities such that $T_k^{(p)} \in \alpha[-\epsilon_k\omega]$ and

$$s_{\epsilon_k} - \delta_k \leq \int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_k\omega)^p \wedge \omega^{n-p} \leq s_{\epsilon_k} =: s_k.$$

Now let θ be a smooth $(1,1)-$form on X representing α. Let $T_{min,k} = \theta + dd^c\phi_{min,k}$ be the current with minimal singularities in $\alpha[-\epsilon_k\omega]$ associated to θ, as described in the remark above. Since $T_k^{(p)} = \theta + dd^c\phi_k^{(p)} \in \alpha[-\epsilon_k\omega]$ this implies $\phi_k^{(p)} \leq \phi_{min,k} \leq 0$. Furthermore the $T_{min,k}$ converge weakly against a current T_{min} with minimal singularities in $\alpha[0]$.
By Demailly's Approximation Theorem 4.4 there exists a decreasing sequence of almost plurisubharmonic functions $\phi_{k,l}$ with analytic singularities converging pointwise and L^1_{loc} against $\phi_{min,k}$ such that $T_{k,l} = \theta + dd^c\phi_{k,l} \in \alpha[-\epsilon_{k,l}\omega]$ for some sequence $(\epsilon_{k,l})_{l\in\mathbb{N}} \overset{\geq}{\to} \epsilon_k$ of positive real numbers. Furthermore $\nu(T_{k,l}, x) \overset{\leq}{\to} \nu(T_{min,k}, x)$ for every point $x \in X$.
Let $\mu : Y \to X$ be a common resolution of the singularities of $T_{k,l}$ and the $T_k^{(p)}$. Then

$$\mu^*T_k^{(p)} = R_k^{(p)} + [D_k^{(p)}], \mu^*T_{k,l} = R_{k,l} + [D_{k,l}]$$

where $R_k^{(p)}, R_{k,l}$ are smooth and $D_k^{(p)}, D_{k,l}$ are effective $\mathbb{R}-$ divisors. Since the $\phi_{k,l}$ form a decreasing sequence, $\phi_k^{(p)} \leq \phi_{k,l}$ and $T_{k,l}$ is less singular than $T_k^{(p)}$. In particular $D_{k,l} \leq D_k^{(p)}$, hence the **class** $\{R_{k,l} - R_k^{(i)}\} = \{D_k^{(i)} - D_{k,l}\}$ is pseudo-effective. Consequently,

$$\int_Y (R_{k,l} + \epsilon_{k,l}\mu^*\omega) \wedge (R_k^{(p)} + \epsilon_{k,l}\mu^*\omega)^{p-1} \wedge \mu^*\omega^{n-p} \geq \int_Y (R_k^{(p)} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p},$$

since the integrals over the compact manifold Y only depend on the cohomology classes, and all factors besides $R_{k,l} + \epsilon_{k,l}\mu^*\omega$ and $R_k^{(p)} + \epsilon_{k,l}\mu^*\omega$ are smooth. Iterating gives

$$\int_Y (R_{k,l} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p} \geq \int_Y (R_k^{(p)} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p}.$$

Noting that

$$\int_Y (R_{k,l} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p} = \int_{X-Sing(T_{k,l})} (T_{k,l} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p}$$

and similarly for $R_k^{(p)}$ and $T_k^{(p)}$ one finally gets

$$\int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p} \leq \int_{X-Sing(T_{k,l})} (T_{k,l} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p}.$$

Since $\epsilon_{k,l} \to \epsilon_k$ the same line of arguments shows

$$\int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p} \to \int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_k\omega)^p \wedge \omega^{n-p}.$$

For l big enough (depending on k) this gives

$$s_k - \delta_k \leq \int_{X-Sing(T_{k,l})} (T_{k,l} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p} \leq s_{k+1}.$$

Combining all these facts one gets a sequence of closed positive $(1,1)-$ currents $T_k = T_{k,l(k)}$ with analytic singularities in $\alpha[-\epsilon_{k+1}\omega]$ such that the T_k converge weakly against T_{min}, and conditions (ii) and (iii) of the theorem are also satisfied. $\square$

Remark. As long as $T_{k,min} \to T_{min}$ weakly for $k \to \infty$, in the construction above it is not necessary that the $T_{k,min}$ are computed w.r.t. the same smooth $(1,1)-$form on α.

The approximation may be used e.g. to prove

LEMMA 4.10. *Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R})$ a pseudo-effective class. Let $\Delta^n \cong U \subset X$ be an open subset, and let $p : \Delta^n \to \Delta^{n-1}$ be the projection onto the last $n-1$ coordinates. Then there is a pluripolar set $E \subset \Delta^{n-1}$ such that for all fibers Δ over points in $\Delta^{n-1} \setminus E$*

$$\lim_{\epsilon \downarrow 0} \inf_T \nu(T_{|\Delta}, x) = \nu(\alpha, x) \text{ for all } x \in \Delta,$$

where the T's run through all currents in $\alpha[-\epsilon\omega]$ with analytic singularities, for which the restriction to Δ is well-defined.

PROOF. The proof is an application of the theory of (L,h)- general curves generalized to almost positive $(1,1)$- currents T on X. As in Chapter 2, a smooth curve C (compact or not) will be called T- general iff the restriction of T on C is well-defined and

(i) C intersects no codim-2-component in any of the Lelong number level sets $E_c(T)$,

(ii) C intersects every prime divisor $D \subset E_c(T)$ in the regular locus D_{reg} of this divisor, C does not intersect the intersection of two such prime divisors, and every intersection point x has the minimal Lelong number $\nu(T,x) = \nu(T,D) := \min_{z \in D} \nu(T,z)$,

(iii) for all $x \in C$, the Lelong numbers

$$\nu(T_{|\Delta}, x) = \nu(T, x).$$

Then Theorem 2.5 can be reformulated in this setting and states that in a family of curves over a smooth base there is a pluripolar subset in the base such that every curve over points outside this pluripolar set is T- general. In particular, this is true for currents T_k approximating T_{min} as in the theorem above. Since the union of countably many pluripolar sets is again pluripolar, this proves the lemma. □

CHAPTER 5

Foliations and Fibrations

In this chapter we want to introduce several constructions on possibly singular holomorphic foliations and describe the connection to fibrations. The facts gathered here will be used in the next chapters.

5.1. Operations on Foliations

Holomorphic foliations on complex manifolds are usually defined as involutive subbundles of the tangent bundle. Then the classical theorem of Frobenius asserts that through every point there is a unique integral complex submanifold [**32**]. Singular foliations may be defined as involutive coherent subsheaves of the tangent bundle which are furthermore saturated, that is their quotient with the tangent bundle is torsion free. In points where the rank is maximal one may use again the Frobenius theorem to get leaves.
Later on we use the following notation:

DEFINITION 5.1. *Let X be a complex manifold and $\mathcal{F} \subset T_X$ a saturated involutive subsheaf. Then the analytic subset*

$$\{x \in X : \mathcal{F}/m_{X,x}\mathcal{F} \to T_{X,x} \text{ is not injective}\}$$

is called the singular locus of $\mathcal{F}$ and is denoted by $\operatorname{Sing} \mathcal{F}$. *The dimension of $\mathcal{F}/m_{X,x}\mathcal{F}$ in a point $x \in X - \operatorname{Sing} \mathcal{F}$ is called rank of $\mathcal{F}$ and denoted by* $\operatorname{rk}(\mathcal{F})$.

Because $\mathcal{F}$ is saturated we have $\operatorname{codim} \operatorname{Sing} \mathcal{F} \geq 2$. The existence of leaves means that around every point $x \in X - \operatorname{Sing} \mathcal{F}$ there is an (analytically) open subset $U \subset X - \operatorname{Sing} \mathcal{F}$ with coordinates $z_1, \ldots z_n$, $n = \dim X$, such that the leaves of $\mathcal{F}$ are the fibers of the projection onto the coordinates $z_{k+1}, \ldots, z_n$ where $k = \operatorname{rk}(\mathcal{F})$. In particular the leaves have dimension $\operatorname{rk}(\mathcal{F})$.
To construct numerically trivial foliations we need a local description of several operations applied on two foliations. We start with the easiest configuration:

PROPOSITION 5.2. *Let $\mathcal{G} \subset \mathcal{F}$ be two foliations on a complex manifold X, $\operatorname{rk}(\mathcal{F}) = k$, $\operatorname{rk}(\mathcal{G}) = l$, $l < k$. Then for all $x \in X - (\operatorname{Sing} \mathcal{F} \cup \operatorname{Sing} \mathcal{G})$ there is an open neighborhood $U \subset X - (\operatorname{Sing} \mathcal{F} \cup \operatorname{Sing} \mathcal{G})$ with coordinates $z_1, \ldots, z_n$ such that the leaves of $\mathcal{F}$ are the fibers of the projection onto the last $n-k$ coordinates and the leaves of $\mathcal{G}$ are the fibers of the projection onto the last $n-l$ coordinates.*

PROOF. Let $z_1, \dots, z_n$ be local coordinates around x such that the leaves of $\mathcal{G}$ are the fibers of the projection onto the last $n-l$ coordinates. Let $z'_1, \dots, z'_n$ be local coordinates around x such that the leaves of $\mathcal{F}$ are the fibers of the projection onto the last $n-k$ coordinates. $\mathcal{G} \subset \mathcal{F}$ implies that the coordinates $z_{l+1}, \dots, z_n$ fix the coordinates $z'_{k+1}, \dots, z'_n$ which consequently do not depend on $z_1, \dots, z_l$. This implies that (after possibly reordering $z_{l+1}, \dots, z_n$) the matrix $\left(\frac{\partial z'_i}{\partial z_j}\right)_{i,j \geq k+1}$ is invertible. The theorem on implicitely defined functions shows then that

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_k, z'_{k+1}, \dots, z'_n)$$

is an invertible map in a neighborhood U of x. The new coordinates $(z_1, \dots, z_k, z'_{k+1}, \dots, z'_n)$ have the required properties since $z'_{k+1}, \dots, z'_n$ only depend on $z_{l+1}, \dots, z_n$. $\square$

Note that neither $\operatorname{Sing} \mathcal{G}$ need to be contained in $\operatorname{Sing} \mathcal{F}$ nor vice versa.

DEFINITION 5.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two foliations on a complex manifold X. Then $\mathcal{F} \cap \mathcal{G} \subset T_X$ is called the intersection foliation of $\mathcal{F}$ and $\mathcal{G}$.*

Note that $\mathcal{F} \cap \mathcal{G}$ is certainly involutive but may be not saturated: the rank of $\mathcal{F} \cap \mathcal{G}$ can even jump in $\operatorname{codim} 1$ subsets. To get a better picture in local coordinates we nevertheless think of it as a foliation and denote by $\operatorname{Sing}(\mathcal{F} \cap \mathcal{G})$ the analytic locus where the rank jumps.

PROPOSITION 5.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be two foliations on a complex manifold X with $\operatorname{rk}(\mathcal{F}) = k$, $\operatorname{rk}(\mathcal{G}) = m$ and $\operatorname{rk}(\mathcal{F} \cap \mathcal{G}) = l$. Let $x \in X$ be a point which is not singular for $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \cap \mathcal{G}$. Then there exists an open neighborhood*

$$U \subset X - (\operatorname{Sing} \mathcal{F} \cup \operatorname{Sing} \mathcal{G} \cup \operatorname{Sing}(\mathcal{F} \cap \mathcal{G}))$$

of x with coordinates $z_1, \dots, z_n$ such that

(i) *the leaves of $\mathcal{F}$ in U are the fibers of the projection on $z_{k+1}, \dots, z_n$,*
(ii) *the leaves of $\mathcal{F} \cap \mathcal{G}$ in U are the fibers of the projection on $z_{l+1}, \dots, z_n$ and*
(iii) *the leaves of $\mathcal{G}$ in U are the fibers of the projection on $z_{l+1}, \dots, z_k, z'_{m+k-l+1}, \dots, z'_n$ where the $z'_{m+k-l+j}$'s are analytic functions with $z'_{m+k-l+j|U_x} = z_{k+j}$ on*

$$U_x = \{z \in U : z_{l+1}(z) = z_{l+1}(x), \dots, z_k(z) = z_k(x)\}.$$

PROOF. Again this results from applying the theorem on implicitely defined functions several times. The geometric essence of the situation may be taken from the figure below.
To start the proof, choose coordinates $z_1, \dots, z_n$ for $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{G}$ in a neighborhood

$$V \subset X - (\operatorname{Sing} \mathcal{F} \cup \operatorname{Sing} \mathcal{G} \cup \operatorname{Sing}(\mathcal{F} \cap \mathcal{G}))$$

of x as in Prop. 5.2. Since the leaves of $\mathcal{G}$ contain the leaves of $\mathcal{F} \cap \mathcal{G}$ we can describe the leaves of $\mathcal{G}$ in V (possibly restricted) as the fibers of the projection given by analytic functions $z'_{m+1},\dots,z'_n$ only depending on $z_{l+1},\dots,z_n$. By construction, the differential of the map

$$(z_1,\dots,z_n)\mapsto(z'_{m+1},\dots,z'_n,z_{k+1},\dots,z_n)$$

has the kernel $T_{\mathcal{F}\cap\mathcal{G},x}$ in x. Hence the rank of the differential in x is $n-l$, and after possibly reordering the $z'_{m+1},\dots,z'_n$ the differential of

$$(z_1,\dots,z_n)\mapsto(z_1,\dots,z_l,z'_{m+1},\dots,z'_{m+k-l},z_{k+1},\dots,z_n)$$

has full rank in x and is invertible. Consequently,

$$z_1,\dots,z_l,z'_{m+1},\dots,z'_{m+k-l},z_{k+1},\dots,z_n$$

are coordinates in V (possibly restricted).
By construction the fibers of the projection onto $z_{k+1},\dots,z_n$ are the leaves of $\mathcal{F}$ and the fibers of the projection onto $z'_{m+1},\dots,z'_n$ are the leaves of $\mathcal{G}$. Since the differentials of

$$(z_1,\dots,z_n)\mapsto(z'_{m+1},\dots,z'_n,z_{k+1},\dots,z_n)$$

and

$$(z_1,\dots,z_n)\mapsto(z'_{m+1},\dots,z'_{m+k-l},z_{k+1},\dots,z_n)$$

have the same rank their kernel in points $y\in V$ is in both cases $T_{\mathcal{F}\cap\mathcal{G},y}$. Consequently the fibers of the projection onto $z'_{m+1},\dots,z'_{m+k-l},z_{k+1},\dots,z_n$ and the leaves of $\mathcal{F}\cap\mathcal{G}$ have the same tangent space in every point $y\in V$, hence are equal in V.
After possibly reordering $z_{k+1},\dots,z_n$ a similar argument as above shows that

$$z_1,\dots,z_l,z'_{m+1},\dots,z'_n,z_{n-m-l+1},\dots,z_n$$

are coordinates in V (possibly restricted). Let

$$a_{m+1}:=z'_{m+1}(x),\dots,a_{m+k-l}:=z'_{m+k-l}.$$

Since the differentials of

$$(z_1,\dots,z_n)\mapsto(z_1,\dots,z_l,z'_{m+1},\dots,z'_n,z_{n-m-l+1},\dots,z_n)$$

and

$$\begin{aligned}(z_1,\dots,z_n)\mapsto\\ &(z_1,\dots,z_l,z'_{m+1},\dots,z'_{m+k-l},\\ &z'_{m+k-l+1}(a_{m+1},\dots,a_{m+k-l}),\dots,z'_n(a_{m+1},\dots,a_{m+k-l}),z_{n-m-l+1},\dots,z_n)\end{aligned}$$

are equal in x, they are both invertible and

$$z_1,\dots,z_l,z'_{m+1},\dots,z'_{m+k-l},$$

$$z'_{m+k-l+1}(a_{m+1},\dots,a_{m+k-l}),\dots,z'_n(a_{m+1},\dots,a_{m+k-l}),z_{n-m-l+1},\dots,z_n$$

are coordinates in V (possibly restricted).
Since $z'_{m+k-l+1}(a_{m+1},\dots,a_{m+k-l}),\dots,z'_n(a_{m+1},\dots,a_{m+k-l})$, $z_{n-m-l+1},\dots,z_n$ and $z'_{m+k-l+1},\dots,z'_n,z_{n-m-l+1},\dots,z_n$ both do not depend on

$z_1, \dots, z_l, z'_{m+1}, \dots, z'_{m+k-l}$ their respective values are fixed by each other. Hence the fibers of the projection onto the two sets of coordinates are equal, and the same is true for the fibers of the two projections onto

$$z'_{m+1}, \dots, z'_{m+k-l}, z'_{m+k-l+1}(a_{m+1}, \dots, a_{m+k-l}), \dots, z'_n(a_{m+1}, \dots, a_{m+k-l}),$$

$$z_{n-m-l+1}, \dots, z_n$$

resp.

$$z'_{m+1}, \dots, z'_n, z_{n-m-l+1}, \dots, z_n.$$

Hence the coordinates

$$z_1, \dots, z_l, z'_{m+1}, \dots, z'_{m+k-l},$$

$$z'_{m+k-l+1}(a_{m+1}, \dots, a_{m+k-l}), \dots, z'_n(a_{m+1}, \dots, a_{m+k-l}), z_{n-m-l+1}, \dots, z_n$$

satisfy all the required properties. $\square$

For our purposes the most important operation on two holomorphic foliations $\mathcal{F}$ and $\mathcal{G}$ on a complex manifold X is the *union* $\mathcal{F} \sqcup \mathcal{G}$. We define it as the foliation given by the smallest saturated involutive subsheaf of T_X containing both $\mathcal{F}$ and $\mathcal{G}$. Such a sheaf exists because saturated foliations contained in each other have different ranks, the intersection of two foliations is again a foliation and T_X is involutive.

Besides this pure existence statement there is an inductive algebraic construction of $\mathcal{F} \sqcup \mathcal{G}$: For $\mathcal{H} \subset T_X$, let $[\mathcal{H}, \mathcal{H}]$ be the subsheaf of T_X generated by all Lie brackets of vector fields in $\mathcal{H}$. Then construct

$$\begin{aligned} \mathcal{H}_1 &:= \text{saturation of } \mathcal{F} + \mathcal{G} \\ \mathcal{H}_2 &:= \text{saturation of } \mathcal{H}_1 + [\mathcal{H}_1, \mathcal{H}_1] \\ &\vdots \end{aligned}$$

and so on until $\mathcal{H}_m = \mathcal{H}_{m+1}$ which means $[\mathcal{H}_m, \mathcal{H}_m] \subset \mathcal{H}_m$. Then $\mathcal{H}_m = \mathcal{F} \sqcup \mathcal{G}$. This is a local construction hence for open subsets $U \subset X$ we have

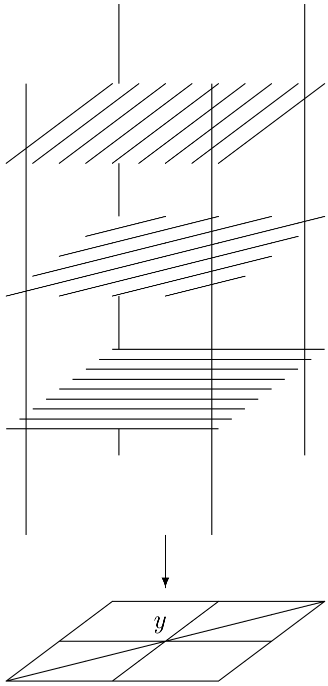
$$\mathcal{F}_{|U} \sqcup \mathcal{G}_{|U} = (\mathcal{F} \sqcup \mathcal{G})_{|U}.$$

We want to describe an inductive geometric construction of $\mathcal{F} \sqcup \mathcal{G}$ on open subsets

$$U \subset X - (\mathrm{Sing}\, \mathcal{F} \cup \mathrm{Sing}\, \mathcal{G} \cup \mathrm{Sing}\, (\mathcal{F} \cap \mathcal{G})) - Z$$

where Z is an analytic subset of $X - (\mathrm{Sing}\, \mathcal{F} \cup \mathrm{Sing}\, \mathcal{G} \cup \mathrm{Sing}\, (\mathcal{F} \cap \mathcal{G}))$. Following the inductive steps of this construction we will construct the maximal numerically α-trivial foliation in Theorem 6.5.

Start with a neighborhood U of a point $x \in X - (\mathrm{Sing}\, \mathcal{F} \cup \mathrm{Sing}\, \mathcal{G} \cup \mathrm{Sing}\, (\mathcal{F} \cap \mathcal{G}))$ having coordinates $z_1, \dots, z_n$ as in Prop. 5.4. Define a foliation $\mathcal{G}'$ on U whose leaves are the fibers of the projection on $z_{l+1}, \dots, z_{n-m+l}$. The figure below illustrates that in general $\mathcal{F} + \mathcal{G}' \neq \mathcal{F} \sqcup \mathcal{G}$ (take the fibers of the vertical projection as leaves of $\mathcal{F}$ whereas the leaves of $\mathcal{G}$ are the horizontal lines twisted around in vertical direction):



Denoting the projection on $z_{k+1}, \dots, z_n$ by $\pi_{\mathcal{F}}$ we examine instead r-tuples of points $x_1, \dots, x_r$ in fibers $\pi_{\mathcal{F}}^{-1}(y)$ of points $y \in \pi_{\mathcal{F}}(U) \subset \mathbb{C}^{n-k}$. If $T_{\mathcal{G}}(x_i) \subset T_{X,x_i}$ indicates the space of directions tangent to $\mathcal{G}$ in x_i we have a sequence of inclusions

$$0 \subset d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_1)) \subset d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_1)) + d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_2)) \subset \cdots \subset \sum_{i=1}^{r} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)) \subset T_{\mathbb{C}^{n-k},y}.$$

There is an $r \in \mathbb{N}$ and a Zariski open subset of the r-fold product

$$\pi_{\mathcal{F}}^{-1}(y) \times \cdots \times \pi_{\mathcal{F}}^{-1}(y)$$

such that

(i) all inclusions in the above sequence are strict and
(ii) $d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x')) \subset \sum_{i=1}^{r} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i))$ for every point $x' \in \pi_{\mathcal{F}}^{-1}(y)$.

Varying $y \in \pi_{\mathcal{F}}(U)$ may change the number r and the dimensions of the vector spaces

$$\sum_{i=1}^{s} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)), \ s = 1, \dots, r.$$

But again there is an analytic subset $Z_U \subset \pi_{\mathcal{F}}(U)$ such that for $y \in V := \pi_{\mathcal{F}}(U) - Z_U$ the dimensions and r remain constant. Since everything is defined intrinsically the sets

$\pi_{\mathcal{F}}^{-1}(Z_U)$ glue together to an analytic subset Z of $X - (\mathrm{Sing}\ \mathcal{F} \cup \mathrm{Sing}\ \mathcal{G} \cup \mathrm{Sing}\ (\mathcal{F} \cap \mathcal{G}))$. Furthermore for every V we can find a covering of V with open subsets $V' \subset V$ and r sections $\sigma_i : V' \to U$ of $\pi_{\mathcal{F}}$ for every V' such that

(i) the points $x_i := \sigma_i(y)$ produce a sequence of tangent subspaces $\sum_{i=1}^{s} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i))$, $s = 1, \ldots, r$, as above, and

(ii) if $\pi : U \to \mathbb{C}^{k-l}$ is the projection onto $z_{l+1}, \ldots, z_k$, the map $\pi \circ \sigma_i$ is constant.

To get the announced inductive construction of $\mathcal{F} \sqcup \mathcal{G}$ on $\pi_{\mathcal{F}}^{-1}(V')$ we need another little observation: Since the holomorphic functions z_j' defining $\pi_{\mathcal{G}}$ do not depend on $z_1, \ldots, z_l$ (see proof of Prop. 5.4) the tangent space

$$d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x))$$

does not change for different x in the intersection of a fixed $\pi_{\mathcal{F}}$- and a π-fiber. Furthermore the fibers of π consist of leaves of $\mathcal{G}$.
Now we construct inductively foliations $\mathcal{F}_i$, $i = 0, \ldots, r$, on $\pi_{\mathcal{F}}^{-1}(V')$. We start with

$$\mathcal{F}_0 := \mathcal{F} \cap \pi_{\mathcal{F}}^{-1}(V').$$

Because of the observation above the leaves of $\mathcal{G}$ in $\pi^{-1}(\pi(x_1))$ map onto the leaves of a smooth foliation $\mathcal{G}_1$ on V' which is induced by a projection $\pi_{\mathcal{G}_1}$. Put

$$\mathcal{F}_1 := \pi_{\mathcal{F}}^{-1}(\mathcal{G}_1)$$

and let $\pi_{\mathcal{F}_1} := \pi_{\mathcal{G}_1} \circ \pi_{\mathcal{F}}$ be the projection whose fibers are the leaves of $\mathcal{F}_1$.
The observation and the properties of the $x_1, \ldots, x_r$ imply that $T_{\mathcal{G}|\pi^{-1}(\pi(x_2))}$ maps onto an involutive **subbundle** of $T_{\pi_{\mathcal{F}_1}(\pi_{\mathcal{F}}^{-1}(V'))}$ and consequently the leaves of $\mathcal{G}$ in $\pi^{-1}(\pi(x_2))$ also map onto leaves of a smooth foliation $\mathcal{G}_2$ on $\pi_{\mathcal{F}_1}(\pi_{\mathcal{F}}^{-1}(V'))$. Define

$$\mathcal{F}_2 := \pi_{\mathcal{F}_1}^{-1}(\mathcal{G}_2)$$

and continue inductively setting

$$\mathcal{F}_i := \pi_{\mathcal{F}_{i-1}}^{-1}(\mathcal{G}_i)$$

where $\mathcal{G}_i$ is the image of the leaves of $\mathcal{G}$ in $\pi^{-1}(\pi(x_i)$ on $\pi_{\mathcal{F}_{i-1}}(\pi_{\mathcal{F}}^{-1}(V'))$.
By construction these foliations $\mathcal{F}_s$ have as tangent space in a point $x \in \pi_{\mathcal{F}}^{-1}(V')$

$$d\pi_{\mathcal{F}}(x)^{-1}(\sum_{i=1}^{s} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i))$$

where $\pi_{\mathcal{F}}(x_i) = \pi_{\mathcal{F}}(x)$ for all i. In addition $\mathcal{F}_r$ contains all leaves of $\mathcal{F}$ and $\mathcal{G}$ in $\pi_{\mathcal{F}}^{-1}(V')$: otherwise there is a point $y \in V'$ and a point $x \in \pi_{\mathcal{F}}^{-1}(y)$ such that

$$d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x)) \not\subset \sum_{i=1}^{r} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)),$$

$\pi_{\mathcal{F}}(x_i) = \pi_{\mathcal{F}}(x)$ for all i.

On the other hand $T_{\mathcal{F}\sqcup\mathcal{G}}(x)$ must contain every tangent subspace

$$d\pi_{\mathcal{F}}(x)^{-1}(T_{\mathcal{G}}(x'))$$

of points x' with $\pi_{\mathcal{F}}(x') = \pi_{\mathcal{F}}(x)$ since $\pi_{\mathcal{F}}^{-1}\pi_{\mathcal{F}}(\pi_{\mathcal{G}}^{-1}\pi_{\mathcal{G}}(x'))$ is contained in a leaf of $\mathcal{F} \sqcup \mathcal{G}$. Consequently, $d\pi_{\mathcal{F}}(x)^{-1}(\sum_{i=1}^{r} d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)) \subset T_{\mathcal{F}\sqcup\mathcal{G}}(x)$ and on $\pi_{\mathcal{F}}^{-1}(V')$ we have

$$\mathcal{F} \sqcup \mathcal{G} = \mathcal{F}_r.$$

5.2. Foliations induced by Fibrations

An important type of foliations are those induced in a unique way by birational maps $f : X \dashrightarrow Y$ from a projective complex manifold X to another projective complex manifold Y. We need some preparations to describe this construction.

LEMMA 5.5. *Let $\mathcal{F}, \mathcal{G} \subset T_X$ be two foliations on a projective complex manifold X and $U \subset X$ a Zariski open subset in X. If $\mathcal{F}_{|U} = \mathcal{G}_{|U} \subset T_U$ then $\mathcal{F} = \mathcal{G}$.*

PROOF. This statement is already true for saturated subsheaves $\mathcal{F}, \mathcal{G} \subset \mathcal{E}$ in a locally free sheaf $\mathcal{E}$ of rank n on X. It is a local statement, so we can prove it on affine open subsets $\mathrm{Spec}(R) \cong V \subset X$ such that R is an integral domain, $\mathcal{E}$ is free on V and $\mathcal{F}, \mathcal{G}$ are the sheafifications of the R-modules $F, G \subset R^n$. Furthermore we can assume that $U \cong \mathrm{Spec}(R_f)$ is a principal open subset of V w.r.t. some non-zero element $f \in R$. Then the assumptions of the lemma tell us that $F_f = G_f \subset (R_f)^n$.
Let $(r_1, \ldots, r_n) \in F$. Because of $F_f = G_f$ there is a $(r'_1, \ldots, r'_n) \in G$ and a $k \in \mathbb{N}$ such that

$$\frac{1}{f^k}(r'_1, \ldots, r'_n) = (r_1, \ldots, r_n).$$

This implies

$$r'_i = f^k \cdot r_i,\ i = 1, \ldots, r_n.$$

If the minimal possible k in these considerations is ≥ 1 we conclude $(r_1, \ldots, r_n) \notin G$ but $f^k \cdot (r_1, \ldots, r_n) \in G$. Consequently, the residue class of $(r_1, \ldots, r_n)$ in R^n/G is a torsion class. This contradicts $G \subset R^n$ saturated.
In the same way we conclude $G \subset F$, and the statement follows. □

LEMMA 5.6. *Let X be a projective complex manifold and $U \subset X$ a Zariski open subset. Then for any foliation $\mathcal{F}_U \subset T_U$ on U there exists a unique foliation $\mathcal{F} \subset T_X$ such that $\mathcal{F}_{|U} = \mathcal{F}_U$.*

PROOF. Since T_X is torsion free the Lie bracket of two vector fields is completely determined on Zariski open subsets. Hence a saturated subsheaf $\mathcal{F} \subset T_X$ that is involutive on a Zariski open subset U is a foliation on X. Consequently it is enough to show: If $\mathcal{F}_U \subset T_U$ is a saturated subsheaf there exists a unique saturated extension $\mathcal{F} \subset T_X$ such that $\mathcal{F}_{|U} = \mathcal{F}_U$.

As in the proof of the previous lemma we can reduce the problem to extending a module F_{R_f} saturated in R_f^n, $f \in R$ and R an integral domain. Set $F := F_{R_f} \cap R^n$. Then F is a saturated submodule of R^n: If $g \cdot m \in F$ for some $0 \neq g \in R$, $m \in R^n$, then $\frac{g}{1} \cdot \frac{m}{1} \in F_{R_f}$. Hence $\frac{m}{1} \in F_{R_f}$ since F_{R_f} is saturated in R_f^n, and consequently $m \in F$. Finally, $F_f = F_{R_f} \subset R_f^n$: If $\frac{m}{f^k} \in F_f$ for some $m \in F \subset F_{R_f}$ then $\frac{m}{f^k} \in F_{R_f}$. Conversely, if $\frac{m}{f^k} \in F_{R_f}$ then

$$\frac{f^k}{1} \cdot \frac{m}{f^k} = \frac{m}{1} \in F_{R_f} \cap R = F.$$

□

Now we can define the foliation induced by a rational map $f : X \dashrightarrow Y$ where X, Y are projective complex manifolds: Let $U \subset X$ be a Zariski open subset where f is regular and smooth. Then the relative tangent bundle $T_{U/Y} \subset T_U$ defines a foliation on U.

DEFINITION 5.7. *The foliation $\mathcal{F}$ induced by $f : X \dashrightarrow Y$ is the extension of $T_{U/Y}$ to a foliation on X.*

This foliation exists by Lemma 5.6 and is unique by Lemma 5.5.
Since the properties defining U above are local the Stein factorization $X \dashrightarrow Y' \dashrightarrow Y$ does not change the induced foliation. Note furthermore that the general fibers of f are irreducible if they are already connected: These fibers are isomorphic to the fibers of the projection $\pi_f : \Gamma_f \to Y$ from the irreducible graph $\Gamma_f \subset X \times Y$ of f to Y. If $\widetilde{\Gamma}_f$ is the desingularization of Γ_f the general fibers of $\widetilde{\Gamma}_f \to Y$ are smooth by [**4**] and birational to the general fiber of π_f. Consequently the general fibers of π_f are irreducible if they are already connected.

PROPOSITION 5.8. *Let X be a projective complex manifold and $f : X \dashrightarrow Y_1$, $g : X \dashrightarrow Y_2$ two rational surjective maps with induced foliations $\mathcal{F}$ and $\mathcal{G}$ on X. Then $\mathcal{F} \sqcup \mathcal{G}$ is also induced by a rational map $h : X \dashrightarrow Z$.*

This proposition is a consequence of the following more general construction:

DEFINITION 5.9. *Let X be a compact Kähler manifold. A covering family $(C_t)_{t \in T}$ of closed complex subspaces in X parametrized by a compact complex base space T is called generically connecting iff for any analytic subset $Z \subset X$ two general points are connected by a finite sequence of elements in (C_t) such that two subsequent elements do not intersect in Z.*
A meromorphic map $f : X \dashrightarrow Y$ is called the generic reduction map with respect to a covering family $(C_t)_{t \in T}$ of closed complex subspaces in X iff the general fibers are generically C_t-connected and every element of (C_t) is contained in a fiber. Here, fibers of f are defined via the graph of f.

THEOREM 5.10. *Let $(C_t)_{t \in T}$ be a covering family of closed complex subspaces in a compact Kähler manifold X given by a closed complex subspace $C \subset T \times X$*

over a compact complex base space T. Suppose that the irreducible components of C correspond bijectively to the connected components of T and project surjectively on X and the associated connected component of T. Suppose furthermore that the general fiber C_t is irreducible.
Then there exists a generic reduction map $f : X \dashrightarrow Y$ for $(C_t)_{t\in T}$.

PROOF. Starting with $C^{(0)} := C$ and $T^{(0)} := T$ we will inductively construct a complex space $T^{(k)}$ and a subspace $C^{(k)} \subset T^{(k)} \times X$ from $C^{(k-1)}$ and $T^{(k-1)}$ such that $C^{(k)}$ and $T^{(k)}$ satisfy the same assumptions as C and T, there exists a finite surjective map $\pi_k : T^{(k)} \to T^{(k-1)}$ and, denoting the fiber over $t \in T$ with $C_t^{(k)}$,

(i) $C_t^{(k)} \supset C_{\pi_k(t)}^{k-1}$ for a general point $t \in T^{(k)}$.
(ii) $C_t^{(k)} = C_{\pi_k(t)}^{(k-1)}$ for general points t in all connected components of $T^{(k)}$ implies that all fibers $C_t^{(k)}$ through a general point $x \in X$ are isomorphic.
(iii) If $Z \subset X$ is a complex subspace (possibly empty) then for general $t \in T^{(k)}$, $x', x'' \in C_t^{(k)}$ there exist $t', t'' \in T^{(k-1)}$, $\overline{x}' \in C_{t'}^{(k-1)} \cap C_{\pi_k(t)}^{(k-1)}$ and $\overline{x}'' \in C_{\pi_k(t)}^{(k-1)} \cap C_{t''}^{(k-1)}$ such that

$$x' \in C_{t'}^{(k-1)}, x'' \in C_{t''}^{(k-1)} \text{ and } \overline{x}', \overline{x}'' \notin Z.$$

Then properties (i) and (ii) imply that after $k \leq \dim X$ steps, $C^{(k)}$ defines a meromorphic map $f : X \dashrightarrow Y$ with the same general fibers as $C^{(k)}$: Let $T^\circ \subset T$ the open locus where the family $C^{(k)} \to T$ is flat. Take Y as the closure and desingularisation of the image of T° under the map to the Douady space of X. The meromorphic map f is induced by the projection from the universal family. Finally (iii) shows that the fibers of f are generically $(C_t^{(k)})$-connected and hence by induction generically (C_t)-connected.
To start the construction, denote the projection of $T \times X$ to T by $p : T \times X \to T$ and the projection from $T \times X$ to X by $q : T \times X \to X$. Denote the restrictions of p and q to C by p_C and q_C. Then the p_C-fiber over $t \in T$ is just $C_t \in X$. Every q_C-fiber over some points $x \in X$ decomposes into not necessarily connected components

$$q_C^{-1}(x) \cap p^{-1}(T_i),$$

where T_i is any connected component of T. For every T_i there exists an $n_i \in \mathbb{N}$ such that for general points $x \in X$ the set $q_C^{-1}(U) \cap p^{-1}(T_i)$ decomposes into n_i irreducible components. Finally, the T_i are irreducible because by assumption the T_i are images of irreducible spaces.
To construct $C^{(1)}$, consider the product $T \times X \times X \times T$ and its projections p_1, p_2, p_3, p_4 onto the subsequent factors. A point (t_1, x_1, x_2, t_2) of the intersection

$$S := (p_1 \times p_2)^{-1}(C) \cap (p_2 \times p_3 \times p_4)^{-1}(C \times_T C) \subset T \times X \times X \times T$$

satisfies $x_1 \in C_{t_1}$, $x_1, x_2 \in C_{t_2}$.

Next, fix a general point $x \in X$, an irreducible component $T_i \subset T$ and enumerate the irreducible components $T_{i,x,k}$ of $q_C^{-1}(x) \cap p^{-1}(T_i)$, $k = 1, \ldots, n_i$. It is possible to extend this enumeration to the irreducible components $T_{i,x',k}$ of $q_C^{-1}(x') \cap p^{-1}(T_i)$ for points x' in a Zariski open subset $U_i \subset X$ in such a way that the $T_{i,x',k}$, $x' \in U_i$, form an irreducible analytic subset of C. Furthermore, the Zariski closure of this subset must be $C \cap p^{-1}(T_i)$, since this is an irreducible component of C.

CLAIM 1. *Each tripel $(T_i, T_{i,x,k}, T_j)$ of two connected components $T_i, T_j \subset T$ and an irreducible component $T_{i,x,k}$, $1 \leq k \leq n_i$, of $q_C^{-1}(x) \cap p^{-1}(T_i)$ determines exactly one irreducible component $S_{ijk} \subset S$ such that $p_2(S_{ij}) = X$, $p_1(S_{ij}) = T_i$, $p_4(S_{ij}) = T_j$, and the fiber over every point $(x', x_2, t_2) \in (p_2 \times p_3 \times p_4)(S)$, $x' \in U_i$, contains an irreducible component isomorphic to $T_{i,x',k}$.*

PROOF. The assumptions imply that over any irreducible component R_j of

$$(p_2 \times p_3 \times p_4)(S) \subset X \times X \times T$$

such that $p_2(R_j) = X$ and $p_4(R_j) = T_j$ there exists exactly one irreducible component S_{ijk} with the required properties.
By the properties of C we have for such R_j's that

$$(p_2 \times p_4)(R_j) = C \cap p^{-1}(T_j) \subset C,$$

and the fiber over $(x_1, t_2) \in (p_2 \times p_4)(R_j)$ in R_j is the irreducible subset $C_{t_2} \subset X$ if $t_2 \in T_j$ is general. Consequently, R_j is uniquely determined by T_j and hence S_{ijk} is uniquely determined by T_i, T_j. $\square$

Set

$$T^{(1)} := \dot{\bigcup_{i,j,k}} T_i,$$

which contains a copy of T_i for every irreducible component S_{ijk} as above. Set

$$C^{(1)} := \dot{\bigcup_{i,j,k}} (p_1 \times p_3)(S_{ijk}) \subset T^{(1)} \times X.$$

By construction, the irreducible components of $C^{(1)}$ correspond bijectively to the connected components of $T^{(1)}$ and project surjectively on X and the associated connected component of $T^{(1)}$. Furthermore there exists a natural finite map $\pi_1 : T^{(1)} \to T^{(0)} = T$.

CLAIM 2. *For a general point $t \in T^{(1)}$ we have $C_t^{(1)} \supset C_{\pi_1(t)}$.*

PROOF. Suppose that t lies in the connected component $T_{ij}^{(1)}$ of $T^{(1)}$ corresponding to S_{ijk}. Then $\pi_1(t) \in T_i$. For a general point $x \in C_{\pi_1(t)}$ choose a general $t' \in q_C^{-1}(x) \cap p^{-1}(T_j)$. Since t, t' and x are general, $(\pi_1(t), x, x, t') \in S_{ijk}$, by construction. Consequently $(\pi_1(t), x) \in (p_1 \times p_3)(S_{ij})$ and $x \in C_t^{(1)}$. Since $C_t^{(1)}$ and $C_{\pi_1(t)}$ are irreducible this implies the claimed inclusion. $\square$

Claim 2 implies property (i) for $C^{(1)}$. Property (ii) follows from

CLAIM 3. *If $C_t^{(1)} = C_{\pi_1(t)}$ for general points $t \in T_{ijk}^{(1)}$ then*

$$C_{t'} \supset C_{t''}$$

for general points $x \in X$ and all points $t' \in p(T_{i,x,k})$ and $t'' \in p(q_C^{-1}(x) \cap p^{-1}(T_j))$.

PROOF. Since π_1 is an isomorphism on $T_{ijk}^{(1)}$ and the component of $C^{(1)}$ over $T_{ijk}^{(1)}$ is irreducible, $C_t^{(1)} = C_{\pi_1(t)}$ holds for all $t \in T_{ijk}^{(1)}$. By construction and assumption,

$$(p_1 \times p_3)(S_{ijk}) \cap p^{-1}(\pi_1(t)) = C_t^{(1)} \subset C_{\pi_1(t)}.$$

Then for all $(\pi_1(t), x, x_2, t'') \in S_{ijk}$ it is true that $x_2 \in C_{\pi_1(t)}$. But the fiber of S_{ijk} over $(\pi_1(t), x, t'')$ is $C_{t''}$ for general points $x \in X$, as the construction of S_{ijk} in Claim 1 shows. Consequently, $C_{t''} \subset C_{t'}$. □

Finally we show property (iii) for $C^{(1)}$: For $t \in T_{ijk}^{(1)}$ and $x', x'' \in C_t^{(1)}$ there exist points

$$(\pi_1(t), \overline{x}', x', t'), (\pi_1(t), \overline{x}'', x'', t'') \in S_{ijk} \subset S.$$

Consequently we only have to assure that for general t, x', x'' the points $\overline{x}', \overline{x}'' \in X$ can be chosen outside Z. But if the fiber of the projection $p_1 \times p_3$ in S_{ijk} over $(\pi_1(t), x')$ is always contained in $Z \times T_j$ then $S_{ijk} \subset T \times Z \times X \times T$, and this contradicts $p_2(S_{ijk}) = X$.
Since the construction of $C^{(1)}$ and $T^{(1)}$ and the proof of properties (i), (ii), (iii) do not rely on properties (i), (ii), (iii) for C and T it is possible to construct $C^{(k)}$ and $T^{(k)}$ from $C^{(k-1)}$ and $T^{(k-1)}$ and prove properties (i), (ii), (iii) in the same way as for $C^{(1)}$ and $T^{(1)}$. □

EXAMPLE 5.11. To illustrate the construction of the generic reduction map consider the family of lines $\{l_t\}_{t \in \mathbb{P}^1}$ in $\mathbb{P}^2$ through a point $p \in \mathbb{P}^2$. Then $T \cong \mathbb{P}^1$ has only one irreducible component, and $S \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ consists of two irreducible components: the closure S' of

$$\{(t, x_1, x_2, t) : x_1 \neq p, x_1, x_2 \in l_t\}$$

and

$$S'' = \{(t_1, p, x_2, t_2) : x_2 \in l_{t_2}\}.$$

The construction above singles out S' because $p_2(S'') = \{p\}$. Hence the generic reduction is already induced by the family of lines $\{l_t\}$, i.e. it is the projection of $\mathbb{P}^2$ from p to $\mathbb{P}^1$. The reduction map is *generic* since the connection of the lines through p is not taken into account.

REMARK 5.12. This construction closely resembles that of Campana's reduction map [**11, 12**]. The difference is that in the above construction for any given analytic subset $Z \subset X$ two general points lie in the same fiber iff they can be connected without touching Z. In the example above the two reduction maps fall apart.
Another difference between the generic and Campana's reduction map is the stability under modifications: Let X be a compact complex manifold and $(C_t)_{t\in T}$ a covering family of complex subspaces of X. Let $f : X \dashrightarrow Y$ be the generic quotient and $g : X \dashrightarrow Z$ Campana's quotient with respect to (C_t). If $\pi : \widehat{X} \to X$ is a modification of compact Kähler manifolds then the generic quotient of $\widehat{X}$ w.r.t. the strict or total transforms of (C_t) is described by $f \circ \pi$ whereas in general Campana's quotient is described by $g \circ \pi$ only w.r.t. the total transforms of (C_t): Consider again the pencil of lines through a point $p \in \mathbb{P}^2$ and its strict transform in the blow up of $\mathbb{P}^2$ in p.

PROOF OF PROPOSITION 5.8. Let $\Gamma_f \subset Y_1 \times X$, $\Gamma_g \subset X \times Y_2$ be the graphs of f and g. Then $\Gamma_f \sqcup \Gamma_g \subset X \times (Y_1 \sqcup Y_2)$ is a covering family of complex subspaces in X. Hence we can apply Theorem 5.10 and get a generic reduction map $h : X \dashrightarrow Y$. In particular, two general points $x, x' \in X$ can be connected by a sequence of f- and g-fibers such that two subsequent fibers do also intersect outside

$$Z = \mathrm{Sing}(\mathcal{F} \sqcup \mathcal{G}) \cup \{\text{indeterminacy loci of } f \text{ and } g\}.$$

Such points x, x' must lie in the same leaf of $\mathcal{F} \sqcup \mathcal{G}$. On the other hand, the f- and g-fibers through a general point $x \in X$ must be contained in the h-fiber of x, by the properties of h.
Consequently h induces $\mathcal{F} \sqcup \mathcal{G}$. □

CHAPTER 6

The numerically trivial foliation of a pseudo-effective class

In this chapter we introduce the notion of numerically trivial foliations with respect to pseudo-effective classes α on compact Kähler manifolds X. Our first goal will be the construction of a maximal numerically α-trivial foliation, and as for the nef reduction map in Chapter 1 and Tsuji's numerically trivial fibrations in Chapter 3 the main tool will be a Key Lemma. This time it is local, according to the nice local structure of foliations.
In section 6.2 we show that the leaf dimension of a numerically α-trivial foliation is bounded from above by the numerical codimension $\dim X - \nu(\alpha)$ of α if the foliation has only isolated singularities. Of course this is a very restrictive assumption, which is in addition difficult to check. To generalize the bound to arbitrary numerically trivial foliations the structure of these foliations around singularities must be analysed much closer. In any case, the philosophy behind the notion of numerical triviality and also the examples in Chapter 7 make the bound plausible.
In section 6.3 we generalize the notion of numerical triviality w.r.t. a pseudo-effective class to α-admissible systems of currents in $\alpha[-\epsilon\omega]$. The most interesting examples are the systems induced by a closed positive $(1,1)$-current Θ representing α, giving rise to numerical Θ-triviality. This notion can be characterized by the Siu decomposition of Θ, as in Theorem 3.8. We also show that the Iitaka fibration of a line bundle L induces the numerically trivial foliation w.r.t. a metric generated by the sections of mL, for $m \gg 0$ appropriately chosen.
In the last section we identify Tsuji's numerically trivial fibrations w.r.t. a pseudo-effective line bundle L and a possibly singular hermitian metric h on L with the pseudo-effective fibration associated to the curvature current Θ_h that is the maximal rational map $f : X \dashrightarrow Y$ inducing a numerically Θ_h-trivial foliation. Finally the nef reduction map of a nef class α also fits into the picture, as the pseudo-effective fibration of α.
These facts yield a sufficient criterion for the Iitaka fibration and the nef reduction map of a nef line bundle L falling apart: The maximal numerically $c_1(L)$-trivial foliation is not a rational fibration. Examples for this phenomenon will be studied in Chapter 7.

6.1. Numerical triviality for pseudo-effective classes

DEFINITION 6.1. *Let X be a compact Kähler manifold with Kähler form ω and pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. A submanifold $Y \subset X$ (closed or not) is* ***numerically α-trivial*** *iff for every immersed disk $\Delta \subset Y$,*

$$\lim_{\epsilon \downarrow 0} \sup_{T} \int_{\Delta' - \mathrm{Sing}\, T} (T + \epsilon\omega) = 0,$$

where the T's run through all currents with analytic singularities in $\alpha[-\epsilon\omega]$ and $\Delta' = \{t : |t| < 1 - \delta\}$ is any smaller disk contained in $\Delta = \{t : |t| < 1\}$.

As a convention set $\int_{\Delta - \mathrm{Sing}\, T}(T + \epsilon\omega) = 0$ if $\Delta - \mathrm{Sing}\, T = \emptyset$. Furthermore note that the restriction to disks Δ' may be replaced by the assumption that it is possible to continue the immersion $\Delta \subset Y$ holomorphically.

DEFINITION 6.2. *Let X be an n-dimensional compact Kähler manifold with a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. A foliation $\mathcal{F}$ of rank k is* ***numerically α-trivial*** *iff for any open subset $\Delta^n \cong U \subset X - \mathrm{Sing}(\mathcal{F})$ such that the leaves of $\mathcal{F}$ are the fibers of the projection $p : U \to \Delta^{n-k}$ onto the last $n - k$ coordinates the following holds:*

(i) *Every fiber of p is numerically α-trivial,*

(ii) *and if $\iota : \Delta^n \hookrightarrow U$ is an extendable immersion with relatively compact image such that the projection $p \circ \iota : \Delta^n \hookrightarrow U \to \Delta^{n-k}$ factors through the projection $q : \Delta^n \to \Delta^{n-1}$ onto the last $n - 1$ coordinates, then for any sequence of currents $T_k \in \alpha[-\epsilon_k\omega]$, $\epsilon_k \to 0$, the integrals $\int_{q^{-1}(a) - \mathrm{Sing}\, T_k}(T_k + \epsilon_k\omega)$ are uniformly bounded from above in $a \in \Delta^{n-1}$ and $k \in \mathbb{N}$.*

Note that no exceptional fibers are allowed: if the fibers are completely contained in the common singularity locus of the $T \in \alpha[-\epsilon\omega]$, then they are numerically trivial by the convention above, otherwise the limit in definition 6.1 is supposed to be 0. The uniform boundedness is essential for the proof of the Local Key Lemma below.
To construct a maximal numerical trivial foliation w.r.t. this notion, we first need to prove an analog for the Key Lemma 3.4 used for constructing Tsuji's numerically trivial fibrations:

LEMMA 6.3 (**Local Key Lemma for pseudo-effective classes**). *Let X be a compact Kähler manifold with a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. Let $W \cong \Delta^n$ be an open subset of X with a projection $p : W \to \Delta^k$ onto the last k factors, and let $V = \{z_1 = \ldots = z_{n-k} = 0\}$ be a complex submanifold of W. If every fiber of p and also V are numerically α-trivial then W will also be numerically α-trivial.*

The proof of this Local Key Lemma for pseudo-effective classes is rather technical but becomes more transparent when looking at the characterization of numerical triviality w.r.t. a single closed positive $(1,1)-$current in Section 6.3: In this case, the numerical triviality of the fibers of the projection implies that the residue current of the Siu decomposition is a pull back of a current on the base (see the Pullback Lemma 6.12). Of course, the Pullback Lemma is not true for pseudo-effective classes. But it is enough to prove that the restriction onto different horizontal sections are almost the same, hence the numerical triviality of V implies the numerical triviality of all horizontal sections, hence that of W. This argument is made exact by

PROPOSITION 6.4. *Let X be a compact Kähler manifold with Kähler form ω, and let $T_k = T'_k + \epsilon_k\omega$, $\epsilon_k \to 0$, be a sequence of closed positive $(1,1)$- currents with analytic singularities on X such that the T'_k represent the same cohomology class. Let $\Delta^2 \hookrightarrow X$ be an immersion (with coordinates z_1, z_2). Let $\Delta' \subset\subset \Delta$ be a disk, and consider the functions $f_k : \Delta' \to \mathbb{R}^+, a \mapsto \int_{(\{z_1=a\}\cap\Delta')-\mathrm{Sing}\ T_k} T_k$ and $g_k : \Delta' \to \mathbb{R}^+, b \mapsto \int_{(\{z_2=b\}\cap\Delta')-\mathrm{Sing}\ T_k} T_k$. Suppose that $\lim_{k\to\infty} f_k(a) = 0$ for all $a \in \Delta$, and that the f_k are uniformly (in a) bounded from above. Suppose furthermore that $\lim_{k\to\infty} g_k(0) = 0$. Then $\lim_{k\to\infty} g_k(b) = 0$ for all $b \in \Delta'$, and the g_k are uniformly (in b) bounded from above.*

PROOF. Since the integrals are always evaluated outside the singularities of T_k, and since the mass of the integration current of a divisor is always concentrated in the divisor, one can assume without loss of generality that the Siu decomposition of T_k does not contain any integration currents of divisors. Consequently, T_k has only finitely many isolated singularities on any compact subset of $p_1^{-1}(\Delta')$ where $\Delta' \subset\subset \Delta$ is any disk and $p_1 : \Delta^2 \to \Delta$ is the projection onto the first coordinate, and T_k may be written on $p_1^{-1}(\Delta')$ as

$$T_k = \theta_{11}^k idz_1 \wedge d\overline{z}_1 + \theta_{12}^k idz_1 \wedge d\overline{z}_2 + \theta_{21}^k idz_2 \wedge d\overline{z}_1 + \theta_{22}^k idz_2 \wedge d\overline{z}_2,$$

where the θ_{ij}^k are smooth functions outside these singularities, and integrable on Δ^2. The current T_k being real implies $\theta_{ij}^k = \overline{\theta_{ji}^k}$.
To prove the proposition it is enough to show that

$$\lim_{k\to\infty} |\int_{\Delta'_b-\mathrm{Sing}\ T_k} T_k - \int_{\Delta'_0-\mathrm{Sing}\ T_k} T_k| = 0.$$

where $\Delta'_b = \{z_2 = b\} \cap \Delta'$. Now, choose a path $\gamma \in \Delta$ from 0 to b. Then,

$$|\textstyle\int_{\Delta'_b-\mathrm{Sing}\ T_k} T_k - \int_{\Delta'_0-\mathrm{Sing}\ T_k} T_k| \quad = \quad |\int_{\Delta'}(\theta_{11}^k(z_1,b) - \theta_{11}^k(z_1,0))idz_1 \wedge d\overline{z}_1|$$

equals (by Stokes and Fubini)

$$|\textstyle\int_{\Delta'}(\int_\gamma d\theta_{11}^k)idz_1 \wedge d\overline{z}_1| \quad = \quad |\int_{\Delta'\times\gamma} d(\theta_{11}^k idz_1 \wedge d\overline{z}_1)|.$$

Since the closedness of T implies

$$d(\theta_{11}^k idz_1 \wedge d\overline{z}_1) = -d(\theta_{12}^k idz_1 \wedge d\overline{z}_2 + \theta_{21}^k idz_2 \wedge d\overline{z}_1 + \theta_{22}^k idz_2 \wedge d\overline{z}_2),$$

this integral equals by Stokes

$$|\int_{\partial(\Delta'\times\gamma)}(\theta_{12}^k idz_1 \wedge d\overline{z}_2 + \theta_{21}^k idz_2 \wedge d\overline{z}_1 + \theta_{22}^k idz_2 \wedge d\overline{z}_2)|,$$

and since z_2 is constant on $\Delta' \times \partial\gamma$, this simplifies to

$$|\int_{(\partial\Delta')\times\gamma}(\theta_{12}^k idz_1 \wedge d\overline{z}_2 + \theta_{21}^k idz_2 \wedge d\overline{z}_1 + \theta_{22}^k idz_2 \wedge d\overline{z}_2)|.$$

Observe that these integrals do not depend on the chosen path γ. Consequently, cover the disk $\Delta_{0,b}$ with center in $b/2$ and radius $|b|/2$ with a family of paths γ_a from 0 to b. Then to prove $\lim_{k\to\infty}|\int_{\Delta_b'} T_k - \int_{\Delta_0'} T_k| = 0$ it is enough to show that

$$\lim_{k\to\infty}\int_a |\int_{(\partial\Delta')\times\gamma_a}(\theta_{12}^k idz_1 \wedge d\overline{z}_2 + \theta_{21}^k idz_2 \wedge d\overline{z}_1 + \theta_{22}^k idz_2 \wedge d\overline{z}_2))|da = 0.$$

The term with θ_{22}^k vanishes since $idz_2 \wedge d\overline{z}_2$ is pulled back to 0 in any chart of $(\partial\Delta')\times\gamma_a$. Since $\theta_{12}^k = \overline{\theta_{21}^k}$ the remaining integral may be bounded from above by

$$C \cdot \int_{\partial\Delta'\times\Delta_{0,b}} |\theta_{12}^k| dV,$$

where C is independent of b and k, and dV is a volume element on $\partial\Delta' \times \Delta_{0,b}$.
Now interpret T_k as a semipositive hermitian form $\langle .\ ,.\rangle$ on every tangent space $T_{X,x}$ (where T has no singularities). Then the Schwarz inequality implies that

$$|\theta_{12}^k| = |\langle\frac{\partial}{\partial\overline{z}_1},\frac{\partial}{\partial\overline{z}_2}\rangle| \le |\langle\frac{\partial}{\partial\overline{z}_1},\frac{\partial}{\partial\overline{z}_1}\rangle|^{\frac{1}{2}} \cdot |\langle\frac{\partial}{\partial\overline{z}_2},\frac{\partial}{\partial\overline{z}_2}\rangle|^{\frac{1}{2}} = |\theta_{11}^k|^{\frac{1}{2}} \cdot |\theta_{22}^k|^{\frac{1}{2}}.$$

Hence the integral above is $\le$ the square root of the product

$$\int_{\partial\Delta'\times\Delta_{0,b}} |\theta_{11}^k| dV \cdot \int_{\partial\Delta'\times\Delta_{0,b}} |\theta_{22}^k| dV,$$

again by the Schwarz inequality.
Claim. There exists a bound $M' > 0$ such that for all k there is a disk $\Delta_k' \subset\subset \Delta$ containing Δ' with

$$\int_{\partial\Delta_k'\times\Delta'} |\theta_{11}^k| dV < M'.$$

PROOF. Suppose that $\Delta' \subset\subset \Delta'' \subset\subset \Delta$, and look at the $(1,1)$- form $\eta = idz_2 \wedge d\overline{z}_2$. There exists a $C > 0$, such that $\eta \le C\cdot\omega$ on $\Delta'' \times \Delta'$. Hence,

$$\int_{(\Delta''-\Delta')\times\Delta'} |\theta_{11}^k| dV = \int_{(\Delta''-\Delta')\times\Delta'} (T_k' + \epsilon_k\omega)\wedge\eta \le C\cdot\int_X (T_k' + \epsilon_k\omega)\wedge\omega,$$

and the last integral only depends on the cohomology class of T_k' (and ω). By Fubini one gets a disk Δ_k' as above. □

For the second term note that the assumptions on the functions f_k imply $\lim_{k\to\infty}\int_{\Delta'} f_k ida \wedge d\overline{a} = 0$, by Lebesgue's dominated convergence, and the measure of the sets $\{a : f_k(a) > \delta\}$ tends to 0, too, for $k \to \infty$.

Hence, as above, for a given $\epsilon > 0$ it is possible to bound the measure of $\{a : f_k(a) > \delta\}$ small enough such that for all k big enough there is a disk $\Delta_k'' \subset\subset \Delta$ containing Δ' with

$$\int_{\partial\Delta_k'' \times \Delta'} |\theta_{22}^k| dV < \epsilon.$$

Choosing δ small enough and M' big enough (but both independent of k!) one can assume that the two disks Δ_k' and Δ_k'' coincide (at least for k big enough). Since M' is independent of ϵ, the difference $\int_{\Delta_{k,b}} T_k - \int_{\Delta_{k,0}} T_k$ tends to 0 for $k \to \infty$, and uniformly in b. Since $\int_{\Delta_{k,0}} T_k \overset{k\to\infty}{\longrightarrow} 0$, this is also true for $\int_{\Delta_b'} T_k - \int_{\Delta_0'} T_k$. Consequently, $\lim_{k\to\infty} g_k(a) = 0$, and the uniformity in b implies the uniform boundedness of the g_k. □

Proof of the Local Key Lemma for pseudo-effective classes. If Δ is a disk immersed in W such that p projects it on a point in Δ^k, there is nothing to prove.
If Δ is a disk immersed in W not intersecting Y which is projected biholomorphically onto Δ^k, then a coordinate change and further cutting down leads to the configuration described in the proposition. Note that it is sufficient to check on any disk $\Delta' \subset\subset \Delta$ that

$$\lim_{k\to\infty} \int_{\Delta' - \mathrm{Sing} T_k} T_k + \frac{1}{k}\omega = 0$$

for arbitrary sequences T_k of currents with analytic singularities in $\alpha[-\frac{1}{k}\omega]$. The assumptions of the Local Key Lemma imply that

$$\lim_{k\to\infty} \int_{\{z_1=a\} - \mathrm{Sing} T_k} T_k + \frac{1}{k}\omega = \lim_{k\to\infty} f_k(a) = 0$$

for all a and $\lim_{k\to\infty} \int_{\{z_2=0\} - \mathrm{Sing} T_k} T_k + \frac{1}{k}\omega = 0$. The definition of a numerically trivial foliation implies the uniform boundedness of the f_k, so it is possible to apply the proposition.
If Δ is a disk immersed in W not satisfying one of the two conditions above, then for any $\Delta' \subset\subset \Delta$ there are disks $\Delta_i'' \subset\subset \Delta_i' \subset \Delta$ such that $\bigcup \Delta_i'' \supset \Delta'$ (hence it is enough to consider finitely many of these disks), and there are projections $p_i : W \to \Delta^{n-k}$ (possibly different from p) such that the restriction onto Δ_i' is a submersion. Since the fibers and sections of these p_i are composed of disks already shown to be numerically trivial, it is possible to apply again the proposition on $\Delta_i'' \subset\subset \Delta_i'$ (by possibly further cutting down and a coordinate change). Since there are only finitely many i's, Δ' is also numerically trivial.
Finally, the uniform boundedness property of the foliation follows directly from the uniform boundedness shown in the proposition. □
Now we construct the *maximal* numerically α-trivial foliation:

THEOREM 6.5. *Let X be a projective complex manifold with Kähler form ω, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. Then there exists a numerically α-trivial foliation $\mathcal{F} \subset T_X$ on X such that $\mathcal{F}' \subset \mathcal{F}$ for all numerically α-trivial foliations $\mathcal{F}'$.*

PROOF. It is enough to prove that for two numerically α-trivial foliations $\mathcal{F}, \mathcal{G} \subset T_X$ the union $\mathcal{F} \sqcup \mathcal{G}$ is also numerically α-trivial. But this follows immediately from the Local Key Lemma 6.3 since it can be applied on the inductive steps of the construction of $\mathcal{F} \sqcup \mathcal{G}$ in Section 5.1. □

6.2. The leaf dimension of numerically trivial foliations

Numerically trivial foliations w.r.t. a pseudo-effective class α have leaves with codimension bounded from below by the numerical dimension of α, provided that the singularities of the foliation are nice enough:

THEOREM 6.6. *Let X be a compact Kähler manifold with Kähler form ω and $\alpha \in H^{1,1}(X, \mathbb{R})$ a pseudo-effective class. Let $\mathcal{F}$ be the numerically trivial foliation w.r.t. α and suppose that the singularities of $\mathcal{F}$ are isolated points. Then the numerical dimension $\nu(\alpha)$ is less or equal to the codimension of the leaves of $\mathcal{F}$.*

PROOF. Applying theorem 4.9 , one gets a sequence of closed $(1,1)$- currents T_k with analytic singularities in $\alpha[-\epsilon_k \omega]$ such that

$$\lim_{k\to\infty} \int_{X-\mathrm{Sing}\ T_k} (T_k + \epsilon_k\omega)^p \wedge \omega^{n-p} = (\alpha^p.\omega^{n-p})_{\geq 0}$$

for all $p = 1, \ldots, n$. In these integrals, the T_k's may be replaced by the residue currents

$$R_k = T_k - \sum \nu(T_k, D)[D]$$

of the Siu decomposition of the T_k.
Now the proof consists of two steps: first, let $\Delta^n \cong U \subset X$ be an open set such that the projection $q : U \cong \Delta^n \to \Delta^l$ on the last l coordinates describes the numerical trivial foliation w.r.t. α locally in U. Then use as in proposition 6.4 that the R_k's get close to pulled back currents from the base Δ^l to show
Claim 1. For $l < p \leq n$ and an open subset $U' \subset\subset U$,

$$\int_{U'} (R_k + \epsilon_k\omega)^p \wedge \omega^{n-p} \to 0.$$

PROOF. Every $R_k + \epsilon_k\omega$ may be written as a sum $\sum_{i,j} \theta_{ij}^k dz_i \wedge d\overline{z}_j$. Then every coefficient of $(R_k + \epsilon_k\omega)^p$ w.r.t. the base $dz_I \wedge d\overline{z}_J$ (I, J multi-indices of length $|I| = |J| = p$) is a product of p of these θ_{ij}^k. If $p > l$, then one of these θ_{ij}^k has index $i \leq n - l$ or $j \leq n - l$.
As in proposition 6.4 one can argue with the Schwarz inequality that

$$|\theta_{ij}^k| \leq |\theta_{ii}^k|^{\frac{1}{2}} \cdot |\theta_{jj}^k|^{\frac{1}{2}}.$$

Furthermore, let F_i be a sufficiently general fiber of the projection $\Delta^n \to \Delta^{n-1}$ onto all but the ith coordinate, $i = 1, \ldots, n-l$. Since R_k is a current with analytic singularities only in codimension 2, a sufficiently general F_i does not hit the singularities of R_k. Then $\theta^k_{ii|F_i \cap U'}$ is smooth and positive, and numerical triviality applied on the 1-dimensional fibers F_i which are leaves of q implies that

$$\int_{F_i \cap U'} |\theta^k_{ii}| dz_i \wedge d\overline{z}_i = \int_{F_i \cap U'} \theta^k_{ii} dz_i \wedge d\overline{z}_i \stackrel{k\to\infty}{\longrightarrow} 0.$$

This leads to the following chain of inequalities: Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_p)$ be two multi-indices of length p such that (without loss of generality) $i_1 \leq n - l$. Then

$$\begin{aligned} \textstyle\int_{U'} |\theta^k_{i_1 j_1} \cdots \theta^k_{i_p j_p}| dV_\omega &\leq \textstyle\int_{U'} |\theta^k_{i_1 i_1} \cdots \theta^k_{i_p i_p}|^{\frac{1}{2}} |\theta^k_{j_1 j_1} \cdots \theta^k_{j_p j_p}|^{\frac{1}{2}} dV_\omega \\ &\leq (\textstyle\int_{U'} |\theta^k_{i_1 i_1}| dV_\omega)^{\frac{1}{2}} \cdot (\textstyle\int_{U'} |\theta^k_{i_2 i_2} \cdots \theta^k_{i_p i_p}||\theta^k_{j_1 j_1} \cdots \theta^k_{j_p j_p}| dV_\omega)^{\frac{1}{2}}. \end{aligned}$$

The second integral of the last term remains bounded for $k \to \infty$ because the $R_k + \epsilon_k \omega$ (weakly) converge against some current according to theorem 4.9. The first integral may be computed via Fubini as

$$\int_{U'} |\theta^k_{i_1 i_1}| dV_\omega = \int_{\Delta^{n-1}} (\int_{F_{i_1}} |\theta^k_{i_1 i_1}| dz_{i_1} \wedge d\overline{z}_{i_1}) dV_{\Delta^{n-1}},$$

hence tends to 0 for $k \to \infty$ since the integrals $\int_{F_{i_1}} |\theta^k_{i_1 i_1}| dz_{i_1} \wedge d\overline{z}_{i_1}$ are uniformly bounded from above by definition of numerically trivial foliations. Consequently, $\int_{U'} |\theta^k_{i_1 j_1} \cdots \theta^k_{i_p j_p}| dV_\omega \stackrel{k\to\infty}{\longrightarrow} 0$ and the claim follows. □

The second step is to give an estimate of the considered integrals around the isolated singularities of the foliation by using the uniform boundedness of the Lelong numbers of (almost) positive currents in the same cohomology class.

Claim 2. There is a sequence of compact sets $K_i \subset X$ exhausting $X - \operatorname{Sing} \mathcal{F}$ and a constant $C > 0$ such that for all $1 \leq p \leq n$

$$\int_{X - K_i} (R_k + \epsilon_k \omega)^p \wedge \omega^{n-p} \leq \delta_i,$$

and $\lim_{i\to\infty} \delta_i = 0$.

PROOF. This is just an expanded version of Boucksom's argument in [**8**, Lem 3.1.11]. Choose a finite covering of X by open charts U_i isomorphic to the unity ball $B \subset \mathbb{C}^n$, such that the balls with half of the diameter still cover X. If $z^{(i)}$ denote coordinates on U_i one may find two constants $C_1, C_2 > 0$ such that

$$C_1 \omega \leq \frac{i}{2} \partial\overline{\partial} |z^{(i)}|^2 \leq C_2 \omega$$

in U_i, for all i.

If $x \in X$ lies in U_i, the Lelong number $\nu((R_k + \epsilon\omega)^p, x)$ is by definition the decreasing limit for $r \to 0$ of

$$\nu((R_k + \epsilon\omega)^p, x, r) := \frac{1}{(\pi r^2)^{n-p}} \int_{|z^{(i)}-x|<r} (R_k + \epsilon\omega)^p \wedge (\frac{i}{2}\partial\overline{\partial}|z^{(i)}|^2)^p.$$

On the one hand, for $r \leq r_0$ one has

$$\nu((R_k + \epsilon\omega)^p, x, r) \leq \nu((R_k + \epsilon\omega)^p, x, r_0) \leq \frac{C_2}{(\pi r_0^2)^{n-p}} \int_X (R_k + \epsilon\omega)^p \wedge \omega^{n-p}.$$

But $\int_X (R_k + \epsilon\omega)^p \wedge \omega^{n-p} \leq \int_X (T_k + \epsilon\omega)^p \wedge \omega^{n-p}$, and the last integral depends only on the cohomology class of T_k, since ω is closed.
On the other hand,

$$\begin{array}{rcl} (\pi r^2)^{n-p}\nu((T_k + \epsilon_k\omega)^p, x, r) & \geq & C_1 \int_{|z^{(i)}-x|<r} (T_k + \epsilon_k\omega)^p \wedge \omega^{n-p} \\ & \geq & C_1 \int_{|z^{(i)}-x|<r} (R_k + \epsilon_k\omega)^p \wedge \omega^{n-p}. \end{array}$$

For $p < n$ the claim follows since $\operatorname{Sing} \mathcal{F}$ is compact, hence consists of only finitely many points. For $p < n$ there is nothing to argue, since $\nu(\alpha) = n$ implies that α is big ([**8**, Thm. 3.1.31]). Hence the numerically trivial foliation coincides with the Iitaka fibration w.r.t. α, because it is the identity map. $\square$

Both claims together show the theorem. $\square$

6.3. Variants of numerically trivial foliations

It is remarkable that the definitions of numerically trivial submanifolds and foliations and the construction of maximal numerically trivial foliations also work when the currents T do not run over all currents with analytic singularities in $\alpha[-\epsilon\omega]$:

DEFINITION 6.7. *Let X be a compact Kähler manifold with Kähler form ω, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. An α-admissible system of currents $(\mathcal{C}_\epsilon \subset \alpha[-\epsilon\omega])_{\epsilon \in \mathbb{R}^+}$ is a system of sets $\mathcal{C}_\epsilon \subset \alpha[-\epsilon\omega]$ such that $\mathcal{C}_\epsilon \subset \mathcal{C}_{\epsilon'}$ for every pair $\epsilon \leq \epsilon'$.*
*A submanifold $Y \subset X$ (closed or not) is **numerically** $(\mathcal{C}_\epsilon)$**-trivial** iff for every immersed disk $\Delta \subset Y$,*

$$\lim_{\epsilon \downarrow 0} \sup_T \int_{\Delta' - \operatorname{Sing} T} (T + \epsilon\omega) = 0,$$

where the T's run through all currents with analytic singularities in $\mathcal{C}_\epsilon$ and $\Delta' = \{t : |t| < 1 - \delta\}$ is any smaller disk contained in $\Delta = \{t : |t| < 1\}$.
*A foliation $\mathcal{F}$ of rank k is **numerically** $(\mathcal{C}_\epsilon)$**-trivial** iff for any open subset $\Delta^n \cong U \subset X - \operatorname{Sing}(\mathcal{F})$ such that the leaves of $\mathcal{F}$ are the fibers of the projection $p : U \to \Delta^{n-k}$ onto the last $n - k$ coordinates the following holds:*

(i) *Every fiber of p is numerically $(\mathcal{C}_\epsilon)$-trivial,*

(ii) *and if* $\iota : \Delta^n \hookrightarrow U$ *is an extendable immersion with relatively compact image such that the projection* $p \circ \iota : \Delta^n \hookrightarrow U \to \Delta^{n-k}$ *factors through the projection* $q : \Delta^n \to \Delta^{n-1}$ *onto the last* $n-1$ *coordinates, then for any sequence of currents* $T_k \in \mathcal{C}_{\epsilon_k}$, $\epsilon_k \to 0$, *the integrals* $\int_{q^{-1}(a)-\mathrm{Sing}\ T_k}(T_k + \epsilon_k\omega)$ *are uniformly bounded from above in* $a \in \Delta^{n-1}$ *and* $k \in \mathbb{N}$.

THEOREM 6.8. *Let* X *be a projective complex manifold with Kähler form* ω, *let* $\alpha \in H^{1,1}(X,\mathbb{R})$ *be a pseudo-effective class and* $(\mathcal{C}_\epsilon \subset \alpha[-\epsilon\omega])_{\epsilon\in\mathbb{R}^+}$ *an* α*-admissible system. Then there exists a numerically* $(\mathcal{C}_\epsilon)$*-trivial foliation* $\mathcal{F} \subset T_X$ *on* X *such that* $\mathcal{F}' \subset \mathcal{F}$ *for all numerically* $(\mathcal{C}_\epsilon)$*-trivial foliations* $\mathcal{F}'$. $\square$

As an immediate consequence of the definition we obtain

PROPOSITION 6.9. *Let* $(\mathcal{C}_\epsilon \subset \alpha[-\epsilon\omega])_\epsilon$, $(\mathcal{C}'_\epsilon \subset \alpha[-\epsilon\omega])_\epsilon$ *be two* α*-admissible systems such that* $\mathcal{C}_\epsilon \subset \mathcal{C}'_\epsilon$ *for each* $\epsilon > 0$. *Then every numerically* $(\mathcal{C}'_\epsilon)$*-trivial submanifold or foliation is also numerically* $(\mathcal{C}_\epsilon)$*-trivial.* $\square$

These generalized notions of numerical triviality are especially interesting when applied to a positive closed $(1,1)$-current $\Theta \in \alpha[0]$.

DEFINITION 6.10. *Let* X *be a projective complex manifold with Kähler form* ω, *let* $\alpha \in H^{1,1}(X,\mathbb{R})$ *be a pseudo-effective class and* $\Theta \in \alpha[0]$ *a positive closed* $(1,1)$*-current. For a sequence* $(\Theta_k)_{k\in\mathbb{N}}$ *of currents* $\Theta_k \in \alpha[-\epsilon_k\omega]$, $\epsilon_k \downarrow 0$, *approximating* Θ *as in Theorem 4.5 consider the* α*-admissible system of sets* $\mathcal{C}_\epsilon = \{\Theta_k : \epsilon_k \leq \epsilon\} \subset \alpha[-\epsilon\omega]$. *Then a submanifold* $Y \subset X$ *or a foliation* $\mathcal{F} \subset T_X$ *is called* ***numerically*** Θ***-trivial*** *iff they are numerically* $(\mathcal{C}_\epsilon)$*-trivial.*

Obviously we have to show the independence of this definition from the approximating sequence $(\Theta_k)_{k\in\mathbb{N}}$. This is done by the following characterization of numerically Θ-trivial submanifolds which is a local analogue of Theorem 3.8:

PROPOSITION 6.11. *With the notation as in the definition, a submanifold* $Y \subset X$ *is numerically* $(\mathcal{C}_\epsilon)$*-trivial iff* $\Theta_{|Y}$ *exists and*

$$\Theta_{|Y} = \sum a_i[D_i]$$

for countably many divisors D_i *in* Y *and real numbers* $a_i > 0$.

PROOF. By the properties of the approximation listed in Theorem 4.5,

$$\lim_{k\to\infty} \int_{\Delta'} (\Theta_k)_{\mathrm{ac}} + \epsilon\omega = \int_{\Delta'} \Theta_{\mathrm{ac}}$$

for every immersed disk $\Delta' \subset\subset \Delta \subset Y$.
If $\Theta_{|Y} = \sum a_i[D_i]$, the absolute continuous part of $\Theta_{|Y}$ vanishes on Y and hence on any disk immersed in Y. Consequently the above integral is always 0, and Y is numerically $(\mathcal{C}_\epsilon)$-trivial.

On the other hand, if Y is numerically $(\mathcal{C}_\epsilon)$-trivial, then $\int_{\Delta'} \Theta_{\mathrm{ac}}$ for every immersed disk $\Delta' \subset\subset \Delta \subset Y$. This can be used as follows:
Let $\Theta = \sum_i \nu_i [D_i] + R$ be the Siu decomposition of Θ. Let $\Delta^n \cong U \subset X$ be an open subset and let $q : \Delta^n \to \Delta^{n-1}$ be the projection onto the first $n-1$ factors. Since the Lelong number level sets $E_c(R)$ contain no codim 1 component, very general fibers F of q do not intersect any of the $E_c(R)$. By the results of [**31**] there is a pluripolar set $N \subset \Delta^{n-1}$ such that the level sets $E_c(R_{|F}) = \emptyset$ for the restriction of R to all fibers F over points outside of N. By assumption $R_{|F} \cong 0$.
By the following lemma there exists a positive closed $(1,1)-$current S on Δ^{n-1} such that $R = q^* S$. Let $D = \Delta^{n-1} \times \{p\}$ be a section of q such that $R_{|D}$ is well defined. By induction $R_{|D} \equiv 0$. Since the projection $q : D \to \Delta^{n-1}$ is an isomorphism $S \equiv 0$ hence $R \equiv 0$. □

LEMMA 6.12. *Let T be a positive closed $(1,1)-$ current on Δ^n and let $q : \Delta^n \to \Delta^{n-1}$ be the projection onto all factors but the last one. If $T_{|q^{-1}(x)} \equiv 0$ for all x outside a pluripolar set $N \subset \Delta^{n-1}$ then there will be a positive closed $(1,1)-$ current S on Δ^{n-1} such that $T = q^* S$.*

PROOF. The positive current T may be written as

$$T = i \sum_{i,j} \Theta_{ij} dz_i \wedge d\overline{z}_j$$

where the Θ_{ij} are complex measures on Δ^n ([**14**, (1.15)]). That T is a real current implies $\Theta_{ij} = \overline{\Theta}_{ji}$. Since T is positive, $\sum \lambda_i \overline{\lambda_j} \Theta_{ij}$ is a positive measure for all vectors $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Hence

$$\lambda_i \overline{\lambda_i} \Theta_{ii} + \lambda_i \overline{\lambda_n} \Theta_{in} + \lambda_n \overline{\lambda_i} \Theta_{ni} + \lambda_n \overline{\lambda_n} \Theta_{nn} \geq 0 \ \ \forall (\lambda_i, \lambda_n) \in \mathbb{C}^2.$$

Claim. As a $(1,1)-$current $i\Theta_{nn} dz_n \wedge d\overline{z}_n = 0$.

PROOF. By definition one has to show that

$$\int_{\Delta^n} i\Theta_{nn} dz_n \wedge d\overline{z}_n \wedge \alpha i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1} = 0$$

for all complex valued functions $\alpha \in \mathcal{C}_c^\infty(\Delta^n)$. Since $T_{|q^{-1}(x)} = i\Theta_{nn} dz_n \wedge d\overline{z}_n$

$$\int_{\Delta^n} i\Theta_{nn} dz_n \wedge d\overline{z}_n \wedge \alpha i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1} =$$
$$\int_{\Delta^n} T \wedge \alpha i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1},$$

and the slicing formula [**14**, (1.22)] implies that this is equal to

$$\int_{\Delta^{n-1}} \left(\int_{q^{-1}(x')} T_{|q^{-1}(x')} \wedge \alpha_{|q^{-1}(x')} \right) i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1}.$$

This is 0 because $T_{|q^{-1}(x)} \equiv 0$ for all x outside a pluripolar set $N \subset \Delta^{n-1}$. $\square$

Consequently,

$$\Theta_{ii} + \overline{\lambda}_n \Theta_{in} + \lambda_n \Theta_{ni} = \Theta_{ii} + \overline{\lambda}_n \overline{\Theta}_{ni} + \lambda_n \Theta_{ni} \geq 0$$

for all $\lambda_n \in \mathbb{C}$. Now suppose that $\Theta_{ni} \neq 0$, i.e. there is a smooth real valued function $\alpha \geq 0$ with compact support such that $\Theta_{ni}(\alpha) \neq 0$. Then there is a $\lambda_n \in \mathbb{C}$ such that

$$\Theta_{ii}(\alpha) + \overline{\lambda}_n \overline{\Theta_{ni}(\alpha)} + \lambda_n \Theta_{ni}(\alpha) < 0.$$

This is a contradiction. Hence $\Theta_{in} = \Theta_{ni} = 0$ for all $i \leq n-1$.
Next, the closedness of T implies

$$\frac{\partial}{\partial z_n} \Theta_{ij} = \frac{\partial}{\partial \overline{z_n}} \Theta_{ij} = 0 \quad \forall i, j \leq n-1.$$

Hence the Θ_{ij} only depend on $z_1, \ldots, z_{n-1}$. One finally gets

$$T = q^* S = i \sum_{i,j \leq n-1} \Theta_{ij} dz_i \wedge d\overline{z_j}$$

and S is a closed positive $(1,1)-$current on Δ^{n-1}. $\square$

Now we can show that the Iitaka fibration of a line bundle L is the maximal numerically trivial foliation w.r.t. a certain metric on L:

PROPOSITION 6.13. *Let L be a holomorphic line bundle on a projective complex manifold X such that $|mL|$ is a non-empty linear system which induces a rational map $\phi_{|mL|} : X \dashrightarrow Y$. Then $\phi_{|mL|}$ is the numerically trivial foliation w.r.t. $h_{|mL|}$.*

PROOF. On every immersed disk $\iota : \Delta \subset\subset X - \mathrm{Bs}(|mL|)$ mapped to a point by $\phi_{|mL|}$ the pulled back curvature current $\iota^* \Theta_{|mL|}$ of $h_{|mL|}$ is $\equiv 0$. Consequently the Iitaka fibration is numerically $\Theta_{|mL|}$-trivial.
On the other hand for every immersed disk $\iota : \Delta \subset\subset X - \mathrm{Bs}(|mL|)$ not mapped to a point by $\phi_{|mL|}$ the disk Δ is **not** numerically trivial w.r.t. $\iota^* h_{|mL|}$: When $|mL|$ has no base points in the image of Δ, the metric $\iota^* h_{|mL|}$ is a smooth metric with smooth positive curvature form different from 0. Consequently no foliation on X not contained in the Iitaka fibration can be numerically $\Theta_{|mL|}$-trivial. $\square$

Note that there is a positive integer m such that the Iitaka fibration of L is induced by the linear system $|mL|$ [**26**, 10.3].

6.4. Pseudo-effective fibrations

In general it is not true that numerically trivial foliations are (rational) fibrations, see the surface examples in chapter 7. This motivates the following

DEFINITION 6.14. *Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R})$ a pseudo-effective $(1,1)$-class. Let $\mathcal{F}$ be the numerically trivial foliation of α. Then the maximal meromorphic map $f : X \dashrightarrow Y$ such that the induced foliation is contained in $\mathcal{F}$ is called the* ***pseudo-effective fibration*** *of α.*

Prop. 5.8 shows that the definition makes sense: There is a maximal fibration contained in a foliation.
If α is a nef class on a projective complex manifold the pseudo-effective fibration of α is given by the nef reduction map of α from Chapter 1:

PROPOSITION 6.15. *Let X be a projective complex manifold and L a nef line bundle on X. Then the nef fibration of L is the pseudo-effective fibration of $c_1(L)$.*

PROOF. We only have to show for every curve $C \subset X$ that

$$C \text{ is numerically } c_1(L)\text{-trivial } \Leftrightarrow \ L.C = 0.$$

So let ω be a Kähler form on X. If $\Delta \subset\subset X$ is an extendable immersion and $T \in c_1(L)[-\epsilon\omega]$ then

$$0 \leq \int_\Delta T_{\text{ac}} + \epsilon\omega \leq \int_C T + \epsilon\omega = L.C + \epsilon\omega.$$

So if $L.C = 0$ the integrals $\int_\Delta T_{\text{ac}} + \epsilon\omega$ tend to 0 for $\epsilon \downarrow 0$. On the other hand, if C is numerically $c_1(L)$-trivial then choose a sequence of smooth forms $T_k \in c_1(L)[-\epsilon_k\omega]$, $\epsilon_k \downarrow 0$, and a covering of C with extendable immersions $\Delta_i \subset\subset C$. Then

$$0 \leq L.C \leq \int_C T_k + \epsilon_k\omega \leq \sum_i \int_{\Delta_i} T_k + \epsilon_k\omega = 0.$$

□

REMARK 6.16. Together with Propositions 6.13 and 6.9 this proposition gives a sufficient criterion for the Iitaka fibration and the nef fibration of a nef line bundle being different: The maximal numerically L-trivial foliation is not a fibration.

The same definition for numerically trivial foliations w.r.t. a single positive current leads to Tsuji's numerically trivial fibrations from Chapter 3:

PROPOSITION 6.17. *Let X be a smooth projective complex manifold and L a pseudoeffective holomorphic line bundle on X with positive singular hermitian metric h. Then the pseudo-effective fibration w.r.t. the curvature current Θ_h is Tsuji's numerically trivial fibration w.r.t. (L,h).*

PROOF. By Definition 3.1, a subvariety $Y \subset X$ is numerically (L,h)-trivial iff $(L,h).C = 0$ for all irreducible curves $C \subset Y$ not contained in the singularity locus of h. The analysis of these intersection numbers in Chapter 2 shows that

$$(L,h).C = (\pi^*L, \pi^*h).\overline{C} = \pi^*L.\overline{C} - \sum_{x \in \overline{C}} \nu(\pi^*h, x),$$

where $\pi : \overline{C} \to C$ is the normalization. In particular, $(L, h).C = 0$ iff the curvature current of $\pi^* h$ on $\overline{C}$ may be written as $\sum_{x \in \overline{C}} \nu(\pi^* h, x)[x]$. Hence proposition 6.11 shows that numerically Θ_h-trivial subvarieties are also numerically (L, h)-trivial.

The converse is also true: By the birational invariance of numerical (L, h)-triviality the normalization and desingularization $\overline{Y}$ of a numerically (L, h)-trivial subvariety Y is also numerically (L, h)-trivial. Hence the curvature current of the pulled back metric is of the form $\sum \nu_i [D_i]$, by Theorem 3.8. But this implies numerical Θ_h-triviality of $\overline{Y}$, and since every holomorphic map $f : \Delta \to Y$ may be lifted to a holomorphic map $f : \Delta \to \overline{Y}$, the numerical Θ_h-triviality of Y follows. $\square$

CHAPTER 7

Three surface examples

7.1. Mumford's example

The first example is due to Mumford and has the property that the nef dimension is bigger than the numerical dimension: Start with a smooth projective curve C of genus ≥ 2 with the unit circle Δ as universal covering and an irreducible unitary representation $\rho : \pi_1(C) \to GL(2, \mathbb{C})$ of the fundamental group of C. This defines a rank 2 vector bundle $E = (\Delta \times \mathbb{C}^2)/\pi_1(C)$ on C of degree 0 where the action of $\pi_1(C)$ is given by covering transformations on Δ and the representation ρ on $\mathbb{C}^2$.
Mumford proved that the nef line bundle $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ on the projectivized bundle $\mathbb{P}(E)$ is stable hence the restriction of L to all curves $D \subset \mathbb{P}(E)$ is positive. On the other hand $\deg E = 0$ hence $L.L = 0$. Hence the numerical dimension $\nu(L)$ is 1, while the nef reduction map is the identity, and the nef dimension is 2.
It seems quite obvious how to explain this deviation: the ruled surface $\mathbb{P}(E)$ carries a foliation induced by the images of the $\Delta \times l$ in $\mathbb{P}(E)$ (where l is a line through the origin in $\mathbb{C}^2$). Furthermore, locally the leaves of this foliation are mapped to points by the morphism induced by $|L|$, which is a kind of numerical triviality.

This intuition is made exact by constructing a smooth closed positive $(1,1)-$current on $L = \mathcal{O}_{\mathbb{P}^1}(1)$ whose maximal numerically trivial foliation is the one described above: Take a measure ω invariant w.r.t. the representation of $\pi(C)$ in $\mathrm{PGL}(2)$. This gives a measure on $(\Delta\times\mathbb{P}^1)/\pi(C)$ transversal to the foliation induced by the images of $\Delta\times\{p\}$. Averaging out the integration currents of the leaves with this transverse measure gives an (even smooth) closed positive $(1,1)-$ current in the first Chern class of $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ which vanishes on the leaves but not in any transverse direction.
We still have to discuss the existence of a measure ω in $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ invariant w.r.t. the unitary representation of $\pi(C)$ in $\mathrm{GL}(2)$ and the smoothness of the metric which results from averaging out the integration currents of the leaves. But this is easy: Take the Haar measure ω on the Lie group $U(2)$ which is absolutely continuous ([**20**, Ch.14]). Since $U(2)$ operates transitively on $\mathbb{P}^1$ this measure induces a $U(2)-$invariant measure on the homogeneous quotient space $\mathbb{P}^1$. Since $U(2)$ is compact it is possible to normalize ω such that $\mathbb{P}^1$ has measure 1. Hence averaging over the integration currents of the leaves w.r.t. ω gives a smooth positive $(1,1)-$ form which is still in the first Chern class of

$L = \mathcal{O}_{\mathbb{P}(E)}(1)$. Since it is smooth it is a current with minimal singularities on L, and obviously, this current is numerically trivial on the leaves.
On the other hand it is strictly positive on the $\mathbb{P}^1$-fibers, hence no submanifold transversal to the leaves of the foliation can be numerically $c_1(L)$-trivial, and the foliation is the maximal numerically $c_1(L)$-trivial foliation.

7.2. A nef line bundle without smooth semipositive metric

This example was already discussed in [**16**]: Let $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\operatorname{Im} \tau > 0$, be an elliptic curve and let E be the rank 2 vector bundle over Γ defined by

$$E = \mathbb{C} \times \mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$$

where the action is given by the two automorphisms

$$\begin{aligned} g_1(x, z_1, z_2) &= (x + 1, z_1, z_2) \\ g_\tau(x, z_1, z_2) &= (x + \tau, z_1 + z_2, z_2), \end{aligned}$$

and where the projection $E \to \Gamma$ is induced by the first projection $(x, z_1, z_2) \mapsto x$. Then $\mathbb{C} \times \mathbb{C} \times \{0\}/(\mathbb{Z} + \mathbb{Z}\tau)$ is a trivial line subbundle $\mathcal{O} \hookrightarrow E$, and the quotient $E/\mathcal{O} \cong \Gamma \times \{0\} \times \mathbb{C}$ is also trivial. Let L be the line bundle $L = \mathcal{O}_E(1)$ over the ruled surface $X = \mathbb{P}(E)$. The exact sequence

$$0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$$

shows that L is nef over X.
Now, in [**16**] all hermitian metrics h (including singular metrics) are determined such that the curvature current $\Theta_h(L)$ is semi-positive (in the sense of currents): These metrics have all the same curvature current

$$\Theta_h(L) = [C],$$

where C is the curve on X induced by $\{z_2 = 0\}$. (This implies in particular that there exists no **smooth** positive hermitian metric on L.) To exclude the possibility that there exist positive currents in $c_1(L)$ which are not the curvature current of a metric on L one proves the following

LEMMA 7.1. *Let X be a projective complex manifold and L a holomorphic line bundle on X. Then for every closed positive current in $c_1(L)$ there is a possibly singular hermitian metric h on L such that the curvature current*

$$\Theta_h(L) = T.$$

PROOF. Let T be any positive current in $c_1(L)$. By [**6**] there exists a line bundle L' on X with a possibly singular hermitian metric h' such that $\Theta_{h'}(L') = T$. (This is just the usual construction of a cycle in $H^1(X, \mathcal{O}^*)$). The line bundle $N = (L')^{-1} \otimes L$ is numerically trivial, hence nef. Consequently there exists a positive singular hemitian metric h_N on N such that the class of the curvature current

$$\{\Theta_{h_N}(N)\} = 0 \in H^{1,1}(X, \mathbb{R}).$$

Now, all closed positive currents in $0 \in H^{1,1}(X, \mathbb{R})$ have the form $dd^c\phi$ for some plurisubharmonic function on X. Since ϕ is upper semi-continuous it attains its supremum. But then the maximum principle implies that ϕ is a constant function. Therefore the only closed positive current in $0 \in H^{1,1}(X, \mathbb{R})$ is the zero form. This implies $\Theta_{h_N}(N) = 0$ (as a current).
Furthermore this gives the hermitian metric $h = h_N \otimes h'$ on $L = N \otimes L'$ with $\Theta_h(L) = T$. $\square$

So $[C]$ really is a positive current with minimal singularities in $c_1(L)$. But then X is numerically trivial w.r.t. $[C]$, and the associated numerical trivial foliation has only one leaf X with codimension 0.
On the other hand, L is certainly not numerically trivial since it intersects a fiber of $X = \mathbb{P}(E)$ with intersection number 1. Consequently, the moving intersection number $(c_1(L))_{\geq 0} = c_1(L)$ is strictly positive, and $(X, c_1(L))$ is a counter example to equality of the numerically trivial foliation w.r.t. the positive closed $(1,1)$- current with minimal singularities and that w.r.t. the associated pseudo-effective cohomology class.
Now there is an obvious candidate for a numerically trivial foliation w.r.t. $c_1(L)$: its leaves are the projection of the curves $\mathbb{C} \times \{p\}$ in $\mathbb{P}_C(E)$. The strategy to show this has two parts: first, one constructs a sequence of currents $T_k \in c_1(L)[-\epsilon_k\omega]$ for some Kähler form ω on X and a sequence ϵ_k of positive real numbers tending to 0 such that the foliation mentioned above is the numerically trivial foliation w.r.t. this sequence of T_k's. Second, one uses that the restriction of the T_k's to any $\mathbb{P}^1$-fiber of $\mathbb{P}_C(E)$ is $\geq c\cdot\omega$, for some fixed number $c > 0$.
The construction of the T_k requires a careful study of almost positive (singular) hermitian metrics h on L: As the total space of L^{-1} is equal to E^* blown up along the zero section, the function

$$\phi(\zeta) = \log \parallel \zeta \parallel^2_{h^{-1}},\ \zeta \in L^{-1}$$

associated to any hermitian metric h on L can also be seen as a function on E^* satisfying the log-homogeneity condition

$$\phi(\lambda\zeta) = \log|\lambda| + \phi(\zeta) \text{ for every } \lambda \in \mathbb{C}.$$

One has

$$\frac{i}{2\pi}\partial\overline{\partial}\phi(\zeta) = \pi^*_{L^{-1}}\Theta_h(L),\ \pi_{L^{-1}} : L^{-1} \to X.$$

Thus $\Theta_h(L)$ is almost positive iff ϕ is almost plurisubharmonic on E^*.
The total space of E^* is the quotient $E^* = \mathbb{C} \times \mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$ by the dual action

$$\begin{aligned} g_1^*(x, w_1, w_2) &= (x+1, w_1, w_2) \\ g_\tau^*(x, w_1, w_2) &= (x+\tau, w_1, w_1 + w_2). \end{aligned}$$

The function ϕ gives rise to a function $\widetilde{\phi}$ on $\mathbb{C} \times \mathbb{C}^2$ which is invariant by g_1^*, g_τ^* and log-homogeneous w.r.t. (w_1, w_2), and $\widetilde{\phi}$ is almost plurisubharmonic iff ϕ is almost psh. Even more is true: Interpret X as the zero section of the total space of L^{-1} and let

$\omega_X, \omega_{L^{-1}}$ be positive $(1,1)$- forms on X, L^{-1}. Then there are constants $C_1, C_2 > 0$ such that

$$0 \le \pi^*_{L^{-1}}\omega_X \le C_1\omega_{L^{-1}},\ 0 \le \omega_{L^{-1}|X} \le C_2\omega_X.$$

Hence $\pi^*_{L^{-1}}\Theta_h \ge -\epsilon\omega_{L^{-1}}$ implies $\Theta_h \ge -\epsilon C_2\omega_X$, and $\Theta_h \ge -\epsilon\omega_X$ implies $\pi^*_{L^{-1}}\Theta_h \ge -\epsilon C_1\omega_{L^{-1}}$. Consequently, instead of constructing currents $T_k \ge -\epsilon_k\omega_X$, $\epsilon_k \to 0$ on X, it suffices to construct currents $\Theta_k \ge -\epsilon'_k\omega_{L^{-1}}$, $\epsilon'_k \to 0$, and functions $\widetilde{\phi}_k$ on $\mathbb{C} \times \mathbb{C}^2$ such that $i\partial\overline{\partial}\widetilde{\phi}_k = \Theta_k$ and the $\widetilde{\phi}_k$ are invariant by g_1^*, g_τ^* and log-homogeneous w.r.t. (w_1, w_2).
This can be done by using a gluing procedure developed in [**19**]: Choosing an appropriate partition of unity which is g_1^*- and g_τ^*- invariant and only depends on the imaginary part of x, one gets the desired almost plurisubharmonic functions $\widetilde{\phi}_k$ from plurisubharmonic functions

$$k\widetilde{\psi}_j = \frac{k}{2}\log(|w_1|^2 + |jw_1 + w_2|^2),\ k \in \mathbb{N},\ j \in \mathbb{Z},$$

defined on stripes of type

$$\{(x, w_1, w_2) : (j-a)\mathrm{Im}\,\tau < \mathrm{Im}\,x < (j-a+1)\mathrm{Im}\,\tau\},\ 0 \le a \le 1,$$

and the associated currents T_k have arbitrary small negative part for $k \to \infty$.
On the other hand, it follows from the construction that the restriction of the induced currents T_k to the $\mathbb{P}^1$-fibers of $X = \mathbb{P}(E)$ remain $> \epsilon\omega$ for some $\epsilon > 0$.
Let $T'_k \in \alpha[-\epsilon_k\omega]$ be another sequence of currents representing α. If $\Delta^2 \cong U \subset X$ is an open subset with coordinates z_1, z_2 such that the lines $\{z_1 = a\}$ belong to $\mathbb{P}^1$-fibers and $\{z_2 = b\}$ are subsets of the leaves of the foliation one can write

$$T_k + \epsilon_k\omega = \sum_{i,j=1}^{2}\theta_{ij}^{(k)} idz_i \wedge d\overline{z}_j,\ T'_k + \epsilon_k\omega = \sum_{i,j=1}^{2}\theta_{ij}^{\prime(k)} idz_i \wedge d\overline{z}_j.$$

By the remark above,

$$(\theta_{22}^{(k)})_{|\{z_1=a\}} idz_2 \wedge d\overline{z}_2 > \epsilon\omega$$

for all a, and

$$\widetilde{\theta}^{(k)} := \theta_{11}^{(k)} idz_1 \wedge d\overline{z}_1 + \theta_{12}^{(k)} idz_1 \wedge d\overline{z}_2 + \theta_{21}^{(k)} idz_2 \wedge d\overline{z}_1 \stackrel{k\to\infty}{\longrightarrow} 0$$

by the numerical triviality (use as before the Schwarz inequality for the terms with $\theta_{12}^{(k)}, \theta_{21}^{(k)}$).
Since the numerical dimension of L is 1, one knows furthermore that

$$\lim_{k\to\infty}\int_{X - \mathrm{Sing}\ T'_k} (T_k + \epsilon_k\omega) \wedge (T'_k + \epsilon_k\omega) = 0.$$

But

$$(T_k + \epsilon_k\omega) \wedge (T'_k + \epsilon_k\omega) = \widetilde{\theta}^{(k)} \wedge (T'_k + \epsilon_k\omega) + \theta_{22}^{(k)} idz_2 \wedge d\overline{z}_2 \wedge \theta_{11}^{\prime(k)} idz_1 \wedge d\overline{z}_1,$$

hence the vanishing of the limits above implies

$$\int_{(\Delta')^2-\mathrm{Sing}\ T'_k} \theta'^{(k)}_{11} i dz_1 \wedge d\overline{z}_1 \wedge i dz_2 \wedge d\overline{z}_2 \overset{k\to\infty}{\longrightarrow} 0,$$

where $\Delta' \subset\subset \Delta$ is any open disk such that $(\Delta')^2 \subset U \cong \Delta^2$.
Consequently, $\int_{\Delta'_b-\mathrm{Sing}\ T'_k}(T'_k + \epsilon_k\omega) \overset{k\to\infty}{\longrightarrow} 0$ for almost all $b \in \Delta'$ (where $\Delta'_b = \{b\} \times \Delta'$). The definition of the numerically trivial foliation requires that $\int_{\Delta'_b-\mathrm{Sing}\ T'_k}(T'_k + \epsilon_k\omega) \overset{k\to\infty}{\longrightarrow} 0$ for **all** $b \in \Delta'$. To prove this one can use the same line of arguments as in the proof of the Local Key Lemma for pseudo-effective classes: One tries to show that

$$\lim_{k\to\infty} \left| \int_{\Delta'_b-\mathrm{Sing}\ T'_k} (T'_k + \epsilon_k\omega) - \int_{\Delta'_0-\mathrm{Sing}\ T'_k} (T'_k + \epsilon_k\omega) \right| = 0.$$

Following the proof of proposition 6.4 one sees that it is enough to show that

$$\lim_{k\to\infty} \int_{\partial\Delta'\times\Delta_{0,b}} |\theta'^{(k)}_{11}| dV \cdot \int_{\partial\Delta'\times\Delta_{0,b}} |\theta'^{(k)}_{22}| dV = 0,$$

where $\Delta_{0,b}$ is the disk with center in $b/2$ and radius $|b/2|$, and dV is a volume element of $\partial\Delta' \times \Delta_{0,b}$.
As in the proof of proposition 6.4 there is a bound $M > 0$ such that for all k there is a disk $\Delta'_k \subset\subset \Delta$ containing Δ' with

$$\int_{\partial\Delta'_k\times\Delta'} |\theta'^{(k)}_{22}| dV < M.$$

For the first term, look at the $(1,1)$- form $\eta = i dz_2 \wedge d\overline{z}_2$ and take a disk $\Delta' \subset\subset \Delta'' \subset\subset \Delta$. Then by the arguments above,

$$\int_{(\Delta''-\Delta')\times\Delta'} |\theta^k_{11}| dV = \int_{(\Delta''-\Delta')\times\Delta'} (T'_k + \epsilon_k\omega) \wedge \eta \overset{k\to\infty}{\longrightarrow} 0.$$

By Fubini, one gets a disk Δ'_k such that

$$\int_{\partial\Delta'_k\times\Delta'} |\theta^k_{11}| dV \overset{k\to\infty}{\longrightarrow} 0,$$

and one concludes that the limit above is indeed 0.

REMARK 7.2. The difference to the previous example is that the unitary group is compact and consequently its Haar measure is finite. This is not the case for the group of linear automorphisms generated by $(z_1, z_2) \mapsto (z_1 + z_2, z_2)$.

7.3. $\mathbb{P}^2$ blown up in nine points

Consider the following situation: Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and let $p_1, \ldots, p_8 \in C$ be sufficiently general points. The aim is to study the numerically trivial foliation w.r.t. the anticanonical bundle $-K_X$ on varieties $X_p = \mathbb{P}^2(p_1, \ldots, p_8, p)$ blown up in points $p \in C$.

Let $E_i = \pi^{-1}(p_i)$ be the exceptional divisor on X over p_i. First of all, $-K_X = \mathcal{O}_{\mathbb{P}^2}(3) + \sum E_i$ is nef and $-K_X^2 = 0$. Next, the pencil of elliptic curves on $\mathbb{P}^2$ through $p_1, \ldots, p_8$ has a base point q. So $X_q = \mathbb{P}^2(p_1, \ldots, p_8, q)$ is an elliptic fibration $\pi_q : X_q \to \mathbb{P}^1$. The pull back of a smooth positive metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ gives a smooth semi-positive hermitian metric on $-K_{X_q}$ which is strictly positive in directions transverse to the fibers. Hence by the same arguments as in the two examples above, the fibration is the numerically trivial foliation w.r.t. $-K_{X_q}$.

For points $p \neq q$ in C there is only one section in $-K_{X_p}$, the strict transform C' of C. But if one considers torsion points (w.r.t. to q) of order m on C then a calculation in [**17**] shows that $-mK_{X_p}$ defines again an elliptic fibration over $\mathbb{P}^1$. This fibration yields a smooth semi-positive hermitian metric on $-mK_{X_p}$, hence on $-K_{X_p}$, and again the fibration is the numerically trivial foliation w.r.t. $-mK_{X_p}$.

The question is: What happens if non-torsion points $p \in C$ are blown up ? In particular: Is there always a smooth semi-positive hermitian metric on $-K_{X_p}$ inducing a holomorphic foliation on X_p, which may be seen as the limit of the fibrations of X_{p_k} where the p_k are torsion points ? (The last question was asked in [**17**].) A strategy to answer it is to use the theory of holomorphic foliations on surfaces, as developed e.g. in [**10**].

DEFINITION 7.3. *A (holomorphic) foliation $\mathcal{F}$ on a compact complex surface X is a coherent analytic rank 1 subsheaf $T_{\mathcal{F}}$ of the tangent bundle T_X (the tangent bundle of the foliation) fitting into an exact sequence*

$$0 \to T_{\mathcal{F}} \to T_X \to \mathcal{J}_Z \otimes N_{\mathcal{F}} \to 0$$

for a suitable invertible sheaf $N_{\mathcal{F}}$ (the normal bundle of the foliation) and an ideal sheaf $\mathcal{J}_Z$ whose zero locus consists of isolated points called the singularities $\operatorname{Sing}(\mathcal{F})$ of $\mathcal{F}$.

Furthermore, one can easily show that $T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}} = K_X$.

Numerically trivial foliations $\{\mathcal{F}, (U_i, p_i)\}$ on surfaces X with $\mathcal{F}$ of rank 1 are such foliations: If $\mathcal{F}$ is not a line bundle then replace it by $\mathcal{F}^{**}$. As a reflexive sheaf on a surface this is a line bundle [**33**, 1.1.10], and dualizing the inclusion $\mathcal{F} \subset T_X$ twice shows that it is still a subsheaf of T_X. Furthermore, $\mathcal{F}$ is locally integrable because it has rank 1, hence the maps p_i exist trivially.

Let $\mathcal{X}$ be $\mathbb{P}^2(p_1, \ldots, p_8) \times C$ blown up in the diagonal

$$\Delta_{C \times C} \subset C \times C \subset \mathbb{P}^2(p_1, \ldots, p_8) \times C.$$

The fibers of $\mathcal{X}$ over $p \in C$ are just the X_p for all p. If there is an algebraic family of foliations on the X_p such that over torsion points, the foliation coincides with the fibration described above, then (at least generically) the conormal line bundles $N^*_{F_p}$ should also fit into a family. But this is impossible, as the following computation shows:

LEMMA 7.4. *Let C, q, X_p be as above, and let p be a torsion point w.r.t. q of order m. Let N_{F_p} be the normal bundle of the foliation induced by the fibration $\pi_p : X_p \to \mathbb{P}^1$. Then*

$$N^*_{F_p} \cong (m+1)K_{X_p}.$$

PROOF. Let D be an irreducible component of a fiber of $\pi = \pi_p$ with multiplicity l_D. If η is a local non-vanishing 1- form on $\mathbb{P}^1$ then $\pi^*(\eta)$ is a local section of $\pi^*(K_{\mathbb{P}^1})$ vanishing of order $l_D - 1$ on D. Hence,

$$N^*_{F_p} = \pi^*(K_{\mathbb{P}^1}) \otimes \mathcal{O}_{X_p}(\sum (l_D - 1)D).$$

The relative canonical bundle formula (for elliptic fibrations, see [**23**]) tells that

$$K_{X_p} = \pi^*(K_{\mathbb{P}^1} \otimes (R^1\pi_*\mathcal{O}_{X_p})^*) \otimes \mathcal{O}_{X_p}(\sum (l_F - 1)F),$$

where the sum is taken over all fibers F occuring with multiplicity l_F in the fibration.
There are two differences between the two formulas: First, in the relative canonical bundle formula occurs the term

$$L := (R^1\pi_*\mathcal{O}_{X_p})^*.$$

Now, $\deg L \geq 0$, and $\deg L = 0$ would imply that L is a torsion bundle on $\mathbb{P}^1$, hence it is trivial, and $X_p = C \times \mathbb{P}^1$ – a contradiction. If L is nontrivial, a short calculation with spectral sequences shows that

$$0 = p_g = \deg L - g(\mathbb{P}^1) + 1,$$

hence $\deg L = 1$, and $L = \mathcal{O}_{\mathbb{P}^1}(1)$ (see again [**23**, Ch.VII]). This shows

$$\pi^*(K_{\mathbb{P}^1} \otimes L) = \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) = mK_{X_p},$$

and together with the relative canonical bundle formula this shows that mC is the only multiple fiber.
The second difference is that some fibers may contain multiple components, but are not multiple themselves. By the classification of singular fibers of elliptic fibrations this is only possible if there are -2-curves ([**23**]). But on $\mathbb{P}^2$ blown up in 9 points in general position, there are no -2-curves. Hence

$$\mathcal{O}_{X_p}(\sum (l_D - 1)D) = \mathcal{O}_{X_p}(\sum (l_F - 1)F),$$

and the claim of the lemma follows. $\square$

The threefold $\mathcal{X}$ is also a counter example to equality of numerical dimension and codimension of the leaves of the numerically trivial foliation w.r.t. some pseudo-effective class: Set

$$L := -\pi^*(p_1^* K_{\mathbb{P}^2(p_1,\ldots,p_8)}) - E_\Delta + p_2^*\mathcal{O}(nr),$$

where p_1 is the projection of $\mathbb{P}^2(p_1, \ldots, p_8) \times C$ onto $\mathbb{P}^2(p_1, \ldots, p_8)$, p_2 is the projection of $\mathcal{X}$ onto C, r is any point on C and $n > 0$ an integer. The restriction of L to any fiber over $p \in C$ is the anti-canonical bundle $K^*_{X_p}$.

For n sufficiently big, L is nef: L is effective, since $D = C \times (C + nX_r)$ is contained in $|L|$. Consequently, to prove the nefness of L it suffices to show that all curves $E \subset C \times C$ have non-negative intersection number with L. To this purpose first get an overview over all curves on $C \times C$: According to the general theory of abelian surfaces the Picard number of $C \times C$ is 4 or 3 depending on whether C has complex multiplication or not ([**5**, 2.7]. Hence it suffices to look at the fibers of the two projections of $C \times C$ onto C, the diagonal, and if necessary, on some other curve constructed as the graph of complex multiplication in $C \times C$. Since it is a graph of an isomorphism, such a curve maps isomorphically to C under both projections.

Now, one has to compute the degree of the restriction of L to E. This restriction may also be seen as the restriction of the divisor $D_{|D}$ to such an E. Let C' be a sufficiently general curve in the pencil $| - K_{\mathbb{P}^2(p_1,\ldots,p_8)}|$. Then the strict transform of $C' \times C$ is an element of $-\pi^*(p_1^* K_{\mathbb{P}^2(p_1,\ldots,p_8)})$ and intersects $C \times C$ in $\{q\} \times C$. Furthermore, E_Δ intersects $C \times C$ in the diagonal $\Delta_{C\times C}$. Therefore,

$$D_{|D} \sim \{q\} \times C + n(C' \times \{r\}) - \Delta_{C\times C} + n(C \times \{r\}),$$

where E_r is the exceptional divisor over r in X_r. And L is nef if n is $\geq$ the maximum of 1 (this is the intersection number of fibers $C \times \{p\}$ with the diagonal) and the intersection number of the curve coming from complex multiplication (if existing) with the diagonal. (The self intersection number of the diagonal is 0 since the tangent bundles on $C \cong \Delta_{C\times C}$ and $C \times C$ are trivial.)

PROPOSITION 7.5. *Let $\mathcal{X}, L$ be as above. Then the numerical dimension $\nu(L)$ of L is 2, but the numerically trivial foliation w.r.t. $c_1(L)$ is the identy map.*

PROOF. To prove $L^2 \neq 0$, observe that L^2 is represented by the cycles in the expression above for $D_{|D}$. This is not $\equiv 0$, since the intersection number with $\{q\} \times C$ is positive for $n \geq 1$.

The numerically trivial foliation w.r.t. $c_1(L)$ cannot be the trivial map onto a point, because in fibers X_p over torsion points p there are curves which are not numerically trivial. Since immersed disks which do not lie in a fiber of the projection onto C are not numerically trivial, the only possible numerically trivial foliation w.r.t. $c_1(L)$ with 2-dimensional leaves is the fibration onto C. But this is impossible by the same reason as above. To exclude the possibility that the numerically trivial foliation has 1-dimensional

leaves, one notes first that over torsion points p, the fibers of $\pi_p : X_p \to \mathbb{P}^1$ are numerically trivial: This is clear since these fibers F are projective, hence $\int_F T_k$ only depends on the cohomology class of the T_k, and $\int_F c_1(L)$ is certainly 0.
This can be used to show that the 1-dimensional leaves of a numerically trivial foliation must lie in the fibers X_p of $\mathcal{X}$: Otherwise, let $\Delta^3 \cong U \subset \mathcal{X}$ be any open subset with coordinates x, z_1, z_2 such that the projection onto C is given by the projection onto the first coordinate, and the foliation is described by the projection onto the two last coordinates. Choose x such that $x = 0$ corresponds to a torsion point p_0. Shrinking U if necessary, one can suppose that the fibers of π_{p_0} are smooth in U. But then the Local Key Lemma for pseudo-effective classes implies that there are 2- dimensional numerically trivial leaves, contradiction.
Next one shows that the 1-dimensional leaves in fibers X_p, where p is a torsion point, must be the fibers of $\pi_p : X_p \to \mathbb{P}^1$: Take an ample line bundle A on $\mathcal{X}$. Since L is nef, $L^k \otimes A$ is also ample, and some multiple is very ample. The global sections of this very ample line bundle generate a smooth metric on $L^k \otimes A$ whose strictly positive curvature form may be written as $k(T_k + \frac{1}{k}\omega_A)$, for some form $T_k \in c_1(L)[-\frac{1}{k}\omega_A]$.
Let $p \in C$ be any torsion point of order m and $\pi_p : X_p \to \mathbb{P}^1$ the induced fibration. Let $T = i\partial\overline{\partial}\log(|z_1|^2 + |z_2|^2)$ be a strictly positive curvature form in $c_1(\mathcal{O}^1_{\mathbb{P}}(1))$. Then

$$(T_k + \frac{1}{k}\omega_A)_{|X_p} \geq \frac{1}{m}\pi_p^* T.$$

But this means in particular that for any disk $\Delta \subset X_p$ not immersed into a fiber of π_p,

$$\int_\Delta T_k + \frac{1}{k}\omega_A \geq \frac{1}{m}\int_\Delta \pi_p^* T > 0.$$

Hence the leaves of the numerically trivial foliation w.r.t. $c_1(L)$ coincide with the fibers of π_p in X_p.
But this is impossible, as shown above. □

REMARK 7.6. This proposition does not exclude the possibility that (some of) the X_p over non-torsion points p have a numerically trivial foliation with 1-dimensional leaves.

Another result dealing with this type of foliations is

PROPOSITION 7.7 (Brunella). *Let $\mathcal{F}$ be a foliation on a compact algebraic surface X and suppose that $\mathcal{F}$ is tangent to a smooth elliptic curve E, free of singularities of $\mathcal{F}$. Then either E is a (multiple) fiber of an elliptic fibration or, up to ramified coverings and birational maps, $\mathcal{F}$ is the suspension of a representation $\rho : \pi_1(\widehat{E}) \to \mathrm{Aut}(\mathbb{CP}^1)$, $\widehat{E}$ an elliptic curve.*

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