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Phase Breaks and Chaos in a Chain of Diffusively Coupled Oscillators

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Abstract—The dynamics of phase disturbances in the 1D uniform medium described by FitzHugh–Nagumo equations was studied. It was found that the system synchronizes its oscillations if the initial disturbances are small. Large disturbances resulted in the appearance of phase breaks. Dependent on the system parameters, phase breaks were found to be either stable or unstable; stable breaks led to target pattern-like sources, unstable to extended chaos.

1. INTRODUCTION

Many processes in solid-state physics, plasmas, theory of lasers, chemistry and biology can be described in terms of the theory of coupled oscillators [1]. The same point of view may be also applied to distributed reaction–diffusion systems. Every point of such a medium is assumed to be a nonlinear oscillator diffusively connected to its neighbours. In the case when the system is described by two equations, the phase of oscillations, ϕ, may be determined by introducing a polar-coordinate system (r, ϕ) [2]. The dynamics of such a system was studied analytically in refs [2–4]. In response to a small initial disturbance of the phase the system evolves to spatially uniform oscillations, or it falls into chaos if the system is unstable to initial perturbations.

The phase analysis approach has also been used to study the spontaneous appearance of the wave sources. Assuming a source occurs at the location of impurities, phase-distribution analysis allows the study of characteristics of the waves irradiated from such sources [2–6].

This paper analyses phase dynamics in the one-dimensional oscillating FitzHugh–Nagumo model, which is widely used as a prototype model for simulation in physics and biology [7–11]. As well as synchronization of oscillations in a spatially uniform medium, there can exist a stable wave-break. This break exhibits a higher frequency of oscillations than the rest of the medium, i.e. it becomes a source of waves. There is also a band of system parameters where a single phase-break loses stability, which results in the chaotic oscillations.

2. MODEL AND CALCULATIONS

We use the modified FitzHugh–Nagumo equations:

$$\partial E/\partial t = \nabla^2 E + f(E) - g \tag{1a}$$

$$\partial g/\partial t = (k(E - a) - g)/\tau(E), \tag{1b}$$

where E and g represent the potential and slow currents that occur in excitable cells [7].

We use the piecewise linear function $f(E)$ consisting of three lines: inclined, $g = E - a$, as $0 < E < 1$, and two vertical ones, $E = 0$ and $E = 1$ (see the nullcline $E_t = 0$ in Fig. 1). The shape of $f(E)$ resembles the classical one [7], but the function used is more convenient for calculations and analysis [11]. The other model parameters are: $k = 2.5$, $\tau(E) = \{4$, as $0 < E < 1$, and 20 otherwise$\}$. The value of the parameter a is given in the figure captions.

Calculations have been performed in a one-dimensional array of 250 elements by using the explicit Euler's method of integration with space- and time-steps, $h_x = 0.5$ s.u. and $h_t = 0.025$ t.u., respectively. On the boundaries the Neumann's 'no flux' conditions were imposed. Test runs with $h_x = 0.25$ s.u. and $h_t = 0.006$ t.u. show a minute deviation in the values of space- and time-parameters measured. This confirms the reliability of the computational method used.

Initial phase distributions were generated by the following procedure: first, in a zero-dimensional medium the period of bulk oscillations, T, was determined. Then, during one more period, two arrays, $E_0[i]$ and $g_0[i]$, each of 1000 elements, were filled with the values of the variables E and g at the time moments of $iT/1000$, $i = 0$–999. The index i of these arrays represents the phase of oscillations, ϕ. Finally, the desired distribution of the phase was created by using these phase-scaled arrays E_0 and g_0 before the calculations started (we used piecewise phase distributions, whose profiles are shown in the lower part of Fig. 2).

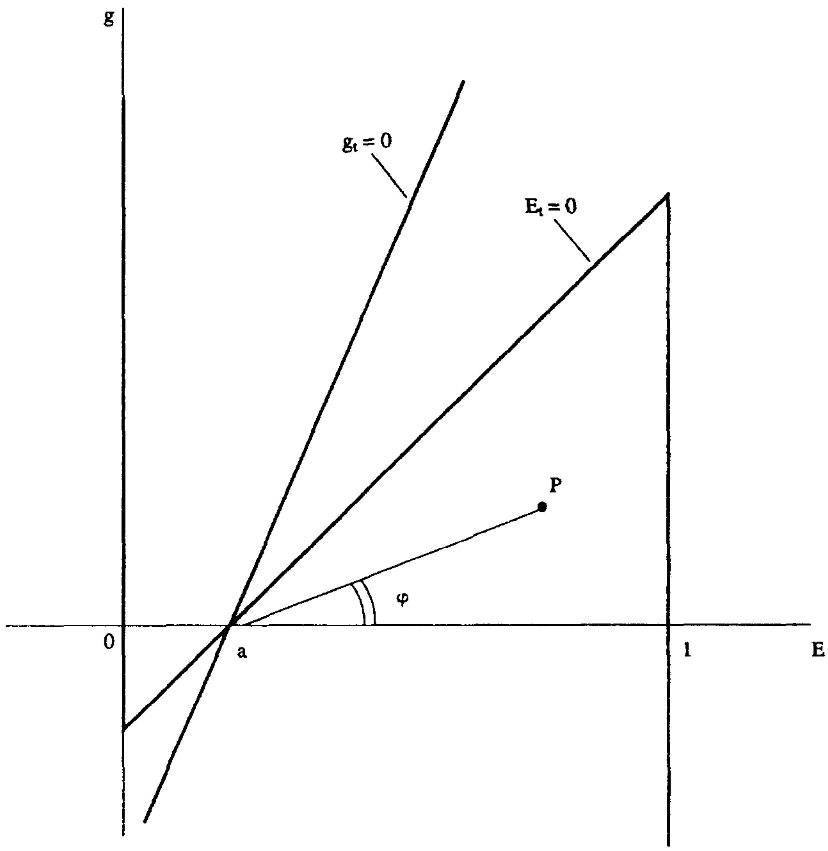

Fig. 1. On the (E, g) coordinate plane of equation (1) nullclines $E_t = 0$ and $g_t = 0$ in the spatially uniform case ($\nabla^2 E = 0$) are depicted. Any point P on the plane can be located by introducing either Cartesian coordinates (E, g), or polar ones (r, ϕ), where angle ϕ presents the phase of oscillations.

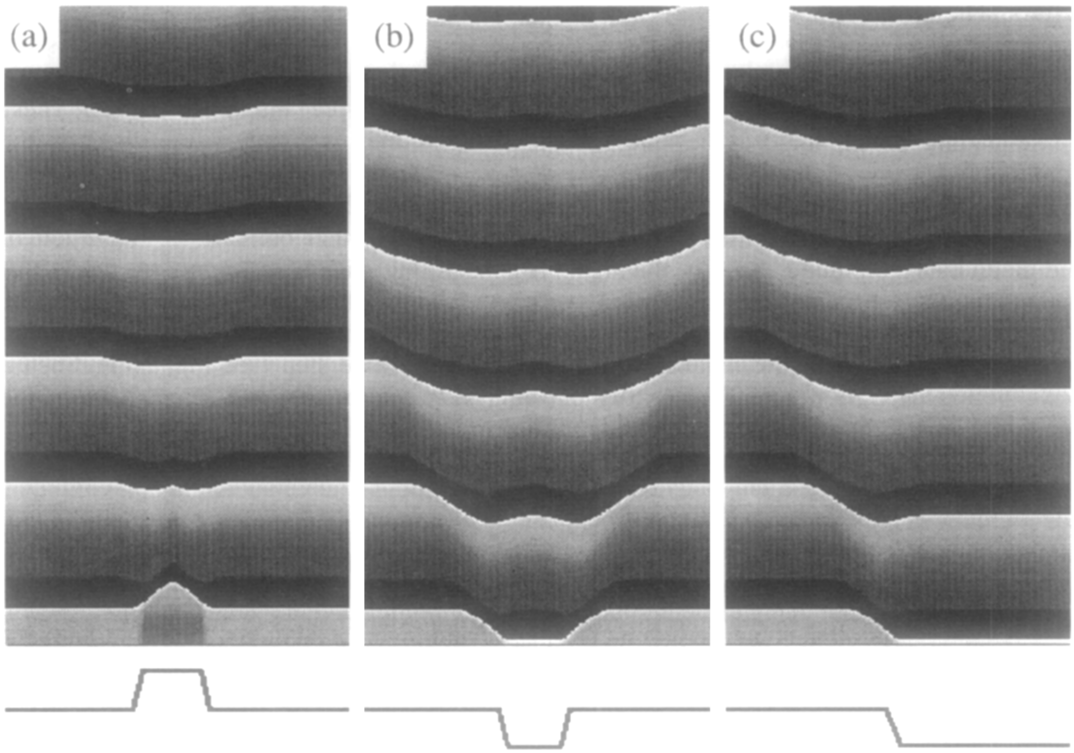

Fig. 2. Evolution of small phase disturbances. The upper part of the figure presents a space-time distribution of the phase ϕ: each row of this chart depicts a phase distribution at the appropriate time moment, t. Time ranges from 0 time units (the lowest row) to 220 t.u. (the upper one). Parts (a), (b) and (c) show the response of the system to different initial disturbances (their profiles are shown in the low part of the figure). Note that in (a) and (b), disturbances identical in shape but different in sign establish the alternative final phases. The period of bulk oscillations was $T = 43$ t.u., $\Delta\phi = 0.25T$, the parameter a in equation (1) was 0.15; the other model parameter values are given in the text.

It should be noted that the procedure given above is convenient for setting up the initial phase distribution, but not for measuring the phase during calculations. For this we use another method of phase determination: the phase at a point $P(E, g)$ is assumed to be numerically equal to the angle ϕ, $0 < \phi < 2\pi$, as shown in Fig. 1 [2]. The angle ϕ at each point of the tested medium was calculated as $\tan^{-1}(g/(E - a))$ (see Fig. 1). This definition of the phase is used in Figs 2–5, where the value of the phase ϕ in range $[0, 2\pi]$ is coded by the darkness of the appropriate part of the figure.

3. RESULTS

3.1. *Small amplitude disturbances*

Figure 2(a) shows the evolution of small local disturbance with positive phase shift, $\Delta\phi = 0.25T$ (i.e. oscillations here are behind oscillations of the remainder of the medium). It is seen that such a phase-lag rapidly vanishes. Note that, because of the nonlinearity of the system, the points of the perturbed region 'overshoot' distant points, forming a site slightly leading in phase (see Fig. 2(a)).

When the imposed disturbance is the same in shape but negative in sign, $\Delta\phi = -0.25T$ (Fig. 2(b)), synchronized oscillations are also established in the system, but their phase differs. The phase established in both cases proved to be that of the phase-leading part of the medium: as a positive disturbance is imposed (Fig. 2(a)), the phase established is the phase of oscillations far from the disturbance (neglecting 'overshooting' described above); as $\Delta\phi < 0$ (Fig. 2(b)) phase-leading points are located inside the perturbation. These phase-leading points impose the phase of oscillations that is finally established in the medium.

The process of phase synchronization is presented in Fig. 2(c): the medium was initially divided into two parts with the phase shift, $\Delta\phi = 0.25T$, so that oscillations in the left of the medium were behind those in the right. It is seen that with time, points on the left of the medium synchronize their oscillations with the leading phase points on the right side. This process may be described in terms of a wave of synchronization, 'dephasing wave', spreading in the left-hand direction from the place of a phase step (see Fig. 2(c)). Such a wave occurs because of the nonlinearity of the system (note, that in a linear system the phase-step would be simply blurred due to diffusion).

3.2. *Phase breaking, wave sources and chaos*

When a large initial phase shift $\Delta\phi$, $\Delta\phi \simeq 0.5T$, was imposed, no synchronization of oscillations was observed, but a stable phase-break arose (Fig. 3). After transient processes have been completed, the medium on the left- and right-hand sides of the break oscillates with a phase-shift equal to half of the period of oscillation. The frequency of oscillation of the break proved to be 1.5 times as great as that of the homogeneous oscillations. So the

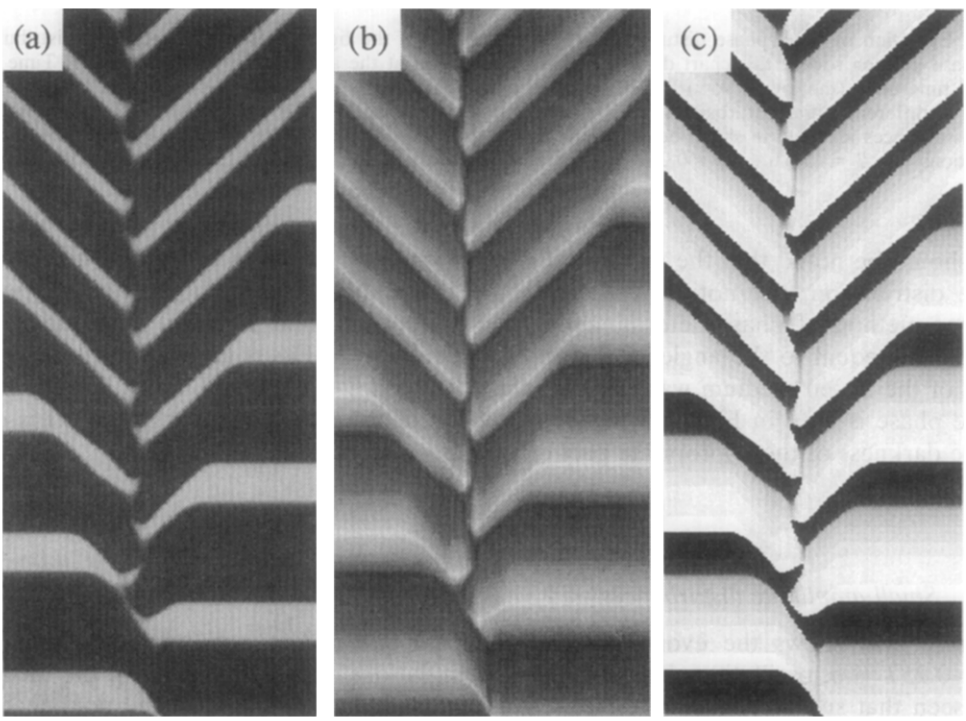

Fig. 3. Break of the phase: (a), (b) and (c) are the space-time distributions of E, g, and phase, ϕ, respectively. The period of oscillations of points adjacent to the break, T_b, is less than the period of bulk oscillations, T, and equals $0.68T$. Because of the high frequency, the break becomes a source of waves irradiated in both directions with the phase-shift $0.5T_b$. The break occurs under the same conditions as in Fig. 2(c), except that $\Delta\phi$ was $0.5T$.

high frequency of the break resulted in anti-phase waves, irradiated by the break in the opposite directions (Fig. 3).

Figure 4 shows two breaks placed close together. This pair of breaks forms a wave source that sends in-phase waves in both directions. Meanwhile, oscillations that occur outside the source are out-of-phase in comparison with those inside. On the boundary of the source, i.e. at the place where the break is situated, oscillations of double frequency were observed. Such double-frequency oscillations are, of course, present in case of single break (Fig. 3) as well.

Evolution of the breaks proved to be affected by the value of the parameter a (see Fig. 1). As a is diminished, a single break loses stability and starts to divide itself. As a result, a stair of breaks appears (Fig. 5) leading to chaotic oscillations: during the time for four bulk oscillations, the entire system is filled with chaotic oscillations of high frequency.

4. DISCUSSION

We have shown that an initially inhomogeneous phase distribution results in either phase-smoothing (Fig. 2) or phase-breaking (Fig. 3). The former phenomenon is well known and has been discussed in the literature [2–5]. On the other hand, phase-breaking in such a distributed system has not been reported previously and seems to be nontrivial. The phenomenon is of interest because the frequency of the phase-break is higher than that of bulk oscillations of the medium. Apparently, this makes the break become a source of waves. Similar wave sources observed in a discrete medium were called 'echo sources' [12]. Although, in contrast to the phase-breaks, echo sources are restricted to discrete media,

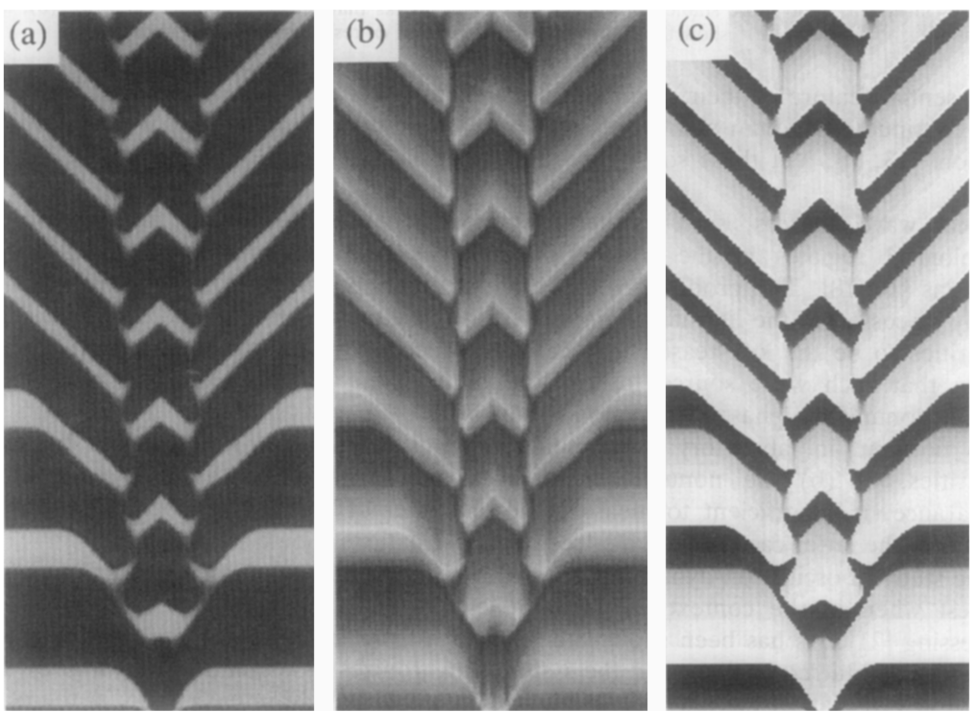

Fig. 4. Wave source in a spatially uniform medium. Symmetric initial conditions, as in Fig. 2(a) with $\Delta\phi = 0.5T$, result in the formation of the source of waves ($T_s = 0.68$, $T = 28$ t.u., $\lambda_s = 72$ s.u.). All system parameters are as in Fig. 3

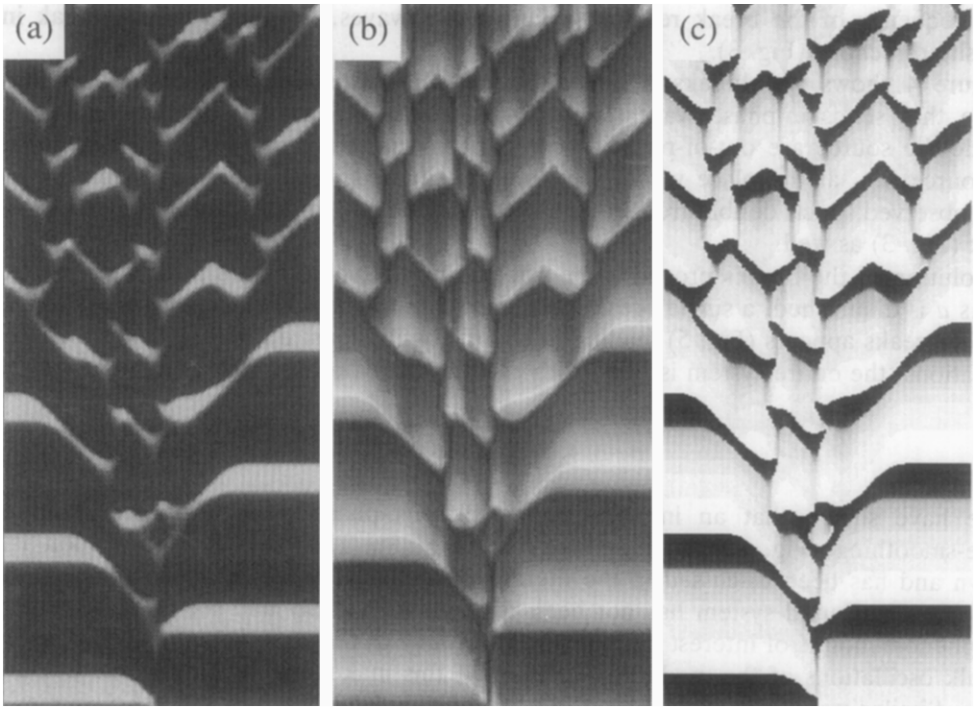

Fig. 5. Transition to chaotic oscillations. The same initial phase-distribution, as in Fig. 2(c) with $\Delta\phi = 0.5T$, causes chaos to appear in the case of small parameter $a = 0.1$ (see equation (1)). Under these conditions single break, as in Fig. 3, is no more stable but divides into many breaks that fill the medium. Parts (a), (b), and (c) show the distributions of E, g, and ϕ, respectively.

the essential feature of their appearance is a finite phase shift between neighbouring cells. In a continuous medium similar regimes were studied by Yakhno and were called 'front division' [13, 14]. Yet these sources require a special symmetry of the system to be stable [14].

Local wave sources, such as 'target patterns' or 'focal sources' are the origin of pathological regimes in the heart [15] and pattern-formation in oscillating chemical reactions [16–18]. The problem of the spontaneous appearance of such sources has been widely discussed in the literature [2–6, 17–20], and most theories consider the presence of impurities to be the key reason for the appearance of wave sources [2–6, 20]. Here it is shown that such wave sources can be initiated in the homogeneous medium when an initially nonuniform phase distribution is imposed.

To find out in laboratory experiments whether (a) a wave source arose because of impurities, or (b) the nonuniform initial phase distribution was the reason for its appearance, it is sufficient to measure the character of oscillations near the centre of a source: in the latter case the double-frequency oscillations should be observed.

The study of oscillation dynamics in systems with nonuniform phase distribution is also of interest when in the context of the light-sensitive BZ reaction as a system for image processing [21]. As has been shown, the phase distribution formed by a short-period light exposition is blurred in time, and the system synchronizes its oscillations (cf. Fig. 2). It was also observed that, at the place of a sharp gradient of the light intensity (i.e. at the place of sharp phase-gradient), wave sources appeared, highlighting this boundary [21]. The origin of such sources has not been investigated in detail, but the mechanism of their initiation at the location of large phase-gradient can be identical to that described above.

The similarities between the processes observed in the Belousov–Zhabotinsky reaction and described here seem to be not accidental. We believe that the initially nonuniform phase distribution plays a significant role in the dynamics of other excitable media as well. It should be also noted that the system, with a piecewise linear nonlinearity, may be studied analytically.

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