Classification of higher dimensional algebraic varieties

Thomas Eckl

University of Liverpool

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INTRODUCTION
Problem: Classify all smooth projective complex algebraic varieties up to isomorphy!
A complex algebraic variety is the solution set of a system of homogeneous polynomial equations in $\mathbb{C}^n (\mathbb{CP}^n)$.

$x^2 + y^2 = 1$
Complex algebraic varieties

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Problem: Classify all smooth projective complex algebraic varieties up to isomorphy!

A projective complex algebraic variety is called smooth if it is a complex manifold. Otherwise the variety is called singular.

\[ x^2 - y^3 = 0 \]
Problem: Classify all projective complex manifolds!

First division by complex dimension of the manifold:

- dimension 0: points
- dimension 1: projective algebraic curves
PROJECTIVE ALGEBRAIC CURVES
1-dimensional projective complex manifold

\[ \|
\]
2-dimensional orientable closed compact real surface with complex structure

Algebraic topology: Homeomorphism classes are characterized by genus \( g \)

genus \( g \) = “number of holes”
Classification of algebraic curves

Division in classes of isomorphic curves by

\textbf{genus:} discrete/numeric invariant

\textbf{moduli space:} continuous invariant

\textbf{Theorem (D. Mumford, Fields-Medal 1974):}
The classes of isomorphic projective algebraic curves of genus $g$ are
parametrized by the points of a projective complex algebraic variety $M_g$, with a
universal property.
Riemann’s Mapping Theorem: There are exactly 3 simple-connected 1-dimensional complex manifolds
Differential geometry on algebraic curves

Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$

- curvature = 1

Euklidian metric on $\mathbb{C}$

- curvature = 0

Poincaré metric on $\Delta$

- curvature = $-1$

Metrics are invariant under deck transformations

↓

Induced metric on covered curve
Trichotomy

\[ g = 0 \]
\[ \text{curvature} > 0 \]

\[ g = 1 \]
\[ \text{curvature} = 0 \]

\[ g \geq 2 \]
\[ \text{curvature} < 0 \]
HIGHER DIMENSIONAL ALGEBRAIC VARIETIES
Trichotomy: curvature $> 0$

$\mathbb{P}^1(\mathbb{C})$ is exceptional: **Unique** curve with positively curved metric

Idea: If a projective complex manifold contains many rational curves $\cong \mathbb{P}^1(\mathbb{C})$, then this implies something for the curvature.
Quadric

\[ x^2 + y^2 - z^2 = 1 \]
A projective complex manifold $X$ is called uniruled, if it is covered by rational curves in $X$. 

Quadric

$$x^2 + y^2 - z^2 = 1$$
A projective complex manifold $X$ is called \textit{rationally connected}, if 2 general points $x, y$ can be connected by chains of rational curves.

\begin{align*}
equivalence relation \sim \Rightarrow \text{quotient map}
\end{align*}
Theorem (Campana; Kollár-Miyaoka-Mori):
For every projective complex manifold $X$ there is a holomorphic map $f : X \to B$ with

- rationally connected fibres and
- a universal property.

$f : X \to B$ is called MRC-quotient.
Problem: Characterisation of the MRC-quotient by “positive directions” of the tangent bundle

Theorem (Graber-Harris-Starr 2000): 
\( X \) projective complex manifold.
\( f : X \rightarrow B \) MRC-quotient \( \implies B \) not uniruled

Resume

Every projective complex manifold can be decomposed in rationally connected and not-uniruled varieties.
Trichotomy: Curvature $= 0$

Metric $g$ on a projective complex manifold.

Ricci curvature $= \text{trace of the curvature tensor of } g$
Theorem (S.-T. Yau, Fields-Medal 1982): There are Ricci-flat, but not flat projective complex manifolds of arbitrary dimension $\geq 2$, so called Calabi-Yau-Varieties.

Fact: Ricci-flat projective complex manifolds are not uniruled.
ABUNDANCE
Two points of view on positivity

$X$, $n$-dimensional projective complex manifold

differential geometry: metric $g$ with Ricci-curvature $\geq 0$

Ricci-curvature $= \text{curvature of the induced metric on the canonical holomorphic line bundle } K_X$

algebraic geometry: global holomorphic sections of $K_X$

global holomorphic sections of $K_X = \text{holomorphic } n\text{-forms, locally: } f(z)dz_1 \wedge \ldots \wedge dz_n \text{ with } f \text{ holomorphic}$
$X$ $n$-dimensional projective complex manifolds

**algebraic geometry**

\[ \exists \text{ sections of } K_X \]

\[ \exists \text{ sections of } K_X^\otimes m \]

\[ \parallel \]

pluri-canonical forms

\[ f(z)(dz_1 \wedge \ldots \wedge dz_n)^\otimes m \]

**differential geometry**

\[ \Rightarrow \exists \text{ singular metric on } K_X \text{ with semi-positive curvature} \]

\[ \Leftarrow \exists \text{ smooth metric on } K_X \text{ with positive curvature} \]
sections of $K_X \rightsquigarrow$ map $X \to \mathbb{P}^N$

$X$ projective complex manifolds
$
\sigma_0, \ldots, \sigma_N$ $m$-canonical forms on $X$

representation in local coordinates:

$(z_1, \ldots, z_n)$ \quad $\sigma_i = f_i(z)(dz_1 \wedge \ldots \wedge dz_n)^\otimes m$

$\psi \downarrow \quad \parallel$

$(w_1, \ldots, w_n) \quad f_i(\psi(w)) \cdot \det\left(\frac{\partial z_k}{\partial w_l}\right)^m \cdot (dw_1 \wedge \ldots \wedge dw_n)^\otimes m$

$\Rightarrow \quad \Phi_{\sigma_0, \ldots, \sigma_N} : X \to \mathbb{P}^N, \ p \mapsto [f_0(p) : \ldots : f_N(p)]$

is a rational map.
The **Kodaira-dimension** $\kappa(X)$ of $X$ is the maximal dimension of the image

$$\Phi_{\sigma_0,\ldots,\sigma_N}(X) \subset \mathbb{P}^N$$

for any $m$-canonical forms $\sigma_0, \ldots, \sigma_N$.

If $\kappa(X)$ equals the dimension of the image of

$$\Phi_{\sigma_0,\ldots,\sigma_N} : X \rightarrow B \subset \mathbb{P}^N,$$

this map is called **Kodaira-Iitaka fibration**.

If there are no $m$-canonical forms $\neq 0$, we set

$$\kappa(X) := -\infty.$$

**Facts:**

- $-\infty \leq \kappa(X) \leq \dim X$.
- $X$ uniruled $\Rightarrow \kappa(X) = -\infty$.
- Fibers of the Kodaira-Iitaka fibration have Kodaira dimension 0.
The Abundance conjecture

Abundance conjecture (Mumford)

$$\kappa(X) = -\infty \Rightarrow X \text{ uniruled.}$$
**Fact:** $\exists$ topological bound for $\kappa(X)$:

$$\kappa(X) \leq \nu(X)$$

**Definition:** Numerical dimension $\nu(X) := \max\{k : c_1(K_X)^k \neq 0\}$

**Generalized Abundance conjecture**

$X$ not uniruled $\Rightarrow \kappa(X) = \nu(X)$. 
A special case of the Abundance conjecture

Idea of proof for a special case of the Abundance conjecture:

**Assumption:** The Kodaira-Iitaka fibration $f : X \to B \subset \mathbb{P}^N$ is everywhere defined.

- $K_X$ is the $f$-pullback of the hyperplane bundle $H$ on $\mathbb{P}^N$.
- The Fubini-Study metric $h_{FS}$ on $H$ has positive curvature.
- The pullback $h_X = f^* h_{FS}$ is a metric on $K_X$, with positive curvature in $f$-transversal directions, $\equiv 0$ in direction of the $f$-fibres.
- $c_1(K_X)^{\dim B} = c_1(K_X, h_X)^{\dim B} > 0$,
- $c_1(K_X)^k = c_1(K_X, h_X)^k = 0$ for $k > \dim B$,
- $\Rightarrow \kappa(X) = \dim B = \nu(X)$. □
A special case of the Abundance conjecture

Observe:

- On $K_X$, a metric with semi-positive curvature is constructed vanishing in direction of the fibres.
- The construction is possible for every holomorphic line bundle.
NUMERICAL TRIVIALITY
**Problem:** Given a semi-positive line bundle on $X$. Construct a maximal fibration with fibers, in whose direction the curvature of the line bundle vanishes!

**Theorem (Tsuji; Bauer, Campana, Eckl, Kebekus et al.):**

$L$ nef line bundle on projective complex manifold $X$. Then there is a rational map $f : X \to Y$, such that:

- $F$ fiber of $f \Rightarrow \forall$ curves $C \subset F$: $c_1(L)|_C = 0$.
- $x \in X$ general, $x \in C \subset X$ curve with dim $f(C) = 1 \Rightarrow c_1(L)|_C$ does not vanish
Definition: $L$ pseudo-effective $:\iff \exists$ singular metric $h$ of $L$ with

$$c_1(L, h) \geq 0.$$ 

Theorem (Tsuji; Eckl): $X$ projective complex manifold, $(L, h)$ pseudo-effective line bundle on $X$. Then there is a map $f : X \to Y$, such that:

- $F$ general fiber of $f$, $C \subset F$ curve with $h|_C \neq +\infty$:
  $$c_1(L, h)|_{C-\text{Sing}(h)} \equiv 0.$$

- $x \in X$ general, $C \ni x$ curve with $\dim f(C) = 1 \implies c_1(L, h)|_{C-\text{Sing}(h)} \neq 0$.

Problem: basis $Y$ can have dimension $\geq \nu(L)$. 

Numerically trivial fibration
**Definition:** $X$ projective complex manifolds, $(L, h)$ line bundle with metric $h$ and curvature $c_1(L, h) \geq 0$. A foliation $\mathcal{F}$ on $X$ is called numerically $(L, h)$-trivial, if

$$c_1(L, h)|_{\text{leaf}} \equiv 0.$$  

**Theorem (Eckl)**

There exists a maximal numerically $(L, h)$-trivial foliation.
Numerically trivial foliations

Properties of the numerically trivial foliation

fibers of the numerically trivial fibration
\[ \cap \]
leaves of the numerically trivial foliation
\[ \cap \]
fibers of the Kodaira-Iitaka fibration.
UPSHOT AND FURTHER PROSPECTS
**Upshot:** Complex differential geometry helps in understanding the Abundance conjecture.

**Strategy for Abundance conjecture:**

1. The numerically $K_X$-trivial foliation has leaves of dimension $\dim X - \nu(X)$.
2. The numerically $K_X$-trivial foliation is a fibration.
3. The numerically $K_X$-trivial foliation is a fibration $f : X \to Y$ with $\dim Y = \nu(X) \Rightarrow X$ is abundant.
**Upshot:** Complex differential geometry helps in understanding the Abundance conjecture.

**Conjecture (Yau):** $X$ compact Kähler manifold, $g$ Kähler metric with non-positive holomorphic bisectional curvature $\Rightarrow X$ is abundant.

**Wu/Zheng:**
Additional assumption on metric $\Rightarrow$ Step (1) and (2)
Step (3) trivial.

**Eckl:** Step (1) $\Rightarrow$ Step (2) and (3)