Classification of higher dimensional algebraic varieties

Thomas Eckl

University of Liverpool

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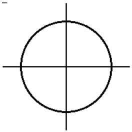
INTRODUCTION

Complex algebraic geometry

Problem: Classify all smooth projective complex algebraic varieties up to isomorphy!

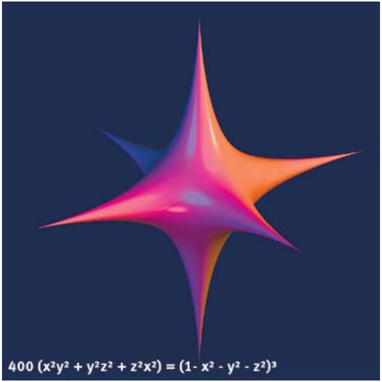
smooth **projective complex algebraic varieties**

A complex algebraic variety is the solution set of a system of homogeneous polynomial equations in $\mathbb{C}^n$ ($\mathbb{CP}^n$).



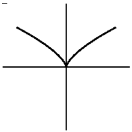
$x^2 + y^2 = 1$

Complex algebraic varieties



Problem: Classify all **smooth** projective complex algebraic varieties up to isomorphy!

A projective complex algebraic variety is called smooth if it is a **complex manifold**.
Otherwise the variety is called **singular**.



$$x^2 - y^3 = 0$$

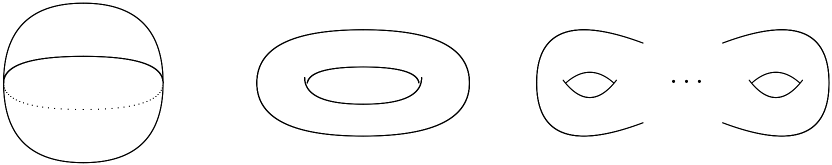
Problem: Classify all projective complex manifolds!

First division by complex **dimension** of the manifold:

dimension 0: points
dimension 1: projective algebraic curves

PROJECTIVE ALGEBRAIC CURVES

1-dimensional projective complex manifold
$\|$
2-dimensional orientable closed compact real surface
with complex structure



Algebraic topology: Homeomorphy classes are characterized by genus g

genus g = "number of holes"

Classification of algebraic curves

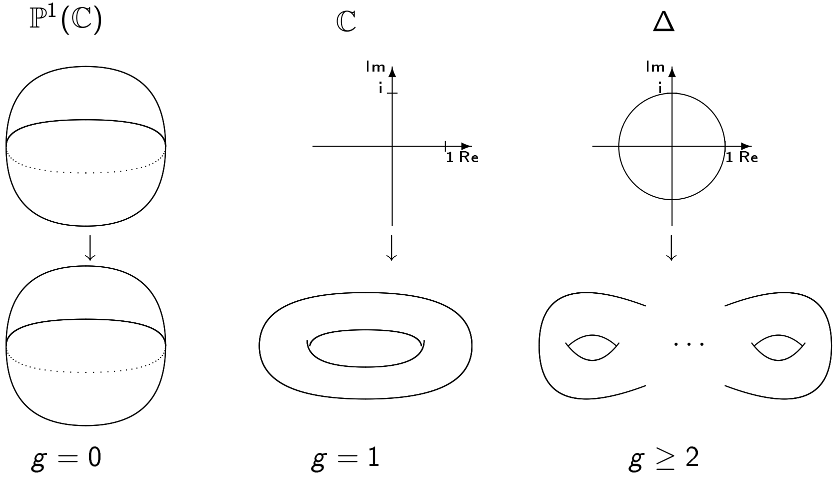
Division in classes of isomorphic curves by

genus: discrete/numeric invariant

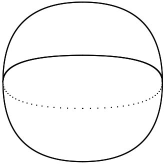
moduli space: continuous invariant

Theorem (D. Mumford, Fields-Medal 1974):
The classes of isomorphic projective algebraic curves of genus g are parametrized by the points of a projective complex algebraic variety M_g, with a universal property.

Riemann's Mapping Theorem: There are exactly 3 simple-connected 1-dimensional complex manifolds

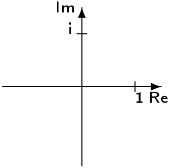


Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$



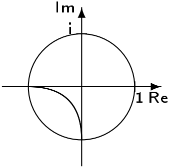
curvature $= 1$

Euklidian metric on $\mathbb{C}$



curvature $= 0$

Poincaré metric on Δ



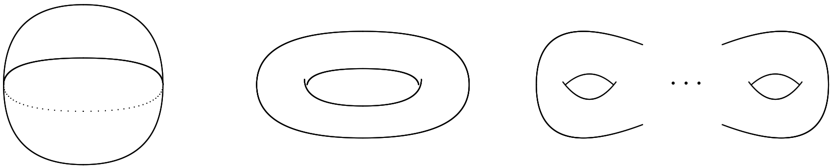
curvature $= -1$

metrics are invariant under deck transformations

$\downarrow$

induced metric on covered curve

Trichotomy



$g = 0$ curvature > 0

$g = 1$ curvature $= 0$

$g \geq 2$ curvature < 0

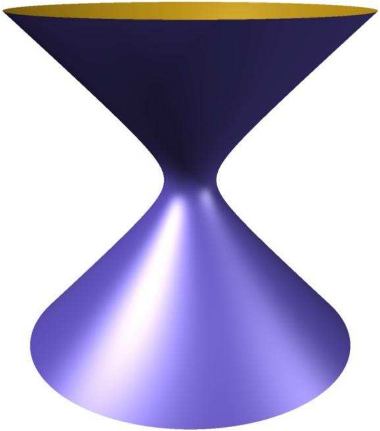
HIGHER DIMENSIONAL ALGEBRAIC VARIETIES

Trichotomy: curvature > 0

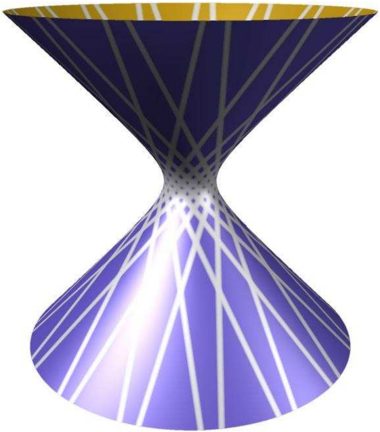
$\mathbb{P}^1(\mathbb{C})$ is exceptional: **Unique** curve with positively curved metric

Idea: If a projective complex manifold contains many rational curves $\cong \mathbb{P}^1(\mathbb{C})$, then this implies something for the curvature.

Quadric
$x^2 + y^2 - z^2 = 1$



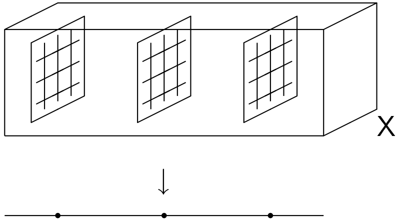
Quadric
$x^2 + y^2 - z^2 = 1$

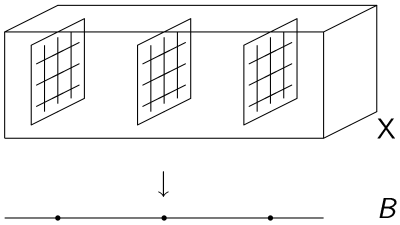


A projective complex manifold X is called **uniruled**, if it is covered by rational curves in X.

A projective complex manifold X is called rationally connected, if 2 general points x, y can be connected by chains of rational curves.

equivalence relation $\rightsquigarrow$ quotient map



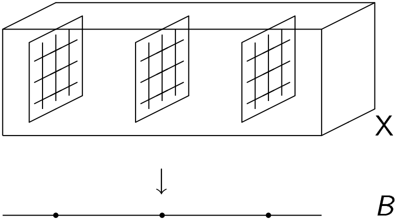


Theorem (Campana; Kollár-Miyaoka-Mori):
For every projective complex manifold X there is a holomorphic map $f : X \to B$ with

- rationally connected fibres and
- a universal property.

$f : X \to B$ is called MRC-quotient.

The MRC-quotient



Problem: Charakterisation of the MRC-quotient
by "positive directions" of the tangent bundle

Eckl (will appear in *Math. Nachr.*): Naive idea not correct.

Theorem (Graber-Harris-Starr 2000):
X projective complex manifold.
$f : X \to B$ MRC-quotient $\Longrightarrow$ B not uniruled

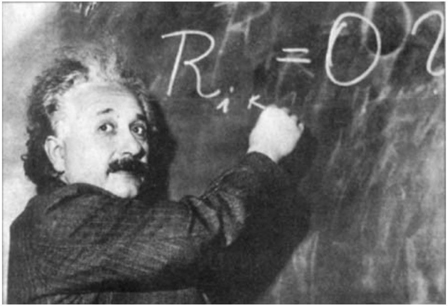
Resume

Every projective complex manifold can be decomposed in **rationally connected** and **not-uniruled** varieties.

Trichotomy: Curvature $= 0$

Metric g on a projective complex manifold.

Ricci curvature $=$ trace of the curvature tensor of g



Theorem (S.-T.Yau, Fields-Medal 1982): There are Ricci-flat, but not flat projective complex manifolds of arbitrary dimension ≥ 2, so called **Calabi-Yau-Varieties**.

Fact: Ricci-flat projective complex manifolds are not uniruled.

ABUNDANCE

Two points of view on positivity

X n-dimensional projective complex manifold

differential geometry: metric g with Ricci-curvature ≥ 0

Ricci-curvature = curvature of the induced metric on the canonical holomorphic line bundle K_X

algebraic geometry: global holomorphic sections of K_X

global holomorphic sections of K_X = holomorphic n-forms,
locally: $f(z)dz_1 \wedge \ldots \wedge dz_n$ with f holomorphic

X n-dimensional projective complex manifolds

algebraic geometry		**differential geometry**
$\exists$ sections of K_X	$\Rightarrow$	$\exists$ singular metric on K_X with semi-positive curvature
$\exists$ sections of $K_X^{\otimes m}$ $\Vert$ pluri-canonical forms $f(z)(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}$	$\Leftarrow$	$\exists$ smooth metric on K_X with positive curvature

sections of $K_X \rightsquigarrow$ map $X \to \mathbb{P}^N$

X projective complex manifolds
$\sigma_0, \ldots, \sigma_N$ m-canonical forms on X

representation in local coordinates:

$$\begin{array}{ccc} (z_1, \ldots, z_n) & & \sigma_i = f_i(z)(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m} \\ \psi \downarrow & & \| \\ (w_1, \ldots, w_n) & & f_i(\psi(w)) \cdot \det(\frac{\partial z_k}{\partial w_l})^m \cdot (dw_1 \wedge \ldots \wedge dw_n)^{\otimes m} \end{array}$$

$\Rightarrow$ $\Phi_{\sigma_0,\ldots,\sigma_N} : X \to \mathbb{P}^N,\ p \mapsto [f_0(p) : \ldots : f_N(p)]$
is a rational map.

The Kodaira-dimension $\kappa(X)$ of X is the maximal dimension of the image

$$\Phi_{\sigma_0,\ldots,\sigma_N}(X) \subset \mathbb{P}^N$$

for any m-canonical forms $\sigma_0, \ldots, \sigma_N$.
If $\kappa(X)$ equals the dimension of the image of

$$\Phi_{\sigma_0,\ldots,\sigma_N} : X \to B \subset \mathbb{P}^N,$$

this map is called Kodaira-Iitaka fibration.
If there are no m-canonical forms $\neq 0$, we set

$$\kappa(X) := -\infty.$$

Facts:

- $-\infty \leq \kappa(X) \leq \dim X$.
- X uniruled $\Rightarrow \kappa(X) = -\infty$.
- Fibers of the Kodaira-Iitaka fibration have Kodaira dimension 0.

Abundance conjecture (Mumford)

$$\kappa(X) = -\infty \Rightarrow X \text{ uniruled.}$$

Fact: $\exists$ topological bound for $\kappa(X)$:

$$\kappa(X) \leq \nu(X)$$

Definition: Numerical dimension $\nu(X) := \max\{k : c_1(K_X)^k \neq 0\}$

Generalized Abundance conjecture

$$X \text{ not uniruled} \Rightarrow \kappa(X) = \nu(X).$$

Idea of proof for a special case of the Abundance conjecture:

Assumption: The Kodaira-Iitaka fibration $f : X \to B \subset \mathbb{P}^N$ is everywhere defined.

- K_X is the f-pullback of the hyperplane bundle H on $\mathbb{P}^N$.
- The Fubini-Study metric h_{FS} on H has positive curvature.
- The pullback $h_X = f^* h_{FS}$ is a metric on K_X,
  with positive curvature in f-transversal directions,
  $\equiv 0$ in direction of the f-fibres.
- $c_1(K_X)^{\dim B} = c_1(K_X, h_X)^{\dim B} > 0$,
- $c_1(K_X)^k = c_1(K_X, h_X)^k = 0$ for $k > \dim B$
- $\Rightarrow \kappa(X) = \dim B = \nu(X)$. □

Observe:

- On K_X, a metric with semi-positive curvature is constructed vanishing in direction of the fibres.
- The construction is possible for every holomorphic line bundle.

NUMERICAL TRIVIALITY

Problem: Given a semi-positive line bundle on X.
Construct a maximal fibration with fibers, in whose direction the curvature of the line bundle vanishes!

Theorem (Tsuji; Bauer, Campana, Eckl, Kebekus et al.):
L nef line bundle on projective complex manifold X.
Then there is a rational map $f : X \to Y$, such that:

- F fiber of $f \Rightarrow \forall$ curves $C \subset F$: $c_1(L)_{|C} = 0$.
- $x \in X$ general, $x \in C \subset X$ curve with $\dim f(C) = 1 \Rightarrow$ $c_1(L)_{|C}$ does not vanish

Definition: L pseudo-effective $:\Leftrightarrow$ $\exists$ singular metric h of L with

$$c_1(L, h) \geq 0.$$

Theorem (Tsuji; Eckl): X projective complex manifold,
(L, h) pseudo-effective line bundle on X.
Then there is a map $f : X \to Y$, such that:

- F general fiber of f, $C \subset F$ curve with $h_{|C} \neq +\infty$:

$$c_1(L, h)_{|C - \mathrm{Sing}(h)} \equiv 0.$$

- $x \in X$ general, $C \ni x$ curve with $\dim f(C) = 1 \Rightarrow$ $c_1(L, h)_{|C - \mathrm{Sing}(h)} \not\equiv 0$.

Problem: basis Y can have dimension $> \nu(L)$.

Definition: X projective complex manifolds,
(L, h) line bundle with metric h and curvature $c_1(L, h) \geq 0$.
A foliation $\mathcal{F}$ on X is called numerically (L, h)-trivial, if

$$c_1(L, h)_{|\text{leaf}} \equiv 0.$$

Theorem (Eckl)

There exists a maximal numerically (L, h)-trivial foliation.

Properties of the numerically trivial foliation

fibers of the numerically trivial fibration

$\cap$

leaves of the numerically trivial foliation

$\cap$

fibers of the Kodaira-Iitaka fibration.

UPSHOT AND FURTHER PROSPECTS

Upshot: Complex differential geometry helps in understanding the Abundance conjecture.

Strategy for Abundance conjecture:

(1) The numerically K_X-trivial foliation has leaves of dimension $\dim X - \nu(X)$.

(2) The numerically K_X-trivial foliation is a fibration.

(3) The numerically K_X-trivial foliation is a fibration $f : X \to Y$ with $\dim Y = \nu(X)$
$\Rightarrow X$ is abundant.

Upshot: Complex differential geometry helps in understanding the Abundance conjecture.

Conjecture (Yau): X compact Kähler manifold,
g Kähler metric with non-positive holomorphic bisectional curvature
$\Rightarrow X$ is abundant.

Wu/Zheng:
Additional assumption on metric $\Rightarrow$ Step (1) and (2)
Step (3) trivial.

Eckl: Step (1) $\Rightarrow$ Step (2) and (3)