A WEAK KAWAMATA-VIEHWEG VANISHING THEOREM ON COMPACT KÄHLER MANIFOLDS

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1. INTRODUCTION

Using a subtle version of the Bochner technique, J.-P. Demailly and Th. Peternell were able to prove one instance of the Kawamata-Viehweg Vanishing Theorem for (even singular) Kähler manifolds:

Theorem 1.1 ([DP02, Thm. 0.1]). *Let X be a normal compact Kähler space of dimension n and L a nef line bundle on X. Assume that $L^2 \neq 0$. Then*

$$H^q(X, K_X + L) = 0$$

for $q \geq n - 1$.

The aim of this paper is to use the same methods to prove a weak version of Kawamata-Viehweg Vanishing on compact Kähler manifolds for all $q > n - \nu(L)$, which also works for pseudo-effective line bundles. It is only a weak version, because we have to tensorize $K_X + L$ with the **upper regularized multiplier ideal sheaf** of a singular hermitian metric on L:

Definition 1.2. *Let X be a compact Kähler manifold of dimension n and L a pseudo-effective line bundle on X. Let $h_{\min}$ be a hermitian metric with minimal singularities among all positive singular hermitian metrics on L. Then the **upper regularized multiplier ideal sheaf** $\mathcal{J}_+(L)$ is defined as*

$$\mathcal{J}_+(L) := \bigcup_{\epsilon \to 0} \mathcal{J}(h_{\min}^{1+\epsilon}).$$

This multiplier ideal $\mathcal{J}_+(L)$ is certainly not optimal: there are examples of nef line bundles where it is not trivial, see [DPS94, Ex.1.7]. At least, it is conjectured (and true in dimension 1 and 2) that it equals the ordinary multiplier ideal $\mathcal{J}(L) := \mathcal{J}(h_{\min})$.

So the main result of this paper will be

Theorem 1.3. *Let X be a compact Kähler manifold of dimension n and L a pseudo-effective line bundle on X of numerical dimension $\nu = \nu(L)$. Then*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}_+(L)) = 0$$

for $q \geq n + 1 - \nu(L)$.

It will be proven as a corollary of

Theorem 1.4. *Let X be a compact Kähler manifold of dimension n and L a pseudo-effective line bundle on X of numerical dimension $\nu = \nu(L)$ with a positive hermitian metric $h_{\min}$ with minimal singularities on L. For every $\epsilon' > 0$ there exists an $0 < \epsilon < \epsilon'$ such that the homomorphism*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}(h_{\min}^{1+\epsilon'})) \to H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}(h_{\min}^{1+\epsilon}))$$

induced by the inclusion $\mathcal{J}(h_{\min}^{1+\epsilon'}) \subset \mathcal{J}(h_{\min}^{1+\epsilon})$ vanishes for $q \geq n + 1 - \nu(L)$.

This theorem implies theorem 1.3, since the ascending chain of ideal sheaves $\mathcal{J}(h_{\min}^{1+\epsilon})$, $\epsilon \to 0$, gets stable at some point by the Noetherian property, hence for ϵ' small enough

$$\mathcal{J}(h_{\min}^{1+\epsilon'}) = \mathcal{J}(h_{\min}^{1+\epsilon}) = \mathcal{J}_+(L).$$

But then the homomorphism between the cohomology groups becomes an isomorphism, and the involved vector spaces must be 0.

The Bochner technique consists of using the Bochner-Kodaira-Nakano inequality to prove the vanishing of certain cohomolgical classes, see [Dem00, §4] for an application to classical vanishing theorems. The main obstacle to use this technique in our case is that the Bochner inequality is only true for *smooth* metrics. This was circumvented by Demailly in [Dem82] who observed that for a compact Kähler manifold X and $Z \subset X$ an analytic subset, $X \setminus Z$ has a complete Kähler metric. Using Hörmander's confirmation of the Bochner inequality for complete metrics (and elements of certain function spaces) in [Hör65] Demailly constructed sequences of complete metrics converging to the original Kähler metric and got vanishing results by going to the limit. This limit process is successful despite of the metrics changing all the time, since there is a uniform bound for all the occuring norms, and because one can compare the different metrics.

So in our case we start with a careful construction of singular hermitian metrics h_ϵ on L which are smooth outside an analytic subset Z_ϵ (section 2). They are composed of a (sufficiently small) part controlling the eigenvalues of the curvature form (they are essential for the Bochner inequality) and another part controlling the multiplier ideal sheaf (and hence the singularities) of the metric. The first is produced by applying the Calabi-Yau theorem as in Boucksom's thesis [Bou02], the latter is constructed with the equisingular approximation theorem of [DPS01].

Then we use the Bochner technique for complete metrics on $X \setminus Z_\epsilon$ converging to the starting Kähler metric ω and go to the limit, using the uniform estimate in section 5 and comparing the different metrics following the results in section 3 (which more or less repeat the inequalities in [Dem82, §3]).

Remark. Since the technical details of the strategy described above are quite intricate and treated in a very summary way in [DP02], the author decided to give all the steps in full details, for his own safety and for the convenience of the not so experienced reader – of course without claiming any originality. The expert may skip section 3 alltogether and skim over the functional analytic details in section 4.

2. The construction of the metrics

As explained in the introduction, we construct metrics $\hat{h}_\epsilon$ for arbitrarily small ϵ composed of two parts, and the first part is produced by using

Theorem 2.1 (Approximative Singular Calabi-Yau theorem). *Let X be a compact Kähler manifold of dimension n with Kähler form ω such that $\int_X \omega^n = 1$, and let*

$\alpha \in H^{1,1}(X, \mathbb{R})$ *be a big class containing a Kähler current $\geq \delta\omega$. Then for every $\epsilon > 0$ there exists a closed positive current $T_\epsilon \in \alpha$ with analytic singularities such that*

$$T_\epsilon(x)^n \geq (1 - \delta_\epsilon)v(\alpha)\omega(x)^n$$

almost everywhere, $\delta_\epsilon \to 0$ if ϵ tends to 0, and the multiplier ideal $\mathcal{J}(T_\epsilon)$ contains $\mathcal{J}(T_{\min})$ for a current $T_{\min}$ in α with minimal singularities in $\alpha[\delta\omega]$.

Here, $v(\alpha)$ denotes the volume of the class α, defined by Boucksom [Bou02, 3.1.6] as

$$v(\alpha) := \sup_T \int_X T_{ac}^n,$$

where the T's run through all closed positive currents in α and T_{ac} is the absolute continuous part of the Lebesgue decomposition of T ([Bou02, 3.1.1]). Since α is big, there is a closed positive current in α bigger than $\epsilon\omega$ for some ϵ, and $v(\alpha)$ is positive.

The proof of the theorem is contained in Boucksom's construction of the current T (with arbitrary singularities) solving the Monge-Ampère equation

$$T_{ac}(x)^n = v(\alpha)\omega(x)^n$$

almost everywhere ([Bou02, Thm. 3.1.23]). His idea was of course to use the ordinary form of the Calabi-Yau theorem (where α contains a Kähler *form* and T will be a form satisfying the Monge-Ampère equation *everywhere*). To be able to do this, he proved a singular version of Fujita's theorem about the approximative Zariski decomposition ([Bou02, Thm. 3.1.24]):

Theorem 2.2 (Singular Fujita decomposition). *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class. For all $\epsilon > 0$ there exists a sequence of blow ups with smooth centers $\mu : \widetilde{X} \to X$ and a decomposition*

$$\mu^*\alpha = \beta + \{E\},$$

where β is a Kähler class and E is an effective $\mathbb{R}-$ divisor such that $|v(\alpha) - v(\beta)| < \epsilon$ and $\mathcal{J}(\mu_*[E_k]) \supset \mathcal{J}(T_{\min})$.

Proof. The additional statement compared to [Bou02, Thm. 3.1.24] is the inclusion of the multiplier ideal sheaves. To get it, we must go through the proof of Boucksom: It starts with a sequence of Kähler currents $T_k \in \alpha$ with analytic singularities such that

$$\int_X T_{k,ac}^n \to v(\alpha).$$

These T_k may be chosen in such a way that $\mathcal{J}(T_k) \supset \mathcal{J}(T_{\min})$: Apply Theorem 3.16 in [Eck03] to $\alpha - \delta\omega$ and get a sequence of currents $T_k' \in \alpha - \delta\omega$ with analytic singularities such that

$$\int_X (T_{k,ac}')^n \to \nu(\alpha - \delta\omega).$$

Note that by construction the potentials of T_k' are not less than the potential of a current T_{min}' with minimal singularities in $(\alpha - \delta\omega)[0]$, hence $\mathcal{J}(T_k') \supset \mathcal{J}(T_{min}')$. Now set $T_k := T_k' + \delta\omega$ and $T_{min} := T_{min}' + \delta\omega$.

Next, we resolve the singularities of T_k and find a sequence of blow ups in smooth centers, $\mu_k : \widetilde{X} \to X$, such that the Siu decomposition ([Dem00, (2.18)])

$$\mu^* T_k = [E_k] + R_k$$

consists of the integration current of an effective $\mathbb{R}$-divisor E_k and a smooth positive residue current R_k. Since T_k is big, $\mu^* T_k$ is also big [Bou02, Prop.1.2.5], hence there is a $\delta > 0$ such that for a Kähler form $\widetilde{\omega}$ on $\widetilde{X}$

$$\mu^* T_k = R_k + [E_k] \geq \delta \widetilde{\omega}.$$

This inequality remains valid, when we subtract an integration current of divisors, hence $R_k \geq \delta \omega$. But R_k is smooth, hence is a Kähler form. Furthermore, we have $\int_X T_{k,ac}^n = \int_{\widetilde{X}} R_k^n$. Finally, as T_k is a current with analytic singularities, its potential may be locally written as

$$\phi_k = \theta_k + c \cdot \log(\sum |f_i|^2),$$

where θ_k is a $\mathcal{C}^\infty$ function, and an easy calculation shows that

$$\mu^* \phi_k = \mu^* \theta_k + c \cdot \log(\sum |f_i'|^2) + c \cdot \log |g_k|^2,$$

where g_k is a local equation of E_k and $\sum |f_i'|^2$ vanishes nowhere. Consequently, $\mathcal{J}(\mu_*[E_k]) = \mathcal{J}(T_k)$, and

$$\mu^* \alpha = \{R_k\} + \{E_k\}$$

is the desired decomposition, if we choose k big enough. $\square$

To prove the approximative singular Calabi-Yau theorem, let us take a modification $\mu : \widetilde{X} \to X$ and a decomposition $\mu^* \alpha = \beta + \{E\}$ belonging to ϵ as above. Let $\widetilde{\omega}$ be a Kähler form on $\widetilde{X}$, and set $\widetilde{\omega}_\delta = \mu^* \omega + \delta \widetilde{\omega}$. For all $\delta > 0$, $\widetilde{\omega}_\delta$ is a Kähler form on $\widetilde{X}$, hence by the usual Calabi-Yau theorem, we can find a Kähler form $\theta_{\epsilon,\delta} \in \beta$ such that

$$\theta_{\epsilon,\delta}(x)^n = \frac{v(\beta)}{\int \widetilde{\omega}_\delta^n} \widetilde{\omega}(x)^n$$

for all $x \in \widetilde{X}$. Now set $T_\epsilon := \mu_*(\theta_{\epsilon,\delta} + [E_\epsilon])$ which is a closed positive current with analytic singularities in $\mu(E_\epsilon)$. Furthermore, choosing δ small enough and using the properties of β and $\theta_{\epsilon,\delta}$, we see that

$$T_\epsilon(x)^n \geq (1 - \delta_\epsilon) v(\alpha) \omega(x)^n$$

almost everywhere, and $\delta_\epsilon \to 0$ if ϵ tends to 0. Since the multiplier ideals only depend on $[E_\epsilon]$, the inclusion of multiplier ideals in the theorem remains true. $\square$

The construction of the second part of $\hat{h}_\epsilon$ uses

Theorem 2.3 (Equisingular Approximation). *Let $T = \alpha + i\partial\overline{\partial}\phi$ be a closed $(1,1)$-current on a compact hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and ϕ a quasi-plurisubharmonic function. Let γ be a smooth real $(1,1)$-form such that $T \geq \gamma$. Then one can write $\phi = \lim_{\nu \to +\infty} \phi_\nu$ where*

  (a) *ϕ_ν is smooth in the complement $X \setminus Z_\nu$ of an analytic subset $Z_\nu \subset X$;*
  (b) *(ϕ_ν) is a decreasing sequence, and $Z_\nu \subset Z_{\nu+1}$ for all ν;*
  (c) *for every $t > 0$*

$$\int_X (e^{-2t\phi} - e^{-2t\phi_\nu}) dV_\omega$$

  is finite for ν large enough and converges to 0 as $\nu \to +\infty$;
  (d) *$\mathcal{J}(t\phi_v) = \mathcal{J}(t\phi)$ for ν large enough ("equisingularity");*
  (e) *$T_\nu = \alpha + i\partial\overline{\partial}\phi_\nu$ satisfies $T_\nu \geq \gamma - \epsilon_\nu \omega$, where $\lim_{\nu \to +\infty} \epsilon_\nu = 0$.*

Proof. See [DPS01] or [Dem00, (15.2.1)] and especially the remark after the proof.
□

Now, let X be a compact n-dimensional Kähler manifold with Kähler form ω and let L be a pseudo-effective line bundle having a hermitian metric h_∞ with a curvature form $\Theta_{h_\infty}(L)$ of arbitrary sign. Let T_ϵ be a closed current with analytic singularities in $c_1(L)[-\epsilon\omega]$ such that

$$(T_\epsilon + \epsilon\omega)^n \geq \frac{v(c_1(L) + \epsilon\omega)}{2}\omega^n,$$

as in the approximative singular Calabi-Yau theorem above. In particular $\mathcal{J}(T_\epsilon + \epsilon\omega) \supset \mathcal{J}(T_{min})$ where T_{min} is a current with minimal singularities in $(c_1(L) + \epsilon\omega - \epsilon\omega)[0] = c_1(L)[0]$.

There exists a hermitian metric $h_\epsilon = h_\infty e^{-2\phi_\epsilon}$ on L, such that

$$\Theta_{h_\epsilon}(L) = T_\epsilon$$

([Bon95],[Eck03, Lem.4.1]). Next, we observe that

$$v(c_1(L) + \epsilon\omega) \geq \epsilon^{n-l} \cdot (c_1(L)^l.\omega^{n-l})_{\geq 0}$$

for all $0 \leq l \leq n$ ([Bou02, p.86]), hence there is a constant $C > 0$ such that

$$v(c_1(L) + \epsilon\omega) \geq C\epsilon^{n-\nu(L)}.$$

Let $h_{min} = h_\infty e^{-2\psi}$ be a metric with $\Theta_h(L) \geq 0$ and minimal singularities, and let $\psi_\epsilon \downarrow \psi$ be an equisingular regularization of ψ, such that

$$\widetilde{h}_\epsilon := h_\infty e^{-2\psi_\epsilon}$$

satisfies $\Theta_{\widetilde{h}_\epsilon} \geq -\epsilon\omega$ in the sense of currents. The metrics considered in the follwing are given by

$$\widehat{h}_\epsilon^{1+s} = h_\infty \exp(-2(\delta(1+s)\phi_\epsilon + (1-\delta)(1+s)\psi_\epsilon))$$

where $\delta > 0$ is a sufficiently small number which will be fixed later. Note that $\widehat{h}_\epsilon^{1+s}$ is really a metric on L, since h_∞ remains unchanged.

$\widehat{h}_\epsilon^{1+s}$ is smooth outside an analytic subset $Z_\epsilon \subset X$. Its multiplier ideal is controlled by subadditivity ([Dem00, (14.2)]):

$$\mathcal{J}(\widehat{h}_\epsilon^{1+s}) \subset \mathcal{J}(h_\epsilon^{\delta(1+s)}) \cdot \mathcal{J}(\widetilde{h}_\epsilon^{(1-\delta)(1+s)}) \subset \mathcal{J}(\widetilde{h}_\epsilon^{(1-\delta)(1+s)}) = \mathcal{J}(h_{min}^{(1-\delta)(1+s)})$$

because of the equisingularity. Locally, the Hölder inequality shows that

$$\int |f|^2 e^{-2[\delta(1+s)\phi_\epsilon + (1-\delta)(1+s)\psi_\epsilon]}dV_\omega \leq$$

$$(\int |f|^2 e^{-2(1+s)\phi_\epsilon}dV_\omega)^\delta \cdot \int |f|^2 e^{-2(1+s)\psi_\epsilon}dV_\omega)^{1-\delta},$$

hence

$$\mathcal{J}(h_\epsilon^{1+s}) \cap \mathcal{J}(h_{min}^{1+s}) = \mathcal{J}(h_{min}^{1+s}) \subset \mathcal{J}(\widehat{h}_\epsilon^{1+s}),$$

where the first equality comes from the properties of h_ϵ.

Finally, we check how the eigenvalues of the curvature form are controlled: By construction,

$$\begin{aligned}
\Theta_{\widehat{h}_\epsilon} + 2\epsilon\omega &= \delta(\Theta_{h_\epsilon}(L) + \epsilon\omega) + (1-\delta)(\Theta_{\widetilde{h}_\epsilon}(L) + \epsilon\omega) + \epsilon\omega \\
&\geq \delta(\Theta_{h_\epsilon}(L) + \epsilon\omega) + \epsilon\omega.
\end{aligned}$$

At each point $x \in X \setminus Z_\epsilon$, we may choose coordinate systems $(z_j)_{1 \leq j \leq n}$ resp. $(w_j)_{1 \leq j \leq n}$ which diagonalize simultaneously the hermitian forms $\omega(x)$ and $T_\epsilon + \epsilon\omega$ resp. $\Theta_{\hat{h}_\epsilon} + 2\epsilon\omega$, in such a way that

$$\omega(x) = i \sum_{1 \leq j \leq n} dz_j \wedge d\overline{z}_j, \ (T_\epsilon + \epsilon\omega)(x) = i \sum_{1 \leq j \leq n} \lambda_j^{(\epsilon)} dz_j \wedge d\overline{z}_j$$

resp.

$$\omega(x) = i \sum_{1 \leq j \leq n} dw_j \wedge d\overline{w}_j, \ (\Theta_{\hat{h}_\epsilon} + 2\epsilon\omega)(x) = i \sum_{1 \leq j \leq n} \widehat{\lambda}_j^{(\epsilon)} dw_j \wedge d\overline{w}_j.$$

Let $\lambda_1^{(\epsilon)} \leq \ldots \leq \lambda_n^{(\epsilon)}$ and $\widehat{\lambda}_1^{(\epsilon)} \leq \ldots \leq \widehat{\lambda}_n^{(\epsilon)}$. Changing from z_j to w_j by a unitary transformation, the $\lambda_j^{(\epsilon)}$'s remain the same, and the inequality between the currents from above implies

$$\widehat{\lambda}_j^{(\epsilon)} \geq \delta\lambda_j^{(\epsilon)} + \epsilon,$$

by Weyl's monotonicity principle [Bha01, p.291]. On the other hand, the Monge-Ampère inequality satisfied by T_ϵ tells us that

$$\lambda_1^{(\epsilon)} \cdots \lambda_n^{(\epsilon)} \geq C \cdot \epsilon^{n - \nu(L)}$$

almost everywhere on X.

3. Comparisons of the metrics

Let Z_ϵ be the analytic subset such that $\widehat{h}_\epsilon$ is smooth on $X \setminus Z_\epsilon$. By [Dem82, Prop.1.6], for every $\epsilon > 0$ there is a sequence of complete Kähler metrics $(\omega_{\epsilon,t})$ on $X \setminus Z_\epsilon$ converging from above against $\omega_\epsilon = \omega$.

Let $\mathcal{D}_{c,\epsilon}^{n,q}$ be the space of all (n,q)- forms with values in L and coefficients in $\mathcal{J}(\hat{h}_\epsilon^{1+s}) \otimes \mathcal{C}^\infty(X)$ and compact support in $X \setminus Z_\epsilon$. Let $L_{\epsilon,t}^{n,q}$ be the L^2- completion of $\mathcal{D}_{c,\epsilon}^{n,q}$ with respect to the norm

$$\|u\|_{\epsilon,t}^2 := \int_{X \setminus Z_\epsilon} |u|_{\bigwedge^{n,q} \omega_{\epsilon,t} \otimes \hat{h}_\epsilon^{1+s}}^2 dV_{\omega_{\epsilon,t}},$$

including the case $t = 0$, where $\bigwedge^{n,q} \omega_{\epsilon,t} \otimes \hat{h}_\epsilon^{1+s}$ denotes the metric on (n,q)- forms with values in L and coefficients in $\mathcal{J}(\hat{h}_\epsilon^{1+s}) \otimes \mathcal{C}^\infty(X)$ naturally induced by $\omega_{\epsilon,t}$ and $\hat{h}_\epsilon^{1+s}$. The volume form $dV_{\omega_{\epsilon,t}}$ equals $\frac{\omega_{\epsilon,t}^n}{n!}$.

The operator $\overline{\partial}$ defines a linear, closed, densely defined operator

$$\overline{\partial}_{\epsilon,t} : L_{\epsilon,t}^{n,q} \to L_{\epsilon,t}^{n,q+1}.$$

An element $u \in L_{\epsilon,t}^{n,q}$ is in the domain $D_{\overline{\partial}_{\epsilon,t}}$ if $\overline{\partial}(u)$, defined in the sense of distribution theory, belongs to $L_{\epsilon,t}^{n,q+1}$. That $\overline{\partial}_{\epsilon,t}$ is closed follows from the fact that differentiation is a continuous operation in distribution theory, and the domain is dense since it contains $\mathcal{D}_{c,\epsilon}^{n,q}$.

Since $\overline{\partial}_{\epsilon,t}$ is densely defined, there is an adjoint operator $\overline{\partial}_{\epsilon,t}^*$, and because $\overline{\partial}_{\epsilon,t}$ is closed,

$$(\overline{\partial}_{\epsilon,t}^*)^* = \overline{\partial}_{\epsilon,t}.$$

(cf. [SN67, p.29]). Let $D_{\overline{\partial}_{\epsilon,t}^*}$ denote the domain of the operator $\overline{\partial}_{\epsilon,t}^*$ in $L_{\epsilon,t}^{n,q}$.

Let Λ be the adjoint of the operator L which multiplicates with ω, that is

$$L\alpha = \omega \wedge \alpha, \quad \langle \Lambda\alpha | \beta \rangle_{\epsilon,t} = \langle \alpha | \ \omega \wedge \beta \rangle_{\epsilon,t}.$$

for all forms $\alpha, \beta \in L^{n,q}_{\epsilon,t}$. (The scalar product above is taken in every point $z \in X \setminus Z_\epsilon$.)

If θ is a real $(1,1)$-form we define for all $q = 1, \ldots, n$ a sesquilinear form θ_q on the fibers of $\Omega^{n,q}_{X \setminus Z_\epsilon} \otimes L$ by setting in every point $z \in X \setminus Z_\epsilon$

$$\theta_q(\alpha, \beta) = \langle \theta \Lambda \alpha | \beta \rangle_{\epsilon,t}$$

for all $\alpha, \beta \in \Omega^{n,q}_{X \setminus Z_\epsilon, z} \otimes L_z$. If $\theta = \Theta_{\epsilon,t}$ is the curvature form of the metric $\hat{h}^{1+s}_\epsilon$ on L, a term of the form $\langle \theta \Lambda \alpha | \beta \rangle_{\epsilon,t}$ occurs in the Bochner-Kodaira inequality:

$$\| \overline{\partial}_{\epsilon,t} u \|^2_{\epsilon,t} + \| \overline{\partial}^*_{\epsilon,t} u \|^2_{\epsilon,t} \geq \int_X \langle \Theta_{\epsilon,t} \Lambda u | u \rangle_{\epsilon,t} dV_{\omega_{\epsilon,t}}.$$

On $\mathcal{D}^{n,q}_{c,\epsilon}$, this inequality is valid by the usual computations ([Dem00, (4.7)]). Hörmander ([Hör65, Lem. 5.2.1]) showed that for the complete metric $\omega_{\epsilon,t}$ ($t > 0$) the forms in $\mathcal{D}^{n,q}_{c,\epsilon}$ are dense in $D_{\overline{\partial}_{\epsilon,t}} \cap D_{\overline{\partial}^*_{\epsilon,t}}$ w.r.t. the graph norm

$$u \mapsto \| u \|^2_{\epsilon,t} + \| \overline{\partial}_{\epsilon,t} u \|^2_{\epsilon,t} + \| \overline{\partial}^*_{\epsilon,t} u \|^2_{\epsilon,t} .$$

Hence we have the Bochner inequality for all $u \in D_{\overline{\partial}_{\epsilon,t}} \cap D_{\overline{\partial}^*_{\epsilon,t}}$.

To really apply the Bochner technique we still need some comparative inequalities between the different metrics $\omega_{\epsilon,t}$:

Lemma 3.1. $\| u \|_{\epsilon,t'} \leq \| u \|_{\epsilon,t}$ for all $u \in L^{n,q}_{\epsilon,t}$ and all $0 \leq t \leq t'$.

Proof. This is just Lemma 3.3 in [Dem82]. $\qquad\square$

Consequently, we have a linear continuous operator $f_q : L^{n,q}_{\epsilon,t} \to L^{n,q}_{\epsilon,t'}$ with norm $\| f_q \| \leq 1$.

Lemma 3.2. *The diagram*

$$
\begin{array}{ccc}
L^{n,q}_{\epsilon,t} & \overset{f_q}{} & L^{n,q}_{\epsilon,t'} \\[4pt]
\overline{\partial}_{\epsilon,t} & \# & \overline{\partial}_{\epsilon,t'} \\[4pt]
L^{n,q+1}_{\epsilon,t} & \underset{f_{q+1}}{} & L^{n,q+1}_{\epsilon,t'}
\end{array}
$$

is commutative.

Proof. Let $v_{\epsilon,t}$ be any element of $L^{n,q}_{\epsilon,t}$ such that $\overline{\partial}_{\epsilon,t}$ is defined. Since $\mathcal{D}^{n,q}_{c,\epsilon}$ is dense in $D_{\overline{\partial}_{\epsilon,t}}$ there is a sequence of smooth forms $(v^{(n)}_{\epsilon,t})_{n \in \mathbb{N}}$ in $\mathcal{D}^{n,q}_{c,\epsilon}$ such that

$$v^{(n)}_{\epsilon,t} \to v_{\epsilon,t}, \quad \overline{\partial} v^{(n)}_{\epsilon,t} \to \overline{\partial}_{\epsilon,t} v_{\epsilon,t}$$

strongly in the (ϵ, t)-norm. Now, the derivation of smooth forms in $\mathcal{D}^{n,q}_{c,\epsilon}$ w.r.t. $\overline{\partial}_{\epsilon,t}$ and $\overline{\partial}_{\epsilon,t'}$ do not differ. So the two limits above exist and are the same for the (ϵ, t')-norm because of lemma 3.1. $\qquad\square$

Again, let θ be any real $(1,1)$-form. If $\alpha \in L^{n,q}_{\epsilon,t}$ we define $|\alpha|_\theta$ in every point $z \in X \setminus Z_\epsilon$ as the smallest number ≥ 0 (perhaps infinite) such that

$$|\langle \alpha | \beta \rangle_{\epsilon,t}|^2 \leq |\alpha|^2_\theta \langle \theta \Lambda \alpha | \beta \rangle_{\epsilon,t}$$

for all $\beta \in L^{n,q}_{\epsilon,t}$.

Lemma 3.3. *The (n,n)-form $|\alpha|^2_\theta dV_{\omega_{\epsilon,t}}$ decreases if t increases.*

Proof. This is just Lemma 3.2 in [Dem82]. $\qquad\square$

Lemma 3.4. *For all $\beta \in L_{\epsilon,t}^{n,q}$, we have*

$$\int_X |\beta|_\theta^2 dV_{\omega_{\epsilon,t}} \le \int_X \frac{1}{\lambda_1 + \ldots + \lambda_q} |\beta|_{\omega_{\epsilon,t}}^2 dV_{\omega_{\epsilon,t}},$$

where $\lambda_1, \ldots, \lambda_q$ are the q smallest eigenvalues of θ with respect to $\omega_{\epsilon,t}$.

Proof. There is an orthonormal base $dz_1, \ldots, dz_n$ of $\Omega_{X \setminus Z_\epsilon, z}^{n,q}$ such that we can write

$$\omega_{\epsilon,t} = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j,$$

$$\theta = \frac{i}{2} \sum_{j=1}^n \lambda_j dz_j \wedge d\bar{z}_j, \quad \lambda_j \in \mathbb{R}.$$

$\beta \in L_{\epsilon,t}^{n,q}$ may be written as

$$\beta = \sum_{|J|=q} \beta_J dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_J \otimes e,$$

where e is any section in L which makes the $dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_J \otimes e$ orthonormal in $L_{\epsilon,t}^{n,q}$. We verify

$$\Lambda\beta = 2 \sum_{|J|=q-1} \sum_{1 \le j \le n} (-1)^{n-j} \beta_{jJ} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n \wedge d\bar{z}_J \otimes e$$

($\widehat{dz_j}$ meaning that we omit dz_j), and

$$\theta_q(\beta,\beta) = \langle \theta\Lambda\beta|\beta\rangle_{\epsilon,t} = 2^{n+q} \sum_{|J|=q-1} \sum_{1 \le j \le n} \lambda_j \beta_{jJ} \overline{\beta_{jJ}} = 2^{n+q} \sum_{|J|=q} (\sum_{j \in J} \lambda_j)|\beta_J|^2.$$

Now,

$$|\beta|_\theta^2 = \sup_u \frac{|\langle\beta|u\rangle_{\epsilon,t}|^2}{\langle\theta\Lambda u|u\rangle_{\epsilon,t}} \le \sup_u \frac{|\beta|_{\epsilon,t}^2 \cdot |u|_{\epsilon,t}^2}{\langle\theta\Lambda u|u\rangle_{\epsilon,t}} = |\beta|_{\epsilon,t}^2 \cdot \sup_u \frac{|u|_{\epsilon,t}^2}{\langle\theta\Lambda u|u\rangle_{\epsilon,t}} =$$

$$= |\beta|_{\epsilon,t}^2 \cdot \sup_u \frac{\sum_{|J|=q} |u_J|^2}{\sum_{|J|=q}(\sum_{j \in J} \lambda_j)|u_J|^2} \le |\beta|_{\epsilon,t}^2 \frac{1}{\lambda_1 + \ldots + \lambda_q},$$

where $u = \sum_{|J|=q} u_J dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_J \otimes e$ as above. The lemma follows. $\qquad\square$

4. The Bochner technique

Let $K_{\epsilon,t}^{n,q}$ be the L^2-completion of $\mathcal{D}_{c,\epsilon}^{n,q}$ w.r.t. the metric

$$\|u\|_{K_{\epsilon,t}^{n,q}}^2 := \int_{X \setminus Z_\epsilon} (|u|_{\bigwedge^{n,q} \omega_{\epsilon,t} \otimes \hat{h}_\epsilon^{1+s}}^2 + |\bar\partial u|_{\bigwedge^{n,q} \omega_{\epsilon,t} \otimes \hat{h}_\epsilon^{1+s}}^2) dV_{\omega_{\epsilon,t}},$$

in the space of all forms with L_{loc}^2 coefficients, and let $\mathcal{K}_{\epsilon,t}^{n,q}$ be the corresponding sheaf of germs of locally L^2 sections on X (the local L^2 condition should hold on X and not only on $X \setminus Z_\epsilon$).

Lemma 4.1. *For all $\epsilon > 0$, the L^2 Dolbeault complex $(\mathcal{K}_{\epsilon,0}^{n,q}, \bar\partial_{\epsilon,0})$ is a fine resolution of the sheaf $K_X \otimes L \otimes \mathcal{J}(\hat{h}_\epsilon^{1+s})$.*

Proof. $X \setminus Z_\epsilon$ can be covered by open subsets $U \subset X$ which are Stein, lie relatively compact in another Stein open subset $U' \subset X$, and on which L is trivial. On these U's we can show the $\overline{\partial}$- Poincaré lemma.

First, as Stein sets, $U \subset\subset U'$ may be embedded as analytic subsets into some $\mathbb{C}^N$. Hence we can find a smooth plurisubharmonic function ψ on U' such that $i\partial\overline{\partial}\psi \geq 2\lambda\omega$ for some constant $\lambda > 0$ on U ($\omega_{\epsilon,0} = \omega$ is smooth on U). Furthermore, $|\psi|$ is bounded on U by some constant $M > 0$. Subtracting M we still have a plurisubharmonic function, which we also call ψ, satisfying

$$-2M \leq \psi \leq 0 \quad \text{and} \quad i\partial\overline{\partial}\psi \geq 2\lambda\omega.$$

This implies that the metrics $\widehat{h}_\epsilon^{1+s}$ and $\widehat{h}_\epsilon^{1+s}e^{-2\psi}$ are comparable on U, and $\Theta_{\widehat{h}_\epsilon^{1+s}e^{-2\psi}} \geq \lambda\omega$, for λ sufficiently big.

Since $\widehat{h}_\epsilon^{1+s}$ and $\widehat{h}_\epsilon^{1+s}e^{-2\psi}$ are comparable, we can interchange them in the metric for $\mathcal{K}_{\epsilon,0}^{n,q}(U)$. So, given $g \in \mathcal{K}_{\epsilon,0}^{n,q}(U)$ with $\overline{\partial}_{\epsilon,0}(g) = 0$, we know that $\int_X |g|_{\epsilon,0}^2 dV_\omega < \infty$, hence also $\int_X |g|_{\epsilon,\psi}^2 dV_\omega < \infty$ with the new metric, and by lemma 3.4

$$\int_X |g|_{\Theta_{\hat{h}_\epsilon^{1+s}e^{-2\psi}}}^2 dV_\omega \leq \int_X \frac{1}{q\lambda}|g|_{\epsilon,\psi}^2 dV_\omega < \infty.$$

Therefore, we can apply theorem 4.1 of [Dem82]: There exists a $(n, q-1)$- form f with L_{loc}^2 coefficients in U such that

$$\overline{\partial}_{\epsilon,0}(f) = g$$

and

$$\int_X |f|_{\epsilon,0}^2 dV_\omega \leq \int_X |f|_{\epsilon,\psi}^2 dV_\omega \leq \int_X \frac{1}{q\lambda}|g|_{\epsilon,\psi}^2 dV_\omega \leq e^{2M} \int_X |g|_{\epsilon,0}^2 dV_\omega.$$

Finally, the L^2 condition forces sections holomorphic on $X \setminus Z_\epsilon$ to extend holomorphically across Z_ϵ ([Dem82, Lem.6.9]). The L^2 condition implies that the coefficients lie in $\mathcal{J}(\hat{h}_\epsilon^{1+s})$. Consequently, the complex is a resolution, and it is fine because of the existence of partition of unities. $\square$

Now, let us take a cohomology class $\{\beta\} \in H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}(h_{\min}^{1+s'}))$. If $\mathcal{U}$ is a covering of X with Stein open subsets U_α, the class $\{\beta\}$ may be represented by a Čech cocycle

$$(\beta_{\alpha_0\cdots\alpha_q})_{\alpha_0\cdots\alpha_q} \in C^q(\mathcal{U}, \mathcal{O}(K_X + L) \otimes \mathcal{J}(h_{\min}^{1+s'})) \subset C^q(\mathcal{U}, \mathcal{O}(K_X + L) \otimes \mathcal{J}(\hat{h}_\epsilon^{1+s})).$$

Let (ψ_α) be a $\mathcal{C}^\infty$ partition of unity subordinate to $\mathcal{U}$. Taking the usual De Rham-Weil isomorphisms between Čech and Dolbeault cohomology, we obtain a closed (n, q)- form in $K_{\epsilon,0}^{n,q}$ of the form

$$\beta = \sum_{\alpha_0,\ldots,\alpha_q} \beta_{\alpha_0\cdots\alpha_q}\overline{\partial}\psi_{\alpha_0} \wedge \ldots \wedge \overline{\partial}\psi_{\alpha_q}.$$

In particular, this form has coefficients in $\mathcal{J}(\hat{h}_\epsilon^{1+s}) \otimes \mathcal{C}^\infty(X)$. We want to show that β is a boundary in $K_{\epsilon,0}^{n,q}$ for some $\epsilon > 0$, hence $\{\beta\} = 0 \in H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{J}(\hat{h}_\epsilon^{1+s}))$. This implies theorem 1.4 because of the inclusion $\mathcal{J}(\hat{h}_\epsilon^{1+s}) \subset \mathcal{J}(h_{\min}^{(1-\delta)(1+s)})$.

The reasoning starts as follows: β is also an element of $K_{\epsilon,t}^{n,q}$ for any $t \geq 0$, because of lemma 3.1. Every L^2 form $u \in D_{\overline{\partial}_{\epsilon,t}^*} \subset K_{\epsilon,t}^{n,q}$ may be written as $u = u_1 + u_2$ with

$$u_1 \in \ker \overline{\partial}_{\epsilon,t} \quad \text{and} \quad u_1 \in (\ker \overline{\partial}_{\epsilon,t})^\perp = \overline{\operatorname{im} \overline{\partial}_{\epsilon,t}^*} \subset \ker \overline{\partial}_{\epsilon,t}^*,$$

since $\overline{\partial}_{\epsilon,t}$ is a closed operator, hence $\ker \overline{\partial}_{\epsilon,t}$ is closed. Using $\beta \in \ker \overline{\partial}$ and the two inequalities in lemma 3.3 and 3.4, we get ($\Theta_{\epsilon,t}$ denotes the curvature form of $\hat{h}_\epsilon^{1+s}$ on $X \setminus Z_\epsilon$, plus $2\epsilon\omega_{\epsilon,t}$)

$$
\begin{aligned}
| \ll \beta, u \gg_{\epsilon,t} |^2 &= \quad | \ll \beta, u_1 \gg_{\epsilon,t} |^2 = | \textstyle\int_{X \setminus Z_\epsilon} \langle \beta, u_1 \rangle_{\epsilon,t} dV_{\omega_{\epsilon,t}} |^2 \leq \\[6pt]
&\leq \quad (\textstyle\int_{X \setminus Z_\epsilon} |\langle \beta, u_1 \rangle_{\epsilon,t}| dV_{\omega_{\epsilon,t}})^2 \leq \\[6pt]
&\leq \quad (\textstyle\int_{X \setminus Z_\epsilon} \beta_{\Theta_{\epsilon,t}} \cdot \sqrt{\langle \Theta_{\epsilon,t} \Lambda u_1 | u_1 \rangle_{\epsilon,t}} dV_{\omega_{\epsilon,t}})^2 \leq \\[6pt]
&\leq \quad \textstyle\int_{X \setminus Z_\epsilon} \beta_{\Theta_{\epsilon,t}}^2 dV_{\omega_{\epsilon,t}} \cdot \int_{X \setminus Z_\epsilon} \langle \Theta_{\epsilon,t} \Lambda u_1 | u_1 \rangle_{\epsilon,t} dV_{\omega_{\epsilon,t}} \leq \\[6pt]
&\leq \quad \textstyle\int_{X \setminus Z_\epsilon} \beta_{\Theta_{\epsilon,0}}^2 dV_{\omega_{\epsilon,0}} \cdot \int_{X \setminus Z_\epsilon} \langle \Theta_{\epsilon,t} \Lambda u_1 | u_1 \rangle_{\epsilon,t} dV_{\omega_{\epsilon,t}} \leq \\[6pt]
&\leq \quad \textstyle\int_{X \setminus Z_\epsilon} \frac{1}{\hat{\lambda}_1^{(\epsilon,0)} + \cdots + \hat{\lambda}_q^{(\epsilon,0)}} |\beta|_{\epsilon,0}^2 dV_{\omega_{\epsilon,0}} \cdot \int_{X \setminus Z_\epsilon} \langle \Theta_{\epsilon,t} \Lambda u_1 | u_1 \rangle_{\epsilon,t} dV_{\omega_{\epsilon,t}}.
\end{aligned}
$$

u_1 is an element of $D_{\overline{\partial}_{\epsilon,t}} \cap D_{\overline{\partial}_{\epsilon,t}^*}$, since $u_1 \in \ker \overline{\partial}_{\epsilon,t}$, $u_2 \in \ker \overline{\partial}_{\epsilon,t}^*$ and $u_1 = u - u_2$. Consequently, we can apply the Bochner inequality on u_1. As $\overline{\partial} u_1 = 0$ we get that the second integral on the right hand side is bounded above by

$$\| \overline{\partial}_{\epsilon,t}^* u_1 \|_{\epsilon,t}^2 + 2q\epsilon \| u_1 \|_{\epsilon,t}^2 \leq \| \overline{\partial}_{\epsilon,t}^* u \|_{\epsilon,t}^2 + 2q\epsilon \| u \|_{\epsilon,t}^2,$$

and finally

$$|\langle \beta, u \rangle_{\epsilon,t}|^2 \leq \int_X \frac{1}{\hat{\lambda}_1^{(\epsilon,t)} + \cdots + \hat{\lambda}_q^{(\epsilon,t)}} |\beta|_{\epsilon,t}^2 dV_{\omega_{\epsilon,t}} (\| \overline{\partial}_{\epsilon,t}^* u \|_{\epsilon,t}^2 + 2q\epsilon \| u \|_{\epsilon,t}^2),$$

where the term $2q\epsilon \| u \|_{\epsilon,t}^2$ comes in because $\Theta_{\epsilon,t}$ differs from the curvature form of $\hat{h}_\epsilon^{1+s}$ by $2\epsilon\omega_{\epsilon,t}$.

Using the uniform bound $C_\epsilon = \int_X \frac{1}{\hat{\lambda}_1^{(\epsilon,0)} + \cdots + \hat{\lambda}_q^{(\epsilon,0)}} |\beta|_{\epsilon,0}^2 dV_{\omega_{\epsilon,0}}$ we apply the Hahn-Banach theorem as in [Dem82]: $\ll \beta, u \gg_{\epsilon,t}$ defines a linear form on the range of the densely defined operator

$$T : L_{\epsilon,t}^{n,q} \to L_{\epsilon,t}^{n,q-1} \oplus L_{\epsilon,t}^{n,q}, \quad u \mapsto \overline{\partial}_{\epsilon,t}^* u + 2q\epsilon u$$

(with domain $D_T = D_{\overline{\partial}_{\epsilon,t}^*}$). Hence there exists an $f_{\epsilon,t} = v_{\epsilon,t} \oplus \frac{1}{2q\epsilon} w_{\epsilon,t} \in L_{\epsilon,t}^{n,q-1} \oplus L_{\epsilon,t}^{n,q}$ such that

$$\ll \beta, u \gg_{\epsilon,t} = \ll f_{\epsilon,t}, \overline{\partial}_{\epsilon,t}^* u + 2q\epsilon u \gg_{\epsilon,t}.$$

Consequently, $\beta = T^* f_{\epsilon,t} = \overline{\partial}_{\epsilon,t} v_{\epsilon,t} + w_{\epsilon,t}$ with

$$\| v_{\epsilon,t} \|_{\epsilon,t}^2 + \frac{1}{2q\epsilon} \| w_{\epsilon,t} \|_{\epsilon,t}^2 \leq C_\epsilon.$$

Furthermore, $\overline{\partial}_{\epsilon,t} w_{\epsilon,t} = \overline{\partial} \beta = 0$, and $v_{\epsilon,t}, w_{\epsilon,t}$ are both contained in $K_{\epsilon,t}^{n,q}$.

The estimates of section 3 tell us that the metrics of the $v_{\epsilon,t}$ and $w_{\epsilon,t}$ in the L^2 space $L_{\epsilon,t_0}^{n,q-1}$ resp. $L_{\epsilon,t_0}^{n,q}$ are uniformly bounded for all $t_0 \geq t > 0$. Since $\| \beta \|_{\epsilon,t_0} \geq | \| \overline{\partial}_{\epsilon,t} v_{\epsilon,t} \|_{\epsilon,t_0} - \| w_{\epsilon,t} \|_{\epsilon,t_0} |$, the same is true for $\overline{\partial}_{\epsilon,t} v_{\epsilon,t}$. Consequently,

the sequences of these elements converge weakly as $t \to 0$, and we have three weak limits in the respective spaces:

$$v_{\epsilon,t} \rightharpoonup v_\epsilon \in L^{n,q-1}_{\epsilon,t_0}, \quad \overline{\partial}_{\epsilon,t} v_{\epsilon,t} \rightharpoonup v'_\epsilon \in L^{n,q}_{\epsilon,t_0}, \quad w_{\epsilon,t} \rightharpoonup w_\epsilon \in L^{n,q}_{\epsilon,t_0}.$$

Note that these weak limits are the same for every choice of $t_0 > 0$: The spaces $L^{n,q}_{\epsilon,t_0}$ all contain the dense subset $\mathcal{D}^{n,q}_{c,\epsilon}$, hence weak convergence is transmitted through the continuous maps between them.

Claim: $\overline{\partial}_{\epsilon,t_0} v_\epsilon = v'_\epsilon$.

Proof. On the one hand, we have

$$\ll \overline{\partial}_{\epsilon,t} v_{\epsilon,t}, u \gg_{\epsilon,t_0} \to \ll v'_\epsilon, u \gg_{\epsilon,t_0}$$

for all $u \in D_{\overline{\partial}^*_{\epsilon,t_0}}$ because of the weak convergence. On the other hand, the commutativity of the diagram in lemma 3.2 and again the weak convergence show that

$$\ll \overline{\partial}_{\epsilon,t} v_{\epsilon,t}, u \gg_{\epsilon,t_0} = \ll \overline{\partial}_{\epsilon,t_0} v_{\epsilon,t}, u \gg_{\epsilon,t_0} = \ll v_{\epsilon,t}, \overline{\partial}^*_{\epsilon,t_0} u \gg_{\epsilon,t_0} \to$$

$$\ll v_\epsilon, \overline{\partial}^*_{\epsilon,t_0} u \gg_{\epsilon,t_0} = \ll \overline{\partial}_{\epsilon,t_0} v_\epsilon, u \gg_{\epsilon,t_0}.$$

Since $D_{\overline{\partial}^*_{\epsilon,t_0}}$ is dense in $L^{n,q}_{\epsilon,t_0}$ the claim follows. $\square$

By standard properties of weak convergence,

$$\| v_\epsilon \|_{\epsilon,t_0} \leq \liminf_{t \to 0} \| v_{\epsilon,t} \|_{\epsilon,t_0},$$

and similarly for $\| \overline{\partial}_{\epsilon,t_0} v_\epsilon \|_{\epsilon,t_0}$ and $\| w_\epsilon \|_{\epsilon,t_0}$. Consequently, these three norms are uniformly bounded by C_ϵ for $t_0 > 0$.

Now, we restrict the integral defining the (ϵ, t_0)- norm to compact subsets $K \subset X \setminus Z_\epsilon$. Of course, we get $\| v_\epsilon \|_{\epsilon,t_0,K} \leq \| v_\epsilon \|_{\epsilon,t_0}$, hence the new (ϵ, t_0, K)-norms of v_ϵ are still uniformly bounded by C_ϵ in $t_0 > 0$. Furthermore, as $\omega_t \downarrow \omega$, we see that $\| v_\epsilon \|_{\epsilon,t_0,K} \to \| v_\epsilon \|_{\epsilon,0,K}$, and monotone convergence tells us that $\| v_\epsilon \|_{\epsilon,0}$ exists and is $\leq C_\epsilon$. The same is true for $\| \overline{\partial}_{\epsilon,0} v_\epsilon \|_{\epsilon,0}$ and $\| w_\epsilon \|_{\epsilon,0}$.

As $\beta = \overline{\partial}_{\epsilon,t} v_{\epsilon,t} + w_{\epsilon,t}$ for all t, $\beta = \overline{\partial}_{\epsilon,0} v_\epsilon + w_\epsilon$ remains true. Furthermore, $\overline{\partial}_{\epsilon,0} w_{\epsilon,t} = \overline{\partial}_{\epsilon,t} w_{\epsilon,t} = 0$. Hence, v_ϵ and w_ϵ belong to $K^{n,q-1}_{\epsilon,0}$ resp. $K^{n,q}_{\epsilon,0}$.

For the last step we note that the almost plurisubharmonic weights ψ_ϵ defining $\tilde{h}_\epsilon$ form a decreasing sequence, and consequently,

$$\| u \|_{\epsilon,0} \leq \| u \|_{\epsilon',0} \quad \forall u \in K^{n,q}_{\epsilon',0},$$

if $\epsilon' < \epsilon$. This implies $\| w_\epsilon \|_{\epsilon_0,0} \leq \| w_\epsilon \|_{\epsilon,0}$ for some fixed $\epsilon_0 > 0$ and $\epsilon < \epsilon_0$. Since

$$\| w_\epsilon \|_{\epsilon,0} \leq C_\epsilon = 2q\epsilon \cdot C_\epsilon,$$

we conclude using the estimate in section 5 that $\| w_\epsilon \|_{\epsilon_0,0} \to 0$ for $\epsilon \to 0$. But the norm of w_ϵ measures the distance of β from the closure of the subspace of boundaries in $K^{n,q}_{\epsilon_0,0}$. So it only remains to show

Lemma 4.2. *The subspace $B^{n,q}_{\epsilon_0,0} \subset K^{n,q}_{\epsilon_0,0}$ of boundaries in the Dolbeault complex $(\mathcal{K}^{n,q}_{\epsilon_0,0}, \overline{\partial})$ is closed.*

Proof. Let $Z^{n,q}_{\epsilon_0,0} \subset K^{n,q}_{\epsilon_0,0}$ be the space of cocycles with respect to $\overline{\partial}$, and let $\mathcal{Z}^{n,q}_{\epsilon_0,0} \subset \mathcal{K}^{n,q}_{\epsilon_0,0}$ be the corresponding sheaf. Let $\mathcal{U}$ be a covering of X with Stein

open subsets as in the proof of exactness of the Dolbeault complex in lemma 4.1. By the usual DeRham-Weil isomorphism,

$$H^q(K^{n,\bullet}_{\epsilon_0,0}) = \frac{Z^{n,q}_{\epsilon_0,0}}{\overline{\partial}K^{n,q-1}_{\epsilon_0,0}} = \frac{Z^0(\mathcal{U}, \mathcal{Z}^{n,q}_{\epsilon_0,0})}{\overline{\partial}Z^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0})}.$$

So we have to prove that $\overline{\partial}Z^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0})$ is closed in $Z^0(\mathcal{U}, \mathcal{Z}^{n,q}_{\epsilon_0,0})$ with respect to the L^2 norms on every set U in $\mathcal{U}$.

Note first that

$$\overline{\partial} : C^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0}) \to C^0(\mathcal{U}, \mathcal{Z}^{n,q}_{\epsilon_0,0})$$

is continuous, by definition of the norms, and surjective, by exactness. Hence $\overline{\partial}$ is an open map, by Banach's open mapping theorem. Its kernel is $C^0(\mathcal{U}, \mathcal{Z}^{n,q-1}_{\epsilon_0,0})$. Next, $C^0(\mathcal{U}, \mathcal{Z}^{n,q-1}_{\epsilon_0,0})$ and $Z^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0})$ are closed in $C^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0})$, since $\overline{\partial}$ is a closed operator, and equality is conserved when going to the limit. Consequently, the sum of these two spaces is closed, too, and its complement is open. But then

$$\overline{\partial}(Z^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0}) + C^0(\mathcal{U}, \mathcal{Z}^{n,q-1}_{\epsilon_0,0})) = \overline{\partial}(Z^0(\mathcal{U}, \mathcal{K}^{n,q-1}_{\epsilon_0,0}))$$

is closed in $C^0(\mathcal{U}, \mathcal{Z}^{n,q}_{\epsilon_0,0})$ and also in $Z^0(\mathcal{U}, \mathcal{Z}^{n,q}_{\epsilon_0,0})$. $\square$

5. The uniform estimate

The aim of this section is to prove

Lemma 5.1. *Let $0 < s < s'$. For every smooth (n,q)-form β with values in L and coefficients in $\mathcal{J}(h^{1+s'}_{\min}) \otimes \mathcal{C}^\infty$,*

$$\int_X \frac{q\epsilon}{\hat{\lambda}^{(\epsilon,0)}_1 + \cdots + \hat{\lambda}^{(\epsilon,0)}_q} |\beta|^2_{\epsilon,0} dV_{\omega_{\epsilon,0}},$$

tends to 0 for $\epsilon \to 0$.

Before starting with the proof, set $\hat{\lambda}_j := \hat{\lambda}^{(\epsilon,0)}_j$ and $\lambda_j := \lambda^{(\epsilon,0)}_j$.

Now, by construction we know that $\hat{\lambda}_j \geq \delta\lambda_j + \epsilon$, and

$$\lambda^q_q\lambda_{q+1}\ldots\lambda_n \geq \lambda_1\ldots\lambda_n \geq C\epsilon^{n-\nu},$$

hence

$$\frac{1}{\lambda_1 + \ldots + \lambda_q} \leq \frac{1}{\lambda_q} \leq C^{-1/q}\epsilon^{-(n-\nu)/q}(\lambda_{q+1}\ldots\lambda_n)^{1/q}.$$

We infer

$$\gamma_\epsilon := \frac{q\epsilon}{\hat{\lambda}_1 + \ldots + \hat{\lambda}_q} \leq \min(1, \frac{q\epsilon}{\delta\lambda_q}) \leq \min(1, C'\delta^{-1}\epsilon^{-(n-\nu)/q}(\lambda_{q+1}\ldots\lambda_n)^{1/q}).$$

We notice that

$$\int_X \lambda_{q+1}\ldots\lambda_n dV_\omega \leq \int_X (\Theta_{h_\epsilon}(L) + \epsilon\omega)^{n-q} \wedge \omega^q \leq (c_1(L) + \epsilon\{\omega\})^{n-q}\{\omega\}^q \leq C'',$$

hence the functions $(\lambda_{q+1}\ldots\lambda_n)^{1/q}$ are uniformly bounded in L^1 norm as ϵ tends to 0. Since $1 - (n-\nu)/q > 0$ by hypothesis, we conclude that γ_ϵ converges almost everywhere to 0 as ϵ tends to 0. On the other hand,

$$|\beta|^2_{\hat{h}_\epsilon} = |\beta|^2_{h_\infty}e^{-2(\delta(1+s)\phi_\epsilon + (1-\delta)(1+s)\psi_\epsilon)} \leq |\beta|^2_{h_\infty}e^{-2(\delta(1+s)\phi_\epsilon)}e^{-2(1-\delta)(1+s)\psi}.$$

Our assumption that the coefficients of β lie in $\mathcal{J}(h^{1+s'}_{\min})$ implies that there exists a $p' > 1$ such that $\int_X |\beta|^2_{h_\infty}e^{-2p'(1-\delta)(1+s)\psi}dV_\omega < \infty$, no matter how small δ is. Let

$p \in (1, \infty)$ be the conjugate exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$. By Hölder's inequality, we have

$$\int_X \gamma_\epsilon |\beta|^2_{\hat{h}_\epsilon} dV_\omega \leq (\int_X |\beta|^2_{h_\infty} e^{-2p\delta(1+s)\phi_\epsilon} dV_\omega)^{1/p} (\int_X \gamma_\epsilon^{p'} |\beta|^2_{h_\infty} e^{-2p'(1+s)(1-\delta)\psi} dV_\omega)^{1/p'}.$$

As $\gamma_\epsilon \leq 1$, the Lebesgue dominated convergence theorem shows that

$$\int_X \gamma_\epsilon^{p'} |\beta|^2_{h_\infty} e^{-2p'(1+s)(1-\delta)\psi} dV_\omega$$

converges to 0 as ϵ tends to 0.

For the first integral, we argue as follows: The ϕ_ϵ may be constructed such that the Lelong numbers $\nu((1+s)\phi_\epsilon, x)$ are bounded from above by the Lelong numbers $\nu((1+s)\psi, x)$ in every point $x \in X$ (see again Theorem 3.16 in [Eck03]). On the other hand, there is a constant C such that $\nu((1+s)\psi, x) < C$ for all points $x \in X$, by [Bou02, Lem.3.11]. Hence, $\nu(\frac{1+s}{C}\phi_\epsilon, x) < 1$, and

$$\int_X e^{-(2/C)(1+s)\phi_\epsilon} dV_\omega < \infty,$$

by Skoda's lemma [Dem00, (5.6)]. Adding sufficiently big constants to ϕ_ϵ we can even reach that the integrals above are *uniformly bounded*. By choosing $\delta \leq 1/(pC)$, the integral $\int_X |\beta|^2_{h_\infty} e^{-2p\delta(1+s)\phi_\epsilon} dV_\omega$ remains bounded and we are done.

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