Time-evolution of age-dependent mortality patterns in mathematical model of heterogeneous human population

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Chapter 1

Introduction

There are many of concepts which need to be introduced in order to study the dynamic of human mortality since they are the basic elements of the way the data is collected and analyzed.The aim of this section is to provide an overview of these definitions.First of all,what does population mean?According to[**?**] Samuel population is the gathering of people who are alive at certain time and facing certain criteria.As demography focus on studying the changes in the size of any population,it also explores the growth rate and its composition.The priority of studying demography is to give an enough explanation of the aggregate process and its implication for individuals.Furthermore,there are two ways to be considered as a member of the populations and two others to leave it,which is represented in(1.1).

$$N(t) = N(0) + B[0,T] - D[0,T] + I[0,T] - O[0,T] \tag{1.1}$$

Where N(0) and N(T) refer to the number of persons alive in the population at time 0 and at time T, being in the population can be either by birth,and that is defined by the number of birth in the population between 0 time and T time,as its notation is B[0,T] or by the number of immigration between 0 time and T time,which has the notation I[0,T].However,leaving the population would be in two-ways as well.Therefore,it is determined by the number of death in the population between 0 time and T time ,or it can be estimated by the procedure of out-migration O[0, T],so that leads to knowing the number of out-migration between 0 time and T time,throughout this equation T refers to the number of years,so the interval in each elements is limited by the length of that year.Lack of evaluation of these elements leads to an imbalance of the equation,it known as (error of closure), as a consequence of applying the data,most of the time this equation is not equal.Kenneth Boulding claims that the relation in this equation is the most basic relation in social sciences because it provides an accurate description.

In addition,the stationary population is distinctive of the general stable population, as it is independent of the life table.However, it is related to the notation of life table in the case of improving it's distinguished.As a consequence of the consecutive operation of three conditions, a stationary population is produced.The conditions are age-specific death rates and the continuity of birth are constants over time (it is not constant over age)and there is no migration.the human population can be considered as a stationary population until the eighteen century,because there was not any noticeable increase during that time. And one of the benefits of considering populations a stationary is that obtained many of expression which related the main demographic parameters to others.for example, the crude death rate is equal to the crude birth rates[19].

1.1 Background and significance

1.1.1 The structure and principal of rates

In addition,from(1.1)the change of the population separate into four parts,and in each one of them it has the sum of events as it appears to individuals.All of the elements in the equation associated with an individual except birth,since it is associated with individual parents.The rate is known as the number of the event appears in specific time divides by the number of person-years at the same specific time.When the population is large that will lead to face the events more common than if the population is small,as the total number of any events affects by how long individuals face that events,therefore,if the person has an experience for any type of events for long time that will lead to have a high number of the total number of that event.For example,when epidemic spreads in large population and for long time this will lead to have a high mortality rate in that population.

Moreover,there is a simple way to represent the impact of events on individual or on a group of members,and that can be represented by lifelines for every individual,as that can be introduced by a set of lifelines for a group of members, as each line describes how many events individual experiences along interval period of time,in the case of studying all the life period for any persons the lifeline will begin from the birth until death.However,it is not necessary to consider all the life period if the study focuses on specific events,such as the risk of death during giving birth,or eating drug,making accident and so on.Therefore,it will be limited by specific time since there are many of lines which describe group of members the mortality rate can be evaluated by taking the sum of individuals lifelines, it is represented as following[19].

$$Rate^G = \frac{\sum_{i \in G} N_i}{\sum_{i \in G} T_i} \tag{1.2}$$

where N_i refers to the total number of occurrences in the lifetimes of individual i,and T_i denoted to the length of the time between the beginning of the interval and the end of it.After our discussion about general rate, we extend that definition and being more particular by providing specific type of rates, such as period rate as this rate is like the general form of rate, but it must be determined through limited period.Furthermore,the denominator of rate includes the person years so there are two methods which can be used to evaluate the number of person years,and that can be done either by Algebraic notation(1.3)or by calculus notation(1.4).

$$PY = \sum_{i=1}^{365} N_i \Delta_i \tag{1.3}$$

$$PY = \int_{firstyear}^{lastyear} N(t).dt \tag{1.4}$$

where $N(t)$ denoted to the number of persons alive at time t,and Δ_i refers to the proportion of a year as the size of the population estimates for every day in (1.3).As we mentioned previously,it becomes clearer when this description is represented through lifelines.Therefore,in this case there is a height which represents years and width which is defined by the proportion of a year,and in order to find the number of person years the value of the area has to be estimated by multiplying its height by its width,so in(1.4)dt refers to the amplitude of the interlude as it is $\Delta_i$in(1.3).

From the balancing of the equation which describes how populations constructed (1.1),as we mentioned it in the introduction,each content in that equation can be represented as a particular crude rate,if it is divided by the number of person-years over a specific time.In addition,there are four of rates which can be obtained,the crude birth rate, the crude death rate,the crude in-migration rate and the crude out-migration,as their notations are CBR,CDR,CRIM,CROM,respectively.However,some of them cannot give an accurate value since they are difficult to know.For example,it might be challenging to estimate the rate of net migration(immigration and out migration).The crude growth rate,it is also called instantaneous growth rate,it

can be produced by dividing (1.1) by person-years lived during period of time$[t, t + \Delta t]$,as the population change can also be estimated over the period of time,between t and $t + \Delta t$,as following

$$\Delta N_t = N(t + \Delta t) - N(t) \tag{1.5}$$

And the another formula can be construct for the instantaneous growth rate by dividing the changing of the population (1.5)by person years which can also be written as $N(t)\Delta(t)$,so the growth rate becomes.

$$r(t) = \frac{\Delta N(t)}{N(t)\Delta t} \tag{1.6}$$

When we take the limitation of the instantaneous growth rate,when Δt tends to 0 that will leads to the following formal.

$$r(t) = lim_{\Delta t \to 0} \frac{\Delta N(t)}{N(t)\Delta t} = \frac{dln[N(t)]}{dt} \tag{1.7}$$

In(1.7)the instantaneous growth rate is estimated for a short period of time.However,in order to evaluate that rate for a long period of time we need to integrate and take the exponential of(1.7) for both sides,that leads to obtaining the change of population for a long period of time(1.8)which is an essential expression,because it is used in several implementations[19].

$$N(t) = N(0)e^{\int_0^t r(t)dt} \tag{1.8}$$

As a result of this (1.8),each growth is an exponential one.Furthermore;this equation(1.8) can be simplified when the instantaneous growth rate is constant over interval term.It can be written as

$$N(T) = N(0)e^{r*T} \tag{1.9}$$

From the previous discussion we conclude that the instantaneous growth rate is an exponential growth,so now we would like to estimate the instantaneous mortality rate among people aged i and$i + 1$.Therefore,the number of people who aged i is defined as follow.

$$\Delta N_i = N_i - N_{i+1} \tag{1.10}$$

Death rate m_i is determined as the number of people who die at age i divided by the number of person years of age i.Consequently,

$$m_i = \frac{\Delta N_i}{py_i} = \frac{\Delta N_i}{0.5(N_i + N_{i+1})} \tag{1.11}$$

In the case of mortality rate the number of person years can be evaluated by the average number of survival during a one year age interval,it is also known as the central death rate.Age i in mortality rate is an integer number which can be a real number when the mortality rate is measured as a continuous procedure.

$$m_i = \frac{\Delta N_i}{N_i - (1 - a)\Delta N_i} \tag{1.12}$$

The different between these two expressions for the mortality rate (1.11) and(1.12)is that,the fraction of years a is 0,5 in(1.11)which means that the person will die after six months from his last birthday,this value is also acceptable for all ages excluding for age 0.In this report([3]),there is a procedure which tried to satisfy this value for age 0 in sub-populations.Furthermore,the instantaneous mortality$\mu(x)$at age x is determined by[19].

$$\mu(x) = lim_{\Delta x \to 0} \frac{-\Delta N(x)}{N(x)\Delta(x)} = \frac{-1}{N(x)}\frac{dN(x)}{dx} \tag{1.13}$$

1.1.2 The concept of cohort

When a group of people exposures to event during a particular time,it is known as a cohort.There are many of cases in the life which can be defined as a type of cohort such as birth,marriages,death and so on.In our case,we are going to study when they have the same experience of death.However,there is a limitation of using cohort data,since it is difficult to have a full detail for each member of the group.This limitation is also related to our study since it is quite difficult to have a complete table of death for all ages during a specific time.As a result of that,many of the data which is used rely on the previous period,and that produces hypothetical cohorts.Therefore,we are going to use period data instead of cohort data for a stationary population which we explained in the introduction(1).

The same method,which was introduced in(1.10),is also used to obtain cohort age-exact rates,and in order to implement this expression,the number of events has to be limited in the associated ages.Moreover,it is clear that there is a corresponding composition between cohort rates and period rates.However,they have separate display portions,this relation represents in Lexis graph(Lexis,1875).He considered that the long of life is a constant value for all ages,but the people who died before that value they face an accident,but his idea is different than other scholar.For example,Pearson (1897)pointed out that there are many of factors which impact on the mortality rate for different ages[19].
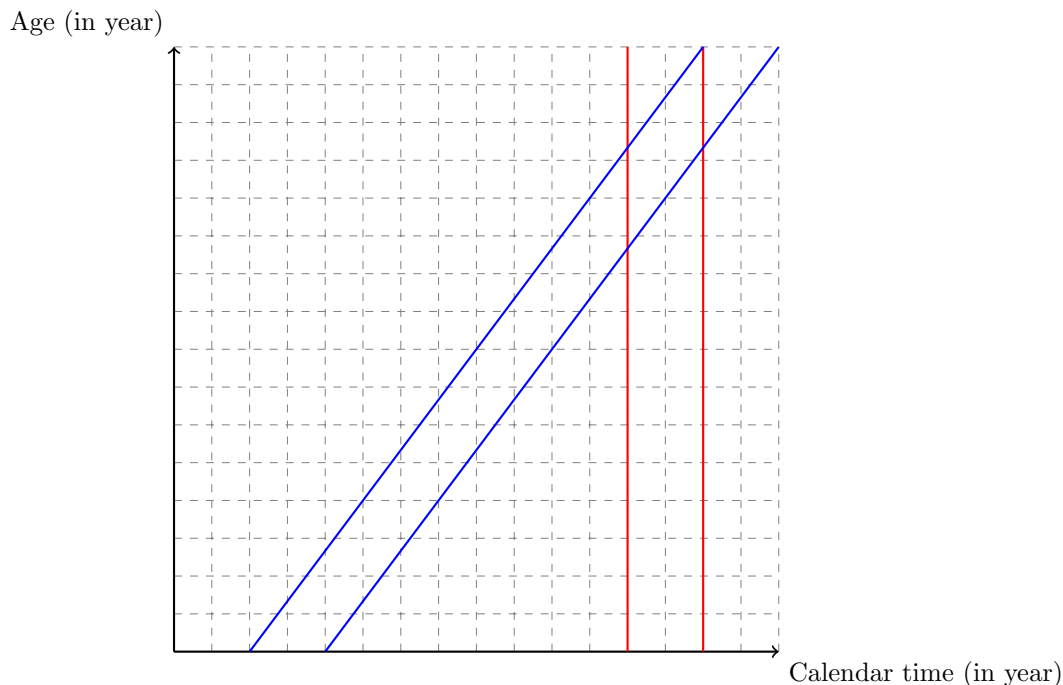
Figure 1.1: This figure represents Lexi-diagram which shows the correlation between cohort data and period data.The two blue diagonal lines refer to the death which is contributed to Cohort mortality,and the two red vertical lines indicate to the death of period mortality.

Figure(1.1)is a two -dimensional graph,the x-axis refers to time and the y-axis refers to age.Furthermore,age and time have the same unit,and the increase of the unites along both of axes are the same as wall.From the graph(1.1),the two horizontal lines represent age span for both of the rates,but their vertical lines structure a parallelogram shape in an age-specific cohort rate and a rectangular shape for an age-specific period rate.In addition,there are several of relation can be represented in this diagram,such as lifeline,counts of events, and figures that defined counts.

1.1.3 The life table and single Decrement processes

There are many ways to represent cohort mortality and period mortality.For example,they can be summarized in life tables,graphical figure which is most common to be represented in a semi-logarithmic graph,and mathematical expression.Generally speaking,using graph is might be a better form than tabulating the data,since the table utilizes a lot of notation functions and excess of columns.Particularly,cohort data cannot be available for all periods,and it is not accurate in some cases.

Consequently,using period table becomes more common which is based on attempting to approximate what will happen,and that can be established by converting age-specific death rates for certain time in cohort table into a probability of dying in a periodic table during the same period of time which was evaluated in cohort tables.It is clear,that cohort data is real observed values,while period data is a synthetic cohort.This transformation can be produced by the set of age-specific death rates for that period,which leads to obtaining relation between age specific death rate and probability of dying,they are(1.14) and(1.15).where $_nd_x$ is the numerator in both of the forms,as it indicates to the number of deaths between age x and $x+n$ its construct is either the number of survivor to age x deduct the number of survivor to age $x+n$,so that can be written $_nd_x = l_x - l_{x+n}$,or the number of survivor to age x multiply by the probability of dying between age x and $x+n$,so its expression is $_nd_x = l_x \times {_nq_x}$,and the mortality rate in the cohort between age x and $x+n$ is divided by the number of persons years lived in the same cohort,while the probability of dying divided by number of survivors to age x in the cohort.

$$_nm_x = \frac{_nd_x}{_nL_x} \tag{1.14}$$

$$_nq_x = \frac{_nd_x}{l_x} \tag{1.15}$$

According to Greville(1943)and Chinanf(1978)one parameter affects this alteration all others are neglected,and that parameter is the fraction of the total number of person years lived in the period between x and $x+n$,and the total number of people who die during the same period of time.There is a question which might arise in this procedure,how we can choose this elements,since its play an essential role in the precision of the data.Moreover,there are four strategies, which is associated with selecting this parameter.The first one is direct observation; it is used when all of the exact data are accessible for all ages in a population,so this makes it feasible to take all of deaths rates within n-years long period,and by computing$_na_x$,this will be relevant in the population instead of cohort.However,this strategy is not a sufficient one,since the distribution of ages in the population during n-year long period impact on the observed value in the population.

Second procedure based on estimating that the distribution of ages obtain from the polynomial function in the time between $x-n$ and $x+n$,this assumption will leads to introduce a formula for$_na_x$.

$$_na_x = \frac{-\frac{n}{24}{_nd_{x-n}} + \frac{n}{2}{_nd_x} + \frac{n}{24}{_nd_{x+n}}}{_nd_x} \tag{1.16}$$

As it is clear from the formula,the approximation of$_nd_x$is important to assume $_na_x$,but it cannot obtain without knowing information about $_na_x$,so by proposing value for $_na_x$ in order to establish the set of $_nd_x$,which are going to be used to produce the set of $_na_x$by using the new set of $_nd_x$ and(1.14).The third procedure relies on obtaining the set of exact data,which has already provided from other population,because the level and shape of mortality curve is similar to different population.Reed and Merrell experienced this methods for US table.

The last and practical strategy which is used to select a,is known as thumbs rules,the priority of this method is based on the amount of error which will obtained.Each of the methods which were discussed leads to some error,and that error will affect in the accuracy of the data.However,this strategy does not give any error which makes it a regular way.The first rule is relied on supposing that the death will appear in the half of the period which means $_na_x = \frac{n}{2}$,so this conjecture related to(1.14).The second rules is supposed that the death rates is a constant value along the interval,then using the following equation.

$$_np_x = 1 - {_nq_x} = e^{-n \cdot {_nm_x}} \tag{1.17}$$

$$_na_x = n + \frac{1}{_nm_x} - \frac{n}{1 - e^{-n \cdot _nm_x}} \tag{1.18}$$

The value of $_na_x$ in second rule has to be less than $_na_x = \frac{n}{2}$, because the number of death in any time in the period will be relative to the number of survivors, and this number is reduced at the beginning of the interval,so the number of death will increase in this stage.Furthermore,second rules works in opposite direction than mortality rate,because mortality rate tend to increase,as a result it can be consider as a neglected method[19].

1.2 Cause of death for the UK database

In this dissertation,we study the combined death rate for male and female in the UK from 1922 to 2014,which is taken from http://www.mortality.org/.It is essential to investigate the cause of the death for this country.According to (Gov UK,2017) the major cause of death for this beginning data set is associated with infectious diseases,as a result of this type of disease,there were many of infant and children died and that affected on people until 2005.In addition,numerous of the improvement which was implemented in order to have a hygienic environment plays a major role in reducing this case of death.Furthermore,aging becomes common in the UK, since the stroke and heart disease was reduced to half for both of the genders in 2001,that which might leads to increase the number of old people.Therefore,it is expected to have more and more elderly in coming years.However,in the same resource,it represents other cases of death as a consequence of aging,and they are Alzheimers and dementia,this two illness is grown during 2001,so 60 percent for male and twice times for female.Sometimes,the cause of death is different as it depends on the genre as well as age.For example,the behavior of young people can be considered as a major cause of their death which means that they tend to harm themselves or committing suicide or using drugs more than any other groups of aging,and elderly usually die because of internal disease.In 1945 until 1983,the road death became a common reason of death for both of the gender and introduced strict law(seat belt) lad to reduce the number of death[7][8].

(a)   (b)   (c)
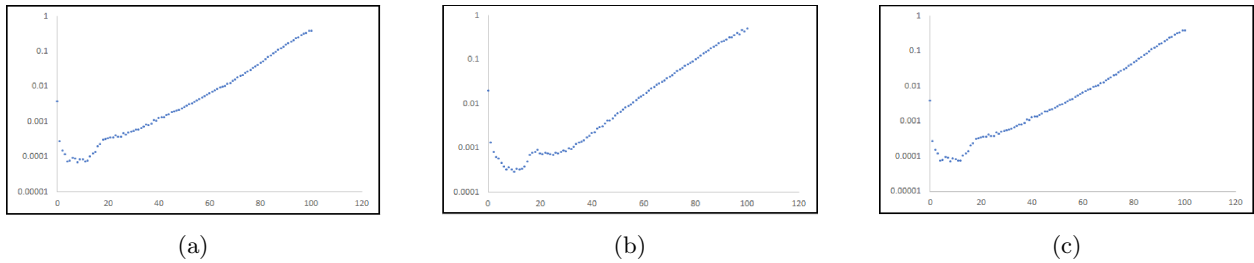
Figure 1.2: Mortality rate for the UK population in 1922(panel A),in 1966(panel B)and in 2014 (panel c)presented in a semi-Logarithmic scale.
The data is taken from the Human Mortality Database,(http://www.mortality.org/)

There are several formats which can represent actual mortality data in human population,as the most common one is by introducing the data as a logarithm of morality rate opposite to age. In Figure(1.2)There are three panels which show the mortality rate for the UK,panel A represents the mortality rate for 1922,panel B for 1966 and panel C for 2014,as the period between these years more than forty ,that large gap between these years to see how the mortality rate changes over a long period of time.it becomes obvious from this three panel that mortality is divided into three key phases.The first phase is between ages 0 to 10 during these ages the mortality rate declines sharply.This is declination might be represented because of the development of an immune system in youngsters.The second phase is represented for the people who aged older than the first phase consequently, the people in this phase aged approximately around 12 to 30. This stage is known as the accidental hump,and its impact on the system until the late 30's to early 40's.The last phase is aging,as in this work we will study the model which explain this stage.

1.3   Analysis of mortality data

Modern techniques play an essential role in gathering and analyzing data-sets,that leads to saving time and examining the ability of models to describe the experimental data.They are also helping to give a good visualization of data as well as simply the data by reducing them to a few parameters.This procedure can be implemented by using regression analysis,and statistical tool.In addition,There are many of software packages which can be used to find the value of parameters and measuring the relationships among variables.One of the most common distributed regression tools is the Solver add-in bundled with Microsoft Excel.This section discusses fitting the set of mortality data,starts from the initial cohort of N individual during a specific period and evaluating the number of the survivors N_t at fixed interval,to the nonlinear function,which is the same as any functions relied on fitting the parameters[21].

It is important to invoke that this procedure has to evaluate not just the fit parameters,but it also approximates the error and examines the fitting.It is clear that quantitative experiments are based on finding the connections between independent variables which is changed over the experiments,and dependent variables which are measured.This relation can be obtained by using the regression analysis,and it needs to know the regression equation,fitting function and applying parameters.Therefore,the analysis is tired to obtain these parameters,as they explain the experimental data-sets,so they are known as the solutions or the best fitting.The least square(lSF) method gives the value of this parameters,and that can be done by reducing the number of residual(SSR)[21][14].

A residual means the sum of the square of the variation between the observed values and fitting function,quantitative experiments are based on finding the connections between independent variables which is changed over the experiments,and dependent variables which are measured.This relation can be obtained by using the regression analysis,and it needs to know the regression equation,fitting function and applying parameters.Therefore,the analysis is tired to obtain these parameters,as they explain the experimental database,so they are known as the solutions or the best fitting.The least square(lSF) method gives the value of this parameters, and that can be done by reducing the number of residual(SSR).A residual means that the sum of the square of the variation between the observed values and fitting function,as it has to include all of the data during the analysis,when the residual is low that means a good approximation has been obtained. Consequently,it gives the experimental error in dependent variables due to a normal distribution[21].

The least squares methods can be used to analyze linear and nonlinear equations,so they are called either linear least-squares fitting (LLSF),it depends on the type of the equations.There is another difference between linear and nonlinear regression since linear regression obtains an accurate solution by relying on the analytical means,but nonlinear regression cannot obtain the same so it depends on repeating methods.In addition,a general nonlinear regression is given by atrial and errors.Therefore,the procedure starts with the initial values for the parameters,these values are going to alter in each step of the procedure,they are also going to be compared with the measured value.Moreover,in order to get efficient amounts that need to use an advanced setting,since the procedure relied on the fitting programming.In this work,Excel solver is used to measuring the parameters which utilizing the generalized diminished gradient algorithm[21].

Although this procedure is implemented in many fields,there is a limitation in this method since its general slope in the fitting neglected the fitting equation which is employed.Besides that,all of the experimental error is caused by the dependent amounts,as there is no relation between the data points,and the value of the independent's variables is accurate.Additionally,the probability of founding adequate parameters is due to the fact that the experimental errors attend normal distribution with a mean of zero,as a result of that the value of square residual method will decrease.However,this values cannot be obtained accept if all parameters are measured so that the reasons its depends on trial and error[21].

1.3.1 General fitting strategy

First of all,data fitting is separated into two parts the first one is by setting and plotting both of the data, experimental data and calculated data which is measured by its regression equation,then evaluating the residual and taking its sum.After that, selecting various values for every parameter that might help to see the impact of each parameter.Secondly,using solver tool in order to evaluate the fitting parameters and minimize the sum of the residual.It might be necessary to repeat this step numerous times,to outline the best fitting and to diminish the sum of the residual,and that can be done by choosing different values for the parameters each time.Therefore, the best fitting relies on repeating the fitting procedure.The quality of fitting is related to the number of the data points and the number of the parameters when the number of parameters large that will give a better fitting.In contrast,when the number of data points is large that leads to reducing the opportunity of having the best fitting.Moreover,most of NLS program has a standard error,so it is an appropriate to explain the feature of the finesses[21][14].

1.4 Likelihood and least squares

The relation between the maximum likelihood and least square method for several variable n,and every one of these variables pursue normal distribution.The probability density of variable for set of variable,it is expressed[5].

$$f(x_j) = \frac{1}{\sqrt{(2\pi\sigma)}} e^{\frac{-(x_j - \mu)^2}{2\sigma^2}} \tag{1.19}$$

The likelihood of the probability distribution is similar to the probability of variable,so it is formed.

$$L(x) = \Pi_{j=1}^n \frac{1}{\sqrt{(2\pi\sigma)}} e^{\frac{-(x_j - \mu)^2}{2\sigma^2}} \tag{1.20}$$

$$L(x) = (\frac{1}{\sqrt{2\pi\sigma}})^n e^{\frac{-\sum_{j=1}^n (x_j - \mu)^2}{2\sigma^2}} \tag{1.21}$$

Taking the logarithm for(2.2)will give the follow.

$$ln(L(x)) = ln(\frac{1}{\sqrt{2\pi\sigma}})^n + ln(e^{-\frac{\sum_{j=1}^n (x_j - \mu)^2}{2\sigma^2}}) \tag{1.22}$$

$$ln(L(x)) = nln(\frac{1}{\sqrt{2\pi}}) - nln(\sigma) - \frac{1}{2\sigma^2} \sum_{j=1}^n (x_j - \mu)^2 \tag{1.23}$$

In order to obtain the maximum likelihood evaluations,the logarithm of the likelihood of the probability distribution needs to be partially derived with respect to both of the parameters.In other word,mean and variance.

$$\frac{dL(x)}{d\mu} = \frac{1}{\sigma^2} \sum_{j=1}^n (x_j - \mu) \tag{1.24}$$

$$\hat{\mu} = \frac{1}{n} \sum_{j=1}^n x_j = \bar{x} \tag{1.25}$$

By inserting($\hat{\mu}$)in the logarithm of the maximum likelihood ,and then deriving its with respect to variance that will give the following,

$$\frac{dln(L(x))}{d\sigma} = \frac{d}{d\sigma}(nln(\frac{1}{\sqrt{2\pi}}) - nln(\sigma) - \frac{1}{2\sigma^2} \sum_{j=1}^n (x_j - \bar{x})^2) \tag{1.26}$$

$$\frac{dln(L(x))}{d\sigma} = -\frac{1}{\sigma}(\frac{1}{\sigma^2} \sum_{j=1}^n (x_j - \bar{x})^2 - n) \tag{1.27}$$

$$\frac{dL(x)}{d\sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{j=1}^{n}(x_j - \bar{x})^2 \tag{1.28}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2 \tag{1.29}$$

It is clear from above expressions that the least square method is appeared meanwhile the evaluation of the maximum likelihood,and both of them give the same value for their parameters in the case of the error is independent and their distribution is identically.In addition,By implementing parameters estimators,the maximum log- likelihood is formed as next,

$$ln(\hat{L}(x)) = nln(\frac{1}{\sqrt{2\pi}}) - nln(\hat{\sigma}) - \frac{1}{2\hat{\sigma}^2}\sum_{j=1}^{n}(x_j - \bar{x})^2 \tag{1.30}$$

$$ln(\hat{L}(x)) = -\frac{n}{2}ln(\hat{\sigma}^2) - \frac{n}{2}ln(2\pi) - \frac{n}{2} \tag{1.31}$$

The last term is a constant so it can be omitted from the log-likelihood,this will leads to an essential estimation.

$$ln(\hat{L}(x)) = -\frac{n}{2}ln(\hat{\sigma}^2) \tag{1.32}$$

In next section Bayesian information criterion is introduced since it has the ability to transform from the least square method analysis into the the maximum estimator of the log-likelihood function.

1.4.1 Bayesian information criterion

There are two formulations to represent (BIC)which play an essential role in analysis.one of them is the likelihood format

$$BIC = -2log(l(\hat{\theta}_-) + klog(n) \tag{1.33}$$

Another formula related to the least square methods.

$$BIC = n_d ln(\hat{\sigma}_e^2) + kln(n_d) \tag{1.34}$$

n_d indicate to the number of data,$\hat{\sigma}_e^2$ refer to the sum of square residual divvied by the number of data,k is the number of free parameters,this number can be written as $k = 3n - 1$for n sub-populations.The Bayesian information criterion is focused on the number of the parameters in the model,as it is considered the fewer parameters in the model the better the model to be used,as well as the lowest the value of Bayesian information criterion the best fit is obtained[1].

Chapter 2

Models of mortality

There is a variety of models which has been introduced in order to give an appropriate represent for mortality rate(Makeham,Siler, Heligman and Pollard, Lee and Carter),most of these models tend to be a simple one ,and some of them are just a develop for previous models.According to the Gompertz law,human mortality rates double over about every 8 years of adult age(Gompertz, 1825).The first and the most common one,which was introduced by Gompertz.it is known as the Gompertz law[2][4][17].

$$m_x = m_0 e^{\beta x} \tag{2.1}$$

m_0 is denoted to the initial mortality rate(t=0)as it exposures to death,and β is denoted to the change of mortality rate,it is also known as Gompertz slope.This equation(2.1) is an age-dependent intrinsic mortality.Moreover, lnm_x is depend on time,therefore it is a linear interpretation.As it appears in the figure ,Gomperts law increases exponentially in the middle age,it also gives a better analytical approximation for observed mortality pattern,at maturity (approximately 30) However,there is an obvious deviation at advance age,at this stage it is called a revolution for aging research,and at earlier stage as well.

Strehler and Mildvan had found the relation between the parameters in Gompertz law.Therefore, it is called Strehler-Mildvan correlation and it is known as conversation law.When they compared among the parameters they conclude that when the initial mortality high that leads to low mortality changing vice versa,if the initial mortality low that leads to high mortality rate.

Furthermore, By combining the Gompertz law(2.1) with the number of death of aged i(1.10) and mortality rate(1.13),that can give the number of individual of aged $i + 1$.

$$N_{i+1} = N_i - \Delta N_i = N_i - \frac{m_i N_i}{1 + 0.5m_i} = \left(\frac{1 - 0.5m_i}{1 + 0.5m_i}\right) N_i = \left(\frac{1 - 0.5m_0 e^{\beta i}}{1 + 0.5m_0 e^{\beta i}}\right) \tag{2.2}$$

It is potential to obtain the size of individual of aged i,as a function of the parameters which introduced in Gompertz law ,from the individual of aged $i + 1$ by deriving N_{i+1}various time[3][4].

$$N_i = \left(\frac{1 - 0.5m_0 e^{\beta(i-1)}}{1 + 0.5m_0 e^{\beta(i-1)}}\right) N_{i-1} = \prod_{k=0}^{i-1} \left(\frac{1 - 0.5m_0 e^{\beta k}}{1 + 0.5m_0 e^{\beta k}}\right) \tag{2.3}$$

2.1   Deviation from Gompertz law

Gompertz claimed that his model cannot be implemented at old ages,and there is a deviation from his law at that stage.Thought,it is an important to define the reason for this deviation,it might be complicated to study the mortality rate at very old age,because there is a number of limitation during that time,such as it is quite difficult to obtain the number of survivors over old ages,due to the fact that the data cannot be accurate at advance ages.Besides that,the fluctuation at very old age emerges the reasons for this is stochastic

influences.After this fluctuation,mortality rate is reduced that was perceived for human and other species.it is differently noticeable for advanced ages,some conclude that at this level it is still increasing,and it is known as deceleration,others believe that level-off might be more appropriate ending and it is known as a plateau,or it is drop down[3][4].

Therefore,several works have been represented in order to explain this deviation.According to(Gavrilov and Gavrilova; Kannisto et al.; Pham),it might be better to use a mathematical function,such as logistic,quadratic or any others function,to describe mortality dynamic at old ages instead of Gompertz law.In addition, heterogeneity can be the reason for the deviation from the straight line(as explained in Gompertz law)for both young and old ages(Vaupel et al.; Vaupel and Yashin).There are some works linked this deviation with stochastic events,but they can not prove that(Vaupel, 2010; Weitz and Fraser, 2001).Although Gompertz function cannot be solved mathematically(Bonneux, 2003),and cannot be explained biologically(Rauser etal.,2006).There is an argument if it can be considered as a root law or it is just an appropriate device to fit the mortality data(Carnes etal.,1996; Olshansky andCarnes,1997).(Carey et al,1992)and Drosophila (Curtsinger et al,1992) do not agree with Gompertz law.Brooks, Lithgow,and Johnson(1994) studied the matter from another point,and they related the variety of rate of aging with the permanence of genetic [2].

The question which can arise here is when the model theorizes as a law.There are many of conditions have to be satisfied in order to deal with a model as a law.[4]Above all,it has to be understood that will help to build a logical analysis and see the changing in each of the elements as well as the reason of this change,besides that it has to be examined for quite acceptable time and in several areas.Furthermore,it can be used to divine what is going to happen,and that might be the main points of a great amount of analysis(Tenenbein and Vanderhoof,1980).This law(2.1)was discovered and enforced on a human by Gomertz.In addition,some researchers applied it on animal and they concluded that it is also satisfied on other species(Strehler, 1978; Finch, 1990).

According to(Jones, 1956, Finch, 1994)several of data which collected to show the mortality rate caused by different illnesses, proved that this data increased exponentially.In other work, it satisfies Gompertz law.In addition,the table of this law has been used by many of companies and there is also proved of its ability to offer a prediction,since it has been utilized for a long time.However,there is a doubt if it can gives an adequate explanation in different fields.

2.2   Models combined with Gompertz law

As it mentions in the introduction of this section,some models are evolved previous one,one of these models is Makeham-Gompertz Law since it attempts to combine between the intrinsic cause of death which has already explained in Gompertz law,with the extrinsic cause of death,and that is established by adding a constant in(2.1)as age-independents.consequently, Makeham-Gompertz law can be written as follow[17].

$$m_x = m_0 e^{\beta x} + c \tag{2.4}$$

Makeham assumed that might give a better fit to the data.Furthermore,The produce of the constant effect in the linearity,in particular at smaller time,but it tends to be linear in the middle segments.

Heligman-Pollard model is another mode which can be considered as an effort to improve the Gompertz model.This model explains each period of ages separately,so the first term represents infinite ages,and because of the developments of health care provides an appropriate environment, which reduced the number of death among children in most countries.The representation of this term is declined curve. In contrast,the second term represents teenagers and young ages and there are many of causes of death for those group,it represents by the accidental hump.The last term represents exponential law and the same as the law it contents adult and increases because of aging.It can be expressed as follow[16].

$$\frac{q_x}{p_x} = A^{(x+B)^c} + De^{-E(lnx-lnf)^2} + GH^x \tag{2.5}$$

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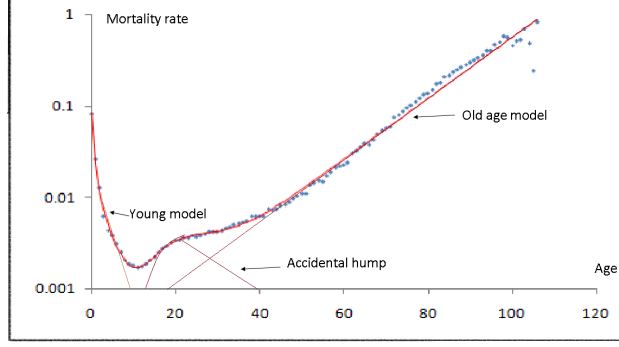
Figure 2.1: Heligman-pollard model set in a semi logarithmic scale,natural logarithm of mortality rate against age.

2.3 Mathematical model of heterogeneous population

It is possible to divide the human population into several sub-populations since they have either different lifestyle or genetic, so the parameters in (1.15)are affected by this variety (Vaupel et al.,1998).In this work,a heterogeneous population is divided into n sub-populations and each one of them follows Gompertz law, that leads to obtaining several parameters for every sub-populations(Vapel,2010).This assumption has been carried out multiplied times to describe the exponential law and its divergence at young and old ages,but in all previous work they do not consider that each sub population follows exponential law(Wrigley-Field, 2014, Chen et al., 2013, Drapeau et al., 2000, Steinsaltz, 2005).A consequence of this separating each sub population has initial (size and mortality)they are definedN_{j0},m_{j0},respectively.As the morality rate in (2.1) becomes(2.6) for j sub-population[16][2].

$$m_{ji} = m_{j0}e^{\beta ji} \tag{2.6}$$

when the whole population is separated into n sub-population of the equation(1.12)

$$m_i = \frac{\Delta N_{1i} + \Delta N_{2i} + ... + \Delta N_{ni}}{N_{1i} + N_{2i} + ... + N_{ni} - 0.5(\Delta N_{1i} + \Delta N_{2i} + ... + \Delta N_{ni})} \tag{2.7}$$

It is also can be written as follow.

$$m_i = \frac{\sum_{j=1}^{n} \Delta N_{ji}}{\sum_{j=1}^{n} N_{ji} - 0.5\sum_{j=1}^{n} \Delta N_{ji}} \tag{2.8}$$

From(1.12)which can be written for j subpopulation.

$$\Delta N_{ji} = N_{ji} - N_{ji+1} \tag{2.9}$$

and substituting it in(2.8)and taking half as a common factor,that will give the following.

$$m_i = \frac{\sum_{j=1}^{n} \Delta N_{ij}}{0.5\left[\sum_{j=1}^{n} N_{ji} + \sum_{j=1}^{n} N_{j,i+1}\right]} \tag{2.10}$$

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And by combining equation(2.8)and (2.6),that leads to obtain.

$$m_i = \frac{\sum_{j=1}^{n} \frac{N_{ji} m_{j0} e^{\beta_j i}}{1 + 0.5 m_{j0} e^{\beta_j i}}}{\sum_{j=1}^{n} N_{ji} - 0.5 \sum_{j=1}^{n} \frac{N_{ji} m_{j0} e^{\beta_j i}}{1 + 0.5 m_{j0} e^{\beta_j i}}} \tag{2.11}$$

In the case of sub-population,the number of death of people aged i for j sub-population ΔN_{ji} is defined as follow.

$$\Delta N_{ji} = \frac{N_{ji} m_{ji}}{1 + 0.5 m_{ji}} \tag{2.12}$$

There is another way to express the mortality rate for sub-populations,and that can be done by inserting the fractions p_{ji} for every j sub-populations in(2.15)rather than the size of sub-populations,so the portion is determined by the number of death of people age i for j sub-populations divide by the total population N_i.

$$p_{ji} = \frac{N_{ji}}{N_i} = \frac{N_{ji}}{N_{1i} + N_{2i} + ... + N_{ni}} \tag{2.13}$$

As the sum of the propitiation is equal to one so one of the parameters can be omitted.

$$p_{ij} = 1 - (p_{i1} + p_{i2} + ... + p_{i(j-1)}) \tag{2.14}$$

Consequently,the mortality rate becomes.

$$m_i = \frac{\sum_{j=1}^{n} \frac{p_{ji} m_{j0} e^{\beta_j i}}{1 + 0.5 m_{j0} e^{\beta_j i}}}{1 - 0.5 \sum_{j=1}^{n} \frac{p_{ji} m_{j0} e^{\beta_j i}}{1 + 0.5 m_{j0} e^{\beta_j i}}} \tag{2.15}$$

As the proportion p_{ji} in(2.15)is determined by the initial fraction,it is performed as follow.

$$p_{ji} = \frac{p_0 \prod_{k=0}^{i-1} \left(\frac{1 - 0.5 m_{j0} e^{\beta_j k}}{1 + 0.5 m_{j0} e^{\beta_j k}} \right)}{p_{s0} \prod_{k=0}^{i-1} \left(\frac{1 - 0.5 m_{s0} e^{\beta_s k}}{1 + 0.5 m_{s0} e^{\beta_s k}} \right)} \tag{2.16}$$

2.4   Strehler and Mildvan theory

Strehler and Mildvan (1960) introduced a sophisticated biodemographic model after they examined 32 countries to provide their general theory of mortality and aging.There are three of consistent in their theory,and they are exponential increasing (Gompertz law),the declination of the vitality which is defined as the ability of the individual organism to be alive at a certain age(t) that can be written as follow[14].

$$V_t = V_0(1 - Bt) \tag{2.17}$$

Where B is the proportion which is reduced every year.Through their work,they also measured the parameters,variation,and magnitude of the environments. In our work,we are interested in their concluded of analyzing the relation between parameters in Gompertz model.They end up by proving that for all countries there is a negative relationship between the logarithm of initial mortality rate and mortality coefficient as that had been examined starting from 1955 to 2003.It can be expressed as following.

$$ln(m_0) = ln(M) - \beta x \tag{2.18}$$

In this expression M indicates the target mortality rate it is also known as the species-specific mortality rate ,and its estimation for the human is $0,5 \ year^{-1}$.While β refers to the target of lifespan it is also known as

species specific lifespan,and its value for human is around 95 years.Strehler and Mildvan found that the value of correlation is between -0.64 and -0.993 due to the fact that the difference of composed of the population in those countries.In addition,they assumed that this correlation works in any human mortality without considering any other factors which might impact on a population.Furthermore,compensation law is not easy to be estimated in many of theories,it is also does not consider the constant value (age-independent)in Makeham model(see Gavrilov and Gavrilova, 1991; Golubev,2004).Moreover,initial mortality and mortality coefficient can be the coordinate for intersecting of all mortality trajectory into one point,as in this case the law has an a strong impact on the parameters which leads to convergence,but there is some cases when it has a weak impact[11].
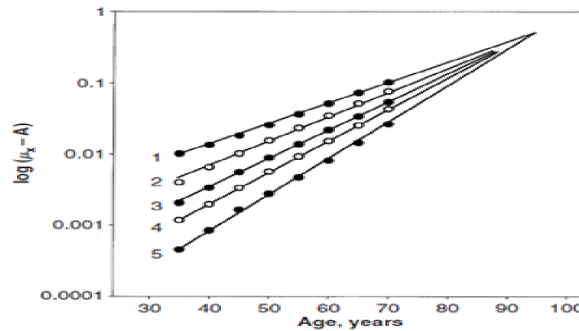
Figure 2.2: it represents Compensation law of mortality ,as all mortality rates for several population Convergence into single point at old ages.mortality rates are plotted in a log scale as a function of age in several countries,and they are India, Turkey, Kenya,England and Wales,Norway .This figure and its estimation is taken from Gavrilov and Gavrilova,2003

Figure(1.4)shows how the compensation effect on different population,and since all trends which represent mortality are for five countries converge into one point that meas the compensation law has a strong affect.In the next chapter we will see the effect of compensation law in the case of population and sub-population.

Chapter 3

Results

There are several of advantages and disadvantages of considering population as a homogeneous.Indeed,one of its advantages is that there are many of developments in different fields as a consequence of this assumption such as demography,actuarial statistics,reliability engineering and epidemiology.Since proposing a population as a homogeneous makes the analysis straightforward rather than when a population is considered as a heterogeneous.However,this assumption cannot be generalized because there are many of the cases which the influence of the individual is much more important than the whole population,and that related to the purpose of the study.In the first section of this chapter,we will consider the population as a homogeneous.

Later on,we will assume another type of populations as we see that there are numerous of benefits from assuming that the population is a homogeneous one,there is also a large number of advantages from dividing the population into a number of sub-populations which is known as a heterogeneity.In addition,a heterogeneous assumption can be used very commonly in biologic perspective,as this assumption is easy to implement for analysis any set of data such as medical data,natural progression and the effect on lifestyle.Furthermore,the main compensation of considering a model as a heterogeneous is that since it can be used for any data,it is also able to provide several of mortality curve for example,the deviation from Gompertz law,deceleration,decline and plateau,as well as it explains each one of this curves.It is also can be used to give a good fitting for mortality data over various period.Consequently,it plays an essential role in analyzing mortality data,since it has the ability to describe and provide mortality prototype and how it changes in an unusual feature.

The following two sections are focused on fitting the UK data,which is taken from(http://www.mortality .org/) to the model which was explained in(2),and therefore explain the evolution of mortality dynamic regarding altering parameters over time as well as show the orientation of every parameter across time,starting from 1922 to 2014 for every four years.The first section is to find the trend for homogeneous population starting from forty age and above,and the second section is for the heterogeneous population has the same aim as the previous section,but the difference is that it will be for two sub-populations starting from age twenty and above.In addition,there is a theory which explained the relationship between parameters,it is known as compensation law,this theory is premised on the assumption that there is a contrary relation between initial mortality and coefficient mortality which means that when the initial mortality increases that leads to decrease the coefficient mortality.Throughout this analysis,we will examine the implementation of this theory in mortality parameters,we also interested in observing the change in these parameters(initial mortality and coefficient mortality)over a long time.

3.1 Mortality dynamics in the model of homogeneous populations

(Estimation of mortality parameters for forty and above) In this section,we analyze the change of parameters for a homogeneous population over 92 years,our aim is to see how they change over a century, but this might be one of the limitations of this study since we found a period set of data which is 8 years less than one century.However,it can not be considered as a weakness of this study,because we have enough set of data as it is still provided a great deal of detail that lead to providing an essential insight of altering parameters.Moreover,the evaluation of these parameters (initial mortality and mortality coefficient)have been measured by implementing the least square method which was explained in(1.3).This evaluation of the parameters for the UK starting from 1922 to 2014 has been tabulated in(4).During this period,the fitting of data can be considered as a good one though it is not very accurate.As a result of that,we omit some part of the data in order to improve the fitting of the data so we can obtain a better fitting when the data starting from age forty and above,in one population.
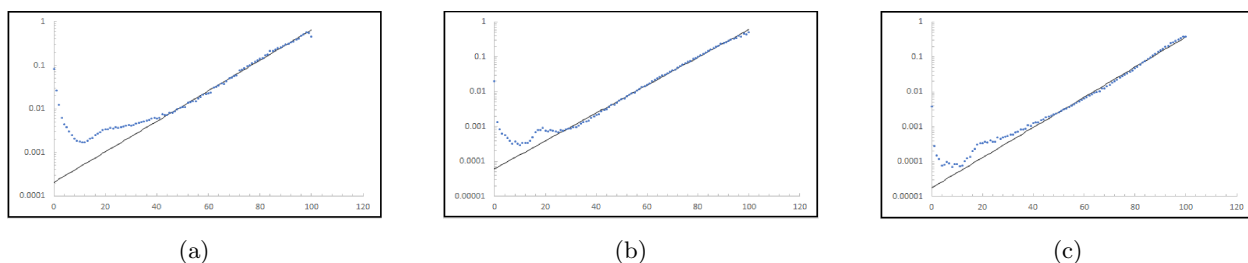

Figure 3.1: Homogeneous model (solid line)fitted to the 1922(panel A),1966(panel B)and 2014(panel C) period the UK data (dots)from age 40 and advance.The mortality parameters as estimated by the least square method are:$m_0 = 0.000207$,$\beta = 0.080583$for panel A,$m_0 = 0.000061693$,$\beta = 0.092063058$ for panel B,and $m_0 = 0.0000178$,$\beta = 0.099547135$ for panel C.

In figure (1.3)we choose this three years from our analysis for the UK data-set which is taken from(http://www.mortality.org/),we found that these three years are able to give us enough information about the changing of parameters over 92 years as well as to prove the compensation law.In figure(1.3)panel(3.1a),it shows that the mortality rate for the year1922,this year has a higher initial mortality rate compared with the other two years since the mortality rate for this year is $m_0 = 0.000207$ and for the two other years are $m_0 = 0.000061$,$m_0 = 0.000017$for 1966 and 2014 respectively,as they appear in panel (3.1b)and(3.1c).But all of these years have the same fact that a newborn baby has a Large number of death comparing with young ages.So at a young age, the number of death is decreased for all of three years,this declination has been changed for teenagers as it is increased over young ages.Especially,in 1966 the hump growth of teenagers is very clear since it starts from age 14 until age 21 and then it declines for a couple of years.However,this hump growth is not obvious in1922 and 2014,because it becomes more constant in 1922 and at the end of 2014.In()claims that this increased is not related to instinct change, it is caused by the behavior of teenagers which is not explained in Gompertz law,he did not represent any element in his law which describe external cause of death1.5.

Moreover,This difference is might be depend on mortality coefficents as this value is close to one in 2014,so it is $\beta = 0,838934$,and it is smaller than that in previous years so it is $\beta = 0.080583$ in 1922,and there is a slightly increase for this value in 1966 as it is$\beta = 0.092063$.Although the growth of the mortality rate is dominated from teenagers term and later one in adulthood.However, it does not increase exponentially before a maturity .therefore, it does not fit a Gompertz law and that what he claimed in his paper(),as that leads to introduce many of models describing this deviation as we explained previously.In order to see wither these three years fitting homogeneous model we need to recall the basin information criterion which was introduced in(1.4.1),and from that we can say that the last year is the less fitting for the data among all the three years, since it is (BIC=-253,255099) as panel c (3.1c) shows that from age 47 the data begin to fit the model which is later than the two earlier years.In addition, on panel b(3.1b) it appears that the fitting of

the data starts from age 28 and according to the value of the Bayesian information criterion this year (1966) is considered the best fitting for the data among these three years since it is(BIC=-301.566631),and in at beginning of the database it is (BIC=-282,165).

(a)   (b)
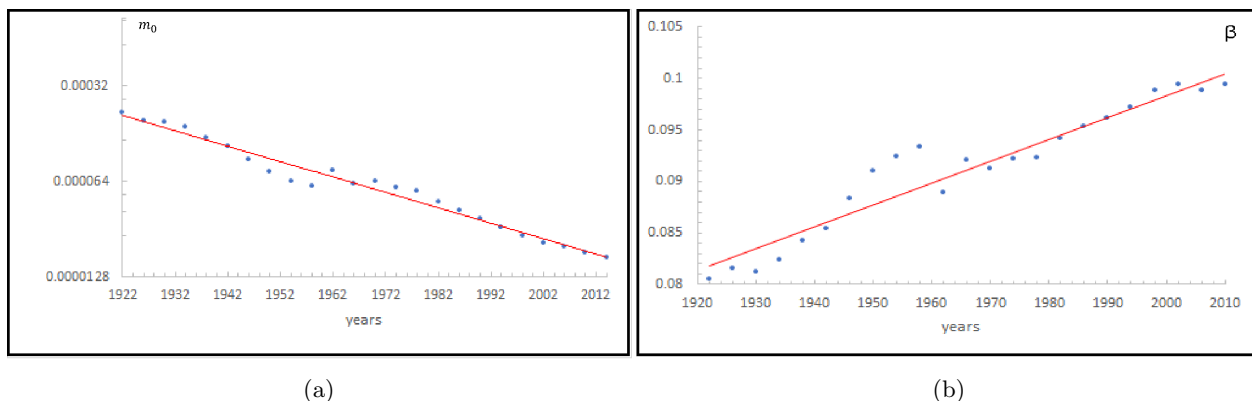
Figure 3.2: Estimation of the parameters of the homogeneous model fitted to the UK mortality rate from age 40 and above, for the periods 1922 to 2014 with four years interval.Panel A represents an exponentially declining for the initial mortality,and panel B shows a linearly increasing trend for the mortality coefficient.In both of plots the parameters are represented by dots and the trendiness by red solid lines.

Furthermore,a homogeneous model satisfies Gompertz law for this database as well as compensation law,referring to the definition of compensation law which is explained the opposite relation between parameters. In this term,we have two parameters for a homogeneous population,and they are initial frailty and the changing of mortality .In addition,it is clear from panel a(3.2a)that there is an exponential decrease for the initial mortality over 92 years ,and the trend line for the logarithm of the mortality reverse to time is linear.While panel b(3.2b) displays the mortality coefficient ,and from that we can say that mortality coefficient is linearly increased over the same period of years.Therefore, compensation law is satisfied for this data,this result is also obtained in [4],since it shows slightly growth between two parameters which means that when the initial mortality decreased at the same time the mortality coefficient increased.

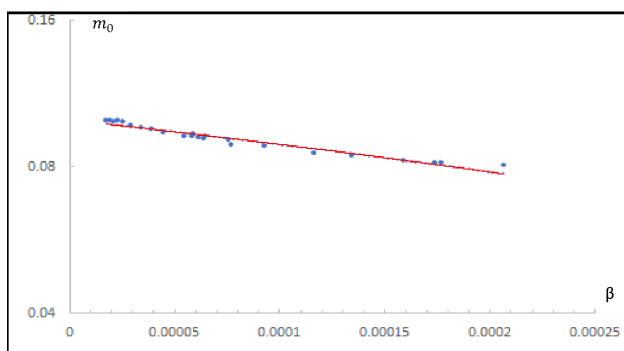Figure 3.3: Compensation effect in 40 and above mortality dynamic represents the reverse association between the parameters of the fitted homogeneous model to the UK mortality rates from age 40 and advance for the period data in the interval 1922-2014.The association between the logarithm of the initial mortality and the mortality change is shown to be linear.

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3.2 Mortality dynamics in the model of heterogeneous populations

In this section,we consider populations as a heterogeneous which was discussed in(2.3),so there are several of free parameters and they are going to be fitted to the UK observation data by using least square method which was explained in(1.3),therefore these parameters leads to giving a close view related to genetic and biological feature.In addition,(Vaupel and Yashi,1985)claimed that heterogeneity can give an explanation of the divergence from Gompertz law.In previous section we estimate that a population is homogeneous and we fit the data in that model in order to judge how well the fitting would be,so we like to keep the same aim in this section and the same data which was used,but the type of population here would be composed of many of sub populations.Firstly,we will divide the population into two sub populations,then into four sub populations,and finally comparing between all of them to see which one gives the best fitting.

3.2.1 Estimation of mortality parameters for twenty and above

we assume that the population is divided into two groups,and they are frailty and robust each one of this group follows Gompertz law(2.1).Therefore,there are two describe which introduce every parameters,and they are initial mortality,initial fraction and changing mortality($N_{10}, p_{10}, \beta_{10}$)for the first sub-populations,and for the second sub-population their notations would be $(N_{20}, p_{20}, \beta_{20})$.And during our analysis,we assume that all parameters are positive values.Therefore, we would like to see how these parameters change over time as if they are still satisfied compensation law.First of all, let see how subpopulations change over time by selecting three different years(1922-1966-2014),then we are going to discuss the estimations of parameters[20].

(a) (b) (c)
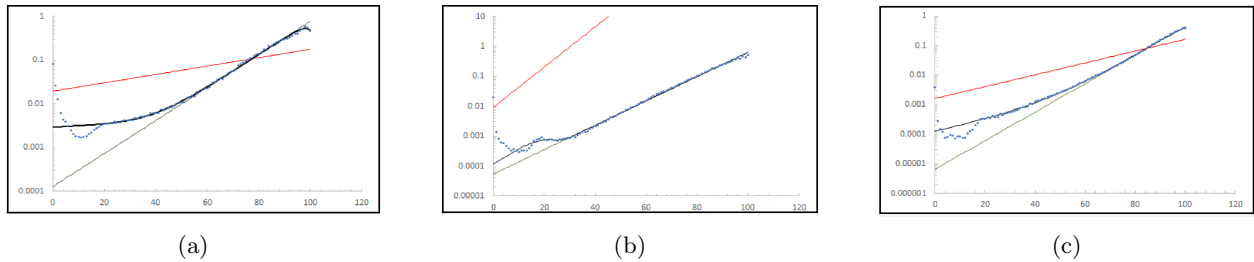
Figure 3.4: Heterogeneous model fitted to mortality data for ages 20 and advance.A two -sub-population model is fitted to the UK data(males and females)data (blue dots)for the periods 1922(Panel A),1966(panel B)and 2014(Panel C),the red line represents the mortality rate for the first sub-population,and green line represent the mortality rate for the second sub-population,while the solid(black)curve is the mortality of the whole population as described by the model of heterogeneous population.The model are evaluated by the least square method.

- panel A.1^{st}subpopulation:$m_{10} = 0.01937, \beta1 = 0.022061, p_1 = 0.145295$.
  2nd subpopulations:$m_{20} = 0.000128, \beta_2 = 0.086905, p_2 = 0.86905$.

- panel B.1^{st}subpopulation:$m_{10} = 0.0092939, \beta1 = 0.154873269, p_1 = 0.0.00681$.
  2nd subpopulations:$m_{20} = 0.00005, \beta_2 = 0.09365612, p_2 = 0.99318254$.

- panel C.1^{st}subpopulation:$m_{10} = 0.00163, \beta1 = 0.0046184, p_1 = 0.073595$.
  2nd subpopulations:$m_{20} = 0.000006, \beta_2 = 0.1113041, p_2 = 0.92640$.

Fitting the heterogeneous model to the UK mortality data using the least squares methods.The data is indicated by blue dots and the mortality rates of modeled sub-population are presented by red solid line for the first sub-population and green solid line for the second sub-population.The mortality of the whole population by the black solid line.This representation is shown for all of the three years which we optimal to see how the sub-population could be established over consecutive time. As we can see in all panels that the accidental hump is still affected in this study, since we began our data from age 20 and above which gives a better fitting for two sub-population model,that is proved when we compared between this model and the one which we discussed in prior section, So BIC in 1922 is -440,094 and for homogeneous model in the same year it was -282,165 as well as for the other two years in 1966 BIC=-381,541and in 1966 BIC=-437,723 So here we have smaller values for BIC than the one which we had in homogeneous model,that means this model fitting better than the previous one.

In addition,Using the least squares methods gives us different entire population for each year,it also gives several values for every parameter.Therefore,the entire population in 1922 is composed of these two initial mortality $m_{01} = 0.019371$ and $m_{02} = 0.000128$,as m_{10}indicate to frailty sub-population which has higher initial mortality than the robust sub-population.Although it is clear that m_{01} has higher initial mortality it affected since there is a divergence at the older age.Also, we can see in panel A there is a cross point in 77 age among trajectories mortality rate for entire population and sub-populations.Furthermore, robust sub-population increases exponentially from age 51 and above.The coefficient parameter in the second sub-population is larger than in the first one since there are $\beta_1 = 0,022061$ for the first sub-population and$\beta_2 = 0.000128$ for the second.Furthermore,the evaluation of the fraction is $p_1 = 0.145295$for the first sub-population, and it is greater in the second sub-population $p_1 = 0,854705$.

In panel(c)shows the fitting of the second sub-population began from age 64 ,as both of the sub-population(frailty and robust) cross the mortality of all population in age 84,this is similar for the behavior which was estimated in 1922 since there was also crossing point between all population and morality rate,but in this case there is no diverge at the end of the ages ,as frailty sub-population began with a high initial mortality and over time ,that will leads to obtain homogeneous population that means the robust sub-population is increased over time and from the panel it also represents a good fitting at advance ages.In particular,the initial population for the first sub-population is $m_{10} = 0.0016286$,while it becomes for the second sub-population$m_{20} = 0.0000066$,as the fraction of the first population is $p_1 = 0.073595$,and for the second $p_2 = 0.926405$,so from these two values it shows that the fraction of the second sub-population is larger than in the first.
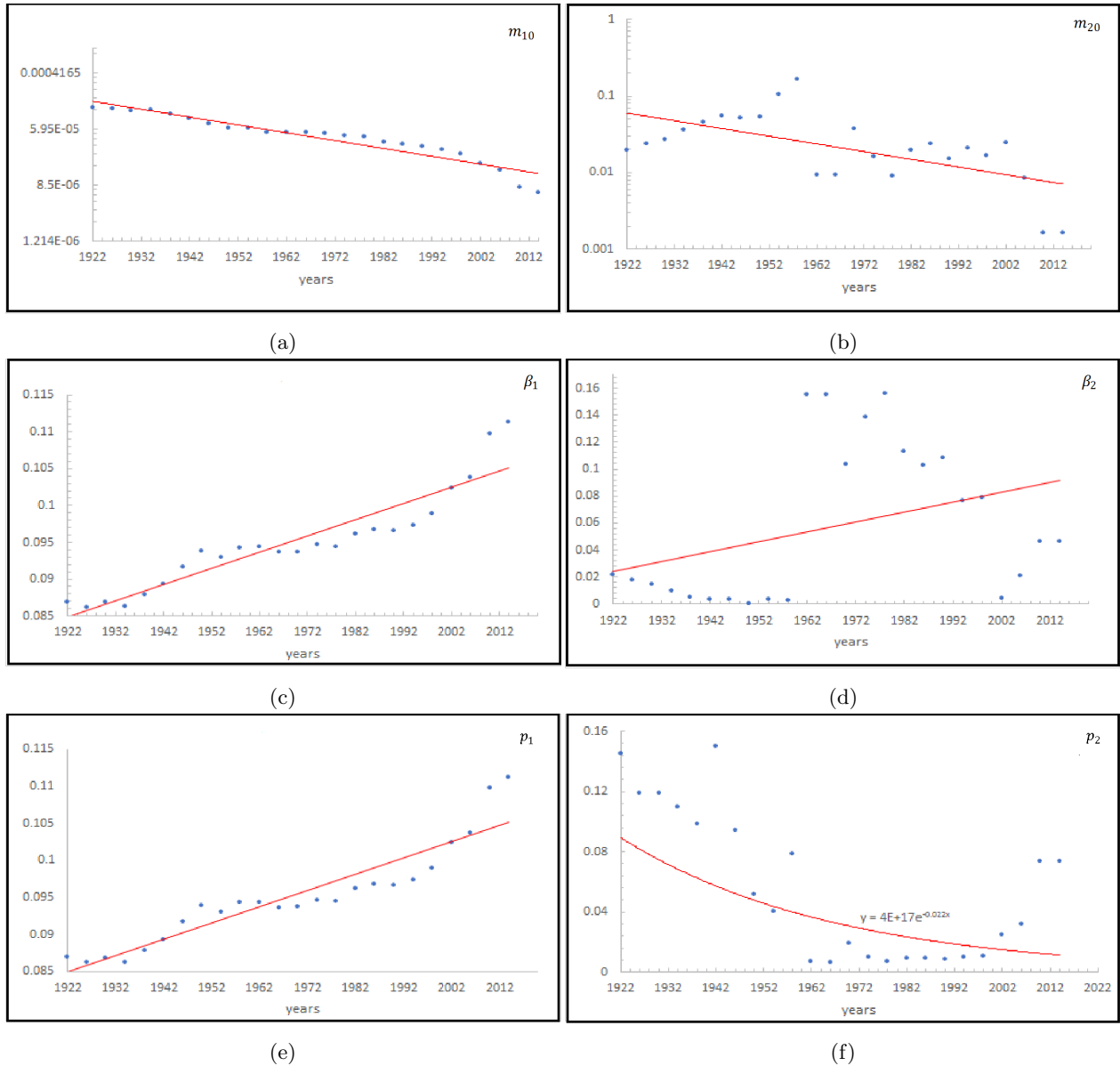
Figure 3.5: Estimation of the parameters of a two-subpopulation model fitted to the UK mortality rates for ages 20 and advance panel A,C and E represents estimation of parameters initial mortality m_0,mortality coefficient$\beta$and initial fraction$p_0$related to the first subpopulation,whereas the panels B,D and F associated to the second subpopulation.The model parameters are evaluated by using the least square method,and the solid curves describe trendiness for the estimation of the parameters over time.

Panel (a),panel(c)and panel(e)shows how the parameters of frailty sub-population change over time,for the UK observation data from 1922 to 2014 for every 4 years, and for the same data and period represents panel(b), panel(d) and panel(f)represents the parameters of robust sub-population.Referring to a panel(a)and panel(b) initial mortality for both of sub-population is decreased exponentially. However, its declination in the first sub-population is better than in other,since there is an random affect which impact on all of the parameters which are estimated for the second sub-population.The mortality confection is also increased in both of the two sub-populations, that classify the fact that compensation law is implemented in every sub-population. The trend of the fraction in the first sub-population is increased.In contrary, the trend of the second sub-population is declined.
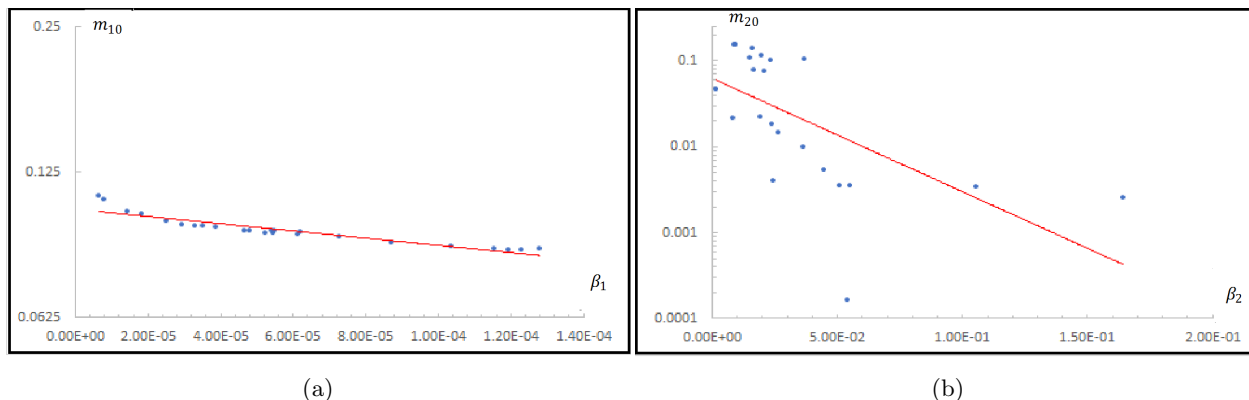
Figure 3.6: Compensation effect in 20 and above mortality dynamic shows the inverse relationship between the parameters of the fitted heterogeneous model for,the first sub-population panel A and for the second sub-population panel B,to the UK mortality rates for age 20 and advance for the period data in the interval 1922-2014 The relationship between the logarithm of the initial mortality and mortality coefficient are represented by linear trained in both of panel A and B.

In order to reflect the association between the two parameters of Gompertz function,which was mentioned previously. We need to set the natural logarithm of the initial mortality sub-population (m_{10}, m_{20}) contrary to morality coefficient (β_1, β_2) for each of sub-population as this represents in figure(3.6), panel a shows the relationship for the first sub-population and panel b for the second sub-population.By perceiving figure a clear link between these parameters ,since theoretical mortality decrease exponentially the demographic slope increase linearly.

3.2.2 Estimation of mortality parameters for all ages

In this section, we assume that the population is heterogeneous similar as we did in the previous case, but we will divide the population into more than two sub-population.According to (Avraam et al.,2013) the best fitting can be obtained by considering the population is separated into four sub-population.So in this case the the first population contain the highest initial mortality as a result of that after a couple of years this population can not be exist.The second sub-population include peoples aged between two to 10 years,and the third sub-population represents an accidental hump,as the last sub-population represents exponential mortality,as this sub-population is the only which is kept over time since it has the lowest initial mortality and the largest initial size, that will leads to transform the heterogeneous population into homogeneous one over consecutive time.Therefore, we would like to see how this changing take place in the UK so we choose three years over 92 years,as they are 1922,1966 and 2014.

(a)   (b)   (c)

Figure 3.7: A four subpopulation model (heterogeneous model) is fitted to the UK (males and fe-males)mortality rate (blue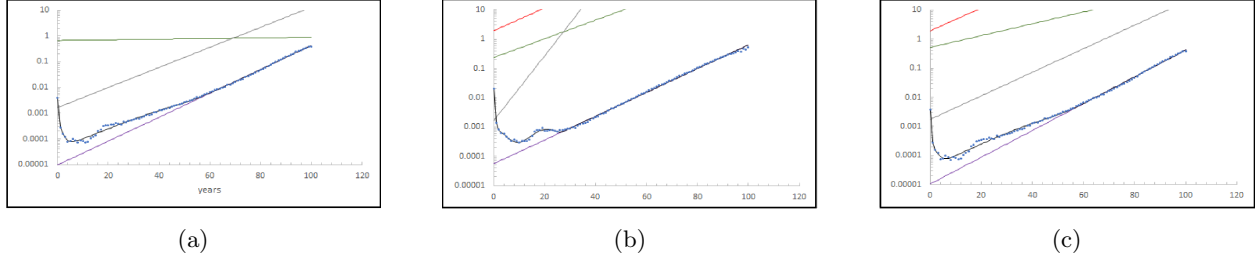 dots)for the periods 1922(panel A),1966(panel B) and 2014(panel C).Red ,green ,gray and purple lines (exponential functions)represent first,second, third and fourth,respectively. While black solid curve displays the mortality of the whole population.The model parameters are estimated by using the least square method,and they are for

- panel A.
  1^{st}subpopulation:$m_{10} = 10.94409, \beta1 = 0.09, p_1 = 0.001987$.
  2nd subpopulation:$m_{20} = 0.666234, \beta_2 = 0.002881, p_2 = 0.000918$.
  3^{nd}subpopulations:$m_{30} = 0.001656, \beta_{30} = 0.089783, p_3 = 0.018365$.
  4^{th}subpopulations:$m_{40} = 0.0000097, \beta_{40} = 0.106832, p_4 = 0.97873$.

- panel B.
  1^{st}subpopulation:$m_{10} = 1.914289, \beta1 = 0.0899, p_1 = 0.019075$.
  2nd subpopulations:$m_{20} = 0.235025, \beta_2 = 0.073438, p_2 = 0.004758$.
  3^{nd}subpopulations:$m_{30} = 0.001653, \beta_{30} = 0.254884, p_3 = 0.004969$.
  4^{th}subpopulations:$m_{40} = 0.0000558, \beta_{40} = 0.0933189, p_4 = 0.971197$.

- panel C.
  1^{st}subpopulation:$m_{10} = 1.942673, \beta1 = 0.08999779, p_1 = 0.003574$.
  2nd subpopulations:$m_{20} = 0.51839575, \beta_2 = 0.046842, p_2 = 0.0007178$.
  3^{nd}subpopulations:$m_{30} = 0.001749, \beta_{30} = 0.093159, p_3 = 0.015927$.
  4^{th}subpopulations:$m_{40} = 0.00001053, \beta_{40} = 0.105909, p_4 = 0.979781$.

In figure(3.7)There are four sub-population the red one represents the first,as it is the most frailty sub-population and the green for the second and the gray for the third and the most robust sub-population purple,as the black trend refers to the morality rate for all of population.It is clear from the figure that the red trend do not exist for long time as in contrary to purple one.For example, in panel (A) which represents 1922 year that the trend line do not exist and that means the frailty age died very fast in this year,but in the two other years we can see from panel (b) and (c) this group has an affect on both of the years.In addition,it is clear that the third sub-population can be impact on all of the three years,as there affect is longer than the second sub-population.
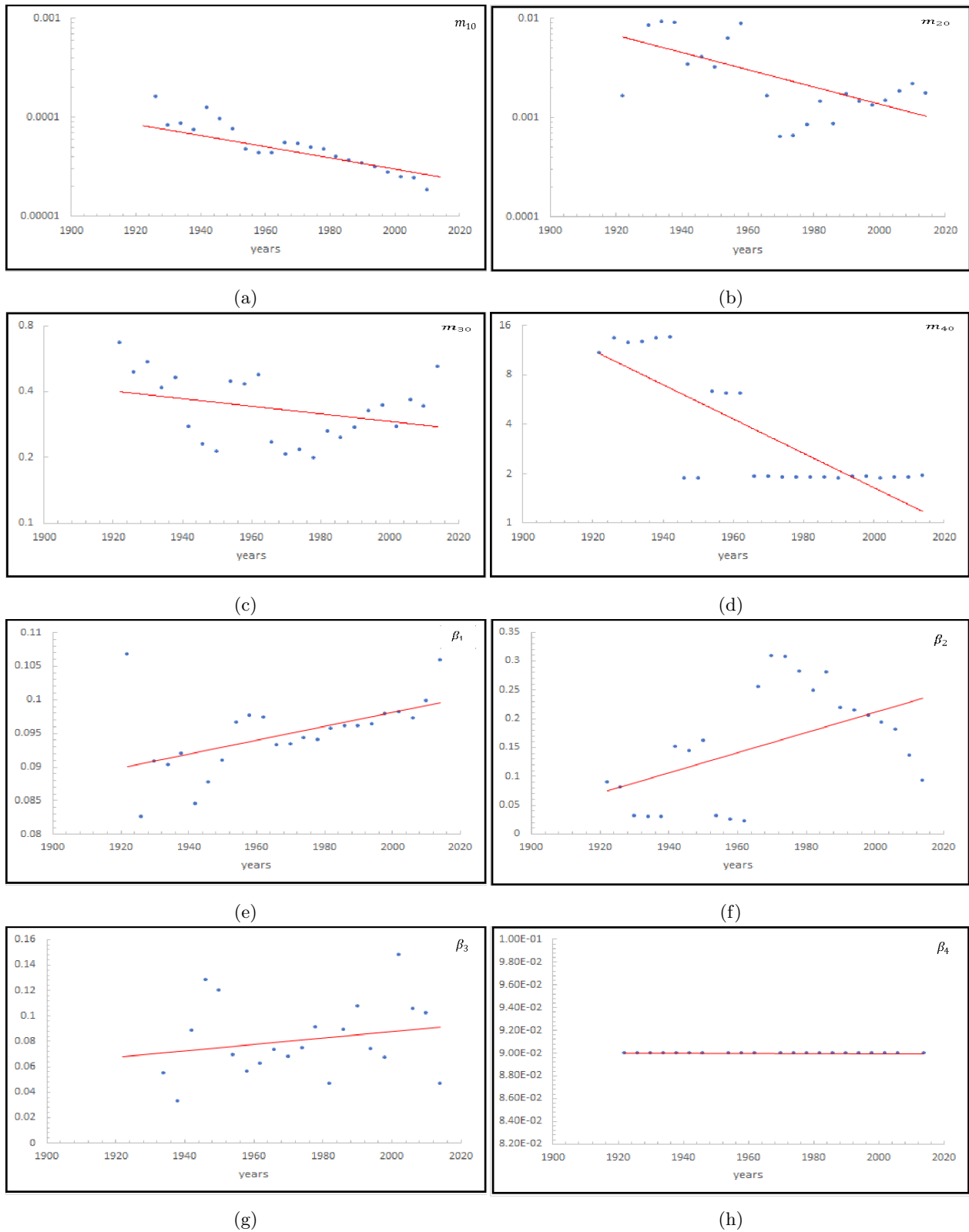
Figure 3.8: Evolution of the parameters of the heterogeneous model(four-sub-population) fitted to the UK mortality for the periods 1922 to 2014 with four years interval.Panels A,B,C,and D represent the initial mortality rates for the first ,second ,third and forth subpopulation,all initial mortality decrease over time .However, there declination are not clear for most of the sub-population except of the first one.And panel E,F,G,and H represent the mortality coefficient for each of the fourth sub-population,respectively,as it is increase in all of the sub-population but it shows great noise over years,and mortality coefficient for the forth sub-population can be a constant when we omit two of the data.

In figure (3.8),There are eight panel,the first for of them show the initial mortality and the rest four represents the mortality coefficient, as we can see both of the parameters is a noisy when it is estimated over time, while this result was not obtain in the heterogeneous model either in the case of the two sub-populations.However, we attempt eliminate that noise by omitting some of the data,and that only make an obvious changing in the mortality coefficient for the fourth sub population.Furthermore, the compensation law is implemented in the case of the four sub populations,since the initial mortality decrease exponentially in all of the population and the mortality coefficient is increased linearly.

(a)   (b)

(c)   (d)
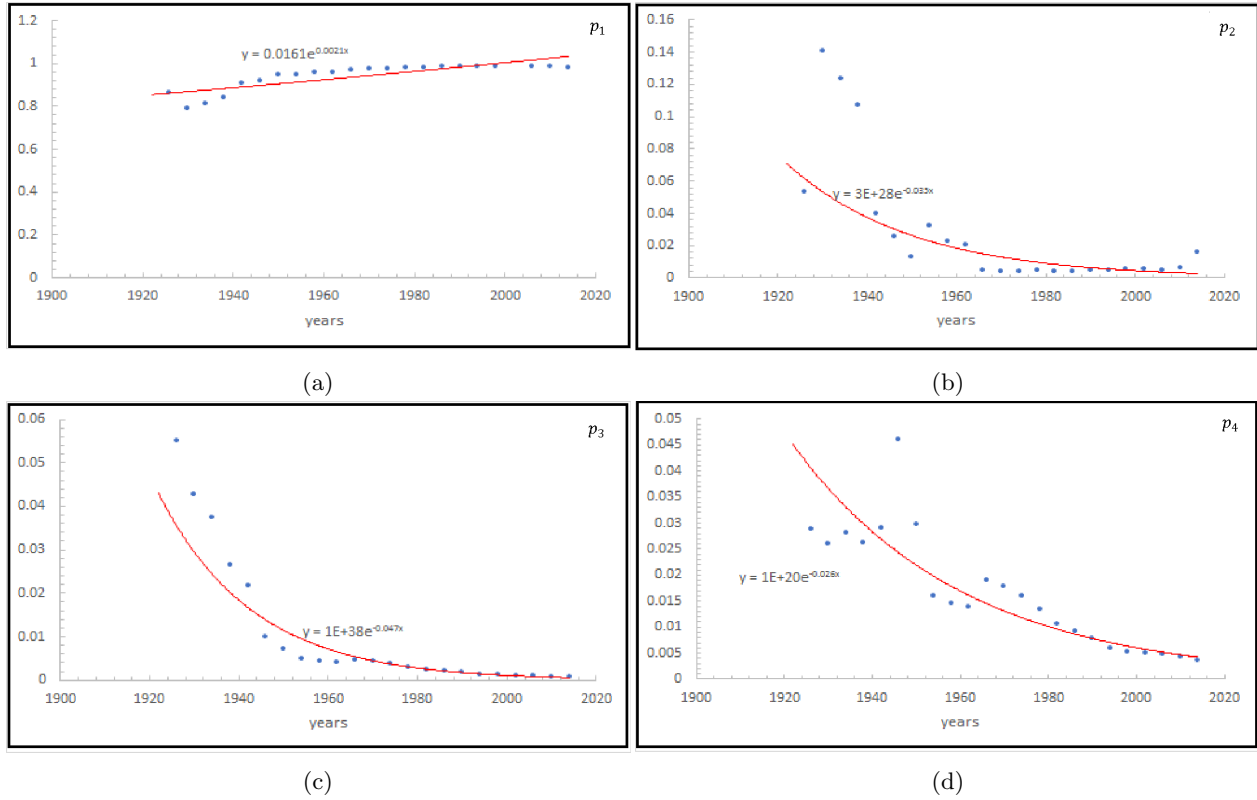
Figure 3.9: Evolution of the fractions of the heterogeneous population in the case of four sub-populations,as it is clear that most of the fraction of the sub-populations are declined linearly.

Figure(3.9) represents the fraction of the heterogeneous model ,when the population is assumed to divide into four sub-populations.Each Panel in the figure refer to one of the sub-population,and by looking at these panels we found that the fraction decreases linearly in all of sub-population,as we also explain previously that the heterogeneous population will be transformed into a homogeneous population over a period of time,and that appears because of the most of populations will not be exist.

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(a)

(b)

(c)

(d)
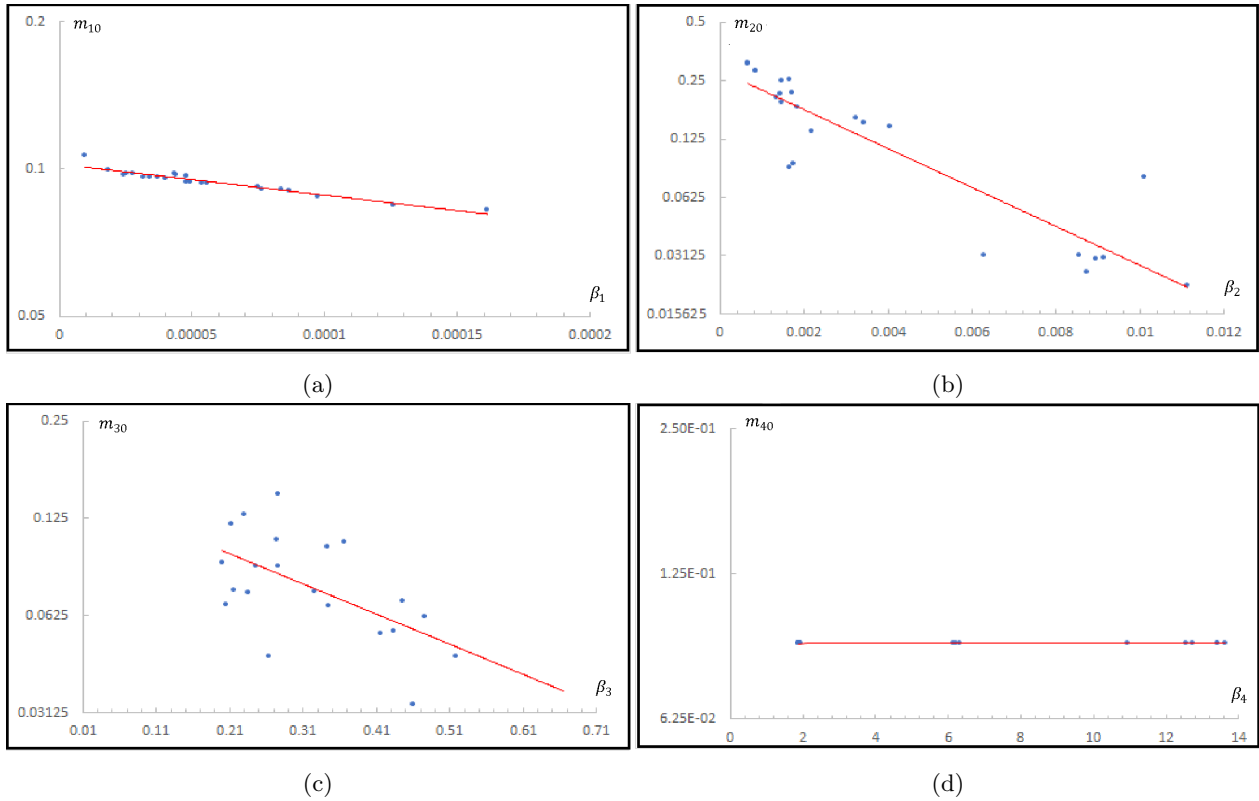
Figure 3.10: The inverse associated among the parameters of the fitted four sub-population to the UK mortality dynamic over all ages for the period in the interval 1922-2014,the relation between the logarithm of initial mortality and the mortality coefficient are shown to be linear for all four sub-population,panel A for the first,panel B for the second,panel D for the third lastly panel D for the fourth.

Figure(3.10) examined the relationship among the parameters in each sub-population.from figure (3.9) and(3.10),we can say that compensation law is implemented in the four supplication consideration,as it the figure (3.10) the relation among the parameters is decreased exponentially.However,this relation quite unclear it is represented for the initial mortality and mortality coefficient for the fourth sub-population

Chapter 4

Discussion and conclusion

Studying mortality dynamic plays an important role in all countries, since it helps to improve health care,economic,environmental and social policies, that improvement based on understanding the structure of any population and how it changes over time. Through studying mortality dynamic, life expectancy can be estimated.Consequently, there are many of models which have been introduced in order to have an accurate view of this changing.In this work, we study Gompertz model and Helgman -pollar model,the methodology behind this paper was described in(Demetris,2014) where the time evolution of mortality dynamic and parameterized was applied to Swedish mortality data, we estimate the evolution of the parameters series of data on mortality rate at five year age intervals.The resulting temporal trajectories on each of the parameters using the least square methods, we would like to use the same method to evaluate the parameters to the UK data instead of Swedish data to see if we can obtain the same result as it was estimated as well as, examined the model which is focused on providing an accurate detail for Gompertz law and the deviation from it weather at earlier age or at advance age.

We begin our analysis by considering a population is an a homogeneous one, then we estimate that the population is separated into two sub-population by this estimation we obtain two kind of parameters which represents each one of the sub-population.After that we analysis heterogeneous population for more than two subpopulation,and through that consideration we can have deeper deviation for the population than which is evaluated by Heligman polar model, since that model divide population into three sub-population,so there are three parts the first one represents young ages and the second indicates to adult as the last one is Gompertz law for advance ages.Therefore, we analysis heterogeneous population for four sub-population,that leads to that three parameters for each sub-population, and they are initial mortality ,mortality coefficient and the fraction, as it represents one more term than polar population by having one more sub-population which represents very young age.

Through our work,we use the basin information criterion to compare between those considerations and the less the value of basin information criterion the better the model. In addition,we found that four sub-population gives the best estimation for all ages than the one which is evaluated in either homogeneous population nor by two sub-populations, as heterogenous population can gives a better result when it start from age forty and above, and that means it just evaluates for Gomertz law since its exponential function represents form adult ages into advance ages.Two sub-population is also provide a better result when its begin from age twenty and above.Both of heterogeneous and homogeneous models satisfies compensation law so in most of the cases we got an increasing for the initial mortality and decreasing for the coefficient for homogeneous and for every sub-population for the heterogeneous.As we found that through time heterogeneous model becomes a homogeneous one.
Despite the fact that the main idea behind produces Gompertz law is to analysis mortality rate which influences by many of factors in demography, it seems there is no greater focused on this law from demography view(Barbi et al.,2003; Carnes andOlshansky,2001; Yashin et al.,1994)

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Appendix

Appendix A

Table 1. Parameter values for homogeneous model (from 1922-1954)

Years	Initial mortality (m_0)	coefficient mortality(β)	Bayesian information criterion(BIC)
1922	0.001225986	0.056282657	-63.26948929
1923	0.001066279	0.057275602	-67.91879237
1924	0.001098479	0.057507733	-62.16377737
1925	0.001074627	0.057718415	-59.72848758
1926	0.000996674	0.058288061	-62.33262432
1927	0.001040742	0.058147935	-64.01934537
1928	0.001042999	0.056938106	-67.81862953
1929	0.001080314	0.05872552	-58.60088151
1930	0.00096279	0.058727072	-65.94428287
1931	0.00088929	0.061004707	-59.92511063
1932	0.000888226	0.060154598	-61.70450169
1933	0.000920553	0.059741064	-68.56552642
1934	0.000929421	0.058206472	-66.80680879
1935	0.000821825	0.060264272	-69.26786066
1936	0.000785482	0.061120959	-65.66319202
1937	0.000767495	0.061807882	-68.13906202
1938	0.000721767	0.061474968	-68.27889191
1939	0.000629012	0.064165146	-74.14234422
1940	0.000913987	0.05943303	-70.43137288
1941	0.000913987	0.05943303	-70.43137288
1942	0.000749169	0.059527515	-67.67299444
1943	0.000677025	0.06249415	-75.65920019
1944	0.000644137	0.062718298	-79.95318647
1945	0.000570987	0.063428626	-82.05895601
1946	0.000459516	0.066669044	-81.12849079
1947	0.000435265	0.067669948	-76.18889904
1948	0.000378212	0.068018157	-87.59678048
1949	0.000326793	0.071305474	-83.5133065
1950	0.000289724	0.072383942	-85.61195555
1951	0.000262321	0.074363156	-80.15500351
1952	0.000229014	0.075076798	-80.50419517
1953	0.000224821	0.074598859	-75.40227772
1954	0.000193498	0.077144711	-82.67999323

Table 2. Parameter values for homogeneous model (from 1955-1999)

Years	Initial mortality (m_0)	coefficient mortality(β)	Bayesian information criterion(BIC)
1955	0.000190807	0.077493519	-78.52877972
1956	0.000174687	0.078433128	-79.66384457
1957	0.000194407	0.076252975	-83.09119892
1958	0.000171547	0.078332531	-80.18309967
1959	0.000180961	0.077040329	-77.98426686
1960	0.000177293	0.077300567	-79.56964266
1961	0.000172543	0.07833109	-78.75564777
1962	0.00017126	0.078216935	-80.30601339
1963	0.000172443	0.078196123	-79.60085695
1964	0.000173662	0.076977539	-86.62814532
1965	0.000178624	0.076551289	-87.64371073
1966	0.000173032	0.077359851	-85.15673498
1967	0.000167215	0.076974375	-87.05482855
1968	0.000160539	0.078261215	-81.95050069
1969	0.000158347	0.078393471	-85.32984355
1970	0.000154613	0.078495461	-86.08622608
1971	0.000156577	0.077927475	-87.1140971
1972	0.00015289	0.078680471	-84.92899527
1973	0.00015313	0.078456444	-87.96411014
1974	0.000143162	0.079422536	-86.73923017
1975	0.000142299	0.079138319	-90.04837901
19776	0.000137307	0.079851099	-91.98254715
1977	0.000131407	0.079787319	-92.79101653
1978	0.000137535	0.079356013	-95.35089309
1979	0.000128792	0.080113458	-97.31988505
1980	0.000123601	0.080342044	-95.50912511
1981	0.00011793	0.080769876	-97.94237513
1982	0.00011465	0.080882015	-99.45361788
1983	9.36439E-05	0.083554803	-190.1519819
1984	0.000108573	0.081010553	-102.5261459
1985	0.00010646	0.081551023	-100.0864146
1986	0.000101704	0.081792023	-101.0295053
1987	0.000101219	0.081562877	-103.4266954
1988	0.000102294	0.081419611	-103.736
1989	0.000100057	0.081837499	-102.269
1990	0.000100102	0.081304769	-108.0829
1991	0.00009851	0.081432609	-108.87439
1992	0.0000908	0.082118731	-115.509855
1993	0.0000893	0.082795782	-115.071187
1994	0.000086	0.0826074330	-119.029897
1995	0.000086	0.0829654856	-119.902209
1996	0.000083	0.0832356731	-115.429899
1997	0.0000824	0.083032203	-116.923252
1998	0.0000803	0.08323882	-116.36854
1999	0.000078	0.083741888	-113.81757

Table 3. Parameter values for homogeneous model (from 2000-2014)

Years	Initial mortality (m_0)	coefficient mortality(β)	Bayesian information criterion(BIC)
2000	0.0000727	0.084489421	-119.677262
2001	0.00007372	0.083645234	-118.497913
2002	0.000072	0.083981959	-118.27193
2003	0.0000693	0.084496859	-113.373879
2004	0.0000653	0.084527822	-117.344518
2005	0.00006287	0.084898734	-115.335647
2006	0.0000679	0.083460522	-117.76543
2007	0.0000657	0.083862529	-116.167435
2008	0.00006368	0.084296333	-114.796448
2009	0.00006071	0.08422912	-120.102816
2010	0.00005668	0.084978501	-118.690276
2011	0.00005207	0.085615879	-115.730908
2012	0.00005001	0.086198554	-116.274812
2013	0.0000495	0.08632268	-118.837929
2014	0.00005	0.08572163	-121.18732

Table 4. Parameter values for homogeneous model from age 40 and above (from 1922-2014)

Appendix B

Years	Initial mortality (m_0)	coefficient mortality(β)	Bayesian information criterion(BIC)
1922	0.000207	0.080583	-282.165
1926	0.000177	0.08152	-287.721
1930	0.000174	0.081261	-285.812
1934	0.000159	0.082369	-295.516
1938	0.000134	0.08424	-297.075
1942	0.000117	0.085484	-322.852
1946	0.000093	0.0884	-340.212
1950	0.000076	0.09103	-297.305
1954	0.000064	0.09244	-298.359
1958	0.000058	0.0933997	-286.492
1960	0.0000589	0.0933	-286.492
1962	0.000076	0.0889	-185.064
1966	0.000062	0.092063	-301.566
1970	0.000064	0.091228	-300.094
19974	0.00006	0.09224	-317.982
1978	0.000055	0.09232	-291.141
1982	0.000045	0.09422	-277.690
1986	0.00004	0.09538	-278.132
1990	0.00003	0.09622	-297.4823
1994	0.00003	0.0972	-326.509
1998	0.00003	0.0988	-387.696
2002	0.00002	0.0993	-365.357
2006	0.00002	0.0989	-318.908
2010	0.000012	0.09939	-277.939
2014	0.000017	0.0995	-253.255