ON THE STRONG FACTORIZATION OF TORIC BIRATIONAL MAPS

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Abstract. A proof of the strong factorization conjecture for toric birational maps is presented, following the ideas of Morelli and Abramovich-Matsuki-Rashid. The main new ingredient is the insertion and deletion of so called trivial cobordisms which allow the formulation of a factorization algorithm.

Contents

1. Introduction

The aim of this paper is to proof

Theorem 1.1 (Strong Factorization of Toric Birational Maps)**.** *Every proper and equivariant birational map $f : X_\Delta \dashrightarrow X_{\Delta'}$ between two nonsingular toric varieties can be factored into a sequence of blowups , immediately followed by a sequence of blowdowns, where all blowups and blowdowns have smooth centers which are the closure of orbits.*

A very nice account of the historical background, explicit examples of the basic ideas and related problems is given in the fundamental article [AMR99]. This paper already contains a proof of the Strong Factorization Theorem, trying to correct errors in an even earlier proof of [Mor96]. Unfortunately, the corrections were not successful, s. [Mat00].

This paper's proof follows closely the main ideas of these earlier works: It starts in section 2 with the explanation of Morelli's wonderful idea of a **cobordism** between toric varieties, then introduces the notions of circuits, bistellar operations, collapsibility and $\pi-$ nonsingularity in section 3 and 4. In this setting the proof of the Weak Factorization Theorem (where the order of the blowups and blowdowns is arbitrary) is possible [AMR99, Mor96]. The only new feature of these sections is

Date: May 8, 2004.
1991 *Mathematics Subject Classification.* Primary 14M25, Secondary 14E05.

the consequent use of cuts through the cobordisms in the direction of the projection π: They allow to draw pictures which help to get a better intuition of these concepts, and also clarify some of the arguments. Especially lemma 4.2 is crucial for understanding the algorithm.

Section 5 introduces the main new tool, the trivial cobordisms. They are the star of a circuit generated only by two rays (with the same projection), and their name comes from the fact that the lower and the upper face of such a cobordism project to the same fan. The theorems of this section show how to insert or delete these trivial cobordisms in arbitrary collapsible cobordisms.

At first sight, the insertion of a trivial cobordism seems to be a useless operation. But what is the main problem of the proof of [Mor96, AMR99]? It tries to transform arbitrary circuits (which correspond to a blowup followed by a blowdown, both with smooth centers) to a pointing up circuit (which corresponds only to a blowup), by smooth star subdivisions. But these star subdivisions also affect the stars of other circuits, and so the algorithm gets out of control.

The insertion of a trivial cobordism at the right place makes it possible to control which stars are affected. Furthermore, the subdivision of a star of a trivial cobordism gives only pointing up and pointing down circuits. All these properties are used to establish the algorithm of section 6.

It starts with replacing an arbitrary circuit by a pair of a pointing up and a pointing down circuit. In the next step the cobordism is modified such that the order of pointing down and pointing up circuits is exchanged until the cobordisms starts with pointing up circuits and ends with pointing down circuits. In the last step, these pointing down circuits are deleted.

Finally a remark on generalizations: The methods of this proof (espacially the insertion and deletion of trivial cobordisms) should be valid for the toroidal case, too. Unfortunately, this is not enough for the general Strong Factorization Theorem: It is still necessary to prove

Conjecture 1.2 (Toroidalization Conjecture)**.** *Let $f : X \to Y$ be a morphism between nonsingular complete varieties. Then there exist sequences of blowups with smooth centers for X and Y such that the induced morphism $f' : X' \to Y'$ is toroidal.*

Note that in [AKMW00] it is only proven that there are sequences of blowups with smooth centers such that X' and Y' are toroidal.

Acknowledgement. This paper is only the last step of the proof of the Strong Factorization Theorem for toric varieties. As already the explanations above should have made clear, it follows very closely the brilliant ideas of Morelli. And it uses abundantly the terminology, the notation and sometimes even the wording of [AMR99]. Of course, I am still responsible for all mistakes in this article.

Recently I learned from new corrections which Morelli offers on his home page [Mor00]. I was not able to check these corrections, but I think that my approach, especially the use of trivial cobordisms, is of its own value.

2. Cobordisms

The notation and terminology concerning toric varieties X_Δ and their corresponding fans Δ follow the presentation in [Dan78, Ful93, Oda88] as adapted in

[AMR99]. Everything is done over an algebraically closed field of characteristic 0, that is, $\mathbb{C}$.

The key operation used over and over again is the star subdivision of a fan Δ. It corresponds to a blow up.

Definition 2.1. *Let ρ be a ray passing in the relative interior of a cone τ in a fan Δ. (Once the ray ρ is fixed, such a $\tau \in \Delta$ is uniquely determined.) Then the **star subdivision** $\rho \cdot \Delta$ of Δ with respect to ρ is defined to be*

$$\rho \cdot \Delta = (\Delta - \mathrm{Star}(\tau)) \cup \{\rho + \tau' + \nu;\ \tau' \text{ a proper face of } \tau, \nu \in \mathrm{link}_\Delta(\tau)\}$$

where

$$\begin{aligned} \mathrm{Star}(\tau) &= \{\zeta \in \Delta : \zeta \supset \tau\} \\ \overline{\mathrm{Star}(\tau)} &= \{\zeta \in \Delta : \zeta \subset \eta \text{ for some } \eta \in \mathrm{Star}(\tau)\} \\ \mathrm{link}_\Delta(\tau) &= \left\{\zeta \in \overline{\mathrm{Star}(\tau)} : \zeta \cap \tau = \emptyset\right\}. \end{aligned}$$

*The inverse of a star subdivision is called a **star assembling**, corresponding to a blowdown.*

*When $\tau = \langle \rho_1, \dots, \rho_l \rangle$ is generated by the extremal rays ρ_i with the primitive vectors $v_i = n(\rho_i)$ and the ray ρ is generated by the vector $v_1 + \ldots + v_l$, the star subdivision is called the **barycentric star subdivision** with respect to τ.*

*When Δ is nonsingular, the barycentric star subdivision with respect to a face τ is called a **smooth star subdivision** and its inverse a **smooth star assembling**. These operations correspond to blowups and blowdowns of smooth varieties in smooth centers.*

The notion of a cobordism as defined below sits in the center of Morelli's idea of proving the toric strong factorization conjecture.

Definition 2.2. *Let Δ and Δ' be two fans in $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ with the same support, where N is the lattice of one-parameter subgroups of the torus. A **cobordism** Σ from Δ to Δ' is a fan in $N^+_{\mathbb{Q}} = (N \oplus \mathbb{Z}) \otimes \mathbb{Q} = N_{\mathbb{Q}} \oplus \mathbb{Q}$ equipped with the natural projection*

$$\pi : N^+_{\mathbb{Q}} = N_{\mathbb{Q}} \oplus \mathbb{Q} \to N_{\mathbb{Q}}$$

such that

(1) *any cone $\tau \in \Sigma$ is π-strictly convex, i.e.*

$$x, y \in \tau,\ \pi(x) = -\pi(y) \Rightarrow\ x = y = 0,$$

(2) *the projection π gives an isomorphism between $\partial_-\Sigma$ and Δ (resp. $\partial_+\Sigma$ and Δ') as linear complexes, i.e., there is a one-to-one correspondence between the cones σ_- of $\partial_-\Sigma$ (resp. σ'_+ of $\partial_+\Sigma$) and the cones σ of Δ (resp. σ' of Δ') such that $\pi : \sigma_- \to \sigma$ (resp. $\pi : \sigma'_+ \to \sigma'$) is a linear isomorphism for each σ_- (resp. σ'_+) and its corresponding σ (resp. σ'). (The map of lattices $\pi : (N \oplus \mathbb{Z}) \cap \sigma_- \to N \cap \sigma$, resp. $\pi : (N \oplus \mathbb{Z}) \cap \sigma'_+ \to N \cap \sigma'$, is **not** required to be an isomorphism.)*

We denote this isomorphism by

$$\pi : \partial_-\Sigma \xrightarrow{\sim} \Delta \text{ (resp. } \pi : \partial_+\Sigma \xrightarrow{\sim} \Delta')$$

where

$$\begin{array}{rcl}\partial_-\Sigma & = & \{\tau \in \Sigma : (x, y-\epsilon) \notin \mathrm{Supp}(\Sigma) \text{ for any } (x,y) \in \tau \\ & & \quad \text{with } x \in N_{\mathbb{Q}},\ y \in \mathbb{Q} \text{ and any sufficiently small } \epsilon > 0\} \\ \textit{resp. } \partial_+\Sigma & = & \{\tau \in \Sigma : (x, y+\epsilon) \notin \mathrm{Supp}(\Sigma) \text{ for any } (x,y) \in \tau \\ & & \quad \text{with } x \in N_{\mathbb{Q}},\ y \in \mathbb{Q} \text{ and any sufficiently small } \epsilon > 0\}\end{array}$$

(3) *the support* $\mathrm{Supp}(\Sigma)$ *of* Σ *lies between the lower face* $\partial_-\Sigma$ *and the upper face* $\partial_+\Sigma$*, i.e., for any* $(x, y) \in \mathrm{Supp}(\Sigma)$*, the point* $x \in N_{\mathbb{Q}}$ *is in* $\mathrm{Supp}(\Delta) = \mathrm{Supp}(\Delta')$*, and there are numbers* $y^x_-, y^x_+ \in \mathbb{Q}$ *such that*

$$\pi^{-1}(x) = \left\{(x, y) \in N^+_{\mathbb{Q}} : y^x_- \leq y \leq y^x_+\right\}$$

and

$$(x, y^x_-) \in \mathrm{Supp}(\partial_-\Sigma), (x, y^x_+) \in \mathrm{Supp}(\partial_+\Sigma).$$

Actually only the third condition is needed for the definition of a cobordism (not the second one as misprinted in [AMR99]). This may be summarized in the following

Proposition 2.3. *A fan* Σ *in* $N^+_{\mathbb{Q}}$ *equipped with the natural projection* $\pi : N^+_{\mathbb{Q}} = N_{\mathbb{Q}} \oplus \mathbb{Q} \to N_{\mathbb{Q}}$ *is a cobordism between the fans* $\pi(\partial_-\Sigma)$ *and* $\pi(\partial_+\Sigma)$ *in* $N_{\mathbb{Q}}$ *iff the support* $\mathrm{Supp}(\Sigma)$ *of* Σ *lies between the lower face* $\partial_-\Sigma$ *and the upper face* $\partial_+\Sigma$*, i.e., for any* $(x, y) \in \mathrm{Supp}(\Sigma)$*, there are numbers* $y^x_-, y^x_+ \in \mathbb{Q}$ *such that*

$$\pi^{-1}(x) = \left\{(x, y) \in N^+_{\mathbb{Q}} : y^x_- \leq y \leq y^x_+\right\}$$

and

$$(x, y^x_-) \in \mathrm{Supp}(\partial_-\Sigma), (x, y^x_+) \in \mathrm{Supp}(\partial_+\Sigma).$$

Proof. Suppose that Σ satisfies the condition. Then, for every $(x, y) \in \mathrm{Supp}(\Sigma)$, there are points

$$(x, y^x_-) \in \mathrm{Supp}(\partial_-\Sigma), (x, y^x_+) \in \mathrm{Supp}(\partial_+\Sigma)$$

with $y^x_- \leq y \leq y^x_+$. Therefore, $\mathrm{Supp}(\pi(\partial_-\Sigma)) = \mathrm{Supp}(\pi(\partial_+\Sigma))$. Furthermore, this implies condition (2) of the cobordism definition.

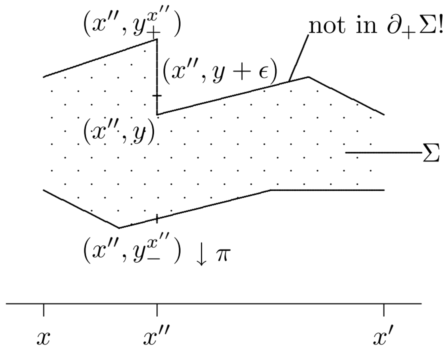
Finally suppose that $\tau \in \Sigma$ is not π-strictly convex, i.e. there are points $(x, y), (-x, y') \in \tau$, $0 \neq x \in N_{\mathbb{Q}}$, $y \in \mathbb{Q}$. By the definition of fans the cone τ of the fan Σ must be strictly convex, i.e. $y \neq -y'$. But then the ray generated by

$$(x, y) + (-x, y') = (0, y + y') \neq (0, 0) \in N_{\mathbb{Q}}$$

lies in τ, and this contradicts the existence of $y^x_-, y^x_+ \in \mathbb{Q}$ as in the assumption. $\square$

Corollary 2.4. *Let* Σ *in* $N^+_{\mathbb{Q}}$ *be a cobordism, let* $x, x' \in \pi(\mathrm{Supp}(\Sigma)) \subset N_{\mathbb{Q}}$*, let* $[x, x']$ *be the line segment connecting* x *and* x'*. Then* $\pi^{-1}([x, x']) \cap \partial_+\Sigma$ *resp.* $\pi^{-1}([x, x']) \cap \partial_-\Sigma$ *is the graph of a continuous, piecewise linear function from* $[x, x']$ *to* $\mathbb{Q}$.

Proof. The piecewise linearity is clear by the definition of cones. The intersections are graphs of continuous functions because there are no points $(x'', y) \in \partial_+\Sigma$ resp. $\partial_-\Sigma$ with $y^{x''}_- < y < y^{x''}_+$.



□

3. Circuits and bistellar operations

In this section it is shown how the circuits of π-nonsingular cobordisms correspond to blowups and blowdowns.

Definition 3.1. *Let Σ be a simplicial fan in $(N \oplus \mathbb{Z}) \otimes \mathbb{Q} = N_{\mathbb{Q}}^+$ with the natural projection $\pi : N_{\mathbb{Q}} \to N_{\mathbb{Q}}^+$. Assume that all the cones in Σ are π-strictly convex.*

A cone $\sigma \in \Sigma$ is π- ***independent*** *if $\pi : \sigma \to \pi(\sigma)$ is an isomorphism. Otherwise σ is π-* ***dependent****.*

A cone $\sigma \in \Sigma$ is called a ***circuit*** *if it is minimal among the π-dependant cones, i.e., if σ is π-**dependent** and any proper face of σ is π-* ***independent****.*

A cone $\sigma \in \Sigma$ is π- ***nonsingular*** *if the projection $\pi(\tau)$ of each π-independent face $\tau \subset \sigma$ is nonsingular as a cone in $N_{\mathbb{Q}}$ in the lattice N. The fan Σ is called π-**nonsingular** if all the cones in Σ are π- nonsingular.*

The following proposition shows how to find cobordisms in an arbitrary simplicial fan.

Proposition 3.2. *Let Σ be a simplicial fan in $(N \oplus \mathbb{Z}) \otimes \mathbb{Q} = N_{\mathbb{Q}}^+$ with the natural projection $\pi : N_{\mathbb{Q}} \to N_{\mathbb{Q}}^+$, which contains only π-strictly convex cones. Let $\sigma \in \Sigma$ be a π-dependent cone. Then σ is a cobordism between $\pi(\partial_-\sigma)$ and $\pi(\partial_+\sigma)$, and $\overline{\mathrm{Star}(\sigma)}$ is a cobordism between $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ and $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$.*

Proof. One problem is to see why $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ and $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ are fans in $N_{\mathbb{Q}}$. But this follows from condition (3) of the cobordism definition, too, so one only has to show this condition:

(a) Let σ be a cone in Σ, let $x \in N_{\mathbb{Q}}$. If $x = 0$, the intersection of the line $L_x = \pi^{-1}(x)$ with σ is $L_x \cap \sigma = \{0\}$ because σ is π-strictly convex. If $x \neq 0$ the intersection $L_x \cap \sigma$ is a closed intervall I_x on L_x. The end points and the interior $\overset{\circ}{I}_x$ are lying in the relative interior of (different) faces τ_+, τ_- and τ of σ. Furthermore, I_x varies continuously with $x \in \pi(\sigma)$. Finally, τ_+ and τ_- are π-independent because a cone τ is π-independent iff for an arbitrary point (x, y) in the relative interior of τ, $(x, y \pm \epsilon) \notin \tau$ for all $\epsilon > 0$.

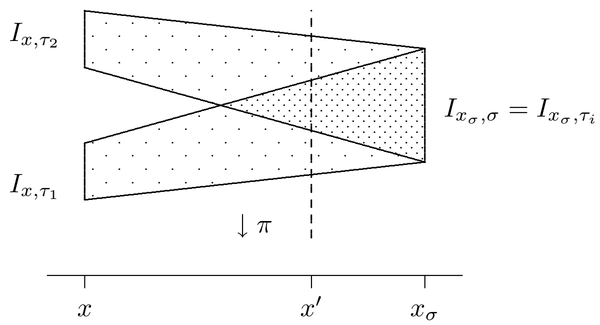
(b) By a characterization of faces, for every face τ of a cone σ there is a hyperplane $H_\tau \subset N_{\mathbb{Q}}^+$ such that $\sigma \cap H_\tau = \tau$, and σ is completely contained in one of the two half spaces seperated by H_τ. If τ is π-independent, H_τ can be chosen π- independent, too. Consequently, every π-independent face of a π-dependent cone σ is contained in $\partial_-\sigma$ or $\partial_+\sigma$.

(c) (a) and (b) imply immediately that every π-dependent cone σ satisfies condition (3) of the cobordism definition and is therefore a cobordism between $\pi(\partial_-\sigma)$ and $\pi(\partial_+\sigma)$.

(d) Since π-independent cones remain π- independent as faces of arbitrary π-dependent cones,
$$\partial_\pm \overline{\mathrm{Star}(\sigma)} = \bigcup_{\tau \supset \sigma} \partial_\pm \tau.$$

(e) For all $x \in \pi(\mathrm{Supp}(\overline{\mathrm{Star}(\sigma)}))$ there is a cone $\tau \supset \sigma$ such that
$$L_x \cap \mathrm{Supp}(\overline{\mathrm{Star}(\sigma)}) = L_x \cap \tau.$$
This is true, because, first of all, $L_x \cap \mathrm{Supp}(\overline{\mathrm{Star}(\sigma)}) = \bigcup_{\tau \supset \sigma} L_x \cap \tau$. If $x \in \sigma$, then $L_x \cap \tau = L_x \cap \sigma$ because of (a). Otherwise, set $I_{x,\tau} = L_x \cap \tau$. Suppose that $\tau_1, \tau_2 \in \Sigma$ are cones containing σ such that neither $I_{x,\tau_1} \subset I_{x,\tau_2}$ nor vice versa. Then $\overset{\circ}{I}_{x,\tau_1} \cap \overset{\circ}{I}_{x,\tau_2} = \emptyset$.

Since σ is π-dependent, there is a point $(x_\sigma, y_\sigma) \in \sigma$ such that $\overset{\circ}{I}_{x,\sigma} \neq \emptyset$. Since τ_1, τ_2 are π-strictly convex, the line segment $[x, x_\sigma]$ is contained in $\pi(\tau_1), \pi(\tau_2)$, and the intervalls $I_{x',\tau_1}, I_{x',\tau_2}$ vary continuously for $x' \in [x, x_\sigma]$ because of (a). This implies the existence of a point $x' \in [x, x_\sigma]$ such that neither $I_{x,\tau_1} \subset I_{x,\tau_2}$ nor $I_{x,\tau_2} \subset I_{x,\tau_1}$ nor $\overset{\circ}{I}_{x,\tau_1} \cap \overset{\circ}{I}_{x,\tau_2} = \emptyset$. This is impossible.



All these points imply that $\overline{\mathrm{Star}(\sigma)}$ satisfies condition (3) of the cobordism definition and consequently, $\overline{\mathrm{Star}(\sigma)}$ **is** a cobordism. □

The following theorem describes the transformation called **bistellar transformation**, from the lower face $\partial_-\sigma$ to the upper face $\partial_+\sigma$ of a circuit σ in a simplicial and π-nonsingular cobordism Σ. More generally, the theorem describes the transformation from the lower face $\partial_-\overline{\mathrm{Star}(\sigma)}$ to the upper face $\partial_+\overline{\mathrm{Star}(\sigma)}$ of the closed star of a circuit σ. It turns out that the bistellar operation corresponds to a smooth blowup immediately followed by a smooth blowdown.

Theorem 3.3. *Let Σ be a simplicial and π-nonsingular cobordism in $N^+_{\mathbb{Q}}$. Let $\sigma = \langle \rho_1, \ldots, \rho_k \rangle \in \Sigma$ be a circuit generated by the extremal rays ρ_i. Each extremal ray ρ_i contains a vector of the form $(v_i, w_i) \in N^+_{\mathbb{Q}} = N_{\mathbb{Q}} \oplus \mathbb{Q}$ where $v_i = n(\pi(\rho_i))$ is the primitive vector of the projection $\pi(\rho_i)$.*

(1) *There is a unique linear relation among the* v_i *(up to renumbering) of the form*

$$\sum r_\alpha v_\alpha = v_1 + \cdots + v_l - v_{l+1} - \cdots - v_k = 0 \text{ for some } 0 \leq l \leq k$$

with

$$\sum r_\alpha w_\alpha = w_1 + \cdots + w_l - w_{l+1} - \cdots - w_k > 0.$$

(2) *All the maximal faces* γ_i *(resp.* γ_j*) of* $\partial_-\sigma$ *(resp.* $\partial_+\sigma$*) are of the form*

$$\gamma_i = \langle \rho_1, \ldots, \widehat{\rho_i}, \ldots, \rho_l, \rho_{l+1}, \ldots, \rho_k \rangle, \ 1 \leq i \leq l,$$

$$(\text{resp. } \gamma_j = \langle \rho_1, \ldots, \rho_l, \rho_{l+1}, \ldots, \widehat{\rho_j}, \ldots, \rho_k \rangle, \ l+1 \leq j \leq k.$$

(3) *Let* l_σ *be the extremal ray in* $N_{\mathbb{Q}}$ *generated by the vector*

$$v_1 + \cdots + v_l = v_{l+1} + \cdots + v_k.$$

The smooth star subdivision of $\pi(\partial_-\sigma)$ *with respect to* l_σ *coincides with the smooth star subdivision of* $\pi(\partial_+\sigma)$ *with respect to* l_σ*, whose maximal faces are of the form*

$$\langle \pi(\gamma_{ij}), l_\sigma \rangle = \left\langle \pi(\rho_1), \ldots, \widehat{\pi(\rho_i)}, \ldots, \pi(\rho_l), \pi(\rho_{l+1}), \ldots, \widehat{\pi(\rho_j)}, \ldots, \pi(\rho_k), l_\sigma \right\rangle.$$

Thus the transformation of $\pi(\partial_-\sigma)$ *to* $\pi(\partial_+\sigma)$ *is a smooth star subdivision immediately followed by a smooth star assembling. This transformation is called a* ***bistellar operation****.*

Similarly, the transformation from $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ *to* $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ *is a smooth star subdivision immediately followed by a smooth star assembling.*

Proof. The only difficult point is to show that the $r_\alpha = \pm 1$. This is done by using the π-nonsingularity. For a complete proof of the theorem s. [AMR99, p.499]. ☐

4. The circuit graph and collapsibility

Let Σ be a simplicial cobordism between simplicial fans Δ and Δ'. Noting that

$$\Sigma = \partial_-\Sigma \cup \bigcup_\sigma \overline{\mathrm{Star}(\sigma)} \cup \partial_+\Sigma$$

where the union is taken over the circuits σ, one may try to factorize the transformation from Δ to Δ' into smooth star subdivisions and smooth star assemblings by replacing $\partial_-\overline{\mathrm{Star}(\sigma)}$ with $\partial_+\overline{\mathrm{Star}(\sigma)}$, if Σ is π-nonsingular. But this is not always possible; for a counterexample s. [Mat00]. The notion of "collapsibility" ensures this possibility.

Definition 4.1. *Let* Σ *be a simplicial cobordism in* $N^+_{\mathbb{Q}}$*. The* ***circuit graph*** *of* Σ *is a directed graph defined as follows: The vertices of the circuit graph consist of the circuits of* Σ*. There is an edge from* σ *to* σ' *if there is a point* $p \in \partial_-\overline{\mathrm{Star}(\sigma)} \cap \partial_+\overline{\mathrm{Star}(\sigma')}$ *such that*

$$p - (0,\epsilon) \in \overline{\mathrm{Star}(\sigma)}, \ p + (0,\epsilon) \in \overline{\mathrm{Star}(\sigma')} \text{ for sufficiently small } \epsilon.$$

Σ *is called* ***collapsible*** *if the circuit graph contains no directed cycle. When* Σ *is collapsible, the circuit graph determines a partial order among the circuits:* $\sigma \leq \sigma'$ *if there is an edge* $\sigma \to \sigma'$*.*

The next lemma gives a better intuition for this notion:

Lemma 4.2. *Let Σ be a simplicial, collapsible cobordism in $N^+_{\mathbb{Q}}$ with circuits σ_i, $1 \leq i \leq k$, such that $\sigma_i \leq \sigma_j$ implies $i \leq j$. Let $x \in \pi(\mathrm{Supp}(\Sigma))$. Then $\pi^{-1}(x) \cap \mathrm{Supp}(\Sigma)$ may be divided into intervalls $I_{x,i} := \pi^{-1}(x) \cap \mathrm{Supp}(\overline{\mathrm{Star}(\sigma_i)})$, such that the relative interiors of $I_{x,i}$ do not intersect. Some of the $I_{x,i}$ may be empty. If $I_{x,i} = [y_i^-, y_i^+]$,*

$$y_1^- \leq y_1^+ = y_2^- \leq \ldots \leq y_{k-1}^+ = y_k^- \leq y_k^+.$$

If $\overset{\circ}{I}_{x,i}, \overset{\circ}{I}_{x,j} \neq \emptyset$ and $y_i^+ = y_j^-$, there is an edge $\sigma_i \to \sigma_j$.

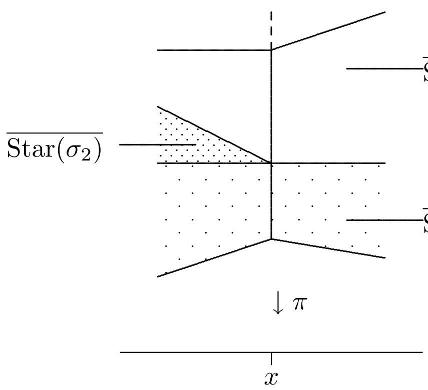
Proof. Point (e) of Proposition 3.2 shows: If $I_{x,i} := \pi^{-1}(x) \cap \mathrm{Supp}(\overline{\mathrm{Star}(\sigma_i)}) \neq \emptyset$ there is a π-dependent cone $\tau \in \overline{\mathrm{Star}(\sigma_i)}$ such that $I_{x,i} = \pi^{-1}(x) \cap \tau$. Since π-dependent cones contain exactly one circuit, the relative interiors of the $I_{x,i}$ do not intersect. The last statement of the lemma is immediate from the definition of an edge: The point (x, y_i^+) satisfies the conditions.

To prove the inequality chain the only cases still to exclude are

$$y_j^- < y_j^+ = y_i^- = y_i^+ \text{ or } y_j^- = y_j^+ = y_i^- < y_i^+ \text{ with } i < j.$$

For the first case, let $\tau_j \in \overline{\mathrm{Star}(\sigma_j)}, \tau_i \in \overline{\mathrm{Star}(\sigma_i)}$ be π-dependent cones such that $I_{x,j} = \pi^{-1}(x) \cap \tau_j$, $I_{x,i} = \pi^{-1}(x) \cap \tau_i$. Since τ_i is π-dependent there is a point x' arbitrarely near to x such that $\pi^{-1}(x') \cap \tau_i$ is an intervall with non-empty interior. Since by corollary 2.4, $I_x = \pi^{-1}(x) \cap \Sigma$ varies continuously with x, there is a π-dependent cone $\tau_j \subset \tau_j' \in \overline{\mathrm{Star}(\sigma_j)}$ with $x' \in \mathrm{Supp}(\tau_j')$. Since $I_{x,j}$ varies continuously with x, $I_{x',j}$ is still an intervall with non empty interior. This contradicts the last statement of the proposition.

The second case is done in the same way. ☐



Corollary 4.3. *Let Σ be a simplicial and collapsible cobordism in $N^+_{\mathbb{Q}}$ and $\sigma \in \Sigma$ a circuit. Let $\tau \in \partial_+\overline{\mathrm{Star}(\sigma)}$ and $\partial_+\sigma \subset \tau$. Then $\tau \notin \overline{\mathrm{Star}(\sigma')}$ for all $\sigma' < \sigma$.*

Similarly, let $\tau' \in \partial_-\overline{\mathrm{Star}(\sigma)}$ and $\partial_-\sigma \subset \tau'$. Then $\tau' \notin \overline{\mathrm{Star}(\sigma'')}$ for all $\sigma < \sigma''$.

Proof. Let $\sigma = \langle \rho_1, \ldots, \rho_k \rangle \in \Sigma$ be generated by the extremal rays $\rho_i = \langle (v_i, w_i) \rangle$, where $v_i = n(\pi(\rho_i))$ is the primitive vector of the projection $\pi(\rho_i)$. Let $v_1 + \cdots + v_l = v_{l+1} + \cdots + v_k$ be the unique relation such that $w_1 + \cdots + w_l - w_{l+1} - \cdots - w_k > 0$. Then $\partial_+\sigma = \langle \rho_1, \ldots, \rho_l \rangle$.

The midray $\rho := \left\langle (\sum_{i=1}^{l} v_i, \sum_{i=1}^{l} w_i) \right\rangle = \langle (v, w) \rangle$ has projection $\pi(\rho) = \langle v \rangle$, and because of the relation, the line segment $I_{v,\sigma} := \pi^{-1}(v) \cap \overline{\mathrm{Star}(\sigma)}$ has non-empty interior with upper bound (v, w). Thus the first statement follows from the lemma.

Similarly one proves the second statement. □

Now it is possible to state the fundamental existence theorem of cobordisms:

Theorem 4.4. *Let Δ and Δ' be two fans in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ with the same support. Then there exists a simplicial, collapsible, π-nonsingular cobordism in $N_{\mathbb{Q}}^{+}$ from Δ to Δ'.*

Proof. The first step to construct this cobordism is easy: Embed Δ "at the level -1" into $N_{\mathbb{Q}}^{+}$ so that the embedding Δ_{-} maps isomorphically back onto Δ by the projection π. Namely, take the fan Δ_{-} in $N_{\mathbb{Q}}^{+}$ consisting of the cones σ_{-} of the form

$$\sigma_{-} = \langle (v_1, -1), \ldots, (v_k, -1) \rangle ,$$

where the corresponding cone $\sigma = \langle \rho_1, \ldots, \rho_k \rangle$ is generated by the extremal rays ρ_i with the primitive vectors $v_i = n(\rho_i)$. Similarly, embed Δ' "at the level $+1$" into $N_{\mathbb{Q}}^{+}$ so that the embedding Δ'_{+} maps isomorphically back onto Δ' by the projection π.

Next, add the cones ζ of the form

$$\zeta = \langle (v, -1), (v, +1) \rangle ,$$

where the v vary among all the primitive vectors for the extremal rays ρ_v such that ρ_v is a generator for some $\sigma \in \Delta$ and $\sigma' \in \Delta'$ simultaneously.

Now, this fan is equivariantly completed, it is simplexified and made collapsible. For a complete account of these steps s. [AMR99].

The last and most subtle step of the construction is the π-desingularization. This done in [AMR99, section 5] and also in the very readable survey [Bon01]. □

To factorize a cobordism it is necessary to define the composition of two cobordisms:

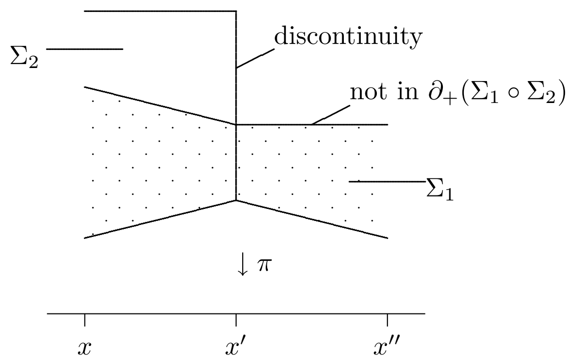
Definition-Proposition 4.5. *Let Σ_1, Σ_2 be cobordisms in $N_{\mathbb{Q}}^{+}$ such that*

(1) *$\Sigma_1 \cup \Sigma_2$ is again a fan in $N_{\mathbb{Q}}^{+}$,*
(2) *For all $x \in N_{\mathbb{Q}}^{+}$, $\pi^{-1}(x) \cap \Sigma_1 = \emptyset$ or $\pi^{-1}(x) \cap \Sigma_2 = \emptyset$ or $\pi^{-1}(x) \cap \Sigma_1 = [y_1^{-}, y_1^{+}]$, $\pi^{-1}(x) \cap \Sigma_2 = [y_2^{-}, y_2^{+}]$ with $y_1^{+} = y_2^{-}$.*
(3) *$\pi^{-1}(x) \cap (\Sigma_1 \cup \Sigma_2)$ varies continuously with $x \in \pi(\Sigma_1 \cup \Sigma_2)$.*

Then the union $\Sigma_1 \cup \Sigma_2$, called the ***composite*** *of Σ_1 with Σ_2 and denoted by $\Sigma_1 \circ \Sigma_2$, is a cobordism.*

Moreover, if both Σ_1 and Σ_2 are simplicial, collapsible and π-nonsingular, then so is the composite $\Sigma_1 \circ \Sigma_2$.

Proof. Condition (3) of the cobordism definition follows from (2) and (3). Condition (3) ensures that the end points of the intervalls $\pi^{-1}(x) \cap \mathrm{Supp}(\Sigma_1 \circ \Sigma_2)$ belong to $\partial_{-}(\Sigma_1 \circ \Sigma_2)$ resp. $\partial_{+}(\Sigma_1 \circ \Sigma_2)$.



The "moreover" part of the proposition is clear. □

In the special case of a simplicial, collapsible and π-nonsingular cobordism the following proposition shows how the factorization works:

Proposition 4.6. *Every simplicial, collapsible and π-nonsingular cobordism*

$$\Sigma = \partial_-\Sigma \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_{l+1})} \circ \overline{\mathrm{Star}(\sigma_l)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+\Sigma$$

in $N^+_{\mathbb{Q}}$ such that σ_i is minimal among the $\sigma_i, \sigma_{i-1}, \ldots, \sigma_1$ may be decomosed in two simplicial, collapsible and π-nonsingular cobordisms

$$\begin{array}{rcl} \Sigma_{k,l+1} & = & \partial_-\Sigma \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_{l+1})} \\ \Sigma_{l,1} & = & \overline{\mathrm{Star}(\sigma_l)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+\Sigma, \end{array}$$

i.e., $\Sigma = \Sigma_{k,l+1} \circ \Sigma_{l,1}$.

When $\pi^{-1}(x) \cap \mathrm{Supp}(\Sigma) = [y^x_-, y^x_+]$, then there is a $y_{l+1,l}$ such that

$$\pi^{-1}(x) \cap \mathrm{Supp}(\Sigma_{k,l+1}) = [y^x_-, y_{l+1,l}] \text{ and } \pi^{-1}(x) \cap \mathrm{Supp}(\Sigma_{l,1}) = [y_{l+1,l}, y^x_+].$$

Proof. This is a direct consequence of the description of a collapsible cobordism in Lemma 4.2 and the continuity statement in Corollary 2.4. □

5. Insertion and deletion of trivial cobordisms

The aim of this secton is to establish a new operation to manipulate a cobordism: the insertion and deletion of trivial cobordisms.

Definition 5.1. *A cobordism Σ in $N_{\mathbb{Q}}$ is called **trivial** iff all circuits are 2-dimensional.*

Obviously, the lower and the upper face $\partial_-\Sigma$ and $\partial_+\Sigma$ are mapped isomorphically to the same fan Δ in $N_{\mathbb{Q}}$. This means that these cobordisms correspond to the identity morphismus. Thus, they seem to be irrelevant. Their importance comes from the fact that a star subdivision can generate circuits whose dimensions are **bigger** than 2, as shown in lemma 6.8.

The next theorem describes how to insert trivial cobordisms:

Theorem 5.2. *Let Δ, Δ' be two nonsingular fans in $N_{\mathbb{Q}}$, let*

$$\Sigma = \partial_-\Sigma \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_{l+1})} \circ \overline{\mathrm{Star}(\sigma_l)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+\Sigma$$

be a simplicial, collapsible and π-nonsingular cobordism in $N^+_{\mathbb{Q}}$ from Δ to Δ' such that σ_i is minimal among the $\sigma_i, \sigma_{i-1}, \ldots, \sigma_1$. Let $\rho_\pm$ be a ray in $\partial_+\Sigma_{k,l+1} \cap \partial_-\Sigma_{l,1}$, *i.e., a generator of a cone in* $\partial_+\Sigma_{k,l+1}$ *and* $\partial_-\Sigma_{l,1}$ *simultaneously. Then there is a simplicial, collapsible and* π*-nonsingular cobordism*

$$\Sigma' = \partial_-\Sigma' \circ \overline{\mathrm{Star}(\sigma'_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma'_{l+1})} \circ \overline{\mathrm{Star}(\sigma_\pm)} \circ \overline{\mathrm{Star}(\sigma'_l)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma'_1)} \circ \partial_+\Sigma'$$

from Δ *to* Δ' *such that* $\sigma_\pm$ *is a 2-dimensional circuit generated by* ρ_-, ρ_+ *with* $\pi(\rho_\pm) = \pi(\rho_-) = \pi(\rho_+)$ *and in the circuit graph, there are no arrows from* $\sigma_\pm$ *to* σ'_i *with* $k \geq i \geq l+1$*, no arrows from* σ'_i *to* $\sigma_\pm$ *with* $l \geq i \geq 1$ *and no arrows from* σ'_i *to* σ'_j *with* $i < j$*.*

Furthermore, σ'_i *is a circuit whose bottom and top have the same dimension as those of* σ_i*.*

This operation is called ***insertion of a trivial cobordism*** *between* σ_{l+1} *and* σ_l *with respect to* $\rho_\pm$*.*

Proof. The main idea is to decompose Σ into $\Sigma_{k,l+1}$ and $\Sigma_{l,1}$ and then to tear apart these two cobordisms in $\rho_\pm$ by splitting $\rho_\pm$ into ρ_- and ρ_+ and "shifting upwards" all rays of $\Sigma_{k,l+1}$ not in the lower boundary. The technical difficulty of this "shifting upwards" is to ensure that the new rays remain linearly independent.

The "shifting upwards" is given by the following data: Attach to each $\rho \in \Sigma_{l,1}$ a rational number $\epsilon_\rho \geq 0$. When $\rho = \langle(v, w)\rangle$ where $v = n(\pi(\rho)) \in N$ is the primitive vector of $\pi(\rho)$, the new ray ρ' is generated by $(v, w + \epsilon_\rho)$.

So the first step is to prove the following

Lemma 5.3. *There is a simplicial fan* $\Sigma'_{l,1}$ *such that*

(a) *there is a 1-1 inclusion preserving correspondence between the cones* τ *of* $\Sigma_{l,1}$ *and* τ' *of* $\Sigma'_{l,1}$*.*

(b) *for each ray* $\rho \in \Sigma_{l,1}$ *there is a rational number* $\epsilon_\rho \geq 0$ *such that* $\rho' = \langle(v_\rho, w_\rho + \epsilon_\rho)\rangle$, $\epsilon_\rho = 0$ *for all* $\rho \in \partial_-\Sigma_{l,1}$, $\rho \neq \rho_\pm$, *and* $\epsilon_\pm := \epsilon_{\rho_\pm} > 0$.

(c) *the only circuits of* $\Sigma'_{l,1}$ *are the cones* σ'_i*; if* $\sigma_i = \langle(v_{i1}, w_{i1}), \ldots, (v_{in}, w_{in})\rangle$ *with*

$$\begin{aligned} v_{i1} + \cdots + v_{im} - v_{i,m+1} - \cdots - v_{in} &= 0 \\ w_{i1} + \cdots + w_{im} - w_{i,m+1} - \cdots - w_{in} &> 0, \end{aligned}$$

then still

$$(w_{i1} + \epsilon_{i1}) + \cdots + (w_{im} + \epsilon_{im}) - (w_{i,m+1} + \epsilon_{i,m+1}) - \cdots - (w_{in} + \epsilon_{in}) > 0.$$

Proof. It is an open and non-empty condition for the ϵ_ρ that the new cones τ' remain simplicial.

Note further that for each circuit σ_i there must be a $\rho_i \in \sigma_i$ such that $\rho_i \notin \partial_-\overline{\mathrm{Star}(\sigma_i)}$ and therefore $\rho_i \notin \sigma_j$ for all $j > i$.

Thus, $\Sigma'_{l,1}$ may be constructed as follows:

(1) Start with $\epsilon_\rho = 0$ for all $\rho \in \partial_-\Sigma_{l,1}$, $\rho \neq \rho_\pm$. Then there is a choice for the rest of the ϵ_ρ, $\rho \in \Sigma_{l,1}$, such that all the new cones τ' in $\Sigma'_{l,1}$ remain simplicial, and $\epsilon_\pm > 0$.

(2) Make successively the ϵ_{ρ_i} big enough such that

$$(w_{i1} + \epsilon_{i1}) + \cdots + (w_{im} + \epsilon_{im}) - (w_{i,m+1} + \epsilon_{i,m+1}) - \cdots - (w_{in} + \epsilon_{in}) > 0,$$

without changing the ϵ_ρ for

$$\rho = \Sigma_{l,i+1} = \partial_-\Sigma_{l,1} \circ \overline{\mathrm{Star}(\sigma_l)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_{i+1})}.$$

$\square$

Back to the proof of the theorem: Σ' consists of all the cones of $\Sigma_{k,l+1}$ and $\Sigma'_{l,1}$ as constructed above and additionally of the cones $\tau' + \rho_\pm$ for all cones $\tau \in \partial_-\Sigma_{l,1} = \partial_+\Sigma_{k,l+1}$ which contain $\rho_\pm = \rho_-$.

This is a simplicial cobordism between Δ and Δ' in $N^+_{\mathbb{Q}}$ which is also π-nonsingular because the primitive vectors of the images of the rays aren't changed. The only new circuit is $\sigma_\pm = \rho_- + \rho_+$.

The following consideration, together with Lemma 4.2, implies the collapsibility and the statements about the arrows:

Let $x \in \pi(\mathrm{Supp}(\Sigma'))$. By Proposition 4.6 there is a unique point $(x, y_\pm) \in \partial_-\Sigma_{l,1} = \partial_+\Sigma_{k,l+1}$. Let $\pi^{-1}(x) \cap \overline{\mathrm{Star}(\sigma_i)} =: I_{x,i} = [y_i^-, y_i^+]$ and $\pi^{-1}(x) \cap \overline{\mathrm{Star}(\sigma_\pm)} =: I_{x,\pm} = [y_-, y_+]$. Then there is an inequality sequence

$$y_k^- \leq y_k^+ = y_2^- \leq \ldots \leq y_{l+1}^+ = y_- \leq y_+ = y_l^- \leq \ldots \leq y_1.$$

Of course, the intervalls $I_{x,i}$ resp $I_{x,\pm}$ are inserted iff they are not empty. If $I_{x,\pm} \neq \emptyset$, then $y_\pm = y_-$. □

Theorem 5.4. *Let Δ, Δ' be two nonsingular fans in $N_{\mathbb{Q}}$, let*

$$\Sigma = \partial_-\Sigma \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_{l+1})} \circ \overline{\mathrm{Star}(\sigma_l)} \circ \overline{\mathrm{Star}(\sigma_{l-1})} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+\Sigma$$

be a simplicial, collapsible and π-nonsingular cobordism in $N^+_{\mathbb{Q}}$ from Δ to Δ' such that σ_i is minimal among the $\sigma_i, \sigma_{i-1}, \ldots, \sigma_1$ and $\sigma_l = \rho_- + \rho_+$ is a 2-dimensional circuit.

Then there is a simplicial, collapsible and π-nonsingular cobordism

$$\Sigma' = \partial_-\Sigma' \circ \overline{\mathrm{Star}(\sigma'_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma'_{l+1})} \circ \overline{\mathrm{Star}(\sigma'_{l-1})} \circ \cdots \circ \overline{\mathrm{Star}(\sigma'_1)} \circ \partial_+\Sigma'$$

from Δ to Δ' such that in the circuit graph, there are no arrows from σ'_i to σ'_j with $i < j$.

Furthermore, σ'_i is a circuit whose bottom and top have the same dimension as those of σ_i.

*This operation is called **deletion of the trivial cobordism** σ_l.*

Proof. There are real non-negative numbers r_+, r_- with $r_+ + r_- = 1$ with the following property:

For any cone τ in Σ of the form $\tau = \rho_1 + \cdots + \rho_{n-1} + \rho_+$, $\tau = \rho_1 + \cdots + \rho_{n-1} + \rho_-$ or $\tau = \rho_1 + \cdots + \rho_{n-2} + \rho_- + \rho_+$, the cone $\tau' = \rho_1 + \cdots + \rho_{n-1} + (r_+\rho_+ + r_-\rho_-)$ will be also a simplicial and π-nonsingular cone in $N^+_{\mathbb{Q}}$.

Now, to get Σ' from Σ, replace all cones $\tau \in \Sigma$ which contain ρ_+ or ρ_-, by τ'.

As in the previous theorem the collapsibility and the statements about the arrows follow from a consideration of the intersection of fibers $\pi^{-1}(x)$ with $\mathrm{Supp}(\Sigma')$. □

6. The algorithm

The purpose of this section is to show the strong factorization theorem, i.e., a proper and equivariant birational map $X_{\Delta_1} \dashrightarrow X_{\Delta_2}$ between smooth toric varieties can be factored into a sequence of smooth equivariant blowups $X_{\Delta_1} \leftarrow X_{\Delta_3}$ followed immediately by smooth equivariant blowdowns $X_{\Delta_3} \to X_{\Delta_2}$.

The starting point is a simplicial, collapsible and π-nonsingular cobordism Σ from Δ_1 to Δ_2. The first step is to identify the circuits which correspond only to a blowing up or down, not to both. Then the aim is to transform Σ into a new simplicial, collapsible and π-nonsingular cobordism Σ' from Δ'_1 to Δ'_2 such that Δ'_i

is obtained from Δ_i by a sequence of smooth star subdivisions, and Σ' contains only circuits corresponding to blowups.

Definition 6.1. *A π-nonsingular simplicial circuit*

$$\sigma = \langle (v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)\rangle \subset N_{\mathbb{Q}}^+$$

is called ***pointing up*** *(resp.* ***pointing down****) if it has exactly one positive (resp. negative) extremal ray, i.e., there is a linear relation among the primitive vectors $v_i = n(\pi(\rho_i))$ of the projections of the extremal rays ρ_i for σ (after renumbering)*

$$\begin{array}{rll} v_1 - v_2 - \cdots - v_k = 0 & \text{with} & w_1 - w_2 - \cdots - w_k > 0 \\ \text{resp. } -v_1 + v_2 + \cdots + v_k = 0 & \text{with} & -w_1 + w_2 + \cdots + w_k > 0. \end{array}$$

An immediate consequence of Theorem 3.3 is

Lemma 6.2. *Let Σ be a simplicial and π-nonsingular cobordism in $N_{\mathbb{Q}}^+$ and $\sigma \in \Sigma$ a circuit which is pointing up. Let*

$$\sigma = \langle (v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)\rangle \subset N_{\mathbb{Q}}^+$$

with the linear relation among the primitive vectors $v_i = n(\pi(\rho_i))$ of the projections of the extremal rays ρ_i for σ

$$v_1 - v_2 - \cdots - v_k = 0 \text{ with } w_1 - w_2 - \cdots - w_k > 0.$$

Then the bistellar operation going from $\pi(\partial_- \overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_+ \overline{\mathrm{Star}(\sigma)})$ is a smooth star subdivision with respect to the ray generated by

$$v_1 = v_2 + \cdots + v_k.$$

If σ is pointing down with the linear relation

$$-v_1 + v_2 + \cdots + v_k = 0 \text{ with } -w_1 + w_2 + \cdots + w_k > 0,$$

then the bistellar operation going from $\pi(\partial_- \overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_+ \overline{\mathrm{Star}(\sigma)})$ is a smooth star assembling, the inverse of a smooth star subdivision going from $\pi(\partial_+ \overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_- \overline{\mathrm{Star}(\sigma)})$ with respect to the ray generated by

$$v_1 = v_2 + \cdots + v_k.$$

Besides the insertion and deletion of trivial cobordism the key operation of the following algorithm is the smooth star subdivision with respect to the "π-barycenter" of a π-independent cone. It is necessary to give a criterion for the $\pi-$nonsingularity of the subdivided cobordism.

Definition 6.3. *Let τ be a cone in a simplicial cobordisms Σ in $N_{\mathbb{Q}}^+$ and l a ray in $\pi(\tau)$. Then the* ***midray*** $\mathrm{Mid}(\tau, l)$ *is defined to be the ray generated by the middle point of the line segment $\tau \cap \pi^{-1}(n(l))$. If $\tau \cap \pi^{-1}(n(l))$ consists of a point, then* $\mathrm{Mid}(\tau, l)$ *is the ray generated by that point.*

Definition 6.4. *Let η be a simplicial, π- dependent and π-strictly convex cone in $N_{\mathbb{Q}}^+$. A π-independent face τ of η is said to be* ***codefinite*** *with respect to η if the set of generators of τ does not contain both positive and negative extremal rays ρ_i of η. That is to say, if $\sum r_i v_i = 0$ is the nontrivial linear relation for η among the primitive vectors $v_i = n(\pi(\rho_i))$, then the generators for τ contain only extremal rays from the set $\{\rho_i : r_i < 0\}$ or from the set $\{\rho_i : r_i > 0\}$, exclusively.*

Lemma 6.5. *Let Σ be a simplicial and π-nonsingular cobordism. Let*

$$\tau = \langle (v_1, w_1), (v_2, w_2), \ldots, (v_l, w_l) \rangle \subset N_{\mathbb{Q}}^{+}$$

*be a π-independent cone of Σ with the $v_i = n(\pi(\rho_i))$ being the primitive vectors of the projections of the extremal rays ρ_i for τ. Let ρ_τ be the midray $\mathrm{Mid}(\tau, l_{r(\tau)})$, where $r(\tau) \in N$ is the vector $r(\tau) = v_1 + \cdots + v_l$, called the **$\pi$-barycenter** of τ. If τ is codefinite with respect to all the circuits $\sigma \in \Sigma$ with $\tau \in \overline{\mathrm{Star}(\sigma)}$, then $\rho_\tau \cdot \Sigma$ stays π-nonsingular.*

Proof. S. [AMR99, Lemma 7.3., resp. Prop. 5.5.]. □

Now, let Σ be a simplicial, collapsible and π-nonsingular cobordism from Δ_1 to Δ_2. To get a new simplicial, collapsible and π-nonsingular cobordism Σ' from Δ_1' to Δ_2' such that Δ_i' is obtained from Δ_i by a sequence of smooth star subdivisions, and Σ' contains only pointing up circuits, one uses an algorithm consisting of three steps:

6.1. **Replacing a circuit by a pointing up and a pointing down circuit.** Let

$$\Sigma = \partial_- \Sigma \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_i)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+ \Sigma$$

be a simplicial, collapsible and π-nonsingular cobordism in $N_{\mathbb{Q}}^{+}$. Let σ_i be a circuit which is not pointing up or down. The aim is to replace $\overline{\mathrm{Star}(\sigma_i)}$ in Σ by $\overline{\mathrm{Star}(\sigma_i^{\uparrow})} \circ \overline{\mathrm{Star}(\sigma_i^{\downarrow})}$, where $\sigma_i^{\uparrow}$ is pointing up and $\sigma_i^{\downarrow}$ is pointing down.

The following simple observation of Morelli is the basis of this step:

Lemma 6.6. *Let σ be a circuit in a simplicial and π-nonsingular cobordism Σ. Let*

$$\sigma = \langle (v_1, w_1), \ldots, (v_m, w_m), (v_{m+1}, w_{m+1}), \ldots, (v_k, w_k) \rangle,$$

where $v_1, \ldots, v_m, v_{m+1}, \ldots, v_k$ are the primitive vectors in N of the projections of the extremal rays for σ, having the unique linear relation

$$v_1 + \cdots + v_m - v_{m+1} - \cdots - v_k = 0 \text{ with } w_1 + \cdots + w_m - w_{m+1} - \cdots - w_k > 0.$$

Let

$$\sigma_+ = \langle (v_1, w_1), \ldots, (v_m, w_m) \rangle \text{ and } \sigma_- = \langle (v_{m+1}, w_{m+1}), \ldots, (v_k, w_k) \rangle.$$

Then the fan $\rho_{\sigma_+} \cdot \overline{\mathrm{Star}(\sigma)}$, where ρ_{σ_+} is the midray $\mathrm{Mid}(\sigma_+, l_{r(\sigma_+)})$ with $r(\sigma_+)$ being the π-barycenter of σ_+, is π-nonsingular and the closed star of a π-nonsingular circuit $\sigma' = \rho_{\sigma_+} + \sigma_-$.

Similarly, the fan $\rho_{\sigma_-} \cdot \overline{\mathrm{Star}(\sigma)}$, where ρ_{σ_-} is the midray $\mathrm{Mid}(\sigma_-, l_{r(\sigma_-)})$ with $r(\sigma_-)$ being the π-barycenter of σ_-, is π-nonsingular and the closed star of a π-nonsingular circuit $\sigma'' = \rho_{\sigma_-} + \sigma_+$.

Proof. S. [AMR99, lemma 7.5.]. □

The problem with this operation (which also spoils the proof of the strong factorization in [AMR99]) is that the star subdivision may possibly also affect other cones than those in $\overline{\mathrm{Star}(\sigma)}$, thus perhaps creating new circuits or even destroying the π-nonsingularity. To control this process one inserts a trivial cobordism:

Proposition 6.7. *Let Σ and σ_i be as in the beginning of this subsection. Let $(v_{i1}, w_{i1}), \dots, (v_{im}, w_{im})$ be the generators of the positive rays of σ_i, i.e., $(\sigma_i)_+ = \langle (v_{i1}, w_{i1}), \dots, (v_{im}, w_{im}) \rangle$, where $(\sigma_i)_+$ is defined as in the lemma above. Let*

$$\Sigma' = \partial_- \Sigma' \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_i)} \circ \overline{\mathrm{Star}(\sigma_\pm)} \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+ \Sigma'$$

be constructed from Σ by inserting the trivial cobordism $\overline{\mathrm{Star}(\sigma_\pm)}$ with respect to $\rho_1 = \langle (v_1, w_1) \rangle$.

Then the star subdivision $\rho_{(\sigma_i)_+} \cdot \Sigma$ only affects the closed stars of σ_i and $\sigma_\pm$ and gives a cobordism which replaces $\overline{\mathrm{Star}(\sigma_i)} \circ \overline{\mathrm{Star}(\sigma_\pm)}$ by $\overline{\mathrm{Star}(\sigma_i^\uparrow)} \circ \overline{\mathrm{Star}(\sigma_\pm)} \circ \overline{\mathrm{Star}(\sigma_i^\downarrow)}$, where $\sigma_i^\uparrow$ is pointing up and $\sigma_i^\downarrow$ is pointing down. Deleting the trivial cobordism $\overline{\mathrm{Star}(\sigma_\pm)}$ finally gives a simplicial, collapsible and π-nonsingular cobordism

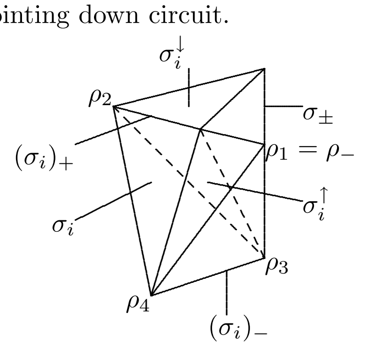
$$\Sigma'' = \partial_- \Sigma'' \circ \overline{\mathrm{Star}(\sigma_k)} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_i^\uparrow)} \circ \overline{\mathrm{Star}(\sigma_i^\downarrow)} \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+ \Sigma''.$$

Proof. First of all, note that the tops and the bottoms of the circuits $\sigma_j' \in \Sigma'$, $\sigma_j'' \in \Sigma''$ have the same dimension as those of $\sigma_j \in \Sigma$. Therefore, the numbers of pointing up and pointing down circuits, besides the inserted ones, remain unchanged. In a sloppy notation, the primes are deleted.

By corollary 4.3, the cone σ_+ is only belonging to $\overline{\mathrm{Star}(\sigma_\pm)}$ and $\overline{\mathrm{Star}(\sigma_i)}$. By Lemma 6.6, $\rho_{(\sigma_i)_+} \cdot \overline{\mathrm{Star}(\sigma_i)} = \overline{\mathrm{Star}(\sigma_i^\uparrow)}$ with $\sigma_i^\uparrow = \rho_{\sigma_+} + (\sigma_i)_-$. The next lemma shows that $\rho_{(\sigma_i)_+} \cdot \overline{\mathrm{Star}(\sigma_\pm)}$ is the union of the (new) closed star of $\sigma_\pm$ and the closed star of the new circuit

$$\sigma_i^\downarrow = \left\langle (v_1, w_+), (v_2, w_2), \dots, (v_m, w_m), \rho_{\sigma_+} \right\rangle,$$

which is a pointing down circuit.



$$v_1 + v_2 - v_3 - v_4 = 0$$

□

Lemma 6.8. *Let $\eta = \langle \gamma_1, \dots, \gamma_m, \rho_1, \dots, \rho_k \rangle$ be a simplicial π-dependent cone such that the unique circuit $\sigma \subset \eta$ is generated by $\rho_1, \dots, \rho_k$, and $\sigma_+ = \partial_+ \sigma = \langle \rho_1, \dots, \rho_l \rangle$, $\sigma_- = \partial_- \sigma = \langle \rho_{l+1}, \dots, \rho_k \rangle$.*

Let $\tau = \langle \gamma_n, \dots, \gamma_m, \rho_1, \dots, \rho_l \rangle \subset \partial_- \eta$ be a face containing σ_-, let ρ_τ be the midray $\mathrm{Mid}(\tau, l_{r(\tau)})$. Then the smooth star subdivision $\rho_\tau \cdot \eta$ is the union of the (new) star of the (old) circuit σ and the closed star of the new circuit $\sigma' = \langle \rho_\tau, \gamma_n, \dots, \gamma_m, \rho_{l+1}, \dots, \rho_k \rangle$, where

$$\sigma'_+ = \langle \gamma_n, \dots, \gamma_m, \rho_{l+1}, \dots, \rho_k \rangle, \ \sigma'_- = \langle \rho_\tau \rangle.$$

A similar statement is true for $\tau' \subset \partial_+ \eta$, $\sigma_+ \subset \tau'$.

Proof. This is a generalization of Morelli's lemma and is proven in the same way. See again [AMR99, lemma 7.5] □

6.2. **Exchanging a pointing down and a pointing up circuit.** Let

$$\Sigma = \partial_-\Sigma \circ \cdots \circ \overline{\mathrm{Star}(\sigma_i)} \circ \overline{\mathrm{Star}(\sigma_{i-1})} \circ \cdots \partial_+\Sigma$$

be a simplicial, collapsible and π-nonsingular cobordism such that σ_i is pointing down and σ_{i-1} is pointing up.

If no arrow in the circuit graph points from σ_i to σ_{i-1} it will be possible to exchange σ_i and σ_{i-1} without destroying the induced preorder.

Otherwise, by definition there will be point p in $\partial_-\overline{\mathrm{Star}(\sigma_i)} \cap \partial_+\overline{\mathrm{Star}(\sigma_{i-1})}$ such that

$$p - (0,\epsilon) \in \overline{\mathrm{Star}(\sigma_i)},\ p + (0,\epsilon) \in \overline{\mathrm{Star}(\sigma_{i-1})} \text{ for sufficiently small } \epsilon.$$

Lemma 6.9. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism containing the circuits σ and σ'. If p is a point in $\partial_+\overline{\mathrm{Star}(\sigma)} \cap \partial_-\overline{\mathrm{Star}(\sigma')}$ such that*

$$p - (0,\epsilon) \in \overline{\mathrm{Star}(\sigma)},\ p + (0,\epsilon) \in \overline{\mathrm{Star}(\sigma')} \text{ for sufficiently small } \epsilon,$$

then p is belonging to a cone $\tau \in \overline{\mathrm{Star}(\sigma)} \cap \overline{\mathrm{Star}(\sigma')}$ such that $\partial_+\sigma \subset \tau$, $\partial_-\sigma' \subset \tau$.

Proof. Let τ be the uniquely determined minimal cone in $\partial_+\overline{\mathrm{Star}(\sigma)} \cap \partial_-\overline{\mathrm{Star}(\sigma')}$ which contains p.

Let $\tau \subset \tau_- \in \overline{\mathrm{Star}(\sigma)}$ be a π-dependent cone containing p and $p-\epsilon$. Let

$$(u_1, w'_1), \ldots, (u_k, w'_k), (v_1, w_1), \ldots, (v_l, w_l), (v_{l+1}, w_{l+1}), \ldots, (v_m, w_m)$$

be generators of τ_-, where $u_1, \ldots, u_k, v_1, \ldots, v_m$ are the primitive vectors in N of the projections of the extremal rays of τ_-, having the unique linear relation

$$v_1 + \cdots v_l - v_{l+1} - \cdots - v_m = 0 \text{ with } w_1 + \cdots w_l - w_{l+1} - \cdots - w_m > 0.$$

Then

$$\begin{array}{rcl} p & = & a_1(u_1, w'_1) + \cdots + a_k(u_k, w'_k) + b_1(v_1, w_1) + \cdots + b_m(v_m, w_m), \\ p - \epsilon & = & c_1(u_1, w'_1) + \cdots + c_k(u_k, w'_k) + d_1(v_1, w_1) + \cdots + d_m(v_m, w_m), \end{array}$$

where all the coefficients are ≥ 0. The cone τ is generated by all (u_i, w'_i), (v_j, w_j) with $a_i \neq 0$, $b_j \neq 0$. Since $p \in \partial_+\overline{\mathrm{Star}(\sigma)}$, there is a $j = 1, \ldots, l$ with $b_j > 0$.

Suppose that $b_1 = 0$. The expressions for p and $p - \epsilon$ above imply the relation

$$a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m = c_1u_1 + \cdots + c_ku_k + d_1v_1 + \cdots + d_mv_m$$

which must be a non vanishing multiple of $v_1 + \cdots + v_l = v_{l+1} + \cdots + v_m$. Therefore,

$$\begin{array}{rclcll} a_i & = & c_i, & & & i = 1, \ldots, k \\ b_i - d_i & = & -d_1 & < & 0, & i = 2, \ldots, l \\ b_i - d_i & = & d_1 & > & 0, & i = l+1, \ldots, m. \end{array}$$

But furthermore,

$$\begin{array}{c} (a_1 - c_1)w'_1 + \cdots + (a_k - c_k)w'_k + (b_1 - d_1)w_1 + \cdots + (b_m - d_m)w_m = \\ (-d_1)(w_1 + \cdots + w_l - w_{l+1} - \cdots - w_m) = \epsilon. \end{array}$$

This contradicts $w_1 + \cdots + w_l - w_{l+1} - \cdots - w_m > 0$. Consequently, $\partial_+\sigma \subset \tau$.

In the same way one proves $\partial_-\sigma' \subset \tau$. □

Now, consider the minimal cone containing $p, \partial_+\sigma_i$ and $\partial_-\sigma_{i-1}$ as in the lemma above. Let

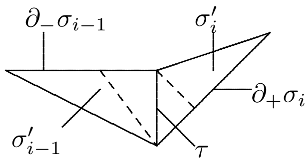
$$(v_1, w_1), \ldots, (v_l, w_l), \ldots, (v_k, w_k), \ldots, (v_m, w_m), \ldots, (v_n, w_n),\ l \leq k \leq m,$$

be generators of this cone, with the usual conventions about the v_i and w_i, such that $(v_1, w_1), \dots, (v_k, w_k)$ generate $\partial_-\sigma_{i-1}$ and $(v_{l+1}, w_{l+1}), \dots, (v_m, w_m)$ generate $\partial_+\sigma_i$.

Suppose first that $k = l$. The first step is to insert a trivial cobordism $\overline{\mathrm{Star}(\tau)}$ between σ_{i-1} and σ_{i-2} with respect to (v_m, w_m). The next step is a smooth star subdivision of $\partial_+\sigma_i$. By corollary 4.3 only the stars of σ_i, σ_{i-1} and τ are involved: σ_i is replaced by a trivial circuit, σ_{i-1} stays as it is (since $\partial_+\sigma_i$ does not intersect σ_{i-1}), and $\overline{\mathrm{Star}(\tau)}$ splits into the star of a trivial cobordism and in the star of a pointing down circuit with peak in $(v_{l+1} + \dots + v_m, w_{l+1} + \dots + w_m)$, as described in lemma 6.8. Deletion of the trivial cobordisms gives the exchange of the pointing up and pointing down circuit.

If $k > l$, the first step is to insert a trivial cobordism $\overline{\mathrm{Star}(\tau)}$ between σ_i and σ_{i-1} with respect to (v_k, w_k). The next step is a smooth star subdivision of $\partial_-\sigma_{i-1}$. Only the stars of σ_{i-1} and τ are involved: σ_{i-1} is replaced by a trivial circuit, and $\mathrm{Star}(\tau)$ splits into the star of pointing up circuit σ'_{i-1} with peak in $(v_1 + \dots + v_k^+, w_1 + \dots + w_k^+)$ and the new star of τ. There is an arrow from σ'_{i-1} to τ.

Afterwards, one makes a smooth star subdivision of $\partial_+\sigma_i$, and the only stars involved are those of σ_i and τ: σ_i is replaced by a trivial circuit, and $\mathrm{Star}(\tau)$ splits into the star of pointing down circuit σ'_i with peak in $(v_{l+1} + \dots + v_k^- + \dots + v_m, w_{l+1} + \dots + w_k^- + \dots + w_m)$ and the new star of τ.



The deletion of the trivial cobordisms gives the exchange of the pointing up and pointing down circuit, because of the arrow from σ'_{i-1} to τ.

6.3. **Deleting pointing down circuits.** By the above two operations it is possible to transform an arbitrary simplicial, collapsible and π-nonsingular cobordism into a simplicial, collapsible and π-nonsingular cobordism which is composed of several pointing up circuits, followed by some other pointing down circuits. A smooth star subdivision of the top of the maximal pointing down circuit replaces this circuit by a trivial one, without affecting the other circuits. Deleting the corresponding trivial cobordism reduces the number of the pointing down circuits. Continuing this process the aim of the algorithm is finally reached.

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