

Mathematical modelling of mortality dynamics



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Abstract

The increase of mortality with age in human populations is approximated by exponent and satisfies the Gompertz equation. However, a deviation from exponential growth is observed in early and late life intervals. We study possible mechanisms to underlie these deviations. Considering a model composed by subpopulations with different mortality parameters, we analyse the effects of heterogeneity on mortality dynamics. We also analyse the influence of stochastic effects which are observed at early and late ages when only few individuals contribute to mortality. Comparisons of our results to real mortality data of US and Swedish populations indicate that the deviations from exponential law at young ages can be explained by the heterogeneity while those at old ages can be modelled as fluctuations and explained by stochasticity.

Introduction

Mortality Rate

If we denote by N_i the number of people who reached age i and ΔN_i the number of deaths of people aged i , then $\Delta N_i = N_i - N_{i+1}$.

The mortality rate m_i of age i is defined as the number of deaths divided by the average number of people of age i which in demography is called person-years:

$$m_i = \frac{\Delta N_i}{PY_i} \quad \text{where} \quad PY_i = N_i - (1 - a_i)\Delta N_i.$$

Parameter a_i indicates the age within one year interval in which the average number of deaths occurs. Observations in mortality for human populations indicate that $a_i \approx 0.5$ for all ages except age zero which is considerably smaller.

Gompertz Law of Mortality

The Gompertz function represents the exponential growth of mortality with age

$$m_i = m_0 e^{i/\tau},$$

where m_0 is the initial mortality and τ the mortality rate coefficient.

Size of Population

Using the definition of mortality rate and the Gompertz equation we express the number of individuals N_{i+1} in terms of N_i :

$$N_{i+1} = N_i - \Delta N_i = N_i - \frac{m_i N_i}{1 + 0.5m_i} = \left(\frac{1 - 0.5m_0 e^{i/\tau}}{1 + 0.5m_0 e^{i/\tau}} \right) N_i.$$

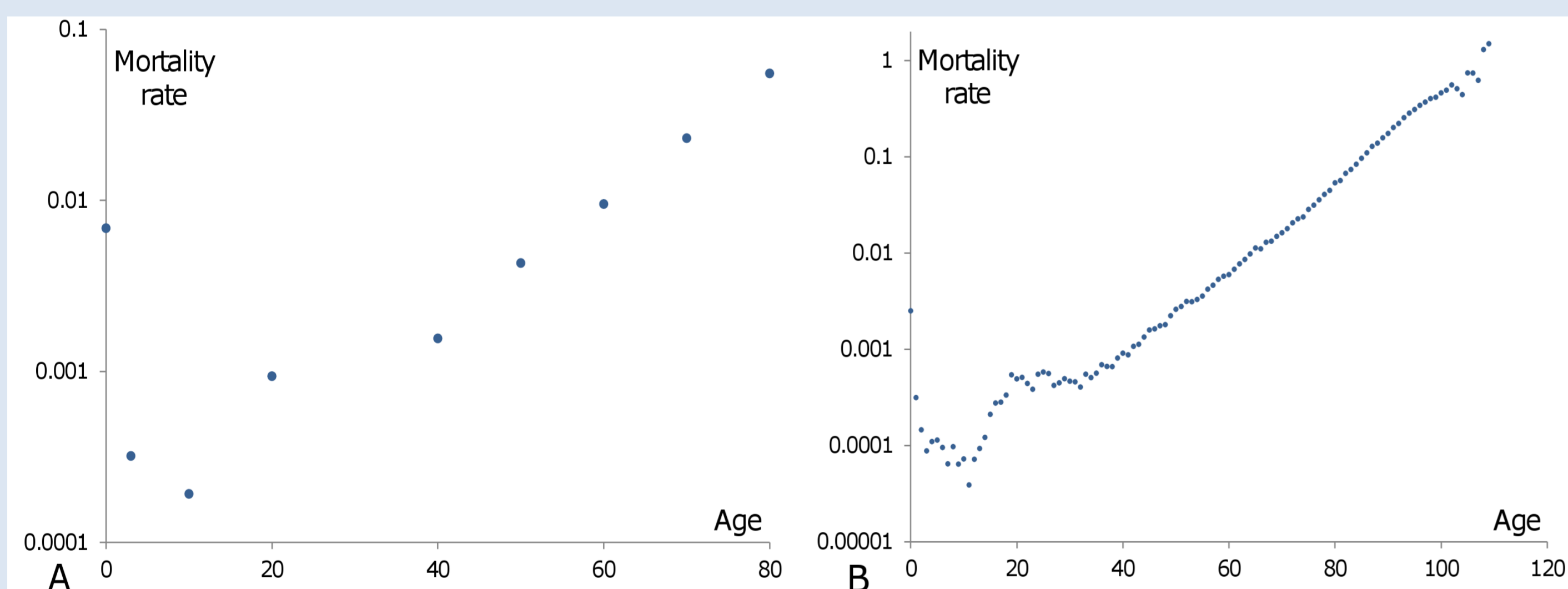


Figure 1: Mortality rate versus age (A) for the US population in the year 2002 (website of Centers for Disease Control and Prevention: <http://www.cdc.gov/nchs/deaths.htm>), and (B) for the Swedish population in the year 2007 (website of Human Mortality Database: <http://www.mortality.org>).

Modelling heterogeneity in mortality dynamics

Heterogeneity in human population means that the population consists of subpopulations with different mortality parameters. Therefore, according to the Gompertz law, we can express the mortality rate of the j -th subpopulation as

$$m_{ji} = m_{j0} e^{i/\tau_j}.$$

Then the mortality rate of the entire population is given by

$$m_i = \frac{\sum_{j=1}^n \frac{\rho_{ji} m_{j0} e^{i/\tau_j}}{1 + 0.5m_{j0} e^{i/\tau_j}}}{1 - 0.5 \sum_{j=1}^n \frac{\rho_{ji} m_{j0} e^{i/\tau_j}}{1 + 0.5m_{j0} e^{i/\tau_j}}},$$

where the fractions ρ_{ji} are defined as

$$\rho_{ji} = \frac{N_{ji}}{N_i} \quad \text{with} \quad \sum_{j=1}^n \rho_{ji} = 1.$$

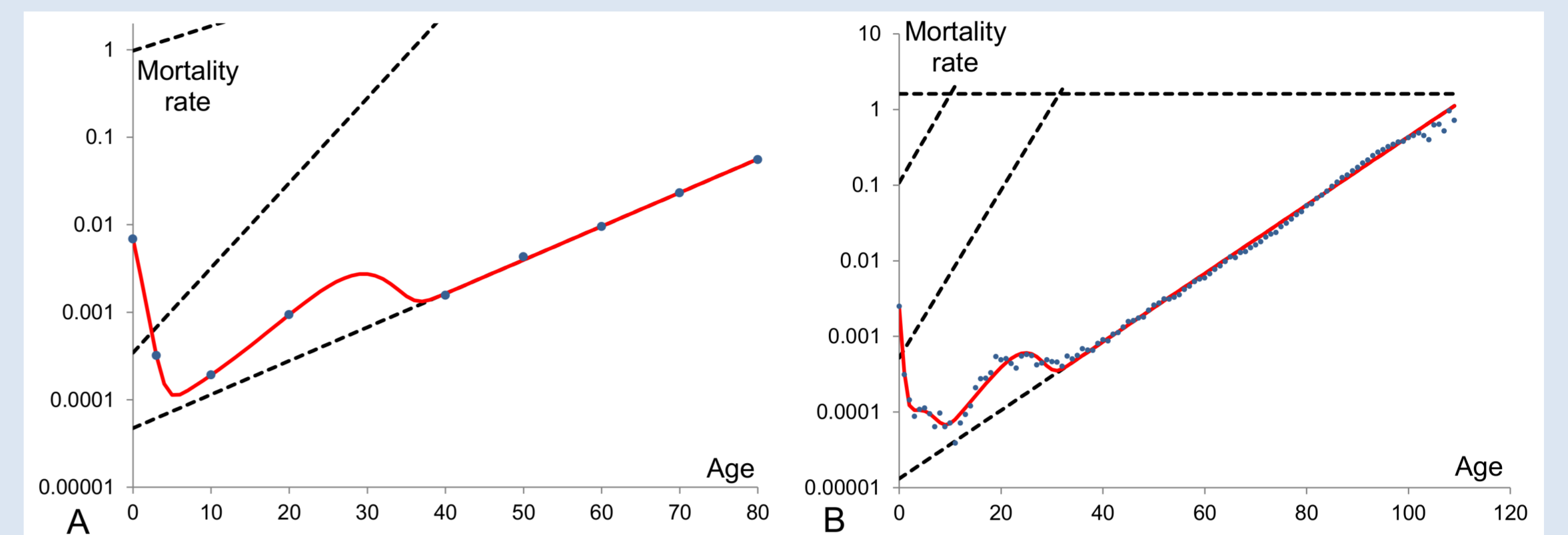


Figure 2: Fitting (A) the US data with the heterogeneous model composed by 3 subpopulations, and (B) the Swedish data with the heterogeneous model composed by 4 subpopulations.

Modelling stochasticity in mortality dynamics

If q_i is the probability of any individual aged i dying before reaching the age of $i + 1$, then the number of deaths of individuals who reached age i is $\Delta N_i = q_i N_i$. Using the definition of mortality rate we have that: $q_i = \frac{m_i}{1 + 0.5m_i}$.

To find the probability that k out of N_i individuals survive within a one-year interval we consider the following binomial expansion:

$$1 = [q_i + (1 - q_i)]^{N_i} = \sum_{k=0}^{N_i} \frac{N_i!}{k! (N_i - k)!} q_i^k (1 - q_i)^{N_i - k}.$$

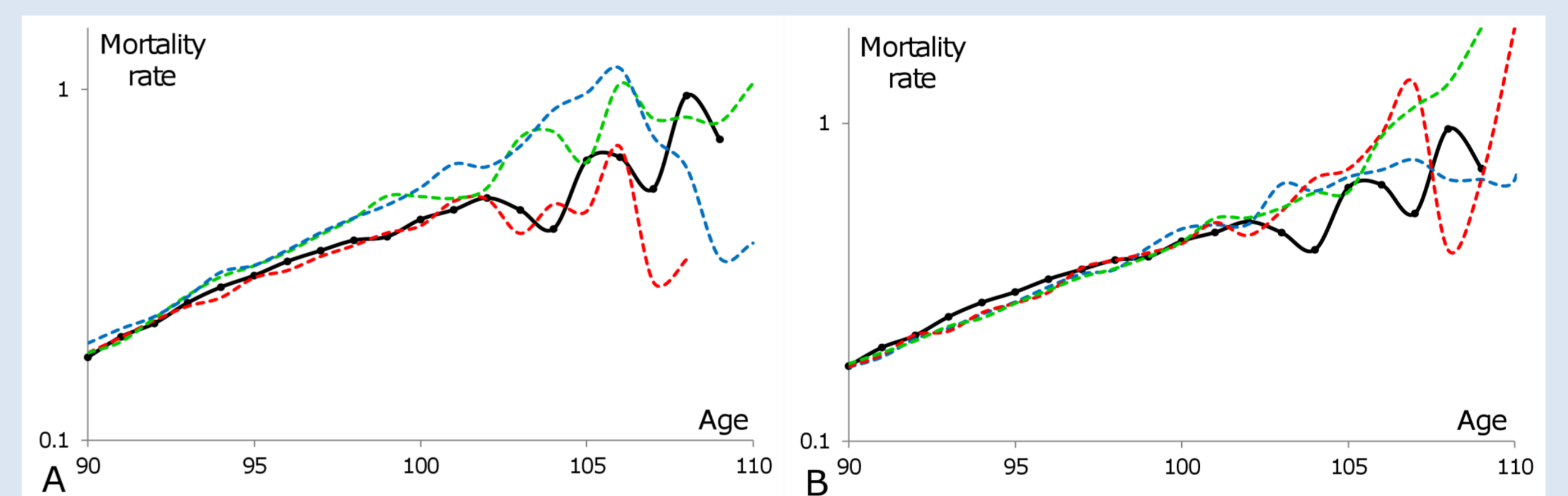


Figure 3: Advanced ages of Swedish population for (A) different real period data, and (B) different simulations of the stochastic model. Both panels show the deviation from exponential growth (deviations are presented by fluctuations).

Combination of heterogeneity and stochasticity

Mortality error Δm_i is defined as the standard deviation of the number of deaths divided by the mean number of person-years lived at age i :

$$\Delta m_i = \frac{\sigma}{\langle PY_i \rangle}, \quad \text{where} \quad \langle PY_i \rangle = N_i - 0.5N_i q_i \quad \text{and} \quad \sigma^2 = N_i q_i (1 - q_i).$$

Relative mortality error is defined as the mortality error divided by the mean mortality:

$$\frac{\Delta m_i}{m_i} = \sqrt{\frac{1 - 0.5m_i}{N_i m_i}}.$$

Fluctuations in mortality are observed when the relative mortality error is above some threshold.

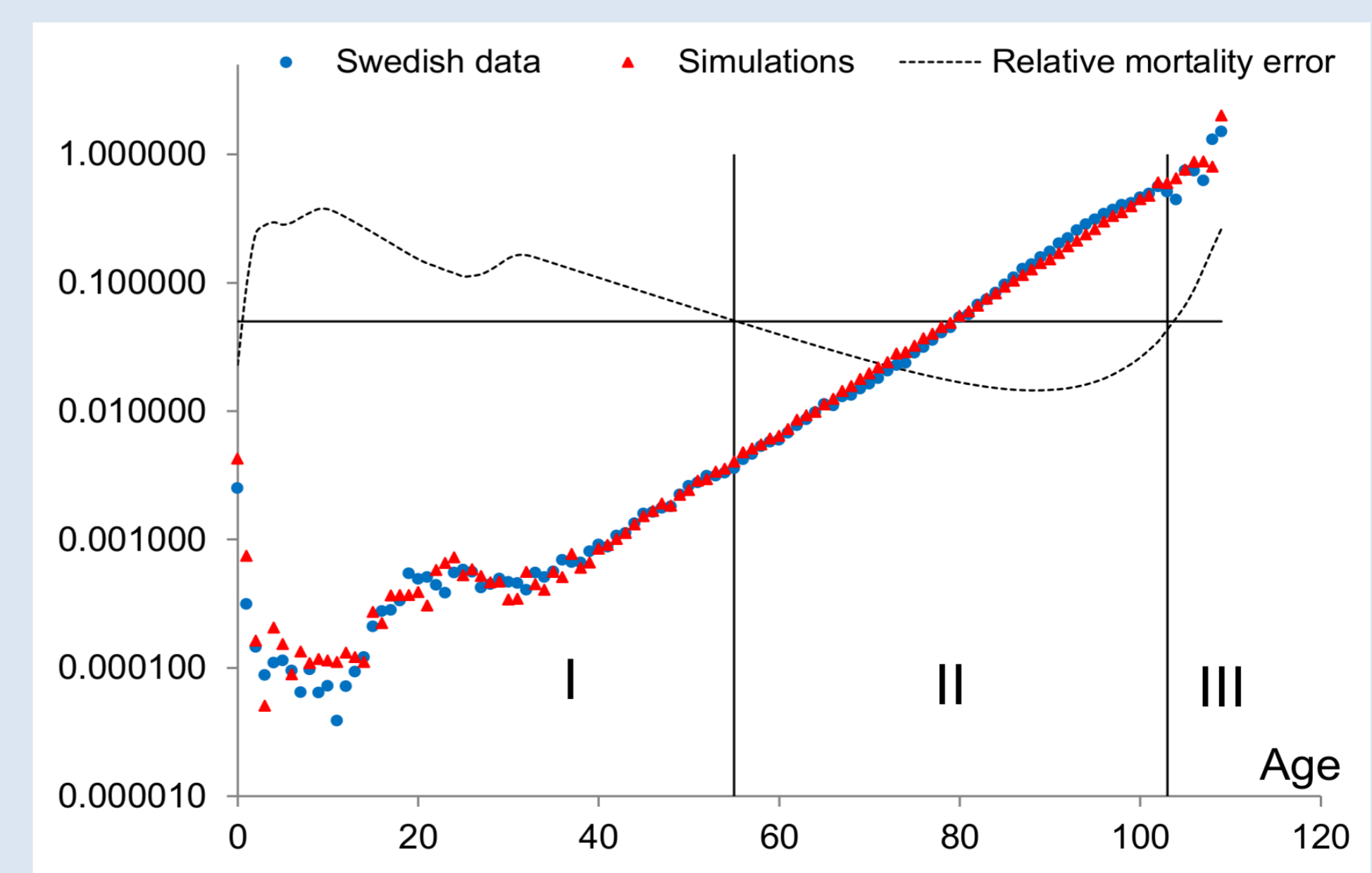


Figure 4: Fitting the mortality data of the Swedish heterogeneous population with stochastic simulations. The fluctuations are observed in sections I and III where the relative mortality error is above the threshold line, $Th = 0.05$.