Pattern formation in a chain of logical elements

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Abstract

Mechanisms of pattern formation have been studied in great details using continuous models, such as FitzHugh-Nagumo system [1]. However, many features of patterns observed in developmental biology suggest that they arise due to interactions of non-continuous nature between biological cells [2]. Here we present a model represented by a chain of logical elements and develop a formalism to consider pattern formation in this model. Different sets of operations reflecting interactions between logical elements can lead to formation of stationary, propagating and oscillating patterns similar to those observed in continuous dynamical systems.

Patterns in continuous systems

Nonlinear systems can exhibit phenomena which are hard to anticipate on the bases of common sense. One of the simplest models to describe such phenomena is represented by FitzHugh-Nagumo equations. By varying model parameters one can get various solutions representing stationary and oscillating patterns as well as propagating waves.

\[ \frac{dA}{dt} = f(A, v) \]
\[ \frac{dv}{dt} = D_A v + \varepsilon g(A, v) \]

Modified FitzHugh-Nagumo system [1].

\( f(A, v) \) and \( g(A, v) \) for reaction kinetics. \( A \) and \( D \) for diffusion.

Phase diagram for FitzHugh-Nagumo equations. Nullclines are represented by the straight and the cubic lines. The intersection of the nullclines is an equilibrium point. In the presented case the equilibrium point is unstable (the dynamic is oscillatory).

Stationary pattern in FHN model (concentration profile of variable \( u \) is shown)

Segmentation pattern in the fly embryo

The body of a fly embryo is made up of a repeated structure called segments which are arranged along the head to tail axis. These segments form at early stages of embryogenesis and associated with mechanisms of identifying the position and identity of differentiating cells [3].

The patterning takes place in four levels:

1. Nonuniform distribution of maternal genes (bicoid, caudal). These control the spatial patterns of transcription of the gap genes (i.e. hunchback, Krüppel, knirps, etc.).
2. Spatial patterns of transcription of the gap genes (Hb, Kr, etc.). The gap genes regulate each other and the next set of genes in the hierarchy, the pair-rule genes (even-skipped, hairy, etc.).
3. Pair-rule genes: form seven stripes of transcription around each embryo. Pair-rule genes determines the initial expression of segment polarity genes.

Illustration of the two-state Model

Time step: \( n = 0 \)

\[ e \]

The Transition is given by a set of 8 rules:

- Rule 0 (000) → (000)
- Rule 1 (001) → (101)
- Rule 2 (010) → (010)
- Rule 3 (011) → (111)
- Rule 4 (100) → (100)
- Rule 5 (101) → (101)
- Rule 6 (110) → (110)
- Rule 7 (111) → (111)

Therefore each transition set can be described by a binary number \( s_0s_1s_2s_3s_4s_5s_6s_7 \) which varies from 0 to 255 (2^8).

Other sets of rules resulting in nontrivial patterns

The transition 15 (000000111) results into fragmentation of triples (triple zero and triple one get an opposing symbol in the middle) and also when the states of two neighbours differ from each other the transition accepts the state of the right neighbour. As a result there appear waves moving to the left and causing a formation of a stationary pattern, which grow on the left side of the medium. Transition 85 is similar to the transition 15 except that it accepts the state of the left neighbour and therefore the waves propagate to the right. There are also other transitions resulting into combinations of stationary patterns with propagating waves. Two transitions 175 and 80 give left and right propagating waves respectively. There are many other transitions (84, 112, etc) resulting into formation of stationary patterns.

Biological implementation of the model

Here we presented our results in patterning due to local interactions of logical elements. Such interactions can represent contact (membrane-to-membrane) interactions between cells in biological tissues resulting into differentiation of cells. For example two-state model can represent chain of locally interacting cells, where cells in state "1" express some particular gene while in state "0" don't. Therefore the modelled interactions can be seen as regulating the differentiation of cells. The periodic stationary pattern forming in a two-state model represents a chain of cells where each second cell expresses the gene. These patterns can form under various interactions (transitions) from wide range of initial conditions. Three-state model doesn't have direct biological implementation, while the four-state model can be viewed as modelling cells whose differentiation is associated with expression of pair of genes. Four-periodic pattern (i.e. 0, 1, 2, 3, ... alternating black, dark grey, light grey and white) can correspond to the following alternation of gene expression:

References